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All Kinds of Mathematics Remind me of You
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- Homogeneity
  - Problem 1: Metrically Homogeneous Graphs
  - Problem 2: Twisted automorphisms; splitting and the PC-property
- Twisted automorphisms of metrically homogeneous graphs
  - Classification
  - Splitting
  - The PC-property

Homogeneity (Klein 1872, Cantor 1895, Urysohn 1924, Fraïssé 1953)

A metric geometry is homogeneous if every congruence on finite parts is induced by a global isometry.

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Examples

1. Some homogeneous graphs:
   - C_n for $n \leq 5$;
   - The random graph (Erdös-Rényi 1963)
2. Some metrically homogeneous graphs:
   - C_n for all n;
   - The random graph of diameter n (the integral Urysohn sphere).

1 Metrically Homogeneous Graphs

2 Twisted automorphisms

3 Let's twist again, twistin' time is here!

4 Splitting problems

Metrically homogeneous graphs

Problem (Moss/Cameron)

Classify the metrically homogeneous graphs

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Classify the metrically homogeneous graphs

Definition

A metrically homogeneous graph Γ is of generic type if

- The induced graph Γ_1 on a neighborhood is primitive; and
- The common neighbors of two points at distance 2 contain an infinite independent set.

Metrically homogeneous graphs

Problem (Moss/Cameron)

Classify the metrically homogeneous graphs

Theorem

The metrically homogeneous graphs of non-generic type are classified, and fall into the following categories.

- *Diameter* ≤ 2 *(classified by Lachlan and Woodrow);*
- *Finite (classified by Cameron);*
- *Tree-like (Dugald Macpherson)*

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Conjecture

The metrically homogeneous graphs of generic type are of the form $\Gamma^{\delta}_{K_1,K_2;C_0,C_1;\mathcal{S}}$ where δ is the diameter and ....

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Twisted automorphisms (Cameron and Tarzi, 2007)

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Examples

An isomorphism of C_5 with its complement; an isomorphism of C_7 as a metric space with distances $1, 2, 3$, to the twist by the cycle $(1, 2, 3)$.

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Language $L_k = \Gamma^k / \mathrm{Aut}(\Gamma)$

Remark (Cameron, Tarzi)

If Γ is homogeneous for the language L_k then the twisted automorphism group is

$$N_{\mathrm{Sym}(\Gamma)}(\mathrm{Aut}\,\Gamma)$$

$$1 \to \mathrm{Aut}(\Gamma) \to \mathrm{Aut}^*(\Gamma) \to \mathrm{Aut}(L/\Gamma) \to 1$$

Natural Questions

1. What is $\mathrm{Aut}(L/\Gamma)$?
2. When does Aut^* split over Aut?
3. Are all automorphisms of Aut given by inner automorphisms of Aut^*?

Definition

A permutation group (G, X) is strictly permutation-complete (PC^+) if $N_{\mathrm{Sym}(X)}(G)$ induces $\mathrm{Aut}(G)$ on G.

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Does Aut^* split, and is Aut^* PC^+?

Cameron, Tarzi: Splitting

Data:
$m = 1$: $\mathrm{Aut}(L/\Gamma) = (1)$, so it splits.
$m = 2$: $\mathrm{Aut}(L/\Gamma) = \mathbb{Z}/2\mathbb{Z}$ so we need a proper twisted automorphism of order 2—but then this would carry edges or non-edges (x, x^α) to themselves: non-split.

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Theorem (Cameron, Tarzi)

$\mathrm{Aut}^*(\Gamma^{(m)}_{random})$ *splits over* $\mathrm{Aut}(\Gamma)$ *iff m is odd.*

Non-splitting: as for $m = 2$, because there is an involution in $\mathrm{Aut}(L/\Gamma)$ without fixed points.
Splitting: more delicate.

PC^+

$\mathrm{Aut}(\Gamma)$ is a topological group (a closed subgroup of $\mathrm{Sym}(\Gamma)$ in fact) and the normalizer in $\mathrm{Sym}(\Gamma)$ acts continuously. So PC^+ implies automatic continuity:

All automorphisms of $\mathrm{Aut}(\Gamma)$ *are continuous.*

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Any homomorphism from $\mathrm{Aut}(\Gamma^{(m)}_{random})$ *to a separable Polish group is continuous.*

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Any homomorphism from $\mathrm{Aut}(\Gamma^{(m)}_{random})$ to a separable Polish group is continuous.

Theorem (Cameron, Tarzi)

All $\Gamma^{(m)}_{random}$ are PC, hence PC^+.

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4. Splitting problems

A context

Γ: homogeneous for a binary language with symmetric relations (self-paired orbits)

Again:

- When does Aut^* split?
- When is Aut PC?

Specifically: for the known metrically homogeneous graphs (mainly, of generic type)?

Twist of Metrically Homogeneous Graphs

Theorem (Rebecca Coulson)

The possible twists of a metrically homogeneous graph of generic type are the permutations $\rho, \rho^{-1}, \tau_0, \tau_1$, where τ_ϵ is the involution $(1, [\delta+\epsilon]-1)(3, [\delta+\epsilon]-3)\cdots$, and

$$\rho(i) = \begin{cases} 2i & i \le \delta/2 \\ 2(\delta - i) + 1 & i > \delta/2 \end{cases}$$

The map ρ is a twisted isomorphism between

$$\Gamma^{\delta}_{C=2\delta+2} \text{ and } \Gamma^{\delta}_{K_1=\delta}$$

The τ_ϵ can act as twisted automorphisms, notably when

$$K_1 = \left\lfloor \frac{\delta+\epsilon}{2} \right\rfloor \quad K_2 = \left\lceil \frac{\delta+\epsilon}{2} \right\rceil \quad C = 2(\delta+\epsilon)+1 \quad C' = C+1$$

Splitting

Theorem

For any known *metrically homogeneous graph of generic type and diameter δ which allows a twist by τ_ϵ the twisted automorphism group splits over the automorphism group iff*

$$\delta + \epsilon \not\equiv 3 \pmod 4$$

τ_ϵ has lots of fixed points

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For any known metrically homogeneous graph of generic type and diameter δ which allows a twist by τ_ϵ the twisted automorphism group splits over the automorphism group iff

$$\delta + \epsilon \not\equiv 3 \pmod 4$$

The splitting part may be stated more precisely.

Theorem

For any known metrically homogeneous graph of generic type and diameter δ which allows a twist by τ_ϵ, if $k, \delta + \epsilon - k$ are fixed points for τ_ϵ differing by at most 1, then there is a twisted automorphism α of order 2 affording τ_ϵ which satisfies

$$d(x, x^\alpha) \in \{k, \delta + \epsilon - k\} \text{ for all } x.$$

The PC-property

Obstruction: Central automorphisms

Theorem

Let Γ be a metrically homogeneous graph and $\alpha \in \mathrm{Aut}^*(\Gamma)$ a non-trivially twisted automorphism α of Γ inducing a non-trivial central automorphism of Γ. Then Γ is of antipodal type, bipartite, and of even diameter; if Γ is not an n-cycle, then α is $(1, \pi)$ or $(\pi, 1)$ with π the antipodal map.

But in the primitive case the situation is very similar to $\Gamma^{(m)}_{\text{random}}$.

The primitive case

Theorem(ish)

Let Γ be a known primitive metrically homogeneous graph of generic type.
Then $\mathrm{Aut}^*(\Gamma)$ induces the full automorphism group of $\mathrm{Aut}(\Gamma)$

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Ingredients as in Cameron and Tarzi—but we haven't actually said what they were ...

Ingredients

- Identify open subgroups (R. Coulson; Aranda, Bradley-Williams, Hubička, Karamanlis, Kompatscher, Konečný, Pawliuk 2017) ;
- Identify setwise stabilizers (Cameron 2005)
- Identify vertex stabilizers (Method of Cameron/Tarzi 2007)

or in terms of methodology

- Finiteness of forbidden partial substructures;
- Strong primitivity;
- Double cosets of $G_{\{A\}}$ count isomorphism types of pairs (A_1, A_2) with $A_i \simeq A$; for vertex stabilizers there are $\delta + 1$ such.

And just one more thing

Problem (Cameron 2002)

Classify the homogeneous structures for a language with finitely many linear relations.

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Classify the homogeneous structures for a language with finitely many linear relations.

Conjecture (Sam Braunfeld, Rutgers, 2017)

Built generically from "sub-quotient orders," over a generalized ultrametric space with values in a distributive lattice. In particular there are only finitely many for a specified finite language, and they have the Ramsey property.

(Also, the isometry group of the g.u.m. has metrizable minimal flow when the lattice is distributive.)

Theorem(ish)

True for 3 orders.

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Problems

The splitting problem for symmetric binary languages.

Problems

- When do involutions in $\mathrm{Aut}(L/\Gamma)$ lift to involutions in $\mathrm{Aut}^*(\Gamma)$?
- If all involutions in $\mathrm{Aut}(L/\Gamma)$ lift to involutions, does Aut^* split?