Generix Begins: Punchlines and Milkshakes (the Weak, the Strong, and the Weyl)

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Topics

- The Algebraicity Problem, in its 4 Flavors
- Weak and Strong embedding
  - Mixed type: strong embedding as a goal
  - Even type: weak embedding as a hypothesis
  - Minimal odd type: bounding the Prüfer rank (3 strategies)
- Torsion and the Weyl group

To be illustrated by Jaligot's thesis and various approaches to minimal connected simple groups
(Jaligot et al 2004, 2007, Altınel–Burdges–Frécon 2013).
A fuller account will be posted on my webpage.

Genesis: b'reshit

Morley (Morley rank) — A countable theory T is categorical in one uncountable power if and only if it is categorical in all uncountable powers.

Marsh, Baldwin/Lachlan (Strong minimality) — dimension theory

Zilber (Groups) — An uncountably categorical but not almost strongly minimal structure involves an infinite definable group of finite Morley rank, either abelian or simple

- And in the simple case, the group itself is almost strongly minimal
- Perhaps even algebraic (Chevalley group over an algebraically closed field)

The Scottish Reformation and the Russian Orthodox view

Theorem (Macintyre)

An infinite $\aleph_0$-stable field is algebraically closed.

I viewed the Algebraicity Problem as a non-commutative version of this.

Borovik proposed to treat this seriously as an analog of **CFSG** (classification of the finite simple groups).

Tame K^*-groups

K-group (Known type) definable simple sections are algebraic;

K^*-group proper definable sections are K-groups;

Tame group No bad fields involved ("pieces" of the multiplicative group)

Philosophy Take the analysis of tame K^*-groups, using ideas of finite group theory, and season with ideas of model theory. If it works, perhaps take out the tameness and look again.

Reference: Axe Soup (La pierre à faire de la soupe)

Vanilla, Chocolate, Stracciatella, or Azuki?

p-Sylow theory in algebraic groups up to finite index:

Char.	Type	Algebraic properties	Model theory
$= p$	unipotent	bounded exponent, nilpotent	definable
$\neq p$	semisimple (toral)	divisible, dense in maximal torus	not definable

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Theorem (Borovik-Poizat, in finite Morley rank)

The connected 2-Sylow subgroup S° is a central product

$$U * T$$

where U is 2-unipotent and T is abelian, divisible.

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Structure	*Type*	*Properties*
Just U	**Even**	bounded exponent, nilpotent, definable
Just T	**Odd**	divisible, abelian, not definable
Both	**Mixed**	Mixed
(1)	**Degenerate**	Trivial

Early Days

Borovik Odd type, locally finite, tame

Altinel Even type, tame K^* with strong embedding

ABC Tame, K^*, and

- mixed type; or
- even type, with weakly embedded subgroup

Strong and weak embedding

$M < G$ (containing some involution)

Strong M contains the normalizer of each nontrivial 2-subgroup of M;

Weak M contains the normalizer of each nontrivial connected subgroup of M

These resist ordinary analysis.

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Criteria

Strong $C(i) \subseteq M$

Weak $N(U), N(T) \subseteq M$
(2-unipotent, 2-torus)

MIXED TYPE: PLAN OF ATTACK

A general strategy:

- Find a weakly embedded subgroup
- Show it is strongly embedded
- Identification or Contradiction

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Possible Punchline: Thompson Rank Formula

- $i \in I_u$, $j \in I_t$ goes to $k \in d(\langle ij \rangle)$ $(k \in C(i, j))$
- $\mathrm{rk}\,(\mathrm{I_u}) + \mathrm{rk}\,(\mathrm{I_t}) = \mathrm{rk}\,$(all k) $+ \mathrm{f}$ (fiber rank)
- $(g - c_u) + (g - c_t) = (g - c_k) + f$ (maybe)
- $g = c_u + c_t - c_k + f$: can it be computed?

—Local data about centralizers
—Expect nonsense answer if G does not exist.

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Never reached stage 2

Toward Mixed Type: $\mathcal{U}(G)$, $B(G)$, $D(G)$

$B(G)$ generated by all U, $D(G)$ generated by all T.

What we expect:

$B(G)$, $D(G)$ should be normal subgroups which commute!

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$\mathcal{U}(G)$ is the graph of all nontrivial unipotent subgroups, edges when they commute.

Fact. $D(C(U))$ is constant on connected components.

Corollary

$\mathcal{U}(G)$ is disconnected.

Let $\mathcal{U}_0(G)$ be a connected component, M its stabilizer. Now what?

Weak Embedding

$\mathcal{U}(G)$: Unipotent groups linked when commuting, disconnected.
M: stabilizer of $\mathcal{U}_0(G)$.

Fact

M is definable, its unipotent subgroups belong to $\mathcal{U}_0(G)$, and M contains their normalizers.

Theorem (Jaligot, 3.22, 3.23)

M is weakly embedded (3.22) and indeed strongly embedded (3.23).

This will end the proof. Strong embedding makes all involutions conjugate but involutions in $U \setminus T$ are not conjugate to involutions in T.

Proof of Weak Embedding

Conguration: $T_0 \le T$, $N(T_0) \not\le M$. K^*-setting:

$Q = B(N(T_0)) \simeq \mathrm{SL}_2$, meeting M in a Borel subgroup ($N_Q(U)$).
So U is elementary abelian and all involutions of U are conjugate.
To kill this: We move toward I_u commutes with I_t
Notation: $i \in U$, $j \in I_t$, not commuting.

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Lemma (Taking Stock)

- $C(i) = C(U)$
- $d(ij)$ *contains a unique involution* k

Knock-Out: Wind-up

$$S = U * T \qquad i \in U \qquad j \in T_j \qquad [i,j] \neq 1 \qquad k \in d(ij)$$

Lemma

(1) $BC(k) \simeq \mathrm{SL}_2$

(2) $i \in BC(k)$ (3) $j \notin BC(k)$ (4) $jk \notin BC(k)$

Proof.

1. j acts on $BC(k)$ and moves U (as i, j do not commute)
2. $i \in U$ 3. j is not conjugate to i
4. $jk \in BC(k) \implies [j, U_{jk}] = 1 \implies [T_j, U_{jk}]] = 1$

$\implies [T_j, k] = 1 \implies [T_j, BC(k)] = 1 \implies [j, i] = 1$ #

□

Punch

Hypothesis	Conclusion
$k \in d(ij)$ unique	
$BC(k) \simeq \mathrm{SL}_2$	$jk \in BC(k)$
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Punch

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Proof.

$$(ij)^2 \in BC(k)$$

$$ij \equiv y \pmod{B} C(k) \cap d(ij) \qquad \text{(2-element)}$$

$$o(y) = 2 \qquad (ij \notin BC(k),\ k \notin BC(k))$$

$$y = k$$

$$ijk \in BC(k)$$

□

Strong Embedding

From weak to strong

Offending involution: $C(\alpha) \not\leq M$.
$C(\alpha)^\circ = \mathrm{SL}_2 \times H$ and H has no involution.

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Offending involution: $C(\alpha) \not\leq M$.
$C(\alpha)^\circ = \mathrm{SL}_2 \times H$ and H has no involution.
Arrive at a similar configuration,
but k becomes a pair of involutions.

Hyp. 1	Hyp. 2	Conclusion
$k \in d(ij)$ unique	$k' \in d(ij)$ unique	
$BC(k) \simeq \mathrm{SL}_2$		jk resp. jk' $\in BC(k)$
$i \in BC(k)$		
$j \notin BC(k)$	$j, k' \notin BC(k)$	

A weak embedding theorem

Theorem (Jaligot 4.1)

G finite Morley rank, even type, K^*, with a weakly embedded subgroup. Then $G \simeq \mathrm{SL}_2$ (char. 2).

- Fundamental for even type
- Harder than the mixed type case
- Much calculation (modeled on Nesin's mad computations)
- Generalized further after Tuna's habilitation

1. The Algebraicity Problem (Background)

2. Jaligot's Thesis (1999)
   Mixed and even type

3. Strong embedding in odd type
   Minimal Connected Simple

Bounding the Prüfer rank

Recall that in the aftermath of Jaligot's thesis and Altınel's habilitation, mixed and even type were disposed of completely. In the meantime Jaligot was looking at minimal connected simple groups of the remaining types.
References: [CJ04], [BCJ07], [ABF13] (3 strategies)

Theorem (Bound on Prüfer rank)

Let G be a minimal connected simple group of odd type. Then the Prüfer rank of G is at most 2.

1. If the Prüfer 2-rank is greater than 2 get a strongly embedded subgroup.
2. From a strongly embedded subgroup get Prüfer rank at most 1

The brief version: the strongly embedded case is key, and leads to a close consideration of the *Weyl group*.

1st strategy: the tame case

Let T be the definable hull of a maximal 2-torus. In the tame case this turns out to be a product of several copies of the multiplicative group of a field, so has constant p-rank for primes p other than the characteristic.

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Let T be the definable hull of a maximal 2-torus. In the tame case this turns out to be a product of several copies of the multiplicative group of a field, so has constant p-rank for primes p other than the characteristic.
The Weyl group $W = N(T)/C(T)$ operates regularly on the involutions of T, and semi-regularly on the nontrivial p-torsion for other primes p. So

$$|W| = 2^n - 1 | p^n - 1$$

(n=Prüfer 2-rank) and by number theory (Feit, big Zsigmondy primes) one comes down to $n = 1, 2, 4, 6, 12$ and one eliminates $n = 4, 6, 12$ via a closer look at W.

2nd Strategy: The milkshake email

```
%From jaligot@logique.jussieu.fr Mon Feb 16 12:39:54 2004
%Received: from shiva.jussieu.fr (shiva.jussieu.fr [134.157.0.129])
% by math.rutgers.edu (8.11.7p1+Sun/8.8.8) with ESMTP id i1GHdr219460
% for <cherlin@math.rutgers.edu>; Mon, 16 Feb 2004 12:39:54 -0500 (EST)
%Received: from mailhost.logique.jussieu.fr (turing.logique.jussieu.fr [134.157.19.1]
%          by shiva.jussieu.fr (8.12.10/jtpda-5.4) with ESMTP id i1GHdrG3021595
%          for <cherlin@math.rutgers.edu>; Mon, 16 Feb 2004 18:39:53 +0100 (CET)
%X-Ids: 166
%Received: from turing.logique.jussieu.fr (turing.logique.jussieu.fr [134.157.19.1])
% by mailhost.logique.jussieu.fr (Postfix) with ESMTP id 323A122EE82
% for <cherlin@math.rutgers.edu>; Mon, 16 Feb 2004 12:39:53 -0500 (EST)
%Date: Mon, 16 Feb 2004 18:39:53 +0100 (CET)
%From: Jaligot Eric <jaligot@logique.jussieu.fr>
%To: cherlin@math.rutgers.edu
%Subject: Milkshake
%Message-ID: <Pine.LNX.4.53.0402161831140.10315@turing.logique.jussieu.fr>
%MIME-Version: 1.0
%Content-Type: TEXT/PLAIN; charset=US-ASCII
%X-Miltered: at shiva.jussieu.fr with ID 40310069.000 by Joe's j-chkmail (http://j-ch
%X-Antivirus: scanned by sophie at shiva.jussieu.fr
%Status: RO
%X-Status:
%X-Keywords:
%
%I made a slight milkshake with your new argument
%and our paper to study the nontame case, with the
%standard Borel nilpotent.
%I think this gives the final bound on the Prufer
%rank, as apparently we don't need to understand
%Borels entirely.
%I did not look at the other case of Prufer rank 2
```

The milkshake

On minimal simple groups

The arguments that follow are intended to eliminate odd order cyclic Weyl groups for nilpotent Borel subgroups; it is not clear if the nilpotence is actually needed, but in general our "B" would have to be replaced by "$C(i)$", at least.
The objective is to eliminate number theory from high Prüfer rank, and also to dispose of at least one, possibly both, of the Prüfer rank 2 cases.

Notations $I = I(G)$. $i \in I$, fixed. $B = C(i)$ is a Borel subgroup (standard, nilpotent). $A = \Omega_1(S)$. $N(B)/B$ is nontrivial and of odd order, and most simply thought of as cyclic of prime order.
Furthermore: **1.** $g = \mathrm{rk}(G)$, $c = \mathrm{rk}(B)$, $c' = \mathrm{rk}(C(\sigma))$ for $\sigma \in N(B) \setminus B$. This is constant, but in any case the generic value along any one coset would be sufficient.

2. $J = \{j \in I : \text{There is } \sigma \in N(B)^{\times}, j \text{ inverts } \sigma\}$.

Facts used:
$\mathrm{rk}(I) = g - c$
The strongly real elements of B are in A.
For $\sigma \in N(B) \setminus B$, $C_B(\sigma) = 1$.
Conjugates of B are disjoint.
For $a \neq 1$ strongly real, $C(a) = C^{\circ}(a)$ is inverted by any involution inverting a.
The elements of $N(B) \setminus B$ are strongly real (this is proved again along the way anyway).

Lemma 1 BI is generic in G.

Lemma 2 J is generic in I.

Lemma 3 $\mathrm{rk}(I) = c + c'$
Now fix $\sigma \in N(B) \setminus B$.

Lemma 4 $BC(\sigma)B$ is generic in G.

Lemma 5 $B[C(\sigma)^{\times} B$ is disjoint from BI.
In particular, we have two disjoint generic subsets of G, and a contradiction.

Proofs

Milkshake: Summary

Treats the case where the involutions are central in a Borel subgroup B with $N(B)$ strongly embedded

Makes two disjoint generic sets

- $B \cdot I$ with I the set of involutions;
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When involutions are not central: bring in Burdges' Bender method and push. (Something similar recurs in Deloro-Jaligot, I believe.)

The 3rd generation

This presupposes the milkshake but handles the other, more obscure half, transparently.

Theorem ([ABF13])

If G is minimal connected simple of odd type, B a nonnilpotent Borel, with $N(B)$ strongly embedded, and involutions of B noncentral, then the Prüfer rank is 1.

This depends partly on [BCJ07], but not when the Weyl group has odd order.

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$$W = N(T)/C(T) = N(Q)/Q$$

T a maximal torus with dense torsion, Q a Carter subgroup. In the strongly embedded setting $W = N(B)/B$ has odd order.

So the main theorem of ABF13 makes W trivial. But W also acts transitively on the involutions in S°. This kills the Prüfer 2-rank.

Generix abides

Genericity arguments: Generic conjugacy (Carter), milkshake (more ad hoc)
The subject remains central to the classification project.

Carter subgroups (Frécon) sufficient for the purposes of classification, feeds back into the Weyl group, (ABCDEF) with other genericity arguments.

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Carter subgroups (Frécon) sufficient for the purposes of classification, feeds back into the Weyl group, (ABCDEF) with other genericity arguments.
The Weyl group remains an active area of investigation, and a powerful tool in the classification enterprise.

The main focus of Éric's own work in recent years with Deloro was an ambitious generalization of the minimal simple case.