CLASSIFYING HOMOGENEOUS STRUCTURES II

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Abstract. On homogeneous ordered graphs

1. The result

Problem 1 (Nguyen van Thé, 2012). *What are the homogeneous ordered graphs?*

The motivation for the question comes from structural Ramsey theory. There is a general conjecture that any amalgamation class for a finite relational language has a natural expansion to a Ramsey class in another finite language. Investigation of the problem is hampered by the absence of nontrivial examples.

From that point of view, systematic classification is a way of picking up the less obvious examples, and one is mainly interested in the list of "sporadic" examples.

As it turns out, there are no sporadic examples in this case, and the classification theorem lists the examples previously considered in the Ramsey theoretic literature, and says that the list is exhaustive.

Given the variety of the list, it is surprising that there is nothing to be added.

Theorem 1. *Let Γ be a homogeneous ordered graph. Then Γ is one of the following.*

- *A homogeneous partial order with strong amalgamation together with a generic linear extension of the order or its reversal (EPO, EPO^c);*
- *An infinite homogeneous tournament together with a generic linear ordering;*
- *A homogeneous graph with strong amalgamation together with a generic linear ordering.*

There are several points to be explained.

- Why are these ordered graphs?
- Strong amalgamation
- What is a generic linear order or a generic linear extension?

We begin with the second and third points.

A strong amalgamation class is one in which all amalgamations can be completed without additional identification of vertices. Equivalently, the algebraic closure operation is trivial. In particular all equivalence classes of nontrivial 0-definable relations are infinite.

When one has a strong amalgamation class the class of all expansions by linear orders is also a strong amalgamation class and its Fraïssé limit is said to be generically ordered. One may do the same with partial orders and linear extensions.

Now, why the first two types are in fact ordered graphs? If one has a partial order with a linear extension $(P, \prec, <)$, one considers the ordered comparability graph $(P, E, <)$. This is the same structure presented with a slightly different language.

If one has a linearly ordered tournament $(\Gamma, \to, <)$ then one considers the ordered graph $(\Gamma, E, <)$ where the edge relation Exy is

$$x \to y \iff x < y$$

Then graph complementation is the same as reversal of either $\to$ or $<$.

One part of the theorem is covered by a result of Dolinka and Mašulović: namely, the case in which Γ arises as some linear extension of a partial order.

One may ask whether there is some very general result concerning the classification of ordered homogeneous structures in terms of homogeneous structures for simpler languages. This seems unlikely in general: it would include a classification of all homogeneous structures in a language with finitely many linear orderings. Probably one should restrict to the primitive case. The problem remains open in that setting but there is no known obstruction to a general result.

One may make the result completely explicit: the homogeneous partial orders, tournaments, and graphs are known [Sch79, Lac84, LW80].

2. Structure of the proof

One can differentiate the various cases which arise according to the allowed substructures of small order (typically order 2 or 3).

We follow the following scheme.

- Omit $\vec{I}_2$ or $\vec{K}_2$: essentially a linear order.
- Embed $\vec{I}_2$ and $\vec{K}_2$ but omit at least one ordered form of $\vec{C}_3$ (a path of length 3, or its complement): up to complementation, this gives linear extensions of partial orders, which are either generic linear extensions of homogeneous partial orders, or generic permutations (with the partial order being the intersection of the two linear orders) [DM12].
- Embed both ordered forms of $\vec{C}_3$. Here the target is generic linear extensions of infinite homogeneous tournaments or homogeneous graphs of Henson type.

The first case is trivial, and the second was treated in [DM12], so we come to the third case. Our first problem is to separate the tournaments from the graphs.

But there are only two tournaments involved, namely $S(2)$ and the generic tournament, and here the generic tournament is equivalent to the random graph. So we just have to isolate the case that corresponds to generic extensions of $S(2)$.

Now $S(2)$ is characterized by omission of $\vec{I}\vec{C}_3$ and $\vec{C}_3\vec{I}$. There are 16 ordered forms of these tournaments. Our catalog suggests that the two main cases should be

- The underlying tournament is a local order: and it is $S(2)$ with a generic ordering;
- All ordered forms of $\vec{I}\vec{C}_3$ and $\vec{C}_3\vec{I}$ embed: and when viewed as a graph it is a Henson graph with generic ordering.

But to get a clean case division we must pick one particular ordered tournament of order 4, and we will focus on $\vec{C}_3^+ \to 1$. So our main claims become the following.

- If the underlying tournament is a local order, then all ordered local orders embed.
- If the configuration $\vec{C}_3^+ \to 1$ is omitted, then the underlying tournament is a local order (16 forbidden configurations).
- If the configuration $\vec{C}_3^+ \to 1$ is realized, then all 16 variants are realized.
- If Γ contains $\mathcal{A}(n) =$
$$\{\vec{I}_1 \perp \vec{P}_3, \vec{P}_3^c, \vec{K}_n\} \cup \{\vec{I}_k \mid k < \infty\}$$
then Γ contains every $\vec{K}_{n+1}$-free ordered graph.

The first three reduce quickly to a finite number of explicit claims about individual finite structures (the first reduces to the case of order 4).

The problem is to reduce the last claim to similarly concrete statements. Here we use Lachlan's idea.

In particular we work in an expanded category of ordered 2-graphs.

Definition 2.1. An ordered 2-graph $\mathbb{H} = (H_1, H_2)$ is *ample* if its second component contains $\vec{I}_1 \perp P_3$, $\vec{P}_3^c$, and all $\vec{I}_k$ for $k < \infty$, and if the first component realizes all initial 1-types (b, I) of the following form:

$I \cong I_k$ is a finite independent subset of H_2, and one of the following holds:
(i) $b \perp I$; (ii) $b \sim I$; or (iii) $|I| = 2$.

(This forces all 1-types $(b, \vec{I}_k)$ over an independent set to be realized.)

We formulate nine statements that we will prove by simultaneous induction on the parameter n, where $n \geq 2$. In these statements, we assume that

> Γ is a homogeneous ordered graph containing all the configurations in $\mathcal{A}(2)$;
> $\mathbb{H}$ is an ample homogeneous ordered 2-graph such that all configurations in $\mathcal{A}(n)$ embed into the the second component H_2 (where n is a parameter occurring in the statement of the proposition).

Configurations denoted A or B are assumed to be finite.

Propositions (I)–(IX)

(I) If $a \in \Gamma$ then the ordered 2-graph $(a^{\perp -}, a^{\perp +})$ is ample.

(II_n) If all elements of $\mathcal{A}(n)$ embed in Γ, and $B = baK$ satisfies
- $K \cong \vec{K}_n$
- $b < a < K$
- $a \perp bK$
- B does not contain $\vec{K}_{n+1}$

then B embeds in Γ.

(III) If $p = (x, A)$ is an $\mathbb{H}$-constrained initial 1-type with $A \in \mathcal{A}(2)$, then p is realized in $\mathbb{H}$.

(IV_n) If $A \in \mathcal{A}(n)$ and $p = (x, A)$ is an initial 1-type over A which is realized in $\mathbb{H}$ with $x \in H_1$, $A \subseteq H_2$, then the ordered 2-graphs $(A^p, A^{\perp -})$ and $(A^p, A^{\perp +})$ are ample.

$\star$ (V$_n$) If $p = (x, \vec{K}_\ell \perp \vec{K}_m)$ is an $\mathbb{H}$-constrained initial 1-type, then p is realized in $\mathbb{H}$.

(VI$_n$) If $p = (x, A)$ is an $\mathbb{H}$-constrained initial 1-type with $A \in \perp \mathcal{A}(n)$, then p is realized in $\mathbb{H}$.

(VII$_n$) Suppose that Γ contains every configuration in $\mathcal{A}(n)$. If $B = A \cup \{b\}$ does not contain $\vec{K}_{n+1}$, and $b < A$, with $A \in \perp \mathcal{A}(n)$, then Γ contains B.

(VIII$_n$) Suppose Γ contains every configuration in $\mathcal{A}(n)$. If A does not contain $\vec{K}_{n+1}$ then A embeds into Γ.

(IX$_n$) If $p = (x, A)$ is an $\mathbb{H}$-constrained initial 1-type and A does not contain $\vec{K}_{n+1}$, then p embeds into $\mathbb{H}$.

Everything comes down to (V_n), and eventually to the case of a type $P{+}P$ over $\vec{K}_n \perp \vec{K}_n$. We let $p = P \restriction \min \vec{K}$.

One then prepares three lemmas.

Lemma. *There is a Ramsey 2-type for* $\mathbb{H}^P = (K^P, K^{\perp +})$ *over* $\mathcal{A}(n-1)$.

Lemma (Cross-type q)**.** *There is an initial cross type q with the following property.*

Assume $C = K \perp A \perp B$, $y = \min K$, and $a = \min A$ satisfy

$$K \cong \vec{K}_n,\ a \to A \setminus \{a\}, \text{ and } A \text{ omits } \vec{K}_{n+1},\ B \text{ omits } \vec{K}_n$$

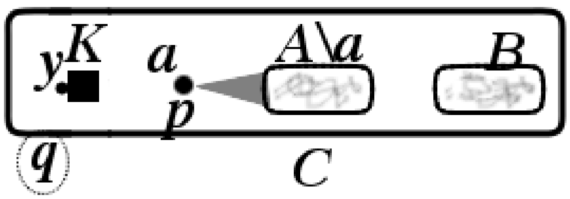
and $Q_1 \perp Q_2 \perp Q_3$ is a 1-type over C with

$$Q_1 \restriction y = q,\ Q_2 \restriction a = p$$

with Q_1, Q_2, Q_3 realized in $\mathbb{H}$.

Then

$$Q_1 \perp Q_2 \perp Q_3 \text{ is realized in } \mathbb{H}$$



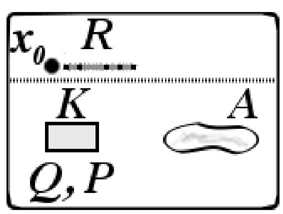
Lemma. *There is a 1-type Q over $\vec{K}_n$ whose restriction to $a = \min \vec{K}_n$ is q, with the following property.*

- For any finite configuration (R, A) realized in $\mathbb{H}$ such that R is r-Ramsey and A omits $\vec{K}_n$, if $x_0 = \min R$, then $\mathbb{H}$ contains the configuration

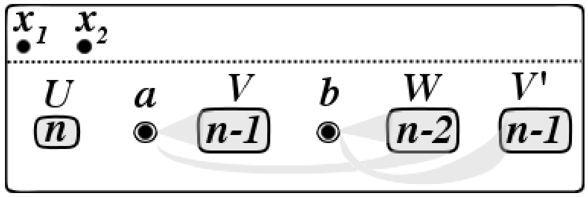
$$(R, K \perp A)$$

where (R, A) is as given, $K \cong \vec{K}_n$, and

$$\mathrm{tp}(x_0/K) = Q$$
$$\mathrm{tp}(x/K) = P \text{ for } x \in R,\ x > x_0$$



And now we amalgamate as follows, with x_1, x_2 Ramsey over $\mathcal{A}(n-1)$.



(*)

$x_1/U, aV, bV' = Q, P, P$
$x_2/U, abW = P, P$

Open Problems

- Homogeneous structures with k linear orders, $k \geq 3$.
- Homogeneous graphs and directed graphs in a language with additional unary predicates
- Homogeneous structures with 2 asymmetric edge relations
- Homogeneous structures with 3 symmetric edge relations

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