CLASSIFYING HOMOGENEOUS STRUCTURES, I

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ABSTRACT. The first of three lectures on some current problems in the classification of homogeneous structures.

1. Introduction

3 lectures on problems involving the classification of homogeneous structures in the sense of Fraïssé:

- some finite structures;
- homogeneous ordered graphs
- metrically homogeneous graphs

Recall *homogeneity.*

The first classification theorem is Fraïssé's characterization in terms of amalgamation classes. We want something more explicit.

Context

- model theory
- combinatorics (Ramsey theory): any class of finite structures with the Ramsey property
$$\mathcal{C} \to (\mathcal{B})^{\mathcal{A}}_k$$
has the amalgamation property (else, color $\mathcal{A}$ by whether it embeds into a given extension in $\mathcal{B}$)
- (KPT2004) Topological dynamics: extremely amenable groups (requires an order) or groups with universal minimal flow metrizable (e.g., the space of all possible orderings).

Of special interest nowadays: ordered structures, metric spaces (Nešetřil: finite metric spaces have the Ramsey property).

My interest in the finite case is much older. I'll say something about this first.

1.1. Homogeneous finite graphs (Sheehan 1974, Gardiner 1976).

- C_5, $E(K_{3,3})$
- $m \cdot K_n$ and its complement

(Can be proved by induction.)

Primitive ones: $C_3, C_5, E(K_{3,3})$

Lachlan: For a fixed finite relational language, there are finitely many homogeneous structures such that the finite ones are the ones that embed in a way that respects the automorphism groups.

E.g., $m \cdot K_n$ reflects $\infty \cdot K_\infty$.

More generally, the stable structures are the downward closure in this sense, and are obtained by adjusting the sizes of certain equivalence classes.

(CFSG)

1.2. **Binary Conjecture.** S_n**,** C_p**,** $AO(q)$ **(anisotropic).** *Reduced to the almost simple case, Cherlin/Wiscons*

Theorem 1. *Finite, affine, binary, primitive implies C_p, D_p, or $AO_2^-(q)$.*

How to visualize $O_2^-(q)$: $\mathbb{F} = \mathbb{F}_{q^2}$, $N = N_{\mathbb{F}/\mathbb{F}_1}$, $G = \ker N \cdot \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$.

Rough Idea of the proof.

AG should be solvable, so aim first to show that this is the case, then that this can be represented as a 1-dimensional semilinear group, and finally that it actually is $AO_2^-(q)$ in nontrivial cases.

A more honest version.

$E(G)$ is the largest normal subgroup which is a central product of quasisimple factors. Show that $E(G) = 1$ and this is close enough to solvable to proceed as indicated.

Use CFSG and the corresponding classification of quasi-simple groups.

The recognition process.

Lemma 1.1. *If AG is a binary affine group acting then G is generated by involutions.*

Lemma 1.2 (Main Step)**.** *AG acting on A, $g \in G$, $g^2 = 1$ (exactly) on A_0, $a \in A$. Then there is an involution t so that t acts like g on $A_0 \cup \{a\}$.*

Proof. Induction on $|A \setminus A_0|$.

$X = A \cup \{a, a^g\}$.

Order X with $a < a^g$.

$$f_1(x) = \begin{cases} x & \text{if } x \geq x^g \\ -x & \text{if } x < x^g \end{cases} \qquad f_2(x) = \begin{cases} x^{g^{-1}} & \text{if } x \geq x^g \\ -x^g & \text{if } x < x^g \end{cases}$$

$$f_1(x)^h = f_2(x)$$

—Because we use g or g^{-1} unless we have the pair (a, a^g) in which case we want $(-a, a^g) \sim (-a^g, a)$, conjugate by a translation.

h agrees with g on X but not on a^g. So by induction t exists for h, hence for g.

□

So in the 1-dimensional semi-linear case the Galois action (if nontrivial) has order 2 and inverts the multiplicative part of G. We just have to argue further that all elements of norm 1 occur.

2. Back to the infinite

Problem (Main Problem (Lachlan)). *Is the relation*

$$A_1 \wedge \cdots \wedge A_k \vdash B_1 \vee \cdots \vee B_\ell$$

decidable?

In all cases studied: definitely. If we ask for the *number* of homogeneous structures satisfying the conditions we get a more subtle combinatorial problem (determining which subsets of a partially ordered set are w.q.o., for example).

We should also take note of the following.

Lemma 2.1 (Nešetřil). *Let $\mathcal{A}$ be a class of ordered finite structures with joint embedding which satisfies the structural Ramsey theorem*

$$\forall A, B \exists C \quad C \to (B)_2^A$$

Then $\mathcal{A}$ is an amalgamation class.

Proof. Take an amalgamation problem $A \to A_1, A_2$ in $\mathcal{A}$ and a structure B containing A_1 and A_2. Take $C \to (B)_2^A$. Given an embedding $f : A \to C$, we give it color 1 if there is an embedding f_1 of A_1 into C extending f, such that f_1 extends to an embedding g_1 of B into C, and otherwise give it color 2.

Any embedding of B into C induces some embedding of A into C with color 1, so in a monochromatically embedded copy B' of B all embeddings of A have color 1. Now take a copy of A_2 in B' and the corresponding embedding of A. This extends to an embedding of A_1 into C. So C is an amalgam of A_1 with A_2 over A. □

2.1. The Case of Tournaments.

The classification (Lachlan 1984)

$$I, \vec{C}_3, \mathbb{Q}, S, \Gamma_\infty$$

The technique introduced by Lachlan in this proof remains the main one.

Easy case: Local Orders. More generally, omit $I\vec{C}_3$:

$T = SL$; if homogeneous, $T = S$ or $T = L$ (first four cases).

Proposition 2.2. *If $\mathcal{A}$ is an amalgamation class of finite tournaments containing the tournament $I\vec{C}_3$, then $\mathcal{A}$ contains all finite tournaments.*

Definition 2.3. $\mathcal{A}'$: 1-point extension property
$\mathcal{A}^*$: linear extension property
$\mathcal{A}^+$: 1-point stack extension property

Lemma 2.4. $\mathcal{A}^*$ *is an amalgamation class.*

(try also possible amalgamation procedures ...)

Lemma 2.5 (Technical Lemmas).

(1) *If $I\vec{C}_3 \in \mathcal{A}$, then $I\vec{C}_3 \in \mathcal{A}^+$*
(2) $\mathcal{A}^+ \subseteq \mathcal{A}^*$

Corollary 2.6. *If $I\vec{C}_3 \in \mathcal{A}$, then $I\vec{C}_3$ belongs to an amalgamation class contained in $\mathcal{A}'$.*

Main Proof. We show by induction on $n = |T|$ that $I\vec{C}_3 \in \mathcal{A}$ implies $T \in \mathcal{A}$.

$n = 1$, clear.

$n + 1$. $T = T' \cup I$.

T' belongs to any amalgamation class containing $I\vec{C}_3$, hence is in $\mathcal{A}'$ by the Corollary. This means T is in $\mathcal{A}$. □

2.2. **From stacks to linear extensions.** $\mathcal{A}^+ \subseteq \mathcal{A}^*$:

Lachlan's Ramsey argument.

2.3. **From $\mathcal{A}$ to $\mathcal{A}^+$.** $L[C_3] \cup I$. How to do induction on the height of the stack?

Problem: If you fix one copy of C_3, the rest falls into two different 1-types.

Change of Category. *Ample 2-tournaments*

$\mathbb{H} = (H_1, H_2)$: $I\vec{C}_3$ in H_1, and both cross types realized in $\mathbb{H}$.

Lemma 2.7. *If $\mathbb{H}$ is ample, then any configuration $L[C_3] \cup I$ embeds, with $L[C_3]$ in H_1, and I in H_2.*

Reduce by induction to $I\vec{C}_3 \implies [\vec{C}_3, \vec{C}_3]$ and triangles (ab, c).

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