Finite Binary Homogeneous Structures

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Edinburgh

The king and the hermit

Another Inadequate Gift
Persian, 1556

The king finally understands that meaningful gifts come from lifelong devotion, the only certain road to heaven.

1. Origins
2. Relational Complexity
3. Binary Affine Groups

A point of departure

Theorem (Macintyre, 1971)

Let F be an infinite field whose theory admits quantifier elimination. Then F is algebraically closed.

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What are the finite primitive structures admitting quantifier elimination in a binary relational language?

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Conjecture

- *Equality* ($\mathrm{Sym}(n)_{nat}$); *or*
- *Oriented p-Cycle* ($\mathbb{Z}/p\mathbb{Z}_{reg}$);
- *Affine space equipped with an anisotropic quadratic form* (AO^-_{nat})

The Affine Case

Fact (OS)

The socle of a primitive permutation group is either elementary abelian, or a direct product of isomorphic nonabelian simple groups.

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An affine primitive binary group is either a p-cycle or affine space with an anisotropic quadratic form.

Stable finite structures

Example (Sheehan, Gardiner)

The homogeneous finite graphs are as follows.

- Pentagon (D_5);
- $(=_3)^2$ $(\mathrm{Sym}_3 \wr \mathrm{Sym}_2)_{\mathrm{pro}}$;
- $K_m^{\pm}[K_n^{\mp}]$ $(\mathrm{Sym}_n \wr \mathrm{Sym}\, n)_{\mathrm{imp}}$

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LACHLAN: The finite homogeneous structures for a finite relational language fall into finitely many families, of two types:

- sporadic finite examples
- Families of smooth approximations to an infinite stable structure, also homogeneous for the same language

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Smooth approximations: the induced automorphism group is the full automorphism group.

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CFSG

Smooth Approximation

LACHLAN: One should be able to do something similar, starting with smooth approximation, and including nontrivial geometries.
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KANTOR, LIEBECK, MACPHERSON, 1989

From smooth approximability—or a bound on 5-types—one gets a classification of the primitive examples.
Grassmannians of classical or semi-classical geometries.

Meanwhile . . .

HRUSHOVSKI 1989: Quasifinite axiomatizability of totally categorical structures
(and $\aleph_0$-categorical, $\aleph_0$-stable).

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MSRI, 1989-1990: affine duality (Hrushovski), connection to simple theories, type amalgamation, etc. (and ACFA)

My question

- Can we do something with finite homogeneous structures in a relational language of bounded complexity?

The meaning of relational complexity

Definition

$$a \sim b \iff b \in G \cdot a \tag{1}$$

$$a \sim_k b \iff a_I \sim b_I \text{ for } |I| = k \tag{2}$$

$$\rho(G, X) = \min(k \mid a \sim_k b \implies a \sim b) \tag{3}$$

I feel this is a natural, and perhaps even fundamental, invariant.

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$\rho \approx$ dimension?

Relational complexity and computational complexity

Example

$$\mathrm{GL}(V)_{\mathrm{nat}}\colon \begin{cases} d+1 & \text{if } F \neq \mathbb{F}_2 \\ \text{else } d \end{cases}$$

Sym_n on $\left[{n \atop k}\right]$: $\lfloor \ln_2 k \rfloor + 2$;
$\mathrm{Alt}(n)$ on $\left[{n \atop k}\right]$: $n-3$ ($k \geq 3$, $2k+2 \neq n$)

$$\rho_S(n,k) \approx \ln_2 k; \qquad \rho_A(n,k) \approx n-3$$

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Base: minimal set with trivial stabilizer.
The base bounds the complexity of group elements; the relational complexity bounds the complexity of the action.
GLUCK-SERESS-SHALEV 1998 Base size is bounded as a function of complexity of composition factors (e.g., 4 in the solvable case).

Orthogonal groups

$AGO(d, q)$ acting naturally.

Anistropic case: $\rho = 2$.
Isotropic case: roughly d
$(e, \lambda(e))$, $(e, \lambda(e) + v)$ with $v \perp e, v$.

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E.g. $AGO^-(6, 2)$: $\rho = 6$ (WISCONS via GAP)
...either linear algebra is irrelevant, or it is essential

Outline

Theorem

An affine primitive binary group (G, V) is either a p-cycle or affine space with an anisotropic quadratic form.

Remark

$G = V.H$, V *acts by translation and* H *acts linearly.*

Target $H = O^{-}(\mathbb{F}_{q^2})$ with quadratic form $N_{q^2/q}$ (dihedral).

Outline of Proof.

- H is solvable
- H embeds into a 1-dimensional semilinear group $\Gamma(1, \mathbb{F})$
- $\mathbb{F} = \mathbb{F}_{q^2}$, $H = K \cdot \langle\sigma\rangle$, $K = \ker N$

□

Identification

The 1-dimensional semilinear case, $G \leq AGL(1, \mathbb{F})$, $\mathbb{F}_+ \leq G$; $G \not\leq \mathbb{F}_+ \cdot \langle \pm 1 \rangle$

- G is generated by involutions

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$c \in K$: $u^\sigma = uc$ (Theorem 90)

$k \neq \pm 1$ in X

$$0,\ u,\ (1+k)u \quad \sim_2 \quad 0,\ u,\ (1+k^{-1})u$$

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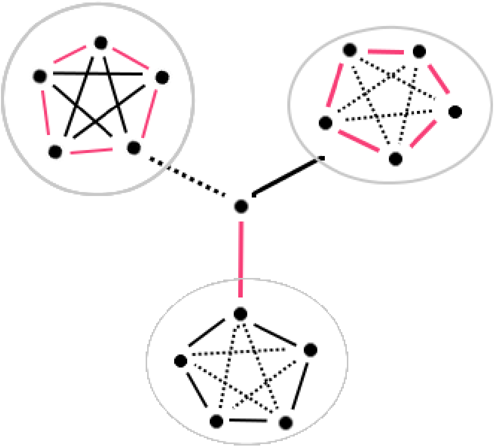
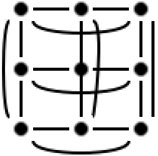
$$0,\ u,\ (1+k)u \quad \sim_2 \quad 0,\ u,\ (1+k^{-1})u$$

The conjugating element must be $c\sigma$ as u is fixed. So $c \in G$.

Examples



$r(3,3,3) > 16$
$AO^-(2,4)$



$L(K_{3,3})$
$AO^-(2,3)$

Solvable case

Induction:

1, $G = VH$, $K \triangleleft H$, V_0 K-irreducible.
Then $\rho(V_0 N_H(V_0)) \leq \rho(G)$

2. K elementary abelian 2-group, $V = \bigoplus V_\lambda$ weight spaces.
Then $\rho(V_\lambda N(V_\lambda)) \leq \rho(G)$

(The restricted group is primitive in both cases.)

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Then $\rho(V_\lambda N(V_\lambda)) \leq \rho(G)$
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A more technical lemma in this spirit ...

Normalization lemma

Lemma (Main Lemma)

Let $G = VH$ affine and binary. Let $K \triangleleft H$, $W \leq V$ an irreducible K-submodule.
$t \in G$ is an involution; $k \in K$, $v \in W$ with

$$v^k \neq \pm v, \quad v^t - v \sim v^{kt} - v^k \text{ under } H \text{ (e.g., } k, t \text{ commute)}$$

Then $W^t = W$.

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Proof.

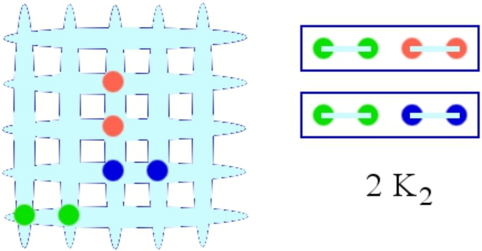
$$(u_1, u_2, u_3, u_4) = (0, v + v^k, v + v^t, v^k + v^t), \qquad u_4' = v + v^{kt}.$$

$$(u_1, u_2, u_3, u_4) \sim (u_1, u_2, u_3, u_4')$$

— but $u_2 \in W$, $u_3 - u_4 \in W$, $u_3 - u_4' \in W^t$.

□

Origin of the Lemma



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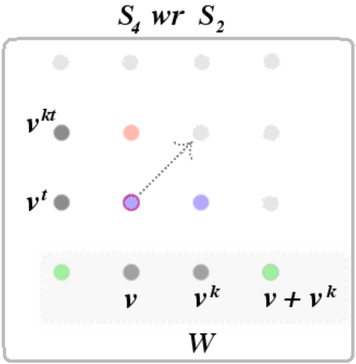

$V = 2^4$, $H = D_3 \wr S_2$, $K = D_3^2$
W = horizontal or vertical, $K_4 = \mathbb{F}_4$ with $\mathbb{F}_4^\times \langle \sigma \rangle$ acting.

Toward Solvability

$G = VH$

Target: $EG = 1$ (no nontrivial quasisimple factor)

- Char 2:
  - Torsion: no elements of order 4
  - (Bender) $H = \mathrm{PSL}_2$, J_1, or ${}^2G_2$
  - Eliminate via action of Borel subgroup and induction
- Odd char:
  - Torsion: no p-elements, complete reducibility
  - Exclude Q_8, Alt_4
  - $L \lhd EG$: PSL_2 or ${}^2B_2$
  - Weight spaces V_λ for max el. abelian 2-group E: $u_\lambda^g = f(\lambda)u_\lambda$
  - $N_G(E)$-orbits on Λ length at most 2
  - $EG = 1$

Problems

- Computational complexity of $\rho(G, X)$ in the primitive case, and in general.
- Qualitative theory of primitive k-ary groups (including the binary non-affine case)
    - Lower bounds on ρ for most OS-types, reducing to affine and almost simple cases (WISCONS, in progress)
    - Affine case: More representation theory
    - Almost simple case: Aschbacher classification should reduce to small maximal subgroups
        - Estimate ρ for classical actions.
- Imprimitive case (model theory)??