The Relational Complexity of a Permutation Group

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Permutation Groups are Automorphism Groups

$$(G, \Omega) \leftrightarrow \mathrm{Aut}(\Omega)$$

Same group iff *interdefinable*

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Imprimitive	Equivalence Relation
k-closed	*k*-ary
ρ_G right regular	Left *G*-action
k-homogeneous	*k*-homogeneous

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Right side: k-ary + orbits determined by isomorphism types
Left side: Orbits determined by k-orbits

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3-homogeneous.
Independent triples: Type (1) common neighbor; Type (2) no common neighbor
I.e. (1) no point in common; (2) unique point in common.

Some Binary (Homogeneous) Permutation Groups

- $(\mathrm{Sym}(n), Nat)$
- $(O_2^-(q), Nat)$
- 1-skeleton of the icosahedron
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Remark (Fourier)

The natural action of an anisotropic orthogonal group is binary.

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Proof.

All isometries are linear □

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The last two examples—the icosahedron and $\mathrm{Sym}(6)$ on 3-sets—are *metrically homogeneous graphs* (Cameron 1980). The edge relation on 3-sets is: $|u \cap v| = 2$.

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The natural action of an anisotropic orthogonal group is binary.

Conjecture

A primitive binary homogeneous group is $\mathrm{Sym}(n)$ or $AO(V)$ (V anisotropic) acting naturally, or the regular action of C_p.

Relational Complexity

Remark

Every permutation group on n points is $(n-1)$*-homogeneous.*

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Every permutation group on n points is $(n-1)$-homogeneous.

$$\rho(G, \Omega) = \text{min degree of homogeneity}$$

Examples

(GL_n, Nat): $n+1$ (or n, over $\mathbb{F}_2$);

$(\mathrm{Sym}\, n, k\text{-sets})$: $\lfloor \ln_2 k \rfloor + 2$;

$(\mathrm{Aut}\, n, k\text{-sets})$: $n-3$ typically, with exceptions for $k \leq 2$ or $n = 2k+2$;

Wreath products

$$(\mathrm{Sym}(n) \wr \mathrm{Sym}(d), n^d)$$

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Proposition (Saracino)

For $n \geq 2\lceil \log_2 d \rceil + 2$,

$$\rho(n^d) = 2\lceil \log_2 d \rceil + 2$$

For $n \leq 2\lfloor \log_2 d \rfloor + 2$,

$$\rho(n^d) = 2\lfloor \log_4 \alpha_n 2^{n/2+1} d \rfloor + \epsilon$$

with $\epsilon = 0$ *or* 1 *unless* $n = d = 3$,

$$\text{and } \alpha_n = \begin{cases} 1 & n \text{ even} \\ (4/3\sqrt{2}) & n \text{ odd.} \end{cases}$$

Two Propositions

Proposition

The relational complexity of $\mathrm{Aut}(n)$ *on* k*-sets* ($2k \leq n$) *is* $n-3$ *with the following exceptions.*

$$\begin{aligned} k = 1 : &\quad n-1 \\ k = 2 : &\quad \max(n-2, 3) \\ k \geq 3, n = 2k+2 : &\quad n-2 \end{aligned}$$

Proposition

Let (G, Ω) *be binary, primitive, and affine. Then* G *is the natural action of* $AO(V)$ *with* V *anisotropic, or the regular action of* C_p*.*

A critical ρ-orbit

$$(X_1, \ldots, X_\rho) \sim_{\rho-1} (Y_1, \ldots, Y_\rho)$$
$$(X_1, \ldots, X_\rho) \not\sim (Y_1, \ldots, Y_\rho)$$

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Let us suppose $\rho > \rho(\mathrm{Sym}(n), k\text{-sets}) = \lfloor \ln_2 k \rfloor + 2$.

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Let us suppose $\rho > \rho(\mathrm{Sym}(n), k\text{-sets}) = \lfloor \ln_2 k \rfloor + 2$. Then $\mathcal{X}^\xi = \mathcal{Y}$ for some $\xi \in \mathrm{Sym}(n)$, necessarily odd.
In particular:

- $\mathcal{X}$ separates points;
- $\mathcal{X}^i = (X_1, \ldots, \widehat{X}_i, \ldots, X_\rho)$ does not separate points.

Show

$$\rho \leq n - 3 \textit{ with specific exceptions}$$

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(u_i, v_i) not separated by X_j $(j \neq i)$
ρ distinct edges: Graph Γ.

Lemma

Γ is acyclic.

Proof.

X_i separates $e_i = (u_i, v_i)$.
X_i does not separate pairs in the same connected component of $\Gamma \backslash e_i$. □

$\rho(\mathrm{Aut}(n), k\text{-sets})$

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So we claim: usually $\gamma \geq 3$.

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Proof.

Component	$\mathcal{C}_1$	$\mathcal{C}_2$
Order	$> k + 1$	the rest
Vertex	leaf u	leaf v
Edge	edge (u, u')	edge (v, v')
Separator	X	X'

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X cannot contain $\mathcal{C}_1 \setminus \{u\}$; so X is $\{u\} \cup \mathcal{C}_2$.
Hence $|\mathcal{C}_2| = k - 1 > 1$

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$|\mathcal{C}_2| > 1$ ... Hence the edge (v, v') exists and X' must meet $\mathcal{C}_2$ and $\mathcal{C}_1$.

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X' must meet $\mathcal{C}_2$ and $\mathcal{C}_1$. ... But then X' contains $\mathcal{C}_1$, a contradiction.

Exceptional cases, $\rho = n - 2$

The previous lemma points toward the case

$$n = 2k + 2$$

with the non-separation graph Γ consisting of two trees of order $k + 1$.

Lemma

If Γ has two components, each of order $k + 1$, then the trees are stars and the separators X_i are

$$\mathcal{C}_\ell \setminus \{u\}$$

with u varying over leaves.

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If X separates the edge (u, v) in the component $\mathcal{C}$, then X is a k-subset of $\mathcal{C}$.

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Corollary

In the exceptional case with $n = 2k + 2$, we have $\rho \geq n - 2$.

Affine Induction

Affine groups: AG acting on A:

A acts by translation—G by automorphisms.

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But we have some useful subquotients . . .

Affine Induction

Lemma

Let AG be affine and binary, $H \triangleleft G$, and $V \leq A$ H-irreducible. Then $VN_G(V)$ is binary.

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Proof.

If $\bar{v} \sim_2 \bar{w}$ then we may suppose $v_1 = w_1 = 0$ and some $v_i \neq 0$. Then

$$\begin{aligned} \bar{v} \sim_{AG} \bar{w} &\implies \bar{v} \sim_G \bar{w} \\ &\implies \bar{v} \sim_{N_G(V)} \bar{w} \end{aligned}$$

since

$$V \cap V^g = V \text{ or } (0)$$

□

Generation

Lemma

Let AG be binary and affine. Then G is generated by involutions.

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$\ldots g \in G$, $a \in A$. Then $\exists t \in I(G)$

$$x^g = x^t \text{ for } x \in C_A(g^2) \cup \{a\}$$

Generation, cont.

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Proof.

$X = C_A(g^2) \cup \{a, a^g\}$ Order X with $a < a^g$.

$$f_1(x) = \begin{cases} x & x \geq x^g \\ -x & x < x^g \end{cases} \qquad f_2(x) = \begin{cases} x^{g^{-1}} & x \geq x^g \\ -x^g & x < x^g \end{cases}$$

$$\text{Binarity: } f_1(x)^h = f_2(x)$$

(g or g^{-1}, except for $(-a, a^g) \mapsto (-a^g, a)$ via $+a - a^g$.)
$0^h = 0$; $h \in G$; and $x^h = x^g$ on $C_A(g^2) \cup \{a\}$ ... □

Recognition

Lemma

Suppose $A = \mathbb{F}^+$, $G \le \mathbb{F}^\# \cdot \mathrm{Aut}(\mathbb{F}/\mathbb{F}_p)$ *is primitive and binary. Then*

- $\mathbb{F} = \mathbb{F}_p$, $G \le \langle \pm 1 \rangle \le \mathbb{F}^\times$; *or*
- $G = K \cdot \langle \sigma \rangle$, $o(\sigma) = 2$, $K = \ker N_{\mathbb{F}/\mathbb{F}_0}$.

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- $G = K \cdot \langle \sigma \rangle$, $o(\sigma) = 2$, $K = \ker N_{\mathbb{F}/\mathbb{F}_0}$.

Proof.

G is generated by involutions.
Case 1. $G \leq \mathbb{F}^\#$: Then $G \leq \langle \pm 1 \rangle$.
Case 2. $\bar{G} \subseteq \mathrm{Aut}(\mathbb{F}/\mathbb{F}_p)$ nontrivial.
$\bar{G} = \langle \sigma \rangle$, order 2, $G \leq K\langle \sigma \rangle$.
$G = (G \cap K)\langle t \rangle$ with $t = a\sigma$.
To show: $K \leq G$.

□

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$$u^g = u,\ (ku)^g = k^{-1}u$$
$$g = c't$$
$$(kc'u)^t = k^{-1}u$$
$$k^{-1}c'^{-1}(cu) = k^{-1}u$$
$$c' = c \in G$$

Characteristic 2

1. $F_2G = 1$
2. $FG \neq 1$
3. FG cyclic
4. $G = FG\langle t \rangle$
5. A is FG-irreducible
6. $\mathbb{F} = C_{\mathrm{End}(A)}(FG)$
7. Recognition

Irreducibility

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Proof.

V FG-irreducible.
$VN_G(V)$ binary.
$N_G(V)$ is generated by involutions.
$N_G(V) = G$ □

$FG \neq 1$

Lemma

G has elementary abelian Sylow 2-subgroups

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Application:

[Bender] If $FG = 1$ then $G = \prod \mathrm{PSL}_2, J_1, \mathrm{Ree}$

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$$L \triangleleft G \text{ simple}$$
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Lemma (Main Lemma)

$N_G(V)$ contains all involutions commuting with L.

$G = L$.

Then one uses the internal structure of L.

Sylow 2-subgroups

Lemma

- *In characteristic 2, G has no element of order 4.*
- *In odd characteristic, G has no element of order p.*

Characteristic 2:

$$o(g) = 4:$$

$$u^g = u + v \qquad v^g = v + w \qquad w^g = w$$

$$u^{g^2} = u + w \qquad (u+v)^{g^2} = u + v + w$$

$$(0, u, v, u+v) \sim_2 (0, u, v, u+v+w)$$

But $(0, u, v, u+v) \not\sim (0, u, v, u+v+w)$.

The Main Lemma

Lemma

$H \lhd G$.
V H-irreducible.
$h \in H$, h^2 nontrivial on V.
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Then $t \in N(V)$.

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$$(0, v + v^t, v + v^h, v^t + v^h) \sim_2 (0, v + v^t, v + v^h, v + v^{ht})$$

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$$v^t - v \sim v^{ht} - v^h \qquad \text{by } h$$

And t (or 1) does the rest.

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$$v + v^h \mapsto v + v^h \qquad v^h - v \mapsto v^{ht} - v^t$$
$$V \mapsto V \qquad V \mapsto V^t$$

□

Review

Even Characteristic:

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2. $FG \neq 1$ (Main Lemma, Syl_2)
3. FG cyclic, $G = FG\langle t\rangle$
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Odd characteristic:

Complete reducibility . . . ; Kill $E(G)$ . . . ;
Eigenspace decomposition relative to elementary abelian 2-subgroups . . .

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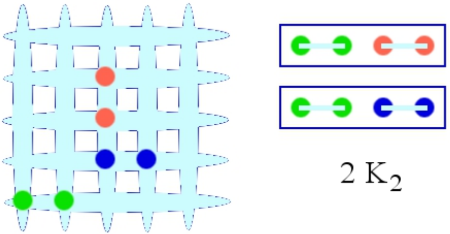
About the Main Lemma:

Homogeneous Graphs: $K_n \otimes K_n$

$$K_5 \otimes K_5 = E(K_{5,5}) = \mathrm{Sym}(5) \wr \mathrm{Sym}(2)$$



Failure of homogeneity in $\mathrm{Sym}(n) \wr \mathrm{Sym}(2)$ ($\rho = 4$)
[Sheehan 1974, Gardiner 1976]

$n = 4$: affine and primitive, $\rho = 4$.
Main Lemma Fails for $n = 4$ (not binary).
But for $n = 3$ this is binary! —$AO_2^-(3)$

Problems

- Non-affine case
  - Reduce to simple socle with maximal subgroup as point stabiliizer
  - Treat geometrically meaningful maximal subgroups
  - Explore with GAP
- Replace binary by k-homogeneous in a qualitative theory.