Homogeneity and Universality, from Urysohn and Ramsey to today

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Münster

Urysohn, 1924

Fréchet: Is there a universal complete separable metric space?

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> *In dieser letzten Hinsicht ist es Urysohn gelungen einen (in Ihrem Sinne) vollständigen metrischen Raum mit abzählbarer dichter Teilmenge, der einen jeden anderen separablen metrischen Raum isometrisch enthält und außerdem eine recht starke* **Homogenitätsbedingung** *füllt, zu konstruieren; letzterer besteht darin, daß man den ganzen Raum (isometrisch) so* **auf sich selbst abbilden** *kann, dass dabei* **eine beliebige endliche** *Menge M in eine ebenfalls beliebige, der Menge M kongruente Menge M*1 *übergeführt wird. [Alexandrov to Hausdorff (Cf. Hušek 2006)]*

Rational Urysohn space

Step 1 Build a countable metric space $\mathbb{U}_{\mathbb{Q}}$ which is universal and homogeneous in the class of $\mathbb{Q}$-valued metric spaces.
Idea: $\mathbb{U}_{\mathbb{Q}} = \lim \mathcal{M}_{\mathbb{Q}}$ where $\mathcal{M}_{\mathbb{Q}}$ is the category of finite rational metric spaces.

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Step 2 Pass to the completion $\mathbb{U} = \hat{\mathbb{U}}_\mathbb{Q}$.

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- Universal for countable graphs, preserving the graph metric.
- Distance-transitive (and even metrically homogeneous).
- Local structure: the random graph

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$$\mathrm{Aut}(\Gamma_\infty) \neq \lim_{G \in \mathcal{G}} \mathrm{Aut}(G)$$

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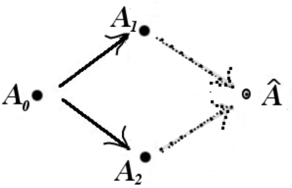
A more fundamental example, in several respects:
$(\mathbb{Q}, \leq) = \lim \mathcal{L}$ (finite linear orders)

Fraïssé 1954

Sur l'extension aux relations de quelques propriétés des ordres

- General definition of homogeneous structure, with $(\mathbb{Q}, \leq)$ as the model;
- Uniquely determined by its finite substructures;
- Universal for the corresponding class of countable substructures;
- Characterized by the amalgamation property

Amalgamation



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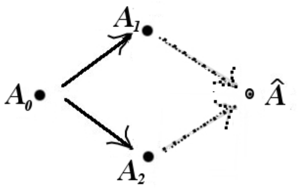
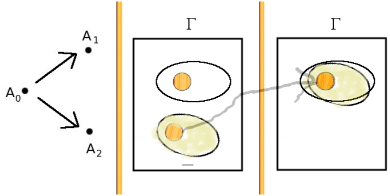


HOMOGENEITY $\Longrightarrow$ AMALGAMATION



The Fraïssé Limit

$$\begin{array}{ccccc} \mathcal{A} & \leftrightarrow & \Gamma & \leftrightarrow & (\Gamma, \mathrm{Aut}(\Gamma)) \\ = \mathrm{Sub}(\Gamma) & & = \lim \mathcal{A} & & \end{array}$$

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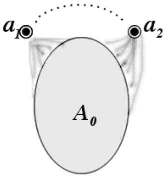
Problem

$$\mathrm{Aut}(\lim \mathcal{A}) = \lim(??(\mathcal{A}))$$

Some Amalgamation Classes and their Limits

2-POINT AMALGAMATION



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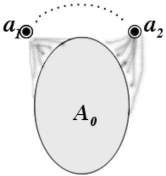

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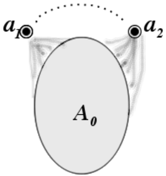

$\mathcal{L}$ (finite linear orders); compare the cuts in A_0, tie-break arbitrarily $[(\mathbb{Q}, \leq)]$
$\mathcal{M}_{\mathbb{Q}}$ (finite rational metric spaces): between $\max(|d_1(a, x) - d(a_2, x))|$ and $\min(d_1(a_1, x) + d(a_2, x))$ $[\mathbb{U}_{\mathbb{Q}}]$

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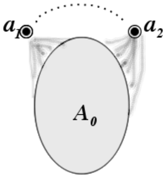

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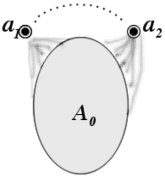

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Bonus: $\mathcal{G}_n$ (K_n-free graphs) —Henson 1971

Superbonus [Hen73]: $\vec{\mathcal{G}}_{\mathcal{T}}$ ($\mathcal{T}$-free digraphs)

uncountably many

Classifications

Lachlan 1984: There are just 5 homogeneous tournaments.

- The 4 homogeneous local orders $\vec{I}$, $\vec{C}_3$, $\vec{\mathbb{Q}}$, $\vec{S}$;
- The random tournament.

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$$\text{Method: } \mathcal{A}^* = \{A \,|\, (\forall A \cup L)(A \cup L \in \mathcal{A})\}$$

Lemma

If $\mathcal{A}$ is an amalgamation class then $\mathcal{A}^*$ is an amalgamation class

Lemma

If the amalgamation class $\mathcal{A}$ contains $[I, \vec{C}_3]$, then so does $\mathcal{A}^*$.

Lachlan's Punch Line

To Prove: If A is a finite tournament, $\mathcal{A}$ an amalgamation class, and $[I, \vec{C}_3] \in \mathcal{A}$, then $A \in \mathcal{A}$.

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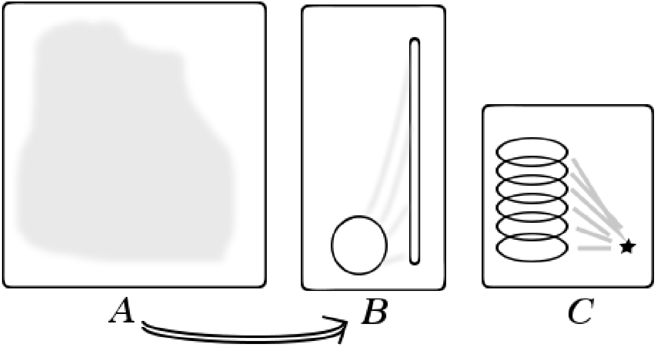
- $A^- \in \mathcal{A}^*$ Induction
- $A \in \mathcal{A}$ Definition of $\mathcal{A}^*$

What Just Happened?



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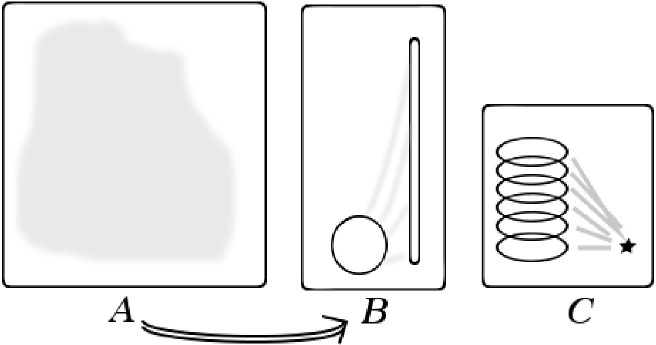

From B to C:

Ramsey's Theorem—any large tournament contains a large linear ordering.

A Conundrum

Henson 1972 There are uncountably many homogeneous directed graphs, of arbitrary complexity
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Cherlin 1998 *Let the force guide you.*

Ramsey 1930

Theorem (Finite Combinatorics)

Any coloring of ordered d-tuples from $([1, N], \leq)$ is monochromatic on a subset of size n, if N is large enough relative to d and n.

$$N \to (n)^d \text{ (Hungarian notation)}$$

Theorem (Logic)

Any universal theory consistent with the theory of $(\mathbb{Q}, \leq)$ has a model definable in $(\mathbb{Q}, \leq)$.

Theorem (Group Actions)

Any closed $(\mathrm{Aut}\,\mathbb{Q})$-invariant subset of $2^{\mathbb{Q}^d}$ contains an $(\mathrm{Aut}\,\mathbb{Q})$-invariant point.

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$\mathcal{A}$	$\leftrightarrow$	Γ	$\leftrightarrow$	$(\Gamma, \mathrm{Aut}(\Gamma))$
Combinatorics		Logic		Group Actions

Structural Ramsey Theory

The Structural Ramsey Property

Combinatorial Version: For all $A, B \in \mathcal{A}$ there is C with

$$C \to (B)^A$$

Logical Version: Any universal theory compatible with the theory of Γ has a Γ-definable model.
Group Theoretic Version: Any closed $\mathrm{Aut}(\Gamma)$-invariant subset of 2^{Γ^d} has a fixed point.

Consequences of Ramsey

Lemma

If $\mathcal{A}$ has the Ramsey property then the following hold.

- *$\mathcal{A}$ is an amalgamation class;*
- *The Fraïssé limit carries an invariant linear order.*

Proof.

Amalgamation: Otherwise, color copies of A_0 according to whether they embed in A_1 or in A_2.
Linear Order: take a fixed point for $\mathrm{Aut}(\Gamma)$ on the space of all possible linear orders. $\square$

Hunting Ramsey Classes

Theorem (Jasinski, Laflamme, Nguyen Van Thé, Woodrow)

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Is homogeneity always just a step away from Ramsey theory?

Topological Dynamics

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Topological group G

Continuous action on a compact set X (flow)

minimal flow: all orbits dense

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For locally compact groups:

- if compact, the universal minimal flow is G
- otherwise, non-metrizable

Some Universal Minimal Flows

Structure Γ	Universal Minimal Flow for $\mathrm{Aut}(\Gamma)$
Unordered Set	All orderings of the set [GlasnerWeiss03]
Random graph	All orderings of the random graph
Generic equivalence relation	Convex orderings of the underlying set
Generic Partial order	Linear extensions
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metrizable

Thesis [G-N-T]: Given a homogeneous structure in a finite relational language, the universal minimal flow is metrizable, and contains a description of the optimal Ramsey theorem for the class (corresponding to a generic orbit).

My projects

Theorem

The homogeneous ordered graphs are as follows.

Source	*Expansion*
Homogeneous Partial Order	*Linear extension*
Their Graph Complements!	*. . .*
Homogeneous Tournament	*Generic Linear Expansion*
Homogeneous Graph	*Generic Linear Expansion*

Metrically Homogeneous Graphs

Conjecture

The metrically homogeneous graphs are as follows.

- *Non-generic type:*
  - *Finite, classified by Cameron*
  - *Diameter ≤ 2, classified by Lachlan/Woodrow*
  - *Tree-like, described by Macpherson in the locally finite case*
- *Generic type:*
  - $\Gamma^{\delta}_{K,C,\mathcal{S}} = \lim \mathcal{A}^{\delta}_{K,C,\mathcal{S}}$ *(new);*
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Non-generic: Defined in terms of the local structure (neighbors of 1 or 2 vertices).

An amalgamation strategy

1. If $C \leq 2\delta + K_1$:
   1. If $r^- \geq K_1$ then take $d(a_1, a_2) = r^-$.
      Otherwise:
   2. If $C' = C + 1$ then:
      1. If $r^+ \leq K_2$ then take $d(a_1, a_2) = \min(r^+, \tilde{r})$
      2. If $r^- < K_1$ and $K_2 < r^+$ then take $d(a_1, a_2) = \tilde{r}$ if $\tilde{r} \leq K_2$ and $d(a_1, a_2) = K_1$ otherwise.
   3. if $C' > C + 1$ then:
      1. If $r^+ < K_2$ then take $d(a_1, a_2) = r^+$;
      2. If $r^- < K_2 \leq r^+$ then take

$$d(a_1, a_2) = \begin{cases} K_2 - 1 & \text{if there is } v \in A_0 \\ & \text{with } d(a_1, v) = d(a_2, v) = \delta \\ K_2 & \text{otherwise} \end{cases}$$

2. And otherwise ...

Supporting Evidence

- The non-generic are all classified.
- The catalog is complete in diameter 3 (joint with Amato, Macpherson).
- The catalog is (conditionally) complete for bipartite graphs, in an inductive setting.

Unversal Graphs with constraints

(with Shelah)
Fix a finite constraint set $\mathcal{C}$ and look for a universal countable $\mathcal{C}$-free graph.

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Example

if $\mathcal{C}$ is closed under homomorphism, then there is a universal countable $\mathcal{C}$-free graph.

Theorem (Nešetřil)

The Ramsey Property holds, with a generic linear order and a suitable base language. So $\mathrm{Aut}(\Gamma)$ *is extremely amenable.*

$\mathcal{C}$-free Graphs

General Expectation: Strong requirements on $\mathcal{C}$, but interesting examples.

Conjecture (1 constraint)

If there is a universal C-free graph then the following hold:

- *All blocks of C are complete;*
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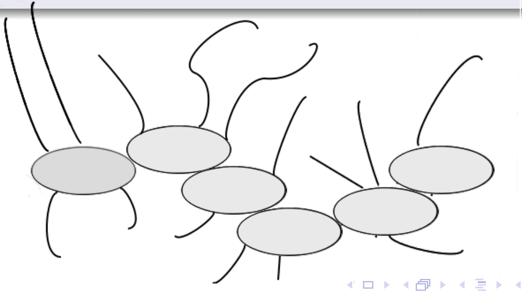

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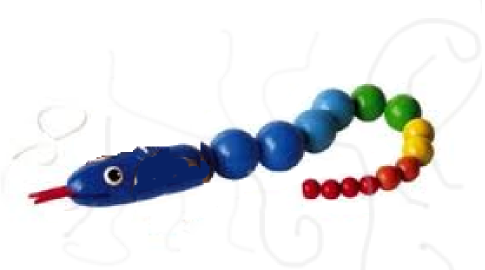

Summing Up

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Ramsey	Colorings	Theories	Fixed Points

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. . . but the Ramsey theory captures some of the richness of the automorphism group, and can identify the universal minimal flow [KechrisPestovTodorcevic 2005]