Finite Groups and Model Theory

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Categoricity in Power

κ-categorical theory:

Determines its models of cardinality κ, up to isomorphism

Countable	$(\mathbb{Q}, <)$	Combinatorics
Uncountable	$(\mathbb{C}, +, \cdot)$	Algebra
Total	$(V_{\mathbb{F}_q})$	Both

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Theorem (Baldwin-Lachlan)

. . . and the models are classified by dimensions—so the countable models form a tower.

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- (Peretyatkin, 1980) Yes: theory of a pseudo-successor.
- (Zilber, 1980) No, if we require total categoricity

Theorem (Zilber, Finite Model Property)

If a model of a totally categorical theory has a first order property ϕ, then ϕ holds in a finite substructure.

Morley rank

$$\bigcup_n \mathrm{Def}(\Gamma^n) \to \mathrm{Ord}$$

Dimension Theory

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Stone Duality: $S \leftrightarrow \hat{S}$ clopen in the Stone dual to $\mathrm{Def}(\Gamma^n)$.
$\aleph_0$-stability: the dual spaces are countable if Γ is countable
$\mathrm{rk}\,(S) = \max_\Gamma \mathrm{CB\text{-}rk}(\hat{S}_\Gamma)$

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Example

$\mathrm{rk}\,0$: Finite

$\mathrm{rk}\,1$, degree 1: "Strongly minimal"—*Every definable subset is finite or cofinite.*

Strongly Minimal Sets (dimension 1)

acl(X) pregeometry on Σ

Example

linear dimension, transcendence degree . . .

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Models of uncountably categorical theories are prime and minimal over a suitable strongly minimal set, hence classified by the corresponding dimension.

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Special Case: "almost strongly minimal"— $\Gamma = \mathrm{acl}(\Sigma)$

Example

$(\mathbb{Z}/p^2\mathbb{Z})^{(\omega)}$: $A[p]$ is a vector space, $A/A[p]$ is fibered by affine spaces of the same type. But $\mathrm{acl}(A[p]) = A[p]$.

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Theorem (Zilber)

If a theory is uncountably categorical but not almost strongly minimal, then some definable permutation group of finite Morley rank acts outside $\text{acl}(\Sigma)$.

Strictly minimal geometries

acl() is locally finite.

$$[\Sigma \setminus \mathrm{acl}(\emptyset)]/\sim \text{ where } \sim \text{ is: coalgebraic}$$

Theorem (Zilber; Mills, Cherlin, Neumann, Kantor; Evans; 1980–1986)

A strictly minimal geometry is degenerate, affine, or projective, over a finite field.

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Jordan Group: $\mathrm{Aut}(\Sigma)$ is transitive on the complement of a subspace.

Local modularity: Two subspaces which meet are independent over their intersection.

The Finite Model Property

Definition (Zilber Envelopes)

$A \subseteq \Sigma$ finite.

E maximal containing A and independent from Σ over A.

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$\mathrm{A} \subseteq \Sigma$ finite.

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Theorem (Zilber)

- *If A is finite then E is finite.*
- $\lim_{\mathrm{A}\to\Sigma} \mathsf{Th}(\mathrm{E}) = \mathsf{Th}(\Gamma)$

Homogeneity

Any isomorphism between finite substructures is induced by an automorphism

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Example (Lachlan/Woodrow)

The countable homogeneous graphs are

- The 5-cycle C_5;
- The 9-point "grid" $K_3 \otimes K_3$ with automorphisms $S_3 \wr C_2$;
- The disconnected graphs $m \cdot K_n$ and their complements;
- The infinite random graph;
- The generic K_n-free graphs, and their complements (Henson).

Finite case: Gardiner and Sheehan, independently.
The infinite ones are classified by Lachlan/Woodrow by a subtle argument.

Smooth Approximation

Observations.

- The graphs $m \cdot K_n$ are smoothly embedded in the graphs $\infty \cdot K_\infty$ in the sense that conjugacy of k-tuples in the smaller graph under its automorphism group is equivalent to conjugacy under the full automorphism group.
- The classifying parameters m, n are the orders of certain "indices" $[E_1 : E_2]$ counting fine equivalence classes contained in a coarse equivalence class.

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Theorem (Lachlan)

For relational systems of a given finite type, the finite homogeneous structures are exactly the smooth approximations to a finite number of maximal homogeneous structures; and these approximations are classified by numerical invariants of the form $[E_1 : E_2]$ *with* E_1, E_2 *nested, invariant equivalence relations.*

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Theorem (Lachlan)

For relational systems of a given finite type, the finite homogeneous structures are exactly . . .
Furthermore, the finite homogeneous structures together with their smooth limits at infinity are exactly the stable homogeneous structures of the given type.

A technical lemma

Theorem (Cherlin/Lachlan)

Any transitive permutation group (Γ, G) for which Γ is sufficiently large relative to $|\Gamma^5/\mathrm{G}|$ contains a large set of indiscernible elements (G induces the full symmetric group). In other words, there is a function $\mu(\mathrm{s}, \mathrm{n})$ such that whenever

- $|\Gamma^5/\mathrm{G}| \leq \mathrm{s}$
- $|\Gamma| > \mu(\mathrm{s}, \mathrm{n})$

then Γ contains n elements on which G induces $\mathrm{S_n}$.

Remark. A good deal can be said in terms of Γ^2/G, but we must avoid projective lines.

Proof outline

A counterexample to the theorem, for a given value of s and n, would be a sequence of finite permutation groups of unbounded size with $|\Gamma^5/G| \le s$ and no indiscernible set of size n. Choose such a counterexample with s minimized.

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The case of nonabelian socle reduces quickly to the study of actions of almost simple groups and then via the classification of the finite simple groups, to a close study of maximal subgroups of simple groups. The bound on Γ^5/G reduces the relevant actions to very classical cases for which indiscernibles are visible, by inspection (linearly independent sets of isotropic vectors and the like).

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The case of abelian socle takes somewhat more analysis. Our analysis was simplified by Kantor along the following lines: Reduce to an irreducible action of a quasisimple group, and handle this case by a direct argument.

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The primitive finite structures with Γ^5/G bounded are all derived from essentially classical structures (one slightly peculiar one in characteristic 2).

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Example

A dual pair (V, V^*): in the finite case, V is just a vector space with no further structure; at infinity, it acquires a topology from V^* (which is a countable dense subset of the full dual).

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Large finite structures with few orbits on Γ^5 are the smooth approximations to "Lie coordinatized" structures.

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Stable embedding: The analog of a strongly minimal set will be the underlying geometry associated with a KLM-structure. These will be stably embedded in the sense that any relation definable "from the outside" is definable "from the inside". (Cf. (V, V^*).)

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Stability theory in a non-stable setting.

Algebraicity Conjecture

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Conjecture (Algebraicity Conjecture)

A simple group of finite Morley rank is algebraic.

Feit-Thompson case: No involutions (or extreme Feit-Thompson: torsion-free).
Open!!!

Characteristic 2

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A simple group of finite Morley rank is said to have **characteristic 2 type** if it contains an infinite elementary abelian 2-subgroup.

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Remark

- *Inspired by CFSG; inductive*
- *No Feit-Thompson theorem*

Themes: strongly embedded subgroups, amalgam method, conjugacy of decent tori, properties of algebraic groups, and various specialized topics from finite group theory.

Application

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If G acts primitively and definably on a set of given rank, then the rank of G can be bounded. In particular, the degree of generic t-transitivity can be bounded.

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Cf.Popov for generically doubly transitive actions in the algebraic category, in characteristic 0.
Open for characteristic p, and for finite Morley rank actions of simple algebraic groups in characteristic 0.

A linearity conjecture

Conjecture

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Known so far only for SL_2 through rank $3\mathrm{f}$ where f is the rank of the field (C-Deloro).

Relational Complexity

(Γ, G) finite.

L_k is the class of G-invariant subsets of G^k, viewed as relations on G.

$\rho(\Gamma, G)$ is the least k such that Γ is homogeneous as an L_k-structure, with automorphism group G.

$$\bar{a} \sim_k \bar{b} \iff \bar{a} \sim \bar{b}$$

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Example

- $(V, GL(V))$: $\dim V + 1$.
- $(V, V \cdot O(V))$: 2, if the form is anisotropic
- The Petersen graph: 3.
  $(\{1,2\},\{1,3\},\{1,4\})$ vs. $(\{1,2\},\{1,3\},\{2,3\})$.

Primitive k-ary Groups

$$\bar{a} \sim_k \bar{b} \iff \bar{a} \sim \bar{b}$$

Generalizing KLM: can we understand all large primitive permutation groups with bounded relational complexity?

The binary case

Conjecture

The finite primitive binary permutation groups are the following:

- (n, S_n) *with natural action;*
- C_p *with natural action;*
- $(V, V \cdot O(V))$ *with* V *anisotropic.*

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Claim (July 2012)

This is true in the affine case.