Connected Groups of Finite Morley Rank: Structure

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Structure; the generic element

Groups without 2-tori:

$$1 \triangleleft O_2(G) \triangleleft U_2(G) \triangleleft G$$

Quotients: unipotent, reductive, $2^\perp$.

Groups with 2-tori:

- *Minimal nonalgebraic:* Prüfer rank at most 2 (cf. Deloro, Burdges). Open problem: an *absolute* bound
- The generic element centralizes a unique maximal 2-torus

Methods

Even type simple: methods of finite group theory, augmented in the early stages by the consideration of maximal p-tori and Wagner's results on fields of finite Morley rank (to remove inductive hypotheses).

Mixed type, simple: methods of finite group theory and the even type result
[ABCJ]

Odd type simple: methods of finite group theory in a strongly inductive setting
Borovik: Altseimer, Berkman, Burdges, Nesin

Degenerate type: black box group theory and genericity arguments
[BBC (UK)]
Nesin, Borovik, Altinel, Altseimer, Berkman, Corredor, . . .

Carter Subgroups

Nilpotent, almost self-normalizing. (Think: maximal torus.)

Solvable case: existence, uniqueness (Wagner, Frécon)

General case: existence (Frécon/Jaligot via Burdges' unipotence theory)

Method: take the least unipotent subgroups available (i.e. the most like tori).

Generosity

Union of the conjugates generic = *generous*

Jaligot: There is at most one conjugacy class of "*generous*" Carter subgroups

Problem Existence.

Conjecture In any connected group of finite Morley rank there is a unique conjugacy class of connected nilpotent subgroups such the union of their conjugates is generic.

Frécon—minimal simple case [Spring 2006]

Toricity

p'-type: No unipotent p-subgroups.

Proposition If G is connected of p'-type then any p-element belongs to a p-torus.

Corollary 1 If G is connected of p'-type then any p-element in the centralizer of a maximal p-torus T lies in T.

Corollary 2 If G is connected of p'-type then the p-Sylow subgroups are conjugate.

Corollary 3 If G is connected and contains p-torsion then it has an infinite Sylow p-subgroup.

Easier for $p = 2$. And powerful.

Applications

Application 1: Poizat's Problem

A connected group of generic exponent 2^n is a 2-group. More generally, generic exponent n reduces to generic exponent n_o, the odd part.

Application 2: Definably Primitive Permutation Groups

If a group acts faithfully and definably primitively on a set of rank r_0, then the rank of the group is bounded as a function of r_0.
($r_0 = 1$: Hrushovski, with sharp bounds)

Problem: Reasonably sharp bounds.

Permutation Groups

Basic ingredient: $\text{rk}\,(G) \leq \text{rk}\,(G_x) + \text{rk}\,(X)$.
Generic t-transitivity.

Step I: Bound t in terms of r_0.

(Weak estimates)

Step II: Bound the rank of G if $t = 1$

(Orbit ranks descend as points are fixed.)

Generic t-transitivity

1. Rank of a maximal p-torus bounded by rank of X.

2. If G is generically highly transitive, then Sym(n) acts on a maximal 2-torus T in some point stabilizer, with n large. At this point either

(a) The action is nontrivial, and n is itself comparable to the rank of T, hence bounded; or

(b) The action is trivial. But we can squeeze Sym(n) into the connected component and contradict our toricity (i.e., Corollary 1).

Challenges

- Generous Carter subgroups.
- Small simple K^*-groups of odd type.
- Simple groups of odd type, absolutely.
- Construction of torsion free simple groups of finite Morley rank.
- Structure of connected p-groups of finite Morley rank (e.g., exponent p).
- Generically highly transitive permutation groups.