Connected groups of finite Morley rank

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Theorem

In a connected group of finite Morley rank, the centralizer of any element is infinite.

Co-conspirators: Borovik, Burdges
Unindicted: Altınel

I. Introduction

Groups of finite Morley rank

Groups of finite Morley rank

- The algebraicity conjecture
- The four types
- Two test problems

Morley rank (dimension)

Algebraic groups: dimension
Finite groups: cardinality
(Link: Lang-Weil, $|X| \approx q^{\dim(X)}$.)

Algebraicity Conjecture
A simple group of finite Morley rank is algebraic.

Free groups: stable, *definably simple*
(Sela; Feighn, Bestvina)

Morley rank

Axioms and Basic Properties

1. $\mathrm{rk}\,(X)$ (dimension); $\deg(X)$ (multiplicity)
2. $K < H$:

   $$[H : K] = \infty \implies \mathrm{rk}\,(K) < \mathrm{rk}\,(H)$$
   $$[H : K] < \infty \implies \deg(K) < \deg(H).$$

3. $\mathrm{rk}\,(G/H) = \mathrm{rk}\,(G) - \mathrm{rk}\,(H)$.
4. $d(X) = d(\langle X \rangle)$

Morley rank

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6. *(strongly) generic*: $\mathrm{rk}\,(G \setminus X) < \mathrm{rk}\,(G)$

N.B.: G connected, $\mathrm{rk}\,(X) = \mathrm{rk}\,(G) \implies X$ generic.

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7. Saturation: a set of bounded cardinality is finite

p-Sylow$^\circ$ subgroups in matrix groups

Characteristic p:

unipotent – [bounded exponent, definable]

Model: Strictly upper triangular matrices.

Other characteristics:

semisimple – [divisible abelian]

Model: Diagonal matrices with entries suitable roots of unity.

$$\begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \dots & 1 \end{pmatrix} \qquad \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & * \end{pmatrix}$$

$$p \qquad \neg p$$

The four types

2-Sylow$^\circ$ structure in groups of FMR

$S = U * T$:

2-Unipotent $\times$ 2-torus

with finite intersection

Types

T \ U	$\neq 1$	$= 1$
$\neq 1$	Mixed	Odd
$= 1$	Even	Degenerate

Even: U	Odd: T	Mixed: $U * T$	Degenerate: 1

Results, by Type

Theorem *A simple group of finite Morley rank of even or mixed type is algebraic.*

Reference: Atlınel, Borovik, Cherlin, book in preparation.

Methods: Finite group theory (2nd and 3rd generations), and the model theory of fields of finite Morley rank (Wagner)

Theorem *A simple group of* **degenerate** *type contains no involutions.*

Methods: Black box group theory, genericity arguments.

Odd type: Borovik, Burdges, Jaligot, ongoing . . .

Poizat's pet problem

Problem *Show that in a connected group of finite Morley rank, if the equation* $x^n = 1$ *holds* generically, *then it holds everywhere.*

In an algebraic group this is trivial, because of the existence of a *topology* on the group (the Zariski topology) for which (1) multiplication is continuous and (2) generic sets are dense.

Until quite recently only the following cases were known:

- If the group is solvable, then the equation holds everywhere and the group is nilpotent.
- The result holds for $n = 2$ (trivially) and $n = 3$ (easily).

In particular the case $n = 4$ was very much open.

Centralizers

Problem *Show that in a connected group of finite Morley rank, any element has an infinite centralizer.*

This may be viewed as a special case of Poizat's Pet.

Lemma *If G is a connected group of finite Morley rank and $a \in G$ an element with finite centralizer, of order n, then the equation*

$$x^n = 1$$

holds generically in G.

Proof. As $a \in C(a)$, the order n is finite.
There is a definable bijection between the right coset space $C(a)\backslash G$ and the conjugacy class $a^G = \{a^g : g \in G\}$.
Hence $\mathrm{rk}\,(a^G) = \mathrm{rk}\,(G)$ and a generic element of G has order n (which is sharper than the claim). ∎

II. On Generic Equations

- The Theorem
- Its proof: The two ingredients

The theorem

Theorem *Let G be a connected group of finite Morley rank satisfying a generic equation*

$$x^n = 1$$

Then a Sylow 2-subgroup U of G is connected, definable, of bounded exponent, and normal in G, and the quotient G/U satisfies the generic equation

$$x^{n_o} = 1$$

where n_o is the odd part of n.

Corollary *Let G be a connected group of finite Morley rank. Then $C(a)$ is infinite for $a \in G$.*

The corollary

Corollary *Let G be a connected group of finite Morley rank. Then $C(a)$ is infinite for $a \in G$.*

Proof. Otherwise, a^G is generic in G, and we get a generic equation

$$x^n = 1 \qquad (n \text{ the order of } a)$$

The theorem applies: a Sylow 2-subgroup U of G is normal.
In $\bar{G} = G/U$ $\bar{a}^{\bar{G}}$ remains generic, so $C_{\bar{G}}(\bar{a})$ is finite.
In consequence we may replace G by $\bar{G}$ and suppose

There are no involutions in G

The generic sets a^G and $(a^{-1})^G$ *intersect*, and $a^x = a^{-1}$ for some $x \in G$. Then $d(\langle x \rangle)$ contains an involution: contradiction. ∎

Proof of the theorem

Outline

1. U is unipotent (= connected, definable, bounded exponent)
2. U is normal
3. G/U satisfies the generic equation $x^{n_o} = 1$.

The third step is immediate.
We mention that the structure of G is actually a little clearer than this:

$$G = U \cdot C_G(U)$$

so that really the even and the odd parts separate out independently.

Preparation: Notation

Objective: Isolate the key ingredients needed from the general theory

First, shift the notation: U stands for a Sylow$^\circ$ 2-subgroup, not a Sylow 2-subgroup, that is we build connectedness into the notation.
The steps are unchanged:

1. U is unipotent (= connected, definable, bounded exponent)
2. U is normal

Then G/U is a group of *degenerate type* and hence by one of the major recent results in the field, contains no involutions.
Hence U is indeed a Sylow subgroup and we are on track.

The two ingredients

- Degenerate type: elimination of involutions.
- Tori: torsion and genericity

Definition *A definable divisible abelian group which is the definable closure of its torsion subgroup is a decent torus*

Model: Multiplicative group of an algebraically closed field, or more generally the group of diagonal matrices over such a field, which is a product of several copies of this group.

Proposition *Let G be a group of finite Morley rank and T a decent torus in G, and $\hat{T} = C^\circ(T)$. Then the conjugates of $\hat{T}$ are generically disjoint and generically cover G.*

Example: $T =$ diagonal matrices, $\hat{T} = T$, and a generic matrix is diagonalizable (indeed, with distinct eigenvalues).

Remarks

1. These ingredients (degenerate type, tori) are general results independent of any classification results.

2. The theory of maximal tori is classical in Lie group theory and algebraic group theory, and motivates a significant chapter of the theory of *Carter subgroups* in finite solvable groups (self-normalizing nilpotent subgroups).

3. A p-torus is a divisible abelian p-group.
The definable closure of a p-torus is a decent torus. It is difficult to work directly with p-tori in a model theoretic context as they are not definable.

U is unipotent

G satisfies $x^n = 1$ generically, and U is a Sylow$^\circ$ 2-subgroup. The key to the entire analysis is the following.

Lemma *G contains no p-torus for any p.*

Taking $p = 2$, we get Step 1 of the proof:

$$U \text{ is unipotent}$$

(Recall the four types.)

Our lemma is much stronger than this and also dominates the analysis in Step 2, and in the strengthening of Step 2 we alluded to at the outset.

Killing p-tori

Now it is time to connect our second ingredient:

Proposition *Let G be a group of finite Morley rank and T a decent torus in G, and $\hat{T} = C^\circ(T)$. Then the conjugates of $\hat{T}$ are generically disjoint and generically cover G.*

with our main lemma

Lemma *G contains no p-torus for any p.*

Killing p-tori

Proposition *Let G be a group of finite Morley rank and T a decent torus in G, and $\hat{T} = C^\circ(T)$. Then the conjugates of $\hat{T}$ are generically disjoint and generically cover G.*

Lemma *G contains no p-torus for any p.*

Proof. Let T be a nontrivial decent torus, $\hat{T} = C^\circ(T)$, and $X_T = \hat{T} \setminus (\bigcup_{g \notin N(\hat{T})} \hat{T}^g)$.
Distinct conjugates of X_T are pairwise disjoint, and generically cover G. The generic equation $x^n = 1$, passes (generically) to X_T, and to $\hat{T}$.

Killing p-tori

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Distinct conjugates of X_T are pairwise disjoint, and generically cover G. The generic equation $x^n = 1$, passes (generically) to X_T, and to $\hat{T}$.

$\hat{T}$ is covered by cosets of T, and some coset kT generically satisfies the same equation.

Then $T = k^{-1}(kT)$ is generically of order n; but the torsion of fixed order in T is finite! ∎

The two (or three) ingredients

- Degenerate type: there are no involutions.
- Decent tori: control the generic element

Both of these theorems have emerged in the past year.
We have not gone into what happens after U is unipotent.
One possibility is to haul out the big guns:

- Classification of simple groups of even type.

But the 200-page argument implicit in this can be reduced to half a page using the “no p-tori” condition!

The treatment of **degenerate type** is startlingly direct . . .

III. Degenerate Type Groups

The theorem

Theorem *Let G be a connected group of degenerate type. Then G contains no involutions.*

Outline We work in a minimal counterexample G.

1. G may be taken to be simple.
2. If for a generic pair of conjugate involutions, $d(ij)$ contains no involution, we reach a contradiction.
3. If for a generic pair of conjugate involutions, $d(ij)$ contains an involution, we reach a contradiction.

Methods used:

- Covariant maps
- Genericity

Covariant maps

Definition *Let G be a group and H a subgroup. A function $\zeta : G \longrightarrow H$ is covariant if*

$$\zeta(hg) = h\zeta(g)$$

for $h \in H$ and $g \in G$.

Lemma *Let G be connected and $\zeta : G \longrightarrow H$ definable and covariant. Then H is connected.*

Covariant maps

Definition *Let G be a group and H a subgroup. A function $\zeta : G \to H$ is covariant if $\zeta(hg) = h\zeta(g)$ for $h \in H$ and $g \in G$.*

Lemma *Let G be connected and $\zeta : G \to H$ definable and covariant. Then H is connected.*

Proof. Let $F_h = \zeta^{-1}(h)$ be the fiber above h and $f_h = \mathrm{rk}\,(F_h)$.
$F_{h'h} = h'F_h$ by covariance, so $f_h = f$ is constant.
Hence for any definable subset X of H, we have

$$\mathrm{rk}\,(\zeta^{-1}(X)) = \mathrm{rk}\,(X) + f$$

Any subset of H of maximal rank pulls back to subset of G of maximal rank, so

$$\deg(G) \geq \deg(H)$$

Our claim follows. ∎

Minimization

Lemma *Let G be a connected group of finite Morley rank of degenerate type, containing an involution, and of minimal Morley rank. Then $G/Z(G)$ is a* **simple** *group of degenerate type containing an involution.*

Here the difficulty lies in showing that $G/Z(G)$ contains an involution. We can reduce to the case in which $Z(G)$ is a 2-group. If $G/Z(G)$ contains no involution, then one defines

$$\zeta : G \to Z(G)$$

as follows.

For $g \in G$, $d(g)$ splits canonically as $A \times Z_g$ with A a 2-divisible subgroup and $Z_g = d(g) \cap Z(G)$ a cyclic 2-group. The projection $\pi_2 : d(g) \to Z_g$ is definable. Set $\zeta(g) = \pi_2(g)$. Covariance is routine, and $Z(G)$ is disconnected (finite)—a contradiction.

Preparation: Dihedral groups

The finite case

A group generated by two involutions i, j is dihedral: setting $a = ij$, the group is

$$\langle a \rangle \cdot \langle i \rangle$$

or in other words a cyclic group with an element of order two inverting it.
If a has odd order then i and j are conjugate in this group, and if a has even order then $\langle a \rangle$ contains a unique involution k commuting with both, and the noncentral involutions fall into two conjugacy classes represented by i and ik.

Preparation: Dihedral groups

The finite case $\langle i, j \rangle = \langle a \rangle \cdot \langle i \rangle$
Odd order: $i \sim j$ under $\langle a \rangle$
Even order: $k \in \langle a \rangle$, $i, j \in C(k)$, $j \sim ik$.

Finite Morley rank

Preparation: Dihedral groups

The finite case $\langle i, j \rangle = \langle a \rangle \cdot \langle i \rangle$
Odd order: $i \sim j$ under $\langle a \rangle$
Even order: $k \in \langle a \rangle$, $i, j \in C(k)$, $j \sim ik$.

Finite Morley rank
The group definably generated by two involutions i, j is generalized dihedral: setting $a = ij$, the group is

$$d(a) \cdot \langle i \rangle$$

with $d(a)$ inverted by $\langle i \rangle$.

Preparation: Dihedral groups

The finite case $\langle i, j \rangle = \langle a \rangle \cdot \langle i \rangle$
Odd order: $i \sim j$ under $\langle a \rangle$
Even order: $k \in \langle a \rangle$, $i, j \in C(k)$, $j \sim ik$.

Finite Morley rank
$d(i, j) = d(a) \cdot \langle i \rangle$
The group $d(a)$ has the form

$$A \times C \qquad (A \text{ divisible, } C \text{ finite cyclic})$$

In degenerate type the divisible group A contains no involutions and thus $d(a)$ contains at most one involution, and the dichotomy holding in the finite case applies here as well.

Black box methods

G is minimal among connected groups of degenerate type with involutions, and is simple. Fix an involution i and let $\mathcal{C} = i^G$.

The first of two cases to take up is:

$d(ij)$ contains no involution, generically

Here i is fixed and j varies over $\mathcal{C}$.

We reach a contradiction by manufacturing a generically defined covariant map

$$\zeta : G \to C(i)$$

The domain is only a generic subset of G but this produces a contradiction as before.

Black box methods

G is minimal among connected groups of degenerate type with involutions, and is simple. Fix an involution i and let $\mathcal{C} = i^G$.

The first of two cases to take up is:

$d(ij)$ contains no involution, generically

We reach a contradiction by manufacturing a generically defined covariant map $\zeta : G \to C(i)$.

Definition: $\zeta(g) \in gd(i \cdot i^g) \cap C(i)$.

- $\zeta(g) = gx$, $x \in d(i \cdot i^g)$, $i^{gx} = i$,
- Unique: otherwise $d(ij)$ meets $C(i)$; but $d(ij)$ is *inverted* by i.

Phase 3

Outline We work in a minimal counterexample G.

✓ G may be taken to be simple.

✓ If for a generic pair of conjugate involutions, $d(ij)$ contains no involution, we reach a contradiction.

(. . .) Genericity Arguments

(3) If for a generic pair of conjugate involutions, $d(ij)$ contains an involution, we reach a contradiction.

Genericity, I

G is minimal among connected groups of degenerate type with involutions, and is simple.

Lemma *Let t be a 2-element. Then for a generic element g of G, the Sylow 2-subgroup of $d(g)$ is generated by a conjugate of t.*

Proof. Let $X = tC^\circ(t)$. For $x \in X$ one finds

$$d(x) = A \times \langle t \rangle \qquad (A = d(x) \cap C^\circ(t))$$

The coset $xA = tA$ contains a unique 2-element, t.

The conjugates of X are pairwise disjoint:

Consider $X \cap X^g$ with $g \notin C(t)$. For $x \in X$ we can recover t from x by examining the structure of $d(x)$; for $x \in X \cap X^g$ this shows that $t = t^g$, a contradiction.

Now by a rank computation $\bigcup X^G$ is generic in G and our claim follows. ∎

Genericity, II

Lemma *Let t be a 2-element. Then for a generic element g of G, the Sylow 2-subgroup of $d(g)$ is generated by a conjugate of t.*

Corollary *A Sylow 2-subgroup of G is elementary abelian.*

Proof. By considering the order of the Sylow 2-subgroup of a generic element of G, one sees that all 2-elements of G have the same order, which must be 2. ∎

Corollary *For i, j involutions of G, if $d(ij)$ contains an involution k then i and j are not conjugate in $C(k)$.*

Proof. We have j conjugate to ik under $d(ij)$ and hence inside $C(k)$. If i is conjugate to j in $C(k)$ then i is conjugate to ik in $C(k)$, and there is a 2-element acting nontrivially on $\langle i, k \rangle$ which contradicts the previous corollary. ∎

End of the proof

G a minimal counterexample. $\mathcal{C} = i^G$.
U a Sylow 2-subgroup

Suppose for generic $j \in \mathcal{C}$ the group $d(ij)$ contains an involution k. (Unique)

For i, j in $\mathcal{C}$ generic and independent over U there are pairs (u, v) of elements $u, v \in U$ such that

(i, k) is conjugate to (u, v) in G.

The pairs (i, j) and (j, i) have the same U-type, the same k. Therefore the pairs in U conjugate to (i, k) or (j, k) are the same.

Hence the pairs (i, k) and (j, k) are conjugate.

This means that i is conjugate to j in $C(k)$, and contradicts our previous claim.

IV. Coda

Generosity

Definition *Let G be a group of finite Morley rank. A Carter subgroup of G is a connected almost self-normalizing nilpotent subgroup H of G.*

Theorem (Frécon-Jaligot) *Every group of finite Morley rank contains a Carter subgroup.*

Definition *A Carter subgroup H of G is generous if $\bigcup H^G$ is generic in G.*

Theorem (Jaligot, Spring 2005) *Any two generous Carter subgroups of G are conjugate.*

The Generosity Conjecture

Conjecture *Every group of finite Morley rank has a generous Carter subgroup.*

The following special case would be very useful in practice.

Conjecture *For every connected group G of finite Morley rank, a generic element of G belongs to a connected nilpotent subgroup.*

This would essentially generalize the theory of decent tori to arbitrary groups of finite Morley rank.

Summary

- The classification program for groups of finite Morley rank has spun off a more intrinsic structural theory
- Centralizers of *decent tori* control the generic element
- Still open, and fundamental: the Generosity Conjecture—Nilpotent subgroups control the generic element.