Countable Universal Graphs with Forbidden Subgraphs

A problem in
model theoretic combinatorics

Hattingen, July 2003

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Does there exist a
universal countable C-*free* graph?

Examples

1. Rado graph (no C)
2. K_n-free (Fraïssé)
3. P_n-free [KMP '88]
4. Bowtie-free [FK]

Non-Examples

1. C_4-free
2. $K_{m,n}$-free [KP '84]
3. C_n-free [CS]
4. C-free (2-connected, incomplete) [FK]
5. T-free, bushy tree [CS]
6. Bridge-free [GK]

Positive Cases

Few
Varied approaches:
Structure theory, amalgamation, more
Frequently associated with:
$\aleph_0$-*categorical* theories

Negative Cases

Uniform approach,
essentially model-theoretic:
acl

Thesis

$\aleph_0$-categoricity is the right "dividing line"

Context

$\mathcal{C}$: Finite collection of finite, connected graphs

Countable $\mathcal{C}$-free graphs

Universal one?

Main Problems

I. Single constraint: *explicit list* of "good" ones

II. General case: *Decidability*

Undecidability?

Generalize: Finite relational structures.

Coding: Graphs with a 2-coloring of the vertices

IIA. Is the generalized problem undecidable?

IIB. Can the 2-colored graphs be coded by graphs?

Decidability?

There are two types of “good” contraint; path-like constraints, and sets of constraints closed under homomorphism.

Is this the whole story?

Known positive cases

1. A path $P_n \in \mathcal{C}$;

2. Augmented path $P_n^+ \in \mathcal{C}$;

3. Fat paths $K_n \underset{\cdot}{+} K_3$, $K_3 \underset{\cdot}{+} K_3 \underset{\cdot}{+} P_n$, $K_m \underset{\cdot}{+} P_n$ (more?)

4. Any class closed under *homomorphism*

5. . . . or an extension of a known class by such a class

Example

$\mathcal{C}$ a finite collection of cycles

Then there is a universal $\mathcal{C}$-free graph if and only if $\mathcal{C}$ consists of all the *odd cycles* up to some length $2N + 1$.

The Model-Theoretic Point of View

$\mathcal{G}_{\mathcal{C}}$: $\mathcal{C}$-free graphs

$\mathcal{E}_{\mathcal{C}}$: *existentially complete* $\mathcal{C}$-free graphs

Bad example

$\mathcal{C}$: all cycles;

$\mathcal{E}_{\mathcal{C}}$: Trees with all vertex degrees infinite

Not an elementary class.

Theorem

For $\mathcal{C}$ a finite set of connected graphs, the class $\mathcal{E}_{\mathcal{C}}$ is elementary. and its theory is complete.

Corollary Tfae:

1. There is a universal countable $\mathcal{C}$-free graph;
2. There is a saturated countable graph in $\mathcal{E}_{\mathcal{C}}$;
3. The theory of $\mathcal{E}_{\mathcal{C}}$ is small ($|S_n| \leq \aleph_0$ all n).

Observation

Most commonly we have the extremes: either $|S_n| = 2^{\aleph_0}$ for some n, or S_n is finite for all n.

Exception

Tallgren's augmented paths, with $\mathbb{Z}$-components (and no additional edges).

Variant: When is $\mathcal{E}_{\mathcal{C}}$ $\aleph_0$-categorical?

Thesis. This is the natural question

I*. $|\mathcal{C}| = 1$

II*. Decidability

acl in $\mathcal{E}_{\mathcal{C}}$

$acl(A)$: Union of the finite A-definable sets

Theorem Tfae:

1. $\mathcal{E}_{\mathcal{C}}$ is $\aleph_0$-categorical;
2. $\mathcal{E}_{\mathcal{C}}$ is *locally finite*:
   for A finite, $acl(A)$ is finite.

Growth rate problem:

$$g(n) = \max(|acl(A)| : |A| = n)$$

"Counterexample":
A "sequence" of equivalence classes
(successor relation $\to$, equivalence relation $\sim$)
S_2 is infinite, *acl* is trivial.

(Why is this not a counterexample?
Hint: the language involved is irrelevant.)

Finiteness Lemma

If A is algebraically closed and finite,
and $k = \max(|C| : C \in \mathcal{C})$, then the type of A is determined by its k-quantifier existential type.

Applications

1. If $\mathcal{C}$ is closed under homomorphism, then $acl(A) = A$ for all A;
   hence $\mathcal{E}_{\mathcal{C}}$ is $\aleph_0$-categorical.

2. If $\mathcal{E}_{\mathcal{C}}$ is $\aleph_0$-categorical, and $\mathcal{C}'$ is closed under homomorphism, then $\mathcal{E}_{\mathcal{C} \cup \mathcal{C}'}$
   is $\aleph_0$-categorical; and $g_{\mathcal{C} \cup \mathcal{C}'} \leq g_{\mathcal{C}}$.

3. *Unarity*
   If $\mathcal{C}$ consists of "solid" graphs then acl is unary: $acl(A) = \bigcup\limits_{a \in A} acl(a)$.

4. Bowtie B:
   $|acl(a)| \leq 4$ (a computation) and hence
   $\mathcal{E}_B$ is $\aleph_0$-categorical.

Conjecture. If $\mathcal{E}_C$ is $\aleph_0$-categorical, then C is a string of complete graphs.

Conjecture. If $\mathcal{E}_C$ is small, then C is a nearstring of complete graphs.

Homomorphisms

$h : G_1 \to G_2$ edge preserving.

Hence if $h(u) = h(v)$ then $u \not\sim v$.

$\mathcal{C}$ is closed under homomorphisms if: $C \in \mathcal{C}$, $C \twoheadrightarrow C'$ implies $\exists C'' \in \mathcal{C}\ C'' \hookrightarrow C'$.

Theorem Tfae:

1. $A \neq acl(A)$;
2. There is $C \twoheadrightarrow C' \subseteq G$ with $C \in \mathcal{C}$, so that $C \hookrightarrow \oplus_A C_i'$.

Proof ($1 \Rightarrow 2$):

$G_1 \oplus_A G_2$, $G_1 \simeq G_2 \simeq G$;

$C \hookrightarrow G_1 \oplus_A G_2 \twoheadrightarrow G_1$; image C'. ∎

Corollary:
Closure under homomorphism implies:
$acl(A) = A$;
hence $\aleph_0$-categoricity.

Example: Cycles

C_{2n} collapses to K_2—not interesting.

C_{2n+1} generates C_{2m+1} for $1 \leq m \leq n$.

Trivial algebraic closure, nontrivial theory.

Why is the amalgamation method harder here?

—It involves an explict calculation of the type structure.

Unarity, growth rates, and other details

Explicit computation of acl

$A \subseteq H \subseteq G$

$cl(A; H)$: A together with the union of all *minimal bases* B for H over A:
H free over $A \cup B$, and B minimal.

Lemma $cl(A; H) \subseteq acl(A)$

Proof: Δ-system lemma

$\mathcal{F}$: $\{(A, H) : H \hookrightarrow C \in \mathcal{C}$, properly$\}$.

Theorem

1. $\mathcal{F} - cl$ generates acl;
2. $\mathcal{F} - cl$ is locally finite

Problem: length of the iteration.

Theorem

1. $\mathcal{F} - cl$ generates acl;
2. $\mathcal{F} - cl$ is locally finite

Proof:

2. $\mathcal{F} - cl(A)$ is a definable subset of $acl(A)$

1. As before, $h : C \hookrightarrow G_1 \oplus X G_2$; $H = h(C) \cap G_2$.

For applications:

Refine $\mathcal{F}$;

Study the length of $\mathcal{F}$-*cl* chains.

If C is solid, then $\mathcal{F}$-closure is generated by unary closures $cl(a; H)$.

Reason: this is where the transition from G_1 to G_2 takes place.

Open problems:

1. Coding and decidability
   General relational systems and graphs.

2. Sets of 2-connected graphs
   Unify Füredi-Komjáth (2-connected graphs) and Cherlin-Shi (sets of cycles)

3. Trees
   Done: bushy trees, bridges, generalized stars

4. Short solid graphs
   Beyond bowties:
   $K_m \mathbin{\underset{\cdot}{+}} K_n$ $(m, n \geq 4)$; $K_3 \mathbin{\underset{\cdot}{+}} K_2 \mathbin{\underset{\cdot}{+}} K_3$.

5. Growth rates
   $g_{P_k}(1) = k$? (Known bound: k^{3k})

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