Simple groups of finite Morley rank

Even Type

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Algebraic groups	**Chevalley groups**
Defined by polynomial equations	Given explicitly (after 50 years)
"$\mathrm{SL}_n(F)$"	A_n, B_n, . . . , G_2
	New finite simple groups

Finite simple groups

$\mathbb{Z}_p$

Alternating

Chevalley

"Twisted" Chevalley

Sporadic (26)

Uncountably categorical (FMR)

¿Algebraic (i.e., Chevalley)**?**

Toward

(*) In any counterexample, the connected component of a Sylow 2-subgroup is divisible abelian.

possibly $= 1$, *however!*

- Altınel, *Habilitation*, June 2001.

ABC/J: True, if *degenerate* infinite simple sections are excluded.

Tame K^*

K^*

L^*

What do we need now?

Various characterizations of SL_2, and notably:

Strong Embedding

$$M : \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \qquad S : \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$$

M is the stabilizer of ∞ under the natural action on the complex projective line $\widehat{\mathbb{C}}$ by *fractional linear transformations* $\frac{az+b}{cz+d}$

So G/M "is" the projective line.

And the stabilizer of two points (e.g., $0, \infty$) looks like

$$T = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

(!) $$S \cap M \cap M^g = \begin{cases} S & g \in M \\ 1 & \text{else} \end{cases}$$

Hypotheses

Strong Embedding:
(i) $S \leq M < G$;
(ii) $S \cap (M \cap M^g) = 1$ for $g \notin M$.

Induction:
Any definable section of even type is a Chevalley group

Target: $G \simeq \mathsf{SL}_2$

Method: Action of G on G/M (then: *Nesin*)

The Case Division

Does G properly contain SL_2*?* Or, more precisely:

Is there a subgroup $L \simeq \mathsf{SL}_2$ containing A, with $H = C^\circ(L) > 1$?

"A" is the subgroup of S generated by its elements of order 2

No: Then it should be SL_2;
Yes: Then it should not exist.

The hard case is (and always has been) the *Yes* side.

In fact, this is probably what earned the Sacks prize for Jaligot.

Tools and Strategy

The strategy has evolved considerably, from Altınel to Jaligot to the current iteration.

Strategy

Data: G, M, A, T
$AT \leq M$; T looks like $F^{\times}$ and A looks like F_{+}

And we consider the family of tori which lie in M:

$$\mathcal{T} = \{T^g : g \in G, T^g \leq M\}$$

M acts on $\mathcal{T}$; we may speak of "orbits" (or conjugacy classes) with respect to this action.

Steps:

	Jaligot	Revised
1.	$\mathcal{T}$ is *one* orbit	. . . *finitely many* orbits
2.	$C(T) \leq M$, all T	$C(T) \leq M$, generically
⋮	Various	About the same
5.	Weird calculation	Weird calculation

1. $\mathcal{T}$ has *finitely many* orbits.

How to do Step 1: **Tools**

Conjugacy theorems: Algebraic Groups
Borel subgroups (maximal solvable connected)
Maximal tori (maximal diagonalizable connected)

Conjugacy theorems: Finite Groups
Sylow
Hall
Carter: *nilpotent, self-normalizing*

Conjugacy theorems: FMR
2-Sylow
Hall
Carter

What's wrong:
Not enough solvable subgroups
(*degenerate sections*)

The story so far

$$HL < G$$

$H = C^\circ(L)$ is connected, degenerate, and an abomination upon the face of the earth.
(Or else a Hrushovski monster.)

$T \leq L$.

HT is *very interesting*

In Altınel's thesis it is nilpotent and self-normalizing.
In Jaligot's thesis it is only solvable at first, and self-normalizing, but eventually it becomes abelian.
In either case it is a **Carter subgroup** of any group containing it.

In our case it is *degenerate* $\times$ abelian, and self-normalizing.
So: we have a problem.

Genericity and conjugacy

Concepts:
Almost self-normalizing:
$N^\circ(H) = H$;
Generically disjoint from its conjugates:
$H \cap \left(\bigcup'_g H^g\right)$ non-generic

Example: maximal tori in simple algebraic groups!

G1. If H has both these properties, then $\bigcup_g H^g$ is generic in G.
G2. If H_1 and H_2 have both these properties, then the union of the conjugates of either generically covers the other.

But what good is that?

Rigid Abelian Groups

Algebraic tori also have *few definable subgroups*, e.g.:

$$T_1 \times \ldots \times T_n; t_1^{d_1} \cdot \cdots \cdot t_n^{d_n} = 1$$

No infinite parametrized families (uniformly definable).

Terminology: Rigid abelian group; rigid torus (connected).

R1. Algebraic tori in positive characteristic are rigid. (Wagner)
R2. A rigid torus is generically disjoint from its conjugates.
R3. A generic covering by rigid tori always involves a *maximal* rigid torus T.

Theorem Self-normalizing rigid tori are conjugate.

Proof:
Let T, T_1 be two such. They are generically disjoint from their conjugates (R2). So the conjugates of T_1 generically cover T (G2).

Then some intersection $T \cap T_1^g$ is a maximal torus in T (R3). This means $T \leq T_1^g$.

And similarly, vice versa. ∎

The real thing™

$T \leq M$; $H \times T \leq G$; T looks like $F^{\times}$, H looks mysterious.

How many conjugacy classes of T?
Let's suppose $HT \leq M$. Then we show:

(1) HT contains an almost self-normalizing subgroup generically disjoint from its conjugates;

(2) All the groups of the form HT in M form a single M-orbit;
(another conjugacy argument)

(3) The set of T_1 such that $H_1T_1 = HT$ (some H_1) is *finite*.

This will do it . . .

(3) The set of T_1 such that $H_1T_1 = HT$ (some H_1) is *finite*.

Checking (3): Let $\hat{T}$ be the maximal rigid torus in $Z(HT)$. Since $T_1 \leq \hat{T}$, and since T_1 varies over a *uniformly definable* family, there are only finitely many of these—rigidity. ∎

Slogan

If you have enough elements of order 2, you don't need the Feit-Thompson Theorem.

This isn't exactly what finite group theory teaches us . . .

Speculations next week, with Borovik.