UNIVERSAL GRAPHS WITH A FORBIDDEN BLOCK PATH V. 2

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Abstract. We work toward the classification of the graphs finite connected graphs C with no trivial blocks having the property that there is a countable universal C-free graph.

Contents

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Introduction

A toy theorem. A highly simplified version of the result of the present paper would be the following.

Proposition 1. *Let C be a Rado constraint which is a solid block path in which all blocks have order at least 4. Then the type of C is one of the following, up to reversal.*

Length 1: (n) *(arbitrary);*
Length 2: (m, n)*,* $\min(m, n) \leq 5$*,* $(m, n) \neq (5, 5)$*;*
Length 3: $(m, 4, n)$

Here we call C a Rado constraint if there is a countable weakly universal C-free graph, and we call a graph a block path if its blocks (maximal 2-connected subgraphs) can be arranged in a sequence $(B_1, B_2, \ldots, B_n)$ with $B_i \cap B_j$ empty for $|i - j| > 1$.

We call a block path solid if the blocks are in fact cliques. It is known that a block path which is a Rado constraint must be solid. This allows us to describe the isomorphism types of the graphs in question by the sequence of clique sizes (up to reversal of the sequence).

In fact we conjecture the following.

Conjecture 1. *Let C be a connected graph in which all blocks have order at least 4. Then C is a Rado constraint if and only if C is a solid block path of one of the types listed above.*

Thus Proposition 1 is one of the main steps in the direction of this conjecture, but only one. One would need to prove further that every constraint listed actually is a Rado constraint (which is known in some cases), and also that every Rado constraint with all blocks of order at least 4 is a block path, a point that we hope to return to fairly soon.

What we really aim to do in the present paper is to prove a similar proposition for the case in which all blocks have order at least 3. The list of examples becomes longer and contains some infinite families. We also make a corresponding conjecture in the spirit of Conjecture 1 for this case, but without insisting on the point that everything on our list should necessarily be a Rado constraint (we think it should not be far off, but it would be miraculous if our thinning-down process gave the exact classification).

At the opposite extreme we have Rado constraints which are trees, that is all blocks have order 2. These are also known explicity [CT07, CS05]: they are just the paths and the trees differing from paths by the addition of one edge.

However the mixed case, where blocks of order 2 are mixed together with larger graphs, is not simply a combination of the two extremes. We conjecture that the Rado constraints in general are close to block paths, and that the list of relevant block paths is of the same general character as the list given here, but it will be seen that the restriction to block size at least 3 brings major simplifications to the analysis.

We remark that when the relevant universal graphs exist, they are canonical (this is most easily seen via model theory) and they have rich and interesting automorphism groups. In particular in the cases listed here, the corresponding automorphism groups should be oligomorphic, and when viewed as topological groups they are anticipated to have metrizable universal minimal flows. However all of this remains conjectural, and deep, and would require a substantial development of structural Ramsey theory, which at present has been worked out in only a few cases, notably the case of a solid block path of type $(3,3)$ [HuNe14(ppt)].

Leaving these broader issues aside, we now begin a more systematic presentation of our subject.

A decision problem. We are interested in the following general decision problem.

Problem 1. *Given $\mathcal{C}$, a finite set of finite, connected graphs, determine whether there is a countable strongly universal $\mathcal{C}$-free graph.*

It remains entirely unclear whether this is a decidable problem, and for us that is the fundamental question, which we have approached from various angles. Regardless of the status of the general problem, we have come to believe that for the case of one constraint the problem should have an explicit solution, and we are working toward that.

There is an extensive literature on the subject, some of which will be recalled below. Much of the literature deals with the case in which $\mathcal{C}$ consists of a single graph, which is also the focus of the present paper. But we consider it very likely that when there is only one constraint, it should be possible to give a complete classification of the constraints C allowing a weakly universal C-free graph, or at the very least to prove the decidability of this restricted form of the decision problem.

The subject begins with Rado's paper [Ra64] and is surveyed in [KP91]. The scope of our decision problem is narrower than that of the survey in two respects. In the first place, by restricting to constraints given by finite data, we arrive at a decision problem. At the same time, this restriction allows for the general theory presented in [CSS99]. In the second place, one may consider the same problem in any cardinality, but the relevant theory changes considerably when one passes to the uncountable case. For the latter, see [Dž05].

Before going into the results of the present paper, let us lay out the definitions more explicitly.

Definition.

1. Let $\mathcal{C}$ be a set of graphs. Then a graph G is *$\mathcal{C}$-free* if it contains no subgraph isomorphic to a graph in $\mathcal{C}$. When $\mathcal{C}$ consists of a single constraint graph C, we say G is C-free.

2. A countable $\mathcal{C}$-free graph G is *weakly universal* if every countable $\mathcal{C}$-free graph is isomorphic to a subgraph of G.

3. A countable $\mathcal{C}$-free graph is *strongly universal* if every countable $\mathcal{C}$-free graph is isomorphic to an induced subgraph of G.

One may vary the notion of $\mathcal{C}$-freeness as well, forbidding only induced subgraphs isomorphic to graphs in $\mathcal{C}$. Then the corresponding decision problem is undecidable [Ch11b]. More precisely, it encodes the domino problem of Hao Wang, which is a 2-dimensional version of the problem of determining an infinite word omitting certain finite subwords. This 1-dimensional problem is decidable, as the existence of such an infinite word implies the existence of a periodic one. The failure of this principle in two dimensions is the basis of the theory of quasicrystals (e.g., Penrose tilings).

In spirit, Problem 1 lies somewhere between the 1- and 2-dimensional tiling problems, and it seems closer to the 1-dimensional problem. To date all problems successively treated obey the periodicity principle; to understand what this means requires familiarity with examples or with the general theory, but it will be amply illustrated here.

These considerations suggest that Problem 1 is natural not only graph theoretically but in regard to the general theory of algorithms.

Notions of universaiity; variations. The preferred notion of universality is the strong form. The notions of strong and weak universality are clearly very different. For example, if there are no constraints, then a countable complete graph is weakly universal, while the construction of a strongly universal graph takes some argument ([ER63] or [Ra64]). At the same time, it is generally the case that when there is a weakly universal countable $\mathcal{C}$-free graph there is also a strongly universal one, and furthermore, a canonical one, with oligomorphic automorphism group: that is, the action of the automorphism group on n-tuples should have finitely many orbits, for each n.

Thus we consider the following variations of the decision problem.

Problem 2. *Given $\mathcal{C}$, a finite set of finite, connected graphs, determine whether there is a countable weakly universal $\mathcal{C}$-free graph.*

Problem 3. *Given $\mathcal{C}$, a finite set of finite, connected graphs, determine whether there is a countable strongly universal $\mathcal{C}$-free graph with oligomorphic automorphism group.*

The theory given in [CSS99] shows that this last version of the problem is very close to the problem of finding an infinite word with finitely many forbidden subwords, though not necessarily close enough to ensure decidability.

Since the strong and weak forms of the decision problem tend to have the same answer in most cases, it is desirable from a purely technical point of view to treat them together. In other words, when a universal graph exists one aims to show strong universality, but in the more common negative cases, one aims to show the nonexistence of weakly universal graphs.

Main results. Since the emphasis on the present paper will be on the identification of negative cases, for the case of a single constraint, we adopt the following terminology for brevity.

Definition. Let C be a graph. We call C a *Rado constraint* if there is a countable weakly C-free graph.

In the present paper we begin the presentation of a general strategy for the identification of all finite connected graphs which are Rado constraints. It is not our intention to complete this classification, but rather to suggest a combination of methods that may be adequate to the task, and some explicit conjectures about the form of the result to be expected.

Any finite connected graph may be analyzed structurally in terms of its associated *tree of blocks*. Here the blocks are the maximal 2-connected components and the associated tree has two sorts of vertices: the cut vertices and the blocks of the original graph. Here edges join cut vertices to the blocks that contain them.

In the long run, we aim to generalize the following result.

Fact 1 ([CT07, CS05]). *Let T be a finite tree. Then T is a Rado constraint if and only if T is either a path or a path with one additional edge attached. The case of a weakly or strongly universal graph with an oligomorphic automorphism group corresponds to the case in which T is a path.*

We consider first the case of *block paths*, that is graphs whose tree of graphs forms a path. We have the following preliminary result. Going forward, all constraints will be assumed to be finite *and connected* without further mention. The point of dealing with connected graphs is that the C-free graphs are closed under disjoint union if and only if C is connected. This is the essential hypothesis. (For sets of more than one constraint it is not necessary to take the constraints connected to achieve this, but one usually does so anyway.)

Fact 2 ([CS14]). *Let C be a block path. If C is a Rado constraint then every block of C is complete.*

The method of proof is to combine an explicit construction of Füredi-Komjáth for the case of one block with two general inductive principles, called *pruning* and *symmetric local pruning* respectively. This is also the plan of attack with respect to the general problem, except that we must enlarge our stock of explicit constructions to deal with the base of the inductive analysis. So [CS14] may be taken as the first of a series, and the present article as a continuation.

We call a graph whose blocks are all complete a *solid* graph. We aim here at a partial determination of the block paths which are Rado constraints, with Fact 2 giving us the point of departure. We are able to give a complete determination of the block paths with all blocks nontrivial which are Rado constraints (a block is trivial if it reduces to an edge). The general problem reduces to some extend to the case of nontrivial blocks, but with a great deal of additional analysis still to be done.

We have mentioned what this looks like when we restrict attention to the case in which all blocks have order at least 4: the block paths then have length at most 3 and satisfy a few constraints on the block sizes.

In this case, it has been checked elsewhere ([**?**], following up on [Ko99]) that the listed types of length 1 or 2 actually are Rado constraints (with the associated automorphism groups oligomorphic). This point has not yet been checked for the cases listed under Length 3, other than some limited and unpublished cases, and it may not be very easy to check. The verification at the end of [CT07] that the listed candidates in length 2 actually are Rado constraints required some detailed analysis. We believe that the graphs listed in Proposition **??** are indeed Rado constraints, but all we claim at present is that we can exclude anything not listed.

The main result of the present paper is the corresponding statement when all blocks are nontrivial. This result has the same general character as the foregoing but both the statement and proof become more elaborate. And we remain in substantial doubt about some of the more extreme candidates still on the list.

The full statement is as follows.

Theorem 1 (Fat Block Paths). *Let C be a finite block path with no trivial blocks. Suppose that there is a weakly universal C-free graph. Let ℓ be the number of blocks in C. Then C has one of the following types.*

ℓ	*Form*
(general)	$(3^{\ell-1}, n)$*; or* $(3^{\ell-2}, n, 3)$ *or* $(3^{\ell-2}, 4, 4)$
2	$(4, n)$ *or* $(5, n)$ *with* $n \geq 6$
3	(n_1, m, n_3) *with* $m = 3$ *or* 4
4	$(n_1, 3, 3, n_4)$ *with* $n_4 \geq n_1 + 2$
″	$(3, n, 3, n)$ *with* $n > 4$
″	$(3, 4, 4, 4)$
″	$(3, 4, 3, n)$ $(4, 4, 3, n)$ *with* $n \geq 4$
5	$(4, 4, 3, 3, n)$ *with* $n \geq 9$
″	$(3, n_2, 3, 3, n_5)$ *with* $n_2, n_5 \geq 4$ *and* $\lvert n_2 - n_5\rvert \geq 2$
″	$(3, 3, n, 3, n)$ *with* $n \geq 5$

For the case of general block paths our result is considerably less satisfactory, and obviously incomplete.

Theorem 2 (General Block Paths). *Let C be a finite block path and suppose there is a countable weakly universal C-free graph. Then C can be obtained from block paths of the types listed in Theorem 1 by connecting them by paths.*

This leaves a great deal more to be analyzed if one aims at an explicit classification comparable to Theorem 1.

But—all of this takes us only so far, and now we enter the realm of conjecture. The point of this close analysis of the case of block paths is to cast light on Rado constraints in general. In particular, we conjecture the following.

Conjecture 2. *If C is a Rado constraint with all blocks nontrivial, then C is a solid block path, as listed above.*

As a proof of this will shed more light on the general problem than a full treatment of the case of block paths, we intend to take this up next. We believe we have all necessary ingredients for a proof of something more general. But again the approach would be inductive and therefore involves a substantial number of specific constructions to deal with the base of the induction. This can be simplified considerably by assuming that all blocks have order at least 4, but even that case requires something substantial.

Now we will lay out our expectations for the final classification more explicitly.

Conjectures. We begin with some general principles that guide our thinking but are not the object of a frontal assault. At this point it is worth recalling explicitly that we are discussing only finite, connected constraint graphs.

Conjecture 3 (Solidity Conjecture)**.** *Let R be a Rado constraint. Then the blocks of R are complete.*

For the next conjecture, we use the following terminology: a *segment* of a graph G is a connected induced subgraph which is a union of some of the blocks of G

Conjecture 4 (Segment Conjecture)**.** *If R is a Rado constraint then any segment of R is a Rado constraint.*

We can read off from the explicit list in Theorem 1 that the Segment Conjecture is valid in the case of block paths with all blocks nontrivial. Conversely, the proof of Theorem 1 would be much simpler if the Segment Conjecture were known a priori.

We note that connected induced subgraphs of Rado constraints need not be Rado constraints; a solid block path of type $(5,5)$ refutes this strong monotonicity conjecture.

Now the essential point is to guess correctly the most general form of a Rado constraint. Our current candidate runs as follows.

Definition. Let C be a connected graph. The *interior* C° of C is the smallest segment containing all nontrivial blocks.

Any constraint graph C which is not itself a tree may be described as a nonempty interior plus a certain number of attached trees. For v in the interior of C, it will be convenient to denote by T_v the tree attached at that point, i.e., the connected component of $(C \setminus (C^\circ \setminus \{v\})$ containing v.

(We notice that the same notions have been used by Gromov with the terminology *core* and *periphery*.)

Conjecture 5 (Block Path Reduction)**.** *Let R be a Rado constraint which is not a tree. Then*

- *The interior of R is a block path; in addition*
- *The attached trees are paths;*
- *The attached trees at cut vertices are trivial.*

We give these three conditions in the order of decreasing confidence, with considerably more stress on the first. But in an inductive proof of the conjecture, a stronger form may be easier to prove.

When the interior consists of a single block we have the following special case.

Conjecture 6. *Let C be a graph with a unique nontrivial block B, Then C is a Rado constraint if and only each vertex of B is attached to a unique path (possibly trivial).*

It is only in this last case that we are prepared at present to conjecture a completely precise description of the Rado constraints. Namely t is reasonable to conjecture that the Rado constraints with a unique nontrivial block are precisely the extensions of a complete graph by at most one path at each vertex.

Remark. *Conjecture 5 (Block Path Reduction) implies the solidity conjecture.*

This holds since the solidity conjecture was proved for block paths in [CS14].

There is no obvious way to derive the Segment Conjecture from a purely qualitative result. That conjecture is a useful principle to keep in mind, but will probably only be verifiable after the fact, if a completely explicit classification of the Rado constraints is found.

The segment conjecture and some additional considerations suggest the following conjecture regarding the case of block paths. First, we rephrase Theorem 2 more explicitly, in the following terms.

Definition. Let C be a block path. A 3-*component* of C is a maximal connected subgraph all of whose blocks are nontrivial.

Then Theorem 2 states that the 3-components of a solid block path which is a Rado constraint are again Rado constraints. We may distinguish *external* 3-components, that is those containing a leaf in the tree of blocks and the *internal* 3-components. Then we conjecture the following.

Conjecture 7 (Internal 3-Components)**.** *The internal 3-components of a Rado constraint consist of single blocks.*

This gives a general indication of what we would expect the explicit classification of block paths which are Rado constraints to look like when trivial blocks are allowed.

The main technical point of the present article from our point of view is the way the two pruning notions presented below, corner pruning and symmetric local pruning, cooperate to reduce the entire classification problem to an explicit collection of base cases. We always try corner pruning first, and when that fails it generally produces a situation in which symmetric local pruning succeeds. In some cases multiple attempts at corner pruning are needed to reach a sufficiently symmetrical set up to apply symmetric local pruning.

The result of this analysis is that one can characterize explicitly the "pruning-minimal" examples (those for which any kind of pruning results in a configuration in the catalog of allowable cases), and one then has the task of dealing with the latter on an individual basis.

Thus we view the main argument as inductive; but the base of the induction consists of the analysis of block paths up through length 5. While even at length 5 and below there are again many inductive reductions available, still there remain about a hundred critical cases to be examined individually (including a few stray cases of length 6 or 7, which prune down to sporadic cases of length 4 or 5). There are also some cases of variable length relating to the "generic" case of the analysis.

We will give the inductive argument in §2; after various preparations we can treat the generic case (length at least 6) efficiently in Proposition 19 (§2.7.

We then indicate how the base of the induction is obtained in §§3-6. As we have a large number of cases to treat and the details vary somewhat unpredictably, we have documented this in considerable detail. After working through the cases of length at most 4 one sees that much of the argument becomes quite routine after the first few steps, so from length 5 onward we mainly give a description of how the argument is to be set up in each case, taking note of any exceptional features that arise here and there.

The final section touches on the case in which trivial blocks are allowed.

We are still looking for an intelligible criterion for the recognition of the Rado constraints, in some form other than an explicit list. For the present, we see the list as consisting essentially of type 3^ℓ with some minor perturbations for length $\ell \geq 6$, and an unintelligible collection of special cases in shorter length, where the loop construction encounters difficulties. This takes us back to the fundamental question of the solvability of the corresponding decision problem.

1. Pruning and Theorem 2

We begin by describing two types of inductive argument which we refer to as *corner pruning* and *symmetric local pruning*. These give us weak forms of the Segment Conjecture adequate to support a variety of inductive arguments. In particular, these principles give us a quick reduction of Theorem 2 to Theorem 1, as we shall see.

Corner pruning was introduced in [CS05] as a method for reducing the classification of the Rado constraints which are trees to a large but manageable number of special cases. Symmetric local pruning was introduced in **??** to prove the solidity conjecture for block path constraints. Life would be much simpler if we had a better approximation to the segment conjecture, but what we have is enough to reduce general statements to a collection of critical cases that can be worked through individually.

1.1. **Corner Pruning.**

Definition 2.

1. A *segment* of a graph C is a connected subgraph which is a union of blocks.

2. A *corner* C_v of a graph C is a segment of the form $\{v\} \cup C'$ where v is a cut vertex and C' is one of the connected components of $C \setminus \{v\}$. Note that C_v contains a unique block B of C with $v \in B$, and that the pair (v, B) determines the corner. We call v the *root* of C_v, and B its *root block*. Note that a corner will frequently be treated as a graph with base point v (or briefly: a *pointed graph*). For pointed graphs we use the notation

$$(v, C)$$

In particular, we may consider embeddings of one corner into another either as a subgraph, or as a pointed subgraph.

Definition 3 (Pruning)**.** Let (v, S) be a pointed graph, C a graph.

1. A corner C_a of C is *pruned by* (v, S) if there is an embedding of (a, C_a) into (v, S) as a pointed subgraph.

2. The (v, S)-pruned graph $C_{(v,S)}$ is the graph obtained from C by deleting the set of vertices in

$$\bigcup \{C_a \setminus \{a\} \mid (a, C_a) \text{ is pruned by } (v, S)\}$$

We do not delete the base point of a pruned corner, only the remainder. With (v, S) specified, we generally write C' for the pruned graph $C_{(v,S)}$.

In its most rudimentary form, we may prune a tree by removing all its leaves. This corresponds to the case in which S consists of an edge. In our context, the most rudimentary form of pruning is the pruning of a minimal block leaf (that is, a block which is a leaf in the tree of blocks). This is the most commonly used form.

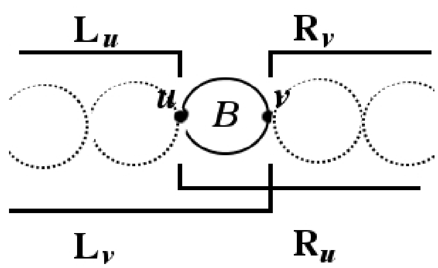
The following can be stated more generally, for sets of constraints and sets of corners, and is proved in that form in [CS14].

Lemma 4 (Pruning Induction). *Let C be a graph, (v, S) a pointed graph, and $C' = C_{(v,S)}$. If there is a countable universal C-free graph (in either the weak or strong sense) then there is a countable universal C'-free graph.*

This is our basic workhorse for inductive arguments. Sometimes we need something more refined, to be described next.

1.2. **Symmetric Local Pruning.** Our second inductive technique is the method of local pruning, also presented in [CS14]. We are interested in the form this takes in the context of block paths, which is the following.

Lemma 5 (Symmetric Local Pruning [CS14, Lemmas 3.2,3.3]). *Let C be a block path allowing a countable weakly universal C-free graph. Let B a block of C containing two cut vertices u, v, and let L_u, R_u, L_v, R_v be the corners rooted at u and v respectively, with R_u and L_v the ones containing the block B. Suppose that $L_v \setminus \{v\}$ embeds into $R_u \setminus \{u\}$. Then there is a weakly universal $(L_v \setminus \{v\})$-free graph.*



1.3. **Theorem 2 (Modulo Theorem 1).**

Definition 6. Let k be fixed. A *k-constituent* of a graph C is a maximal segment in which all blocks have order at least k.

When C is a block path, for any k it may be viewed as having some k-constituents linked by chains of smaller blocks. The case of interest is $k = 3$, when the constituents have no trivial blocks and are joined by paths.

Lemma 7. *Let C be a finite block path which is a Rado constraint. Then for any k, each k-constituent of C is a Rado constraint.*

Proof. We argue by induction on the length of C. We suppose C_0 is not C.

If the constituent C_0 does not contain either block leaf of C, then we prune a minimal block leaf of C to get C', in which C_0 is again a constituent. Then there is again a countable weakly universal C'-free graph, and induction applies.

The alternative is that C_0 is a corner of C. Let the blocks of C be denotes $(B_1, \ldots, B_\ell)$ and choose the notation so that C_0 consists of the first i blocks of C. Let C_1 be corner consisting of the remaining blocks $(B_{i+1}, \ldots, B_\ell)$.

Prune the corner C_1: if C_0 remains, then there is a countable weakly universal C_0-free graph. So suppose that pruning C_1 also removes a part of C_0. Then some reversed segment $(B_j, B_{j-1}, \ldots, B_1)$ must embed as a pointed graph into C_1, and in particular B_j must go into B_{i+1}. But

$$|B_{i+1}| < k$$

so $j \geq i+1$ and C_0 embeds into $(B_{i+2}, \ldots, B_\ell)$. Now according to Lemma 5, taking $v = v_{i+1}$

□

Proof of Theorem 2 (Modulo Theorem 1). Theorem 2 is essentially Lemma 7 with $k = 3$, apart from the fact that the result of Theorem 1 makes the statement informative. □

Now we take up the proof of Theorem 1. Here the consideration of many critical configurations seems to be unavoidable.

2. The inductive argument

In the present section, we will state without proof the base of the inductive proof of Theorem 1, and check that it suffices for an inductive argument based on pruning—for the most part, the simplest kind of pruning: pruning of minimal block leaves. We return to the particular constructions needed to deal with the critical configurations which form the base of the induction afterward.

There are many critical configurations of length at most 6 (that is, with at most 6 blocks), a couple such configurations of length 7, as well as some key configurations of variable length. These critical configurations cannot be reduced by pruning to simpler ones (though many of them contain proper segments which are also critical). Furthermore, we found it necessary to treat each of the critical configurations individually. With a few exceptions, the same method applies to each, but with specific variations that vary in a not entirely predictable manner.

2.1. **Critical Configurations.**

Proposition 8. *Let C be a solid block path of length ℓ and type $(n_1, \ldots, n_\ell)$. Under any of the following conditions, C is not a Rado constraint.*

- **Length 2**
  (1) $n_1, n_2 \geq 6$;
  (2) $n_1 = n_2 = 5$.
- **Length 3**
  (1) $n_1 = n_2 = 5 \leq n_3$;
  (2) $n_1 = n_2 = n_3 = 6$;
  (3) $4 \leq n_1 = n_3 < n_2$;
  (4) $5 \leq n_2 < n_1 = n_3$;
  (5) $3 \leq n_1 < n_2 < n_3$, $n_2 \geq 5$;
  (6) $3 \leq n_2 < n_1 < n_3$;
  (7) $3 \leq n_1 < n_3 < n_2$;
- **Length 4**
  (1) $n_1 = n_2 = n_4 < n_3$;
  (2) $n_1, n_4 < n_2, n_3$;
  (3) $n_1 \geq n_4 > n_2, n_3$; $n_2, n_3 \geq 3$; $n_1 < n_3 + n_4 - 1$;
  (4) $n_4 > n_1 > n_2 > n_3 \geq 3$;
  (5) $n_4 > n_2 > n_1 > n_3 \geq 3$;
  (6) $n_2 \geq n_4 > n_1 > n_3 \geq 3$;
  (7) $n_3 < n_1 = n_4 = 5 < n_2$;
  (8) $n_4 > n_1 > n_3 > n_2 \geq 3$;
  (9) $n_1 = n_3 = 4 < n_4 < n_2$;
  (10) $n_1 = n_3 = 4 < n_2 = n_4$;
  (11) $n_1 = n_3 = 4 < n_2 < n_4$;
  (12) *Type* $(4, 4, 4, 4)$;

(13) $n_1 = n_2 = n_3 = 4 < n_4$;
(14) $n_4 > n_1 > n_2 = n_3 \geq 4$;
(15) $n_2 \geq 5$, $n_1 = n_4 = 4$, $n_3 = 3$;
(16) $n_2 \geq n_4 \geq 5$, $n_1 = 4$, $n_3 = 3$;
(17) $n_1, n_3 < n_2 < n_4$, $n_1 \neq n_3$;
(18) $n_1 = 3$, $n_3 = 4 < n_2 = n_4$;
(19) $n_1 = 3$, $n_3 = 4 < n_4 < n_2$;
(20) $n_2 \geq 5$, $n_1 = 3$, $n_3 = n_4 = 4$;
(21) *Type* $(3, n_2, 3, n_4)$, $n_4 > n_2 \geq 5$;
(22) *Type* $(3, n_2, 3, n_4)$, $n_2 > n_4 \geq 4$;
(23) *Type* $(3, 4, 4, n)$, $n \geq 5$;
(24) *Type* $(3, 3, 4, n)$, $n \geq 5$;
(25) *Type* $(4, 3, 4, n)$, $n \geq 5$;

- **Length 5**

(1) *Type* $(n_1, n_2, 3, n_2, n_1)$, $n_1, n_2 \geq 4$, $n_1 \neq n_2$;
(2) $n_3 = n_4 = 3$, $n_5 > n_1 > n_2 > 3$;
(3) $n_3 = n_4 = 3$, $n_5 > n_2 > n_1 > 3$;
(4) *Type* $(4, 4, 4, 5^+, 4)$;
(5) *Type*$(4, 4, 4, 4, 4)$;
(6) *Type* $(3, n_2, 4, n_4, 3)$, $n_2, n_4 \geq 5$;
(7) *Type* $(3, 4, 4, n_4, 3)$, $n_4 \geq 5$;
(8) *Type* $(3, 4, 4, 4, 3)$, $n_4 \geq 5$;
(9) *Type* $(4, 4, 4, 3, 3)$;
(10) *Type* $(n, 4, 4, 3, n)$;
(11) *Type* $(4, 4, 4, 3, 4)$;
(12) *Type* $(4, 4, 4, 3, n)$, $n \geq 5$;
(13) *Type* $(3, 4, 4, 3, 5)$, $n_5 \geq 3$;
(14) *Type* $(3, n_2, 4, 3, 3)$, $n_2 \geq 5$;
(15) *Type* $(4, 4, 3, n, 3)$, $n \geq 5$;
(16) *Type* $(4, 4, 3, 4, 3)$;
(17) *Type* $(4, 4, 3, 3, n)$, $4 \leq n \leq 8$;
(18) *Type* $(n_1, 4, 3, 3, n_1)$, $n_1 \geq 4$;
(19) *Type* $(4, 4, 3, 4, 4)$;
(20) *Type* $(n_1, 3, n_3, 3, n_5)$, $n_1, n_5 < n_3$;
(21) *Type* $(n_1, 3, n_3, 3, n_3)$, $n_3 > n_1 \geq 4$;
(22) *Type* $(3, 3, 4, 3, 4)$, $n \geq 4$;
(23) *Type* $(n, 3, n, 3, n)$, $n \geq 4$;
(24) *Type* $(n_1, 3, 4, 3, n_5)$, $n_5 > n_1 \geq 5$;
(25) *Type* $(4, 3, 4, 3, n)$, $n \geq 5$;
(26) *Type* $(3, 3, 4, 3, n)$, $n \geq 5$;
(27) *Type* $(n, 3, n_3, 3, n)$, $n > n_3 \geq 4$;
(28) *Type* $(3, n_2, 3, n_4, 3)$, $n_4 > n_2 \geq 4$;
(29) *Type* $(3, n, 3, n, 3)$, $n \geq 4$;

- **Length 6**

(1) *Type* $(3, 4, 4, 4, 3, 3)$;
(2) *Type* $(4, 4, 4, 3, 3, n)$, $n \geq 5$;
(3) *Type* $(4, 4, 4, 3, 3, 4)$;
(4) *Type* $(4, 4, 4, 3, n_5, 4)$, $n_5 \geq 5$;
(5) *Type* $(4, 4, 4, 3, 4, 4)$;
(6) *Type* $(3, 4, 4, 3, 3, n)$, $n \geq 3$;
(7) *Type* $(3, 4, 4, 3, n, 3)$, $n \geq 3$;
(8) *Type* $(4, 4, 3, 3, n, 3)$, $n \geq 5$;
(9) *Type* $(4, 4, 3, 4, 3, 4)$, $n \geq 3$;
(10) *Type* $(n_1, 3, n_3, 3, 3, n_6)$, $n_6 > n_1, n_3$ *and* $n_1, n_3 \geq 4$;
(11) *Type* $(n_1, 3, n_3, 3, 3, n_6)$, $n_3 > n_6 > n_1 \geq 4$;
(12) *Type* $(n, 3, n, 3, 3, n_6)$, $n_6 > n \geq 4$;
(13) *Type* $(3, 3, n_3, 3, 3, n_6)$, $n_6 > n_3 \geq 4$;
(14) *Type* $(3, 3, n_3, 3, 3, n_6)$, $n_3 > n_6 \geq 4$;
(15) *Type* $(n_1, 3, 3, n, 3, n)$, $3 \leq n_1 < n$;
(16) *Type* $(n, 3, n_3, 3, 3, n)$, $n_3 > n \geq 4$;
(17) *Type* $(n, 3, n_3, 3, 3, n)$, $n > n_3 \geq 4$;
(18) *Type* $(n, 3, n, 3, 3, n)$, $n \geq 4$;
(19) *Type* $(3, n_2, 3, 3, n_5, 3)$, $n_5 > n_2 \geq 4$;
(20) *Type* $(3, 3, n, 3, n, 3)$, $n \geq 4$;
(21) *Type* $(3, 3, 4, 3, n, 3)$, $n \geq 5$;

- **Length 7**
  (1) *Type* $(3, 4, 4, 3, 3, n, 3)$, $n \geq 3$;
  (2) *Type* $(n, 3, 3, 3, n, 3, n)$, $n \geq 4$;
  (3) *Type* $(3, 3, n_3, 3, 3, n_6, 3)$, $n_3 \neq n_6$, $n_3 \geq 4$;
  (4) *Type* $(3, 3, 3, n, 3, n, 3)$;
- **Variable Length**
  (1) $n_1, n_\ell > n_i$ $(1 < i < \ell)$, $n_2 = 4$;
  (2) *Type* $(3, 4, 4, 3, 3, \ldots, 3)$;
  (3) *Type* $(4, 4, 3, 3 \ldots, 3, 4)$;
  (4) *Type* $(n, 3, \cdots, 3, n)$, $\ell \geq 6$;
  (5) *Type* $(n_1, 3, \cdots, 3, n_\ell)$, $n_\ell > n_1 \geq 4$, $\ell \geq 6$;
  (6) *Type* $(3, 3, \ldots, n, 3, 3)$, $n \geq 4$;

These points will all be proved by explicit constructions. First we will carry out the inductive proof of Theorem 1, based on Proposition 8.

An efficient way of arriving at this list would be to assume the Segment Conjecture initially. Given the known restrictions on Rado constraints of length 2, one adds to the list all configurations of length 3 that contain a non-Rado segment of length 2, but which cannot be reduced by pruning to non-Rado segments. Then one examines the remaining configurations of length 3 that are not settled by this.

One may then proceed similarly with length 4, with considerably sharper constraints inherited from lengths 2,3, and so on. After the fact one must eventually show by construction that every putatively "non-Rado" constraint so identified is indeed non-Rado. But this method does produce by a reasonably efficient process both the statement of Theorem 1 and the statement of Proposition 8, as well as a proof that the latter implies the former.

Modulo Proposition 8, what needs to be shown to prove Theorem 1 is that a solid block path which cannot be pruned by corner pruning or local pruning down to one of the configurations ruled out by Proposition 8 must be of one of the forms listed in Theorem 1.

For length 1 or 2, Proposition 8 and Theorem 1 say the same thing. For length 3, some pruning is needed to complete the argument For lengths 4 and 5 we have a large number of critical configurations, but also a large number of configurations which must be reduced to shorter length by pruning. And from length 6 onward we begin to be in the general case, with relatively few critical configurations, so that the argument deals mainly with the relevant types of pruning.

2.2. **Length up to 3.** Proposition 1 is vacuous for length 1. The case of length 2 was treated earlier, and in a sharper form, in [CT07]: in this case, the configurations listed in Theorem 1 are *precisely* the Rado constraints, a claim that we are not making in general.

Fact 3 (Length 2)**.** *Let C be a solid block path of length 2 of type (n_1, n_2). Then C is a Rado constraint if and only if the following hold.*

- $\min(n_1, n_2) \leq 5$
- $(n_1, n_2) \neq (5, 5)$

So our discussion begins in a serious way with length 3.

Proposition 9 (Length 3)**.** *Let C be a solid block path which is a Rado constraint, of length 3, with all block sizes at least 3. Then*

$$n_2 \leq 4 \text{ or } n_1 = n_3 = 3$$

Proof. We choose notation so that $n_1 \leq n_3$.

We may assume toward a contradiction that

$$n_3 \geq 4 \qquad\qquad n_2 \geq 5$$

Case 1. $n_1 = n_3$.

We may put the argument in tabular form, listing the relevant clauses of Proposition 8 under appropriate hypotheses.

Hypothesis	Clauses
$n_1 = n_2$	3.1,2
$n_1 \neq n_2$	3.3,4

Here we must remark that when $n_1 = n_2 = n_3$, we apply Lemma 5 to conclude that a block path of type $(n_1, n_2 - 1)$ is a Rado constraint, and hence $n_2 \leq 6$.

Case 2. $n_1 < n_3$.

Then by pruning, a solid block path of type (n_2, n_3) is a Rado constraint. This forces $n_2 \neq n_3$, and if $n_2 < n_3$ then $n_2 = 5$.

If n_1, n_2, n_3 are distinct then one of clauses 3.5,6,7 of Proposition 8 applies.

So there remains the case $n_1 = n_2 < n_3$, and $n_2 = 5$, which is covered by clause 3.1 of Proposition 8. □

2.3. **Forbidden segments:** 5^+5^+ **and** $4^+4^+4^+4^+$**.** It will be useful to prove some forbidden segment results concerning segments of length 2. To begin with, we want to show that the non-Rado solid block path constraints of length 2 cannot occur as segments of Rado solid block path constraints. But we will strengthen this for the case of solid block paths of length at least 4: in this case we claim that whenever $n_i \geq 5$, then $n_{i\pm 1} = 3$ (where defined).

Lemma 10. *Let C be a solid block path of length ℓ in which all block sizes are at least 5. If C is a Rado constraint then $\ell \leq 2$.*

Proof. Supposing the contrary, let C be a counterexample of minimal length $\ell \geq 3$. By Proposition 9, $\ell \neq 3$. So we suppose $\ell \geq 4$.

Then pruning a minimal block leaf will provide a counterexample of shorter length unless $\ell = 4$ and $n_1 = n_4$. In this case, pruning the block leaves will give a solid block path of type (n_2, n_3), and we may suppose

$$n_2 = 5$$
$$n_3 \geq 6$$

If $n_1 > 5$ then pruning the reversed initial segment (B_2, B_1) leaves the segment (B_3, B_4) with $n_3, n_4 \geq 6$, contradicting the case of length 2.

If $n_1 = 5$ then clause 4.1 of Proposition 8 applies and gives a contradiction. □

Corollary 11. *Let C be a solid block path of length ℓ which is a Rado constraint. Suppose $i < \ell$ and $n_i, n_{i+1} \geq 5$. Then*

- *$n_{i-1}, n_{i+2} \leq 4$, when defined;*
- *$\min(n_i, n_{i+1}) = 5$ and $\max(n_i, n_{i+1}) \geq 6$.*

Proof. By Lemma 7, the 5-constituent containing the block B_i is a Rado constraint. So Lemma 10 and Fact 3 apply. □

Lemma 12. *Let C be a solid block path of length $\ell \geq 2$ with all block sizes at least 3, and suppose that for some $i < \ell$ we have*

$$n_i, n_{i+1} \geq 5$$

If C is a Rado constraint, then $\ell = 2$.

Proof. We suppose C is a counterexample with ℓ minimal. The case of length 3 is covered by Proposition 9. So we suppose

$$\ell \geq 4 \tag{1}$$

If $1 < i < \ell - 1$, then prune a minimal block leaf to reduce the length by at most 2. This forces $\ell = 4$, $i = 2$, and $n_1 = n_4$. By Corollary 11 we have $n_1 < 5$ and clause 4.2 of Proposition 8 applies.

So $i = 1$ or $\ell - 1$, and we may choose the numbering so that

$$i = 1 \tag{2}$$

By Corollary 11 we have

$$\min(n_1, n_2) = 5 \qquad \max(n_1, n_2) \geq 6 \tag{3}$$

$$n_3 \leq 4 \tag{4}$$

If $n_\ell < n_1$ then prune the last block to reach a contradiction. So we suppose

$$n_\ell \geq n_1 \tag{5}$$

Now we need to treat the cases $\ell = 4$, $\ell > 4$ separately.

Case 1. $\ell = 4$

If $n_1 \geq 6$ then $n_2 = 5$ and either clause 4.3 or 4.4 of Proposition 8 applies. So $n_1 = 5$, and $n_2 \geq 6$.

If $n_4 > n_1$ then depending on the relative size of n_2 and n_4, either clause 4.5 or 4.6 of Proposition 8 applies.

On the other hand, if $n_1 = n_4 = 5$, then clause 4.7 applies.

This disposes of the case $\ell = 4$. The analysis in the second case is longer and will again involve some special considerations for $\ell = 5$.

Case 2. $\ell \geq 5$

We show first that

$$n_5 \geq 5 \tag{6}$$

Suppose on the contrary $n_5 \leq 4$. Prune the terminal segment $R_5 = (B_5, \ldots, B_\ell)$. If this leaves a segment containing (B_1, B_2, B_3) then we contradict the minimality of ℓ. So there must be some reversed initial segment $(B_j, B_{j-1}, \ldots, B_1)$ embedding into R_5 over a basepoint of B_5. Since $j \geq 3$, this gives an embedding of (B_4, B_3, B_2, B_1) into $(B_4, B_5, \ldots, B_\ell)$. Then by Lemma 5, a solid block path of type $(n_1, n_2, n_3, n_4 - 1)$ must be a Rado constraint. This again contradicts the minimality of ℓ. So (6) follows.

Next we show

$$n_4 \leq 4 \tag{7}$$

Otherwise, we have $n_4, n_5 \geq 5$, and pruning the first block will yield a contradiction unless $\ell = 5$ and $n_1 = n_5$. In this case, we may choose the numbering so that $n_4 \leq n_2$. If $n_4 < n_2$ we prune the terminal segment (B_4, B_5) for a contradiction. So we arrive at the symmetric case $\ell = 5$, $n_1 = n_5 = 5$, $n_2 = n_4 \geq 6$, $n_3 \leq 4$. Then Lemma 5 says that a solid block path of type $(n_1, n_2, n_3 - 1)$ is a Rado constraint, which in light of Proposition 9 forces $n_3 = 3$. In this case, if $n_4 > 3$ then clause 5.1 applies.

So (7) holds.

Now prune the terminal segment $(B_4, B_5, \ldots, B_\ell)$. Again Lemma 5 will apply, and gives a contradiction unless

$$n_3 = 3 \tag{8}$$

At this point, we examine the possibility $\ell = 5$ separately.

Pruning the terminal segment (B_4, B_5) will give a contradiction unless

$$n_5 \geq |B_1 B_2| = n_1 + n_2 - 1 > n_1, n_2$$

In this case, pruning the initial block B_1 leaves a solid block path of type (n_2, n_3, n_4, n_5) with

$$n_5 > n_2 > n_4 > n_3 = 3$$

If $n_4 > n_3$ then clause 4.8 applies, and if $n_4 = n_3 = 3$, then clause 5.2 or 5.3 applies.

So we may suppose

$$\ell \geq 6 \tag{9}$$

We summarize the conditions established so far.

$$n_1, n_2, n_5 \geq 5 \qquad\qquad n_3 = 3 \qquad\qquad n_4 \leq 4$$

To conclude, it will suffice to consider n_6.

If $n_6 \geq 5$ then pruning the reversed initial segment $L_3 = (B_3, B_2, B_1)$ will give a contradiction unless the terminal segment $(B_4, B_5, \ldots, B_\ell)$ embeds into L_3. In this case Lemma 5 gives a contradiction unless $\ell = 6$ and $n_4 = 3$. Applying this to the reversal of C, it then follows that C is fully symmetric. So pruning the block leaves B_1, B_6 leaves a solid block path of type (n_2, n_3, n_4, n_5) with $n_2 = n_5 \geq 5$ and $n_3 = n_4 = 3$. So clause 4.3 of Proposition 8 applies.

On the other hand, if $n_6 \leq 4$, then we prune the terminal segment $R_6 = (B_6, B_7, \ldots, B_\ell)$, getting a contradiction unless some reversed initial segment $(B_j, B_{j-1}, \ldots, B_1)$ embeds into R_6 over a basepoint in B_6. Since $n_4 < n_5$ this forces the reversed initial segment (B_4, B_3, B_2, B_1) to embed into $(B_5, B_6, \ldots, B_\ell)$ and then Lemma 5 gives a Rado constraint of type $(n_1, n_2, n_3, n_4 - 1)$, contradicting the minimality of ℓ. □

Lemma 13. *Let C be a solid block path in which four consecutive block sizes $n_i, n_{i+1}, n_{i+2}, n_{i+3}$ are at least 4. Then C is not a Rado constraint.*

Proof. We take a counterexample C of minimal length ℓ. By Lemma 7, all block sizes of C are at least 4.

As pruning a minimal block leaf reduces the length by at most 2, we find

$$\ell \leq 5 \tag{1}$$

Case 1. $\ell = 4$

Suppose first that $n_2 > 4$. By Lemma 12 we have $n_1 = n_3 = 4$. If $n_4 = 4$ then clause 4.1 applies to the reversal of C. If $n_4 > 4$ then one of clauses 4.9,10,11 applies.

As the case with $n_3 > 4$ is the same up to reversal, the remaining alternative is $n_2 = n_3 = 4$. We may suppose $n_1 \leq n_4$.

If $n_1 = 4$ then clause 4.12 or 4.13 applies. If $n_1 > 4$ then clause 4.14 applies.

This completes the analysis when $\ell = 4$.

Case 2. $\ell = 5$

In this case we must have

$$n_1 = n_5 \tag{2}$$

Pruning B_1, B_5 gives a solid block path of type (n_2, n_3, n_4). By Proposition 9 we conclude

$$n_3 = 4 \tag{3}$$

We may suppose that $n_2 \leq n_4$. Then the reversed initial segment (B_3, B_2, B_1) embeds into (B_3, B_4, B_5) and Lemma 5 says that a solid block path of type $(n_1, n_2, n_3 - 1)$ is a Rado constraint. By Proposition 9 we find

$$n_2 = 4 \tag{4}$$

If $n_4 > 4$ then by Lemma 12 we have $n_5 = 4$, and clause 5.5 or 5.4 applies.

On the other hand, if $n_4 = 4$ then the variable length clause V.1 applies (we will use the label V for "variable").

□

2.4. **Length 4.**

Proposition 14 (Length 4)**.** *Let C be a solid block path which is a Rado constraint, of length 4, with all block sizes at least 3.*

Then up to reversal, C has one of the following types.

- $n_3 \geq 5$
  (1) $(n_3, 3, n_3, 3)$
  (2) $(3, 3, n_3, 3)$
- $n_2 = n_3 = 4$
  (1) $(3, 4, 4, 4)$
- $n_2 = 4, n_3 = 3$
  (1) $(4, 4, 3, n_4)$ *($n_4 \geq 3$)*
  (2) $(3, 4, 3, n_4)$ *($n_4 \geq 3$)*
- $n_2 = n_3 = 3$
  (1) $(n_1, 3, 3, n_4)$*, $n_1 \leq n_4$; if $n_1 \geq 4$ then $n_4 \geq n_1 + 2$*

Proof. We treat first the special cases in which $n_1 = n_4$, or $n_2 \geq 5$.

Case 1. $n_1 = n_4$

Suppose first that $n_1 = n_4 = 3$. Then by clause 4.2, we have $\min(n_2, n_3) = 3$ and we may suppose $n_2 = 3$. This is one of the allowed types and there is nothing more to prove in this case.

So now suppose that $n_1 = n_4 \geq 4$. Then by Lemma 13 we may suppose $n_3 = 3$.

If $n_2 < n_1$ then clause 4.3 applies.

If $n_1 \leq n_2$ then by Lemma 12 we find $n_1 = 4$. As type $(4, 4, 3, 4)$ is allowed, we may suppose $n_2 \geq 5$. Then clause 4.15 applies.

Case 2. $n_2 \geq 5$, $n_1 \neq n_4$

By Lemma 12 we have $n_1, n_3 \leq 4$. Suppose first

- $n_1 = 4$

If $n_4 = 3$ then we prune the block B_4 and contradict Proposition 9. So $n_4 \geq 4$; but $n_4 \neq n_1$, so $n_4 \geq 5$.

Now Lemma 13 yields $n_3 = 3$. Then clause 4.16 or 4.17 applies.

Now suppose

- $n_1 = 3$, $n_3 = 4$

If $n_4 \geq 5$ then clause 4.17, 18, or 19 applies.

If $n_4 \leq 4$ then clause 4.2 or 4.20 applies.

Finally, suppose

- $n_1 = n_3 = 3$

The claim is that $n_4 = 3$ or n_2. In all other cases, clause 4.21 or 4.22 applies.

Since the case $n_3 \geq 5$ involves only a change of notation, the remaining case is the following.

Case 3. $n_2, n_3 \leq 4$, *and* $n_1 \neq n_4$

We may suppose $n_2 \geq n_3$.

If $n_2 = n_3 = 4$ then by Lemma 13 we may suppose $n_1 = 3$. If $n_1 = 3$ then clause 2 applies, and if $n_1 = 4$ then we have one of the allowed configurations, of type $(3, 4, 4, 4)$. If $n_1 \geq 5$ then clause 4.23 applies.

So we may suppose $n_3 = 3$.

If $n_2 = 4$, then as the configurations with $n_1 \leq 4$ are allowed, we may suppose $n_1 \geq 5$. Then clauses 4.24,25 force $n_4 \geq 5$. If $n_4 > n_1$ then clause 4.8 applies to C. If $n_1 > n_4$ then the variable length clause V.**??** applies.

Finally, suppose $n_2 = n_3 = 3$. Then we may suppose $n_1 < n_4$. If $n_1 = 3$ we have no constraint, and if $n_1 \geq 4$ then by clause 4.3 we have $n_4 \geq n_1 + 2$.

This completes the analysis in all cases. $\square$

2.5. **Segments of type 4^+4^+.** We now consider solid block paths containing two consecutive blocks of size 4 or more. We begin with the case in which three consecutive blocks have order at least 4. Rather than establishing an absolute result, we embed this into the context of an inductive proof of Theorem 1.

Lemma 15. *Let C be a solid block path of length ℓ in which all blocks have size at least 3. Suppose that three consecutive block sizes n_i, n_{i+1}, n_{i+2} are all at least 4, and C is a a Rado constraint. Suppose further that all solid block paths of length less than ℓ which are Rado constraints satisfy the conditions of Theorem 1. Then $\ell \le 4$, and one of the following applies.*

- $\ell = 3$ *and* $n_2 = 4$
- $\ell = 4$ *and C or its reversal is of type* $(4, 4, 4, 3)$

Proof. Suppose the contrary.

By Lemma 13 the 4-constituent C_0 containing the blocks B_i, B_{i+1}, B_{i+2} consists of just these three blocks, and by Lemma 7 it is a Rado constraint. In particular

$$n_{i+1} = 4 \tag{1}$$

If $\ell \le 4$ then it suffices to apply Proposition 9 or 14. So we suppose

$$\ell \ge 5 \tag{2}$$

We choose notation so that $i \le \ell/2$. Our first claim is

$$i = 1 \tag{3}$$

Supposing the contrary, notice that by Lemma 13 we have $n_{i-1} = n_{i+3} = 3$. Furthermore, after pruning a minimal block leaf we have one of the configurations listed above, so $\ell \le 6$ and indeed $\ell \le 5$ if $n_1 \ne n_\ell$.

Suppose first that $n_1 \ge 4$. Then $i \ge 3$ so $\ell \ge 6$. It follows that $\ell = 6$ and $n_1 = n_\ell$. But here $i = 3$, $n_{i+3} = 3$, and we have a contradiction.

So $n_1 = 3$. If $n_\ell > 3$ then pruning the initial block B_1 gives a contradiction. So $n_1 = n_\ell = 3$. Thus we are dealing with type $(3, n_2, 4, n_4, 3)$ or $(3, 4, 4, 4, 3, 3)$ with $n_2, n_4 \ge 4$, and one of clauses 5.6,7,8, 6.1 applies.

Thus claim (3) holds.

In particular,

$$n_2 = 4 \qquad\qquad n_4 = 3 \tag{4}$$

Next we dispose of the case $\ell = 5$. If $n_5 < n_1$ we prune the final block B_5 and find $n_1 = n_2 = n_3 = 4$, hence $n_5 = 3$ and clause 5.9 applies.

If $n_5 = n_1$ then again we prune the block B_5 and we find $n_3 = 4$. Then clause 5.10 or 11 applies.

Now suppose $n_5 > n_1$. Then we prune the initial block B_1 to find $n_3 = 4$. If $n_1 \ge 5$ then the variable length clause V.1 applies. If $n_1 = 4$ then clause 5.12 applies.

So we may suppose

$$\ell \geq 6 \tag{5}$$

If $n_\ell < n_1$ we have an immediate contradiction by pruning the last block B_ℓ, since the pruned block path should satisfy Theorem 1.

Similarly if $n_1 < n_\ell$ and we prune the first block B_1, then since $\ell \geq 6$ the only possibility for the type of the pruned block path C' is $(4, 4, 3, 3, n_6)$ with $n_6 \geq 9$, so the type of C is $(n_1, 4, 4, 3, 3, n_6)$ with $4 \leq n_1 < n_6$. Then pruning the terminal segment (B_5, B_6) shows $n_1 = 4$ and clause 6.2 applies.

So we suppose

$$n_1 = n_\ell \tag{6}$$

If $n_{\ell-1} = 3$ we prune the terminal segment $(B_{\ell-1}, B_\ell)$ and the pruned block path must have type $(4, 4, 4, 3)$, which implies $\ell = 6$ and the type of C is $(4, 4, 4, 3, 3, 4)$. So clause 6.3 applies.

So we suppose

$$n_{\ell-1} \geq 4 \tag{7}$$

In this case, after pruning the minimal block leaves and applying Theorem 1 to the residue, we find

$$\ell = 6 \tag{8}$$

and the type is $(n_1, 4, 4, 3, n_5, n_1)$.

If $n_1, n_5 \geq 5$ we prune the reversed initial segment (B_2, B_1) and arrive at a contradiction. So n_1 or n_5 equals 4.

If $n_1 = 4 < n_5$ then clause 6.4 applies. If $n_5 = 4 < n_1$ then the variable length clause V.1 applies. If $n_1 = n_5 = 4$ then clause 6.5 applies. □

We continue this analysis for the case of two successive blocks of size at least 4, a crucial case for the analysis in general.

Lemma 16. *Let C be a solid block path of length $\ell \geq 5$ in which all blocks have size at least 3. Suppose that two consecutive block sizes n_i, n_{i+1} are both at least 4, and C is a a Rado constraint. Suppose further that all solid block paths of length less than ℓ which are Rado constraints satisfy the conditions of Theorem 1. Then*

$$n_i = n_{i+1} = 4$$

and C or its reversal has one of the following types.

(1) $(4, 4, 3^{\ell-2})$
(2) $(4, 4, 3, 3, n)$ *with* $n \geq 9$ *(length 5).*

Proof. We suppose that C is a counterexample of length ℓ, and that i is minimized. In particular since C may be reversed, we have $i \leq \ell/2$.

By Lemma 15 we have

$$n_{i-1}, n_{i+2} = 3$$

when defined.

We divide into cases according to the value of i.

Case 1. $i \geq 3$

In this case, $\ell \geq 6$. If we prune a minimal block leaf then the length of the residue is at least 4 and contains the pattern $3, n_i, n_{i+1}, 3$, so we reach a contradiction.

Case 2. $i = 2$

Then

$$n_1 = n_4 = 3$$

Suppose first

- $n_\ell > 3$

Then pruning the first block B_1 leaves a residue C' of the form $(4, 4, 3, n)$ with $n \geq 4$ or $(4, 4, 3, 3, n)$ with $n \geq 9$ (bearing in mind that $n_\ell \geq 4$).

We find that $\ell = 5$ or 6 and C has type $(3, 4, 4, 3, n_5)$ or $(3, 4, 4, 3, 3, n_6)$. Then clause 5.13 or 6.6 applies.

Now suppose

- $n_1 = n_\ell = 3$

Then pruning the block leaves shows that $n_3 = 4$. If $\ell = 5$ then the type is $(3, n_2, 4, 3, 3)$ and clause 5.14 or 5.13 applies. So suppose $\ell \geq 6$ and the residue C' has length at least 4.

Then C has one of the following types.

(1) $(3, 4, 4, 3, n_5, 3)$ with $n_5 \geq 4$ $(\ell = 6)$
(2) $(3, 4, 4, 3, 3, n_6, 3)$ with $n_6 \geq 9$ $(\ell = 7)$
(3) $(3, 4, 4, 3^{\ell-3})$ $(\ell \geq 5)$

Now clause 6.7, 7.1 or V.2 applies.

Case 3. $i = 1$

In this case, we subdivide further by comparing n_1 and n_ℓ.

Case 3A. $n_\ell < n_1$

Then we prune the final block B_ℓ, leaving a residue of length at least 4 with $n_1 = 4$, hence $n_\ell = 3$; so C is of one of the following types.

(1) $(4, 4, 3^{\ell-2})$
(2) $(4, 4, 3, n, 3)$, $n \geq 4$ $(\ell = 5)$
(3) $(4, 4, 3, 3, n, 3)$, $n \geq 9$ $(\ell = 6)$

As type $(4, 4, 3^{\ell-2})$, is one of the allowed configurations, it suffices to deal with the other two.

C has type $(4, 4, 3, n, 3)$, with $n \geq 4$ then clause 5.15 or 5.16 applies.

If C has type $(4, 4, 3, 3, n, 3)$ with $n \geq 9$then clause 6.8 applies.

Case 3B. $n_\ell > n_1$

We prune the initial block B_1, leaving a residue C' of length $\ell - 1$ whose first and last block sizes are both at least 4. It follows that $\ell \le 6$ and C is of one of the following forms.

(1) $(n_1, n_2, 3, 3, n_5)$
(2) $(n_1, 4, 4, 3, 3, n_6)$, $n_6 \ge 9$

The type $(n_1, n_2, 3, 3, 4, 4)$ cannot occur here as $n_\ell > n_1 \ge 4$.

Pruning the terminal segment $(B_{\ell-1}, B_\ell)$ shows that $n_2 = 4$, and if $\ell = 6$ then $n_1 = 4$. That is, C has one of the following types.

(1) $(n_1, 4, 3, 3, n_5)$, $n_5 > n_1 \ge 4$
(2) $(4, 4, 4, 3, 3, n_6)$, $n_6 \ge 9$

In the first case, if $n_1 \ge 5$ then the variable length clause V.1 applies, and if $n_1 = 4$ then by clause 5.17 we arrive at the allowed configuration $(4, 4, 3, 3, n_5)$ with $n_5 \ge 9$.

In the second case, clause 6.2 applies.

Case 3C. $n_1 = n_\ell$

We may suppose $n_2 \ge n_{\ell-1}$, since otherwise we reverse C.

Suppose first that

- $n_2 > n_{\ell-1}$

Then we prune the terminal segment $(B_{\ell-1}, B_\ell)$, leaving a residue C' of length $\ell - 2 \ge 3$ of the form $(n_1, n_2, 3, \ldots, n_{\ell-2})$. If $\ell \ge 6$ then this forces $n_1 = n_2 = 4$ and hence $n_{\ell-1} = 3$. So C is of one of the following forms.

(1) $(4, 4, 3^{\ell-3}, 4)$
(2) $(n_1, 4, 3, 3, n_1)$, $n_1 \ge 5$ $(\ell = 5)$;
(3) $(4, 4, 3, n_4, 3, 4)$, $n_4 \ge 4$ $(\ell = 6)$

In the first case, the variable length clause V.3 applies.

In the second case, clause 5.18 applies.

In the last case, pruning the block leaves shows $n_4 = 44$. and clause 6.9 applies.

Now suppose that

- $n_2 = n_{\ell-1}$

Then we have $n_3 = n_{\ell-2} = 3$.

If $\ell = 5$, then C has the symmetric type $(n_1, n_2, 3, n_2, n_1)$. For $n_1 \ne n_2$ this is covered by clause 5.1. If $n_1 = n_2$ then by Lemma 12 we have $n_1 = 4$ and clause 5.19 applies.

If $\ell \ge 6$, then pruning the block leaves results in a residue C' of type $(n_2, n_3, \ldots, n_{\ell-1})$ with $n_2 = n_{\ell-1} \ge 4$, $n_3 = n_{\ell-3} = 3$, which contradicts the relevant instance of Theorem 1. □

2.6. **Length 5.** Our target now is the following.

Proposition 17. *Let C be a solid block path of length 5 with all block sizes n_i at least 3. If C is a Rado constraint, then up to reversal the type of C is of one of the following forms.*

(1) $(3, 3, 3, 3, n)$*;*
(2) $(3, 3, 3, n, 3)$*;*
(3) $(3, 3, 3, 4, 4)$*;*
(4) $(4, 4, 3, 3, n)$ *with* $n \geq 9$
(5) $(3, n_2, 3, 3, n_5)$ *with* $n_2, n_5 \geq 4$ *and* $|n_2 - n_5| \geq 2$
(6) $(3, 3, n, 3, n)$ *with* $n \geq 5$

Proof. In Lemma 16 we disposed of the case in which two consecutive block sizes are at least 4, so we suppose here that this is not the case.

Case 1. $n_3 \geq 4$
In this case, $n_2 = n_4 = 3$. We may suppose $n_1 \leq n_5$.
If $n_5 < n_3$, then clause 5.20 applies.
Suppose $n_5 = n_3$. If $n_1 < n_5$ then clause 5.21 applies when $n_1 \geq 4$, and when $n_1 = 4$ clause 5.22 says that $n_3 \geq 5$, at which point we have an allowed configuration. If $n_1 = n_3 = n_5$ then clause 5.23 applies.
Finally, suppose $n_5 > n_3$.
If $n_1 < n_5$ then we prune the initial block B_1 leaving a residue of type $(3, n_3, 3, n_5)$ with $4 \leq n_3 < n_5$. This forces $n_3 = 4$: the type of C is $(n_1, 3, 4, 3, n_5)$ with $n_1 < n_5$ and $n_5 \geq 5$. Then clause 5.24, 25, or 26 applies.
If $n_1 = n_5 > n_3$ then clause 5.27 applies.

Case 2. $n_2 \geq 4$
Now we have $n_1 = n_3 = 3$.
If $n_4 \geq 4$ then the type is $(3, n_2, 3, n_4, 3)$ and we may suppose $n_2 \leq n_4$. Then clause 5.28 or 5.29 applies.
Suppose $n_4 = 3$, so the type is $(3, n_2, 3, 3, n_5)$ with $n_2 \geq 4$. If $n_5 = 3$ the configuration is allowed, so suppose $n_5 \geq 4$. Pruning the first block we find $|n_5 - n_2| \geq 2$ and now the configuration is allowed.

Case 3. $n_2 = n_3 = n_4 = 3$
We may suppose $n_1 \leq n_5$.
If $n_1 = 3$ this is an allowed configuration, so we suppose $n_1 \geq 4$. Then one of the variable length clauses V.4, 5 applies. □

So at this point we have the following.

Proposition 18. *Theorem 1 holds for solid block paths of length at most 5.*

Proof. This is vacuous for length 1, and the other possibilities are treated in Fact 3 and Propositions 9, 14, 17. □

With this, the base of an inductive proof of Theorem 1 is complete, and we may now treat the general case inductively.

2.7. **The generic case.**

Proposition 19. *Let* C *be a solid block path of length* $\ell \geq 6$ *with all block sizes* $n_i \geq 3$*. If* C *is a Rado constraint then up to reversal the type of* C *has one of the following forms.*

(1) $(3, 3, \ldots, n, 3)$*;*
(2) $(3, 3, \ldots, 3, n)$*;*
(3) $(3, 3, \ldots, 3, 4, 4)$*;*

Proof. Let C be a counterexample of minimal length ℓ. By Proposition 18 we have

$$\ell \geq 6$$

Now Lemma 16 applies.

If $n_i \geq 4$ *then* $n_{i\pm 1} = 3$*, when defined.*

We may suppose $n_1 \leq n_\ell$.

Case 1. $n_1 < n_\ell$

In particular $n_\ell \geq 4$ and $n_{\ell-1} = 3$. We prune the initial block to get a residue C' of length $\ell - 1 \geq 5$ with terminal block of size at least 4. The possible types for C' are the following.

(1) $(3, \ldots, n_\ell)$
(2) $(3, \ldots, 4, 4)$
(3) Length 5: $(3, n_3, 3, 3, n_6)$ with $n_3, n_6 \geq 4$ and $|n_3 - n_6| \geq 2$
(4) Length 5: $(3, 3, n, 3, n)$ with $n \geq 5$

We list the corresponding possibilities for the type of C.

(1) $(n_1, 3, \ldots, 3, n_\ell)$ with $n_1 < n_\ell$, $\ell \geq 6$
(2) $(3, \ldots, 4, 4)$ (allowed)
(3) Length 6: $(n_1, 3, n_3, 3, 3, n_6)$, $|n_3 - n_6| \geq 2$, $n_1 < n_6$, $n_3 \geq 4$
(4) Length 6: $(n_1, 3, 3, n, 3, n)$ with $n_1 < n$, $n \geq 5$

In type (1), if $n_1 \geq 4$ then the variable length clause V.5 applies, while if $n_1 = 3$ the configuration is allowed.

Type (2) is allowed.

In type (3), we have $n_3 \neq n_6$. If $n_1 \neq n_3$ and $n_1 \geq 4$ then clause 6.10 or 6.11 applies. If $n_1 = n_3$ then clause 6.12 applies. If $n_1 = 3$ then clause 6.13 or 6.14 applies.

In type (4), clause 6.15 applies.

Case 2. $n_1 = n_\ell \geq 4$

Then $n_2, n_{\ell-1} = 3$.

We prune the block leaves of C and as the residue C' has length at least 4 and initial and terminal blocks of order 3, and no consecutive blocks of order at least 4, applying Theorem 1 to the residue shows that up to reversal it may be taken to have the following type.

$$(3^{\ell-4}, n_{\ell-2}, 3)$$

That is, the type of C is

$$(n_1, 3^{\ell-4}, n_{\ell-2}, 3, n_1)$$

with $n_1 \geq 4$.

If $n_{\ell-2} = 3$ then the variable length clause V.4 applies. So suppose $n_{\ell-2} \geq 4$.

Then we prune the reversed initial segment (B_3, B_2, B_1). If $\ell \geq 8$ this gives a contradiction. We consider the case $\ell = 6$ or 7 separately.

If $\ell = 6$ then clause 6.16, 17, or 18 applies.

If $\ell = 7$, C is of type $(n_1, 3, 3, 3, n_5, 3, n_7)$, then the residue of the last pruning has type $(3, n_5, 3, n_1)$ with $n_1, n_5 \geq 4$. So $n_1 = n_5$ and clause 7.2 applies.

Case 3. $n_1 = n_\ell = 3$

Now after pruning the leaf blocks we arrive at the following possible types for C. In each case we have indicated the relevant clauses from Proposition 8, apart from the first case, which is an allowed configuration.

(1) $(3, \ldots, 3, n_{\ell-1}, 3)$, $n_{\ell-1} \geq 3$; (allowed)
(2) $(3, \ldots, 3, n_{\ell-2}, 3, 3)$, $n_{\ell-2} \geq 4$; variable length clause V.6
(3) Length 6: $(3, n_2, 3, 3, n_4, 3)$ $n_4 \geq n_1 + 2 \geq 6$; clause 6.19
(4) Length 6: $(3, 3, n_3, 3, n_3, 3)$ clause 6.20
(5) Length 6: $(3, 3, 4, 3, n_5, 3)$ clause 6.21
(6) Length 7: $(3, 3, n_3, 3, 3, n_6, 3)$, $|n_3 - n_6| \geq 2$ clause 7.3
(7) Length 7: $(3, 3, 3, n, 3, n, 3)$, $n \geq 5$ clause 7.4

This completes the analysis

□

Modulo the critical configurations listed in Proposition 8, this completes the proof of Theorem 1. Length at most 5 is covered by Proposition 18, and length at least 6 is covered by Proposition 19.

3. Critical Configurations of Length 3

Now we take up the proof of Proposition 8.

There is a standard method for proving that a constraint C is not a Rado constraint, which was given a theoretical justification in [CSS99]: when the key step in the usual construction fails, there is always a universal countable C-free graph, and even one with oligomorphic automorphism group. However we will not concern ourselves further with the general theory here. We will however lay out the general form of the construction here, as it is typically encountered in practice.

3.1. C-rigid graphs.

If the reader is not familiar with this type of construction it may be better to return to this general discussion later, with one of the concrete cases discussed below in mind.

Definition 20. Let C be a constraint graph. Let G be a countable C-free graph and A a fixed finite set of vertices in G.

1. Two extensions G_1, G_2 of G are *C-incompatible* over A if there is no C-free graph G^* for which there are isomorphisms

$$f_i : G_i \to G^* \ (i = 1, 2)$$

with subgraphs of G^*, and with f_1, f_2 agreeing on A.

2. G is a *C-skeleton* if it contains a finite subset A such that there is an uncountable family $\mathcal{G}$ of C-free graphs which are pairwise C-incompatible over A.

Once we have found a C-skeleton we will be done.

Lemma 21. *Let C be a constraint graph. If there is a C-skeleton, then there is no countable weakly universal C-free graph.*

Proof. If G^* is a countable weakly universal C-free graph then we have an uncountable family of embeddings $f_X : X \to G^*$ where X varies over the associated family $\mathcal{G}$ of extensions of G.

As A is finite and G^* is countable, there must be distinct $X, X' \in \mathcal{G}$ for which the associated functions agree on A. But this contradicts our definitions. □

The way one gets a C-skeleton is first to construct G so that some finite set A "controls" an infinite subset $\hat{A}$ in a manner we will make precise, and then to make various extensions by attaching fragments of C to pairs of vertices in $\hat{A}$ in as independent a manner as possible.

The issue of control is formalized as follows.

Definition 22. Let C be a constraint graph, G a C-free graph, and A, B two subsets of G. Then B is *C-controlled* by A in G if for any C-free graph G^* and any two embeddings $f_1, f_2 : G \to G^*$ of G as a subgraph of G^*, if f_1, f_2 agree on A then they agree on B.

G is *C-rigid* if it contains an infinite subset which is controlled by a finite subset.

C-rigidity is the fundamental issue, and the bulk of our argumentation below will be aimed at the construction of C-rigid graphs G for a variety of specific constraints C.

To reach the point where Lemma 21 applies, starting with G C-free and C-rigid, with B the infinite subset controlled by a finite subset A, one shows that one can attach additional graphs C_0, C_1 to certain pairs of vertices in B in an essentially arbitrary way, so that the resulting extension is always C-free, but any two such are C-incompatible over B (and hence over the finite subset A).

In most cases we deal with here, C_0 will be just a pair of vertices v_1, v_2 connected by an edge, and C_1 will be the result of deleting an edge from some complete graph. Then $C_0 \cup C_1$ will be a complete graph of some specified size, taken to be some particular block of the constraint graph C whose insertion would produce an embedding of all of C.

What we wish to take away from the discussion so far is the following.

> *The main goal of any of the following constructions is to produce a C-free graph G in which some finite set of vertices C-controls some infinite set.*

In very general terms, the way to get a C-rigid graph is by first forming some particular homomorphic image $\bar{C}$ of C in which two non-adjacent vertices are identified, and then gluing together multiple copies of $\bar{C}$ along some "template" which may be thought of s a k-hypergraph, where $k = |\bar{C}|$ (approximately, or a little larger to allow for embellishments).

It will be instructive to consider three examples of this type of construction, since this is where the main decisions are actually made. One has the choice of $\bar{C}$ and the choice of template to consider.

Example 1. Füredi-Komjáth [FK97a]

Here the result to be proved is that if C consists of a single incomplete block, then C is not a Rado constraint.

What is noteworthy here is that the focus of attention is exclusively on the template.

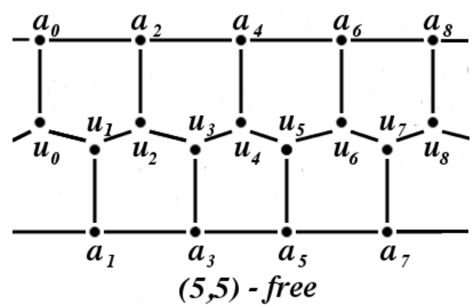
Indeed, the hypothesis on C tells us that there is a nontrivial homomorphic image $\bar{C}$ formed by identifying two non-adjacent vertices, and in this case it does not matter which such homomorphic image is taken.

The main ingredient in the construction is the identification of a suitable k-hypergraph with $k = |C| = 2$, such that hyperedges meet in at most one point and the girth is large. These two conditions ensure that C will not embed in the graph that comes from placing a copy of C on the vertices of each hyperedge (with 2 additional vertices kept in reserve for a later part of the argument). They do not ensure rigidity, so a third condition is imposed on the hypergraph, and a little more care is taken with the precise placement of the "duplicated" vertex of $\bar{C}$.

In the Füredi-Komjáth construction, the whole vertex set plays the role of the controlled part.

Example 2. A solid block path of type $(5,5)$

Here the constraint graph C consists of two complete graphs K_5 with one point in common, and $\bar{C}$ will consist of two copies of K_5 with two points in common. Now the template has a particular form, and in fact there will be considerable overlap between various copies of $\bar{C}$, as shown here.



Here $\{a_i, u_i, u_{i+1}, u_{i+2}, a_{i+2}\}$ is a clique of order 5.

This is a very different pattern from the "almost disjoint" pattern of our first example.

Here the important controlled set is the sequence $(u_i : i \in \mathbb{Z})$ running along the middle, and the template is roughly a $\mathbb{Z}$-template, as is more commonly the case.

The rigidity argument is based on the impossibility of squeezing another copy of K_5 into this picture without creating a copy of C; another way of looking at is that we have not made any unnecessary identifications of vertices.

Here both the choice of $\bar{C}$ and the general form of the template are trivial, and everything comes down to choosing the right additional identifications (overlaps).

We will make some further use of this construction below.

Example 3. Loop Constructions

Our third construction is the workhorse for the proof of Proposition 8, and will be presented formally a little later.

This applies when we have a solid block path constraint C with at least three blocks, and it applies more easily when there are more blocks (ideally, it would be nice to have at least 6, but this will not generally be the case here, so our proofs will involve a substantial amount of background noise—which may indeed be why these are the critical cases).

In this case the choice of $\bar{C}$ is largely standardized: we identify a vertex in the initial block with a vertex in the terminal block. This gives us one distinguished vertex $\bar{v}$ in the quotient $\bar{C}$. We need a second distinguished vertex $\bar{u}$ and for this we select some cut vertex of C (generally chosen with care).

The template is a $\mathbb{Z}$-template, and we use something like an almost disjoint construction. Namely, we may take copies L_i of $\bar{C}$ for $i \in \mathbb{Z}$ and identify $\bar{v}_i$ with $\bar{u}_{i+1}$ to get a chain of loops L_i.

If this turns out to be C-free then it is roughly what we are looking for. More often than not in the cases that actually us concern us there will be "unintended" embeddings of C into this chain of loops and they will be killed by introducing some additional identifications (overlaps).

Generally speaking, by the time one has worked out exactly which additional identifications are needed, the proof of rigidity is fairly clear (at least, once one has the full details as we give them below). It is the sequence $(u_i : i \in \mathbb{Z})$ which is controlled by a finite set.

It is not so easy to say which cut vertex should play the role of $\bar{u}$ here; this depends on what sorts of unintended embeddings of C one gets, and how they can be blocked.

We will develop a tabular representation of these constructions which gives the essential data for the construction in a few lines, once we have seen a few worked through.

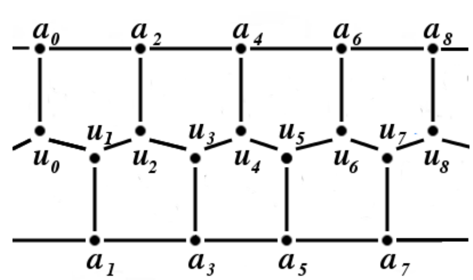
Before taking up the loop constructions, we deal with a few cases that are treated more in the manner of our second example.

3.2. **Length 3, cases 1 and 2.** We take up the first few solid block paths listed in Proposition 8. There are none of length 1, and the case of two blocks was handled in [CT07]. So the analysis begins with length 3. In this case, loop constructions may already be appropriate, but some cases behave more like the case of length 2. So we take up those exceptional cases first.

The solid block paths in question have type $(5, 5, n_3)$ with $n_3 \geq 5$ or $(6, 6, 6)$ (clauses 5.1,2).

Lemma 23 (5.1)**.** *Let C be a solid block path of type $(5, 5, n)$ with $n \geq 5$. Then there is no countable weakly universal C-free graph.*

Proof. Let G_0 be the graph introduced in Example 2 above.



(5,5) - free

The vertices are labeled a_i, u_i ($i \in \mathbb{Z}$) and

$$K_i = \{a_i, u_i, u_{i+1}, u_{i+2}, a_{i+2}\}$$

is a clique of order 5.

Then the graph G_0 contains no block path of type $(5,5)$, and its maximal cliques are just the K_i.

Now adjoin a set U of $(n-1)$ additional vertices forming a clique, and adjacent to all vertices a_i with

$$i \equiv 0, 1 \pmod 4$$

Let $\hat{U}_i = U \cup \{a_i\}$ be the corresponding clique of order n, for $i \equiv 0, 1 \pmod 4$. Let G_1 be the resulting graph.

We claim that G_1 is C-free and C-rigid, but it will be more efficient to complete our description of the construction by going through the extension process that produces an uncountable family of C-free extensions first.

So for any $\varepsilon \in 2^{\mathbb{Z}}$ (that is, $\varepsilon : \mathbb{Z} \to \{0,1\}$) we extend G_1 to a graph G_ε as follows.

> Let C_0 be an edge, and let C_1 be a clique of order 5 with one edge deleted. For each i, attach a copy of $C_{\varepsilon(i)}$ to the pair $(u_{4i}, u_{4(i+1)})$ (identified with the vertices on the deleted edge of C_1 when $\varepsilon(i) = 1$).

This completes the construction. Now we check the relevant properties.

Claim 1. The graphs G_ε are C-free.

The passage from G_1 to G_ε adds no cliques of order 5, so it suffices to consider G_1. It is clear that an embedding of C into G_1 would embed a block path of type $(5,5)$ into G_0, and G_0 is $(5,5)$-free.

This proves the claim.

Since we consider embeddings of graphs as (not necessarily induced) subgraphs of other graphs, it is useful to know in some cases that non-edges are preserved.

Claim 2. Let G be a C-free graph containing G_1. Then (a_i, a_{i+1}) is a nonedge in G.

Otherwise, the set $Q = \{a_i, u_i, u_{i+1}, a_{i+1}, u_{i+2}\}$ is a clique of order 5 meeting K_{i+2} in the unique vertex u_{i+2}, and then together with a clique $\hat{U}_j$ with $j = i+2$ or $i+3$, we embed C into G.

The main point in the proof of rigidity is the following.

Claim 3. Let G be a C-free graph containing G_1 and let K be a clique of order 5 containing u_i and disjoint from U. Then K is one of K_{i-1}, K_i, or K_{i+1}.

One just inspects the possibilities. But we give the details.

Let i^-, i^+ be respectively the least and greatest i such that $u_i \in K$. Our claim is that $K = K_{i^-}$.

Suppose first that there are $j^- < i^-$, $j^+ > i^+$ so that $a_{j^-}, a_{j^+} \in K$, and let j^-, j^+ be respectively minimal and maximal. If $j^- \not\equiv 0, 1 \pmod 4$ we embed C into G as $(K, K_{j^- -2}, \hat{U}_{j-2})$ for a contradiction. So suppose $j^- \equiv 0$ or $1 \pmod 4$, and similarly $j^+ \equiv 0$ or $1 \pmod 4$. Then $\hat{U}_{j^-}$ and $\hat{U}_{j^+}$ are defined, and C embeds into $(\hat{U}_{j^-}, K, \hat{U}_{j^+})$.

So we may suppose that there is no $j < i^-$ such that $a_j \in K$. If $K \cap K_{i^- -2} = \{u_{i^-}\}$ then we embed C into G as $(K, K_{i^- -2}, \hat{U})$ where $\hat{U}$ is attached to $K_{i^- -2}$.

So K must contain another vertex of $K_{i^- -2}$, which can only be a_{i^-}. By the previous claim, it then follows that $a_{i^- \pm 1} \notin K$. Now considering $K \cap K_{i^- -1}$, we see that $u_{i^- +1} \in K$.

In particular, $i^+ > i^-$ and we have similarly $a_{i^+}, u_{i^+ -1} \in K$. As K has order 5, we conclude that $i^- + 1 = i^+ - 1$ and $K = K_{i^-}$, as claimed.

Claim 4. The graph G_1 is C-rigid: the set $\{u_i \mid i \in \mathbb{Z}\}$ is controlled by the set $\{u_{-1}, u_0, u_1\}$.

We suppose G^* is C-free and G_1 is embedded as a subgraph into G^* by the function $f : G_1 \to G^*$, with $f(u_{-1})$, $f(u_0)$, $f(u_1)$ known. It suffices to show that $f(u_2)$ is then known, and to proceed inductively.

In G_1, the three cliques of order 5 disjoint from U and containing u_1 are K_0, K_1, K_2. By considering their intersections with $\{u_{-1}, u_0\}$ we can distinguish them. Then considering $K_i \cap K_1$ we determine u_2.

Since the images of these three cliques in G^* are the only cliques of order 5 containing $f(u_1)$, we determine $f(u_2)$ the same way.

Now the graphs G_ε come into play in the manner of Lemma 21.

Claim 5. G_1 is a C-skeleton.

We know that the various extensions G_ε of G_1 are C-free, and it suffices to show that they are pairwise C-incompatible over the finite set $\{u_{-1}, u_0, u_1\}$; since this set controls $\{u_i \mid i \in \mathbb{Z}\}$ we may work over the latter set.

So suppose $\varepsilon \neq \varepsilon'$ and $f : G_\varepsilon \to G^*$, $f' : G_{\varepsilon'} \to G^*$ are embeddings as subgraphs of a C-free graph G^*. We fix an index i so that $\varepsilon(i) \neq \varepsilon'(i)$ We may suppose that $\varepsilon(i) = 1$ and we may view f as the identify map, f' as some other map which is the identity on $\{u_i \mid i \in \mathbb{Z}\}$.

In G_ε there is a copy of C_1 (K_5 with an edge deleted) attached to $(u_{4i}, u_{4(i+1)})$ and the missing edge is supplied by $f'[C_0]$. So writing K for the resulting clique of order 5, we get an embedding of C into $(K_{4i-2}, K, \hat{U}_{4(i+1)})$, and a contradiction.

This proves the claim, and now Lemma 21 applies. □

The next argument differs from the foregoing in the insertion of an additional stage of *decoration by anti-edges.* This guarantees that certain non-edges in our original skeleton graph must remain non-edges in any larger C-free graph, and will be a standard feature of our loop constructions as well.

Lemma 24 (3.2)**.** *Let C be a solid block path of type $(6,6,6)$. Then there is no countable weakly universal C-free graph.*

Proof. We begin with the graph G_0 considered in the previous proof. Recall that the vertices of G_0 are labeled a_i and u_i with $i \in \mathbb{Z}$.

Let (v, P) be a clique P of order 6 with a distinguished basepoint v. Let (v_{ij}, K_{ij}) be disjoint copies of (v, P) for $i, j \in \mathbb{Z}$. We let G_1 be the disjoint sum of G_0 and the cliques K_{ij}, with additional edges joining the vertex v_{ij} to the clique K_i in G_0. We now have overlapping cliques $\hat{K}_{ij} = K_i \cup \{v_{ij}\}$ of order 6. In particular we have many configurations $(\hat{K}_{ij}, K_{ij})$ of type $(6,6)$, with v_{ij} the cut vertex.

Now we extend G_1 further by the addition of *anti-edges.*

Let A be the graph derived from a clique of order 6 by removing one edge $e = (a, a')$. We extend G_1 to the graph G_2 formed by freely attaching a copy A_i of A to G_0 with the pair (a, a') identified with the vertices (a_i, a_{i+1}) in G_0.

This construction *immunizes* the nonedge (a_i, a_{i+1}) in the following sense.

Claim 1. If G is a C-free graph containing G_2, then (a_i, a_{i+1}) is a nonedge in G.

Otherwise, the induced graph on A_i would be a clique, and then $(A_i, \hat{K}_{ij}, K_{ij})$ would have type $(6,6,6)$ for any j.

We will prove that G_2 is a C-skeleton. So we now prepare an uncountable family of extensions of G_2. For $\varepsilon \in 2^{\mathbb{Z}}$, form the extension G_ε by freely adding either edges or anti-edges (copies of A) over the pairs $(u_{6i}, u_{6(i+1)})$ according to the value of $\varepsilon(i)$.

The construction is complete, and we must verify various properties of the graphs constructed.

Claim 2. The graphs G_ε are C-free.

As the decoration by anti-edges is clearly harmless, and the decoration by edges equally so, it suffices to check that our point of departure G_1 is C-free.

The cliques of order 6 in G_1 are those we have explicitly defined, namely the K_{ij} and the $\hat{K}_{ij}$. The claim follows.

Now we prepare for a rigidity argument.

Claim 3. Let G be a C-free graph containing G_2, and K a clique of order 5 in G with the following properties.

- $a_i, u_i \in K$;
- $a_j, u_j \notin K$ for $j < i$;
- G contains infinitely many extensions of K to a block path of type $(6,6)$, disjoint over K, with cut vertex outside K

Then $K = K_i$.

We show first that

$$K \cap K_{i-1} = \{u_i, u_{i+1}\}$$

Since $a_i \in K$, we have $a_{i\pm 1} \notin K$. So by our assumptions $K \cap K_{i-1} \subseteq \{u_i, u_{i+1}\}$. If u_i is a cut vertex in $K \cup K_{i-1}$ then we extend K to one of the block paths of type $(6, 6)$ and we extend K_{i-1} to one of the $\hat{K}_{i-1,j}$ so that u_i is a cut vertex of the union, and we get a copy of C, and a contradiction. This proves the claim.

In particular K meets K_{i+1}. Take j maximal so that K meets K_j. Then $K \cap K_j \subseteq \{a_j, u_j\}$.

Again, if $|K \cap K_j| = 1$ we extend to an embedding of C and get a contradiction. So

$$K \cap K_j = \{a_j, u_j\}$$

As $a_i \in K$, the vertex a_{i+1} is not in K. So $j > i + 1$ Thus

$$K = \{a_i, u_i, u_{i+1}, u_j, a_j\}$$

If $j > i + 2$ consider $K \cap K_{j-1}$ to reach a contradiction. So $j = i + 2$ and $K = K_i$.

Claim 4. G_2 is C-rigid, with $\{u_{-1}, u_0, u_1\}$ controlling $\{u_i \mid i \in \mathbb{Z}\}$.

This follows from the preceding claim just as in the previous argument.

Claim 5. The extensions G_ε of G_2 are pairwise C-incompatible over $\{u_i \mid i \in \mathbb{Z}\}$.

This is clear by the construction (compare the proof of the corresponding claim in the previous lemma, where we wrote out the point explicitly).

Putting all of these claims together, we find that G_2 is a C-skeleton, and hence Lemma 21 applies. □

We take the previous two arguments as representative of the genre. Going forward, we will introduce loop constructions, which give us a different sort of graph G_0 as a point of departure, after which we continue much as above with some further elaborations: in addition to the use of anti-edges there is an amalgamation process involved in most cases. And the actual construction of G_0 has some delicate features, and some choices of identifications (overlaps) to be fine-tuned according to the type of the configuration C.

3.3. **Loop constructions.** We turn now to the typical instance of Proposition 8. The first order of business is to describe in detail the general form of the loop constructions alluded to above.

In very general terms, we proceed as follows.

Definition 25 (Loop Construction)**.** Suppose C is a solid block path of length ℓ, with B_i its i-th block and v_i the cut vertex of $B_i \cup B_{i+1}$ for $i < \ell$.

- **The chain G_0:** Identify a vertex of B_1 with a vertex of B_2 to form a loop $\bar{C}$, and let G_0 be a chain consisting of an infinite sequence of such loops.
- **Identifications:** Make some identifications among vertices in successive loops of G_0 to form a quotient G_1.
- **Rigidification:** Freely amalgamate infinitely many copies of the G_1 over a suitable base set to get our main graph G_2.
- **Prophylaxis:** Add some anti-edges as a prophylactic device, getting a slight extension G_3 of G_2.
- **Extensions:** Decorate the graph G_3 in uncountably many pairwise incompatible ways $(G_\varepsilon : \varepsilon \in 2^{\mathbb{Z}})$.

Thus we have a 4-stage construction of a skeleton G_3. We will need mainly to check that G_1 as constructed is C-free; when anything goes wrong, this is the usual place for it—we must block unintended embeddings of C into the graph G_1. And one must take a little care with the passage from G_2 to G_3, where we glue together multiple copies of the graph over a fixed base.

One might expect that one would have to pay equal attention to the issue of rigidity, but the point is that if one makes only the minimal identifications necessary to block unintended embeddings of C, rigidity generally follows. And since we begin with no such identifications and only add them as they are needed to block unintended copies of C, any slips will simply result in a graph which is not quite C-free.

So as we work through the examples we will tend to focus mainly on the construction of the graph G_1. But the first few examples will be given in some detail.

Now we begin again, with more precise notation. We are concerned with the construction of G_0 and G_1, and we should warn the reader that our description of G_0 above reflects our intent, but turns out not to match our final notation exactly. In any case what will matter is the graph G_1, as a point of departure for a construction which is quite standardized past that point.

Definition 26. A *loop* L of length ℓ is a graph of the form $\bigcup_{i \in \mathbb{Z}/\ell\mathbb{Z}} B_i$ where B_i is a complete graph, $|B_i \cap B_{i\pm1}| = 1$, and $B_i \cap B_j = \emptyset$ for $|i - j| \geq 2$.

The type of the loop is the sequence $(n_1, n_2, \ldots, n_\ell)$ with $n_i = |B_i|$. (We index from 1 to retain compatibility with our path notation.)

(These should be called *solid loops*, but they are the only loops that interest us here.)

For our purposes, a loop L should always be viewed as a homomorphic image of a solid block path C of the same type, with two vertices v, v' in the first and last blocks, respectively, identified.

Therefore we adopt the otherwise inappropriate terminology *block* for the maximal cliques B_i, and *cut vertex* v_i for the unique vertex in $B_i \cap B_{i+1}$. This terminology refers largely to the situation in the original solid block path, except that v_ℓ is the vertex that is covered by two vertices of C.

In particular, we always view a loop as coming with a particular indexing, and in particular the vertex v_ℓ will be treated as a marked vertex.

To make a chain of loops requires marking a second vertex, as follows.

Definition 27. Let L be a loop of length ℓ and type $(n_1, n_2, \ldots, n_\ell)$. Let ℓ' be a fixed index with $1 \le \ell' < \ell$.

1. The *marked loop* (L, ℓ') is the loop L with two distinguished vertices $a = v_{\ell'}$, $b = v_\ell$. Its type will be denoted

$$(n_1, n_2, \ldots, n_{ell'}; n_{\ell'}, \ldots, n_\ell)$$

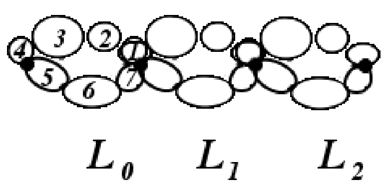
(The only indication of the marking is the semicolon in position ℓ'.)

2. A *chain* G_0 of marked loops (L, ℓ') is defined as follows.

Let $(L_i \,|\, i \in \mathbb{Z})$ be a $\mathbb{Z}$-indexed sequence of disjoint copies of L. Let a_i, b_i be the vertices in L_i corresponding to the marked vertices a, b.

Let G_0 be the graph obtained from the disjoint union of all L_i by identifying b_i with a_{i+1}.

We illustrate with a chain of loops formed from a block path of length 7 divided between blocks 4 and 5; the indices of the blocks in one of the loops are also shown.



We leave for later the description of the identification process that produces G_1, and the associated notation. By the time we are done with that, we will realize that it is more convenient to look at a chain of loops L which are not derived from the original constraint C, but which instead gives a subgraph of the desired graph G_1. Then rather than *identifying vertices* we can view the process as one of *adding edges.*

This will make it easier to think about the maximal cliques in G_1, and that will facilitate the verification that G_1 is C-free.

But we should first consider a concrete example.

3.4. **Length 3: cases 3 and 4.** Our first application of a loop construction will be the following. We give this proof in considerable detail; later proofs along the same lines will involve very similar considerations.

Lemma 28 (3.3)**.** *Let C be a solid block path of length 3 satisfying the following conditions.*

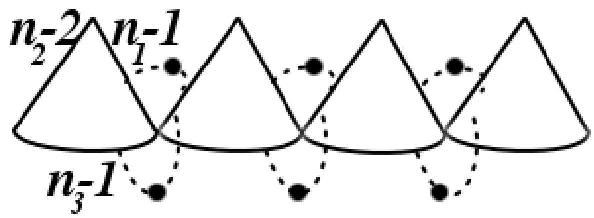
$$4 \leq n_1 = n_3 < n_2$$

Then there is no countable universal C-free graph.

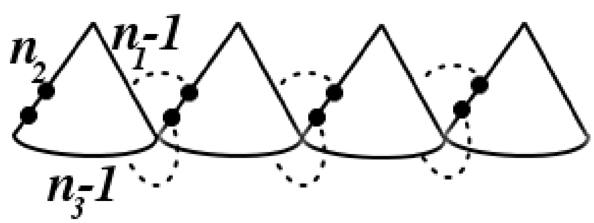
Before we begin the proof, we have some comments to make about the notation we will adopt at the beginning. In the language of §3.3, we begin with a chain of loops of type $(n_1, n_2; n_3)$ (where here $n_3 = n_1$) and make the following identifications: one vertex in the first block of L_i is identified with one vertex in the second block of L_{i+1}; one vertex in the third block of L_i is identified with one vertex in the second block of L_{i+1}.

Since we need to avoid the cut vertices, we require $n_1 \geq 3$, $n_2 \geq 4$ for this to make sense (and we require somewhat more eventually).

It would be a little awkward to draw a picture of the result, but the following schematic diagram is adequate. Here we show the vertices which arise from identifications separately. So the basis of the construction is a chain of loops of type $(n_1 - 1, n_2 - 2, n_3 - 1)$, to which some vertices are added. This may be shown schematically as follows.



We prefer a less symmetrical representation in which all the vertices occur in the original chain of loops. So we place the two extra vertices back into the second block of L_{i+1} and represent the construction as follows.



Thus our point of view becomes the following: starting with a chain of loops of type $(n_1 - 1, n_2; n_3 - 1)$, we add some edges to extend the cliques of order $n_1 - 1$ and $n_3 - 1$ to cliques of order n_1 and n_3, respectively, meeting the next loop.

Thus we revise our original description of G_0: rather than beginning with a loop of type $(n_1, n_2; n_3)$, we begin with a loop of type $(n_1 - 1, n_2; n_3 - 1)$.

One final word about the notation: we will refer to the (so-called) blocks of the i-th loop L_i systematically as (P_i, Q_i, R_i), with similar notation for longer loops. Thus in the case at hand, two vertices of Q_{i+1} (not cut vertices) will be singled out for attention.

In addition we have the distinguished cut vertices (a_i, a_{i+1}) in L_i (recall $b_i = a_{i+1}$).

Proof of Lemma 28. Let G_0 be a chain of loops $L_i = (P_i, Q_i, R_i)$ of type $(n_1 - 1, n_2; n_1 - 1)$. Select vertices $u_{i,P}, u_{i_R} \in Q_i$; this actually means we take vertices in $Q_i \setminus (P_i \cup R_i)$, but for the sake of brevity we will omit this specification in the future. Form the extension G_1 by adjoining edges to make the following vertex sets be cliques.

$$\hat{P}_i = P_i \cup \{u_{i+1,P}\}$$
$$\hat{R}_i = R_i \cup \{u_{i+1,R}\}$$

Take a subset $X_i \subseteq Q_i \setminus P_i$ containing the vertices $a_i, u_{i,P}, u_{i,R}$ with $|Q_i \setminus X_i| = n_1 - 2$ (recall $n_2 > n_1$).

Let G_2 be the free amalgam of infinitely many copies of G_1 over the base

$$A = \bigcup_i X_i$$

Attach

- edges $(u_{i,P}, u_{i+1,P})$ and $(u_{i,R}, u_{i+1,R})$;
- anti-edges $(K \backslash e)$ with K a clique of order n_1 to all pairs (u, u') such that $u \in X_i$, $u' \in X_{i+1}$, and (u, u') is not any of the following.

$$(a_i, a_{i+1}), (a_i, u_{i+1,R}), (u_{i,P}, u_{i+1,P}), (u_{i,R}, u_{i+1,R})$$

to get a graph G_3.

Then for $\varepsilon : \mathbb{Z} \to \{0, 1\}$ extend G_3 to G_ε by adding edges and anti-edges $(K \backslash e)$ at $(a_{3i}, a_{3(i+1)})$.

This concludes the construction. Now we have the usual claims to verify.

Claim 1. The nontrivial maximal cliques of G_ε (i.e., those with at least 3 vertices) are of the following forms.

(a) The cliques of order $n_1 - 1$ in an attached anti-edge;
(b) Copies of the cliques $\hat{P}_i$, Q_i, or $\hat{R}_i$ in a copy of G_1 in G_2;
(c) Copies of the triangles $T_i = \{a_i, a_{i+1}\} \cup (P_i \cap Q_i)$ in a copy of G_1 in G_2.

In particular, the cliques of order at least n_1 are contained in copies of the cliques $\hat{P}_i$, Q_i, or $\hat{R}_i$.

This reduces quickly to the consideration of cliques in G_2, and since these lie in one of the copies of G_1 used to form G_2, it suffices to consider the maximal cliques of G_1, where there is in fact something to check.

If the clique K contains no vertex of the form $u_{i,P}$ or $u_{i,R}$ then we are working in G_0 where everything is clear; though as the loop has length 3 we have to take note of some cliques of order 3 that come from the loop structure itself.

So it suffices to look at the neighbors of $u_{i,P}$ or $u_{i,R}$ in G_1, i.e.

$$Q_i \cup P_{i-1}$$
$$Q_i \cup R_{i-1}$$

The claim is then clear.

So for a claim of this type, the analysis reduces more or less immediately to checking neighborhoods of the special vertices, as long as one keeps track of the "unintended" triangles coming from the loop structure.

Claim 2. The graphs G_ε are C-free.

If $j : C \cong C' = (P', Q', R') \subseteq G_\varepsilon$ is an embedding as a subgraph, then the clique Q' must be a copy of some Q_i and we may suppose $Q' = Q_i \subseteq G_1$.

The cliques P', R' cannot lie in a second copy of Q_i, since $|Q_i \setminus X| < n_1 - 1$. Nor can P' or R' be a copy of $\hat{P}_{i-1}$ or $\hat{R}_{i-1}$, in view of the overlap with Q_i. So the only candidates for P', R' are the clique $\hat{P}_i$ in G_1 together with copies of $\hat{R}_i$ in G_2. But these have the common vertex a_{i+1}.

This proves the claim.

Now we work toward rigidity.

Claim 3. Let $G_3 \subseteq G$ with G C-free, and let K be a clique of order n_1 containing a_i and not a_{i-1}, and free in G over a subset X with $|X| = 3$ and $X \cap X_i = \{a_i\}$. Then $X = \{a_i, u_{i+1,R}, a_{i+1}\}$.

Here the condition of *freeness* means that there are infinitely many copies of K in G with pairwise intersections equal to X.

We may suppose that

$$K \cap \hat{P}_i Q_i \subseteq X \cap (X_i \cup \{u_{i+1,P}, a_{i+1}\}) = \{a_i\} \cup (X \cap \{u_{i+1,P}, a_{i+1}\})$$

If this intersection reduces to $\{a_i\}$ then we get an embedding of C into G, and a contradiction. So K contains at least one of $u_{i+1,P}$ and a_{i+1}. Now $a_i \in K$ and $(a_i, u_{i+1,P})$ is not an edge, so we find

$$a_{i+1} \in X$$

Now we may suppose

$$K \cap \hat{P}_{i+1} Q_{i+1} \subseteq X \cap (X_{i+1} \cup \{u_{i+2,P}, a_{i+2}\})$$

As $a_i \in X$ and X is a clique, we find

$$X \cap X_{i+1} \subseteq \{a_{i+1}, u_{i+1,R}\}$$

So if we show that the vertices $u_{i+2,P}$ and a_{i+2} are not in X, then the claim will follow.

Now X consists of a_i, a_{i+1} and exactly one further vertex, so consideration of $K \cap \hat{P}_{i+1}Q_{i+1}$ eliminates $u_{i+2,P}$ and a_{i+2}.

At this point we have seen that

$$a_{i-1},\ a_i,\ X_i \text{ determine the set } \{u_{i+1,R}, a_{i+1}\}$$

If in addition we have identified the parameter $u_{i,R} \in X_i$ then we can distinguish the vertices $u_{i+1,R}$ and a_{i+1}, and thus each is determined separately.

Claim 4. Let $G_3 \subseteq G$ with G C-free. If K is a clique of order n_2 meeting X_i and free over a subset X of order $|X_i|$, then $X = X_i$.

We may suppose that i is maximal so that K meets X_i.

We may also suppose that

$$K \cap \hat{P}_i Q_i = X \cap (X_i \cup \{u_{i+1,P}, a_{i+1}\} = X \cap X_i)$$

If $X \neq X_i$ we may take a clique K' of order n_1 contained in K so that

$$|K' \cap (X \cap X_i)| = 1$$

As $K' \cap \hat{P}_i Q_i = K' \cap (X \cap X_i)$ we then have an embedding of C into G, and a contradiction.

This proves the claim.

Claim 5 (Rigidity). If G is a C-free graph and f_0, f_1 are embeddings of G_3 into G in such a way that the images of a_{i-1}, a_i, $u_{i,R}$ agree, then the images of a_{i+1}, X_{i+1}, and $u_{i+1,R}$ agree.

It follows that the set $\{a_0,\} \cup X_1$ controls $\{a_i \,|\, i \in \mathbb{Z}\}$, and thus G_3 is a C-skeleton. So Lemma 21 applies. $\square$

The next loop construction introduces one more feature. In addition to the "local" identifications which are reflected in the choice of the special vertices $u_{i,P}$, $u_{i,R}$ in the preceding argument, we will have some global identifications involving a fixed finite set of additional vertices, and these will be included in the finite controlling set in the resulting C-skeleton.

Thus the next proof represents the complete set of ingredients in a general loop construction.

Lemma 29 (3.4)**.** *Let C be a solid block path of length 3 satisfying the following conditions.*

$$5 \leq n_2 < n_1 = n_3$$

Then there is no countable universal C-free graph.

Proof. We may give the construction in tabular form, as follows.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(2, n_2; 2)$
G_1	Clique Ext'n	$\hat{P}_i = P_i \cup \{u_{i+1,P}\} \cup P^*$, $\hat{R}_i = R_i \cup \{u_{i+1,R}\} \cup R^*$, $u_{i+1,P}, u_{i+1,R} \in Q_{i+1}$, $\lvert P^*\rvert = \lvert R^*\rvert = n_1 - 3$
G_2	Amalg'n	$X = P^* \cup R^* \cup \bigcup_i X_i$, $X_i = \{a_i, u_{i,P}, u_{i,R}\}$
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+1,P})$, $(u_{i,R}, u_{i+1,R})$; anti-edges $(K \setminus e)$ at $(u_{i,R}, a_{i+1})$, $(K' \setminus e)$ at $(r, u_{i,P})$, $r \in R^*$; $\lvert K\rvert = \lvert Q\rvert$, $\lvert K'\rvert = \lvert P\rvert$
G_ε	Ext'n Family	Edges, anti-edges $(K \setminus e)$ at $(a_{3i}, a_{3(i+1)})$

This follows the general outline of the previous argument, but we will now write out what it means in more detail.

- G_0 is a chain of loops of type $(2, n_2; 2)$
- G_1 has some additional vertices P^*, R^* used along with suitable $u_{i,P}$, u_{iR} to complete the cliques $\hat{P}_i$, $\hat{R}_i$
- We amalgamate copies of G_1 over the specified amalgamation base X to get G_2
- We add both edges and anti-edges. This allows us to use certain parameters in L_i to distinguish parameters in L_{i+1}, after the resulting graph G_3 is embedded in a larger C-free graph
- The extensions G_ε are obtained by adding an edge or anti-edge to pairs of the form $(a_{3i}, a_{3(i+1)})$, according to the value of $\varepsilon(i)$. This part of the construction is generally invariable apart from the form of the anti-edge.

Now we may deal with the usual claims that lead to the conclusion that G_3 is a C-skeleton, and hence C is not a Rado constraint.

Claim 1. The maximal cliques of order at least n_2 in G_ε are either copies of Q_i or are contained in cliques of the form $\hat{P}_i$, $\hat{R}_i$. or the anti-edges attached to pairs $(r, u_{i,P})$ above.

In particular, the cliques of order n_1 are the cliques $\hat{P}_i$ and $\hat{R}_i$.

This is proved as before. In addition to neighborhoods of the special vertices $u_{i,P}$, $u_{i,R}$ one considers neighborhoods of the vertices in P^*, R^*.

Claim 2. The graphs G_ε are C-free.

If $j : C \cong C' = (P', Q', R') \subseteq G_\varepsilon$ is an embedding as a subgraph, then each of P' and R' must be a copy of some clique $\hat{P}_i$ or $\hat{R}_i$ in some copy of G_1 in G_2.

As the copies of $\hat{P}_i$ or $\hat{R}_i$ intersect pairwise, while P' and R' are disjoint, we may suppose that $P' = \hat{P}_i$ for some i and that $R' = \hat{R}_j$ for some j.

Now if Q' meets P^* then $Q' \cap P' = Q' \cap P^*$ and $Q' \setminus P^*$ is a clique of order $n_2 - 1$ which is contained in $\bigcup_i (\hat{P}_i \setminus P^*)$. But the maximal cliques in $\bigcup_i \hat{P}_i \setminus P^*$ have order 3 so $n_2 \leq 4$, a contradiction. So Q' is disjoint from P^*, and similarly Q' is disjoint from R^*.

Thus Q' is not contained in any clique of the form $\hat{P}_k$ or $\hat{R}_k$. If Q' is contained in one of the anti-edges of order n_1 attached to a pair $(r, u_{k,P})$ with $r \in R^*$, then there is no clique of order n_1 meeting Q' in a single vertex. Thus Q' must be a copy of some Q_k which meets $\hat{P}_i$ and $\hat{R}_j$ in a single vertex. This forces $i, j = k$ and hence P' meets Q', a contradiction.

Claim 3. If G is a C-free graph containing G_3 and K is a clique of G of order n_1 containing a_i and R^*, but not $u_{i,R}$, then K is $\hat{R}_i$.

We may suppose that

$$K \cap (\hat{P}_i Q_i) = K \cap (X_i \cup \{u_{i+1,P}, a_{i+1}\})$$

Since K contains R^* it does not contain any vertex of the form $u_{j,P}$ and since $u_{i,R} \notin K$ we find $K \cap (\hat{P}_i Q_i) \subseteq \{a_i, a_{i+1}\}$. If $K \cap (\hat{P}_i Q_i) = \{a_i\}$ then we embed C in G, a contradiction. Thus we may suppose

$$a_{i+1} \in K$$

Again, we may suppose

$$K \cap (\hat{P}_{i+1} Q_{i+1}) \subseteq K \cap (X_{i+1} \cup \{u_{i+2,P}\})$$

and conclude that $K \cap (\hat{P}_{i+1} Q_{i+1}) \subseteq \{a_{i+1}, u_{i+1,R}, a_{i+2}\}$. Now if $a_{i+2} \in K$ we have identified K completely and we then get an embedding of C into G as $(K Q_{i+2} \hat{P}_{i+2})$. So $a_{i+2} \notin K$ and thus $u_{i+1,R} \in K$. Again we have identified K: it is $\hat{R}_i$.

Claim 4. In G_3, the set $R^* \cup \{a_0, u_{i,R}\}$ controls $\{a_i \mid i \in \mathbb{N}\}$.

The previous claim shows that $R^* \cup \{a_i, u_{i,R}\}$ controls $\{a_{i+1}, u_{i,R}\}$ and we conclude by induction.

So G_3 is a C-skeleton, as required. □

This concludes our presentation of the loop construction method. We will follow the same scheme in verifying the remaining clauses of Proposition 8.

3.5. **Length 3: Cases 5, 6, 7.** We continue to use loop constructions, with the construction presented in tabular form. We will compress the notation a little more.

Lemma 30 (3.5)**.** *Let C be a solid block path of length 3 and type (n_1, n_2, n_3) satisfying*

$$n_1 < n_2 < n_3$$
$$n_1 \geq 3,\ n_2 \geq 5$$

Then there is no countable universal C-free graph.

Proof. We make the following loop construction.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(n_1, 2, ; 2)$
G_1	Clique Ext'n	$\hat{R}_i = R_i \cup \{u^Q_{i+1,R}\} \cup R^*$
G_2	Amalg'n	$X' = \bigcup_i X_i$, $Q_i - X_i\| < n_1 - 1$
G_3	Anti-edges	Edges $(a_i, u_{i+2,R})$ Anti-edges at $(u_{i-1,R}, a_{i+1})$, (a_i, a_{i+2}), (a_i, x) $(x \in X_{i+1} \setminus \{a_{i+1}, u_{i+1,R}\})$ $\|K\| = \|P\|$
G_ε	Ext'n Family	$\|K'\| = \|Q\|$

For the general interpretation of this table see the previous example, in the proof of Lemma 29. Since we have further compressed the notation, we elucidate further.

In the description of G_1 we have indicated the provenance of $u_{i+1,R}$ by a superscript; but below we continue to refer to this vertex simply as $u_{i+1,R}$, as in the previous case.

Also, we have not specified the size of R^*, but one understands that the clique $\hat{R}_i$ should have order n_3.

Turning to G_3, the amalgamation base X should be $R^* \cup X'$ with $X' = \bigcup_i X_i$. Here $X_i \subseteq Q_i$, and furthermore $a_i, u_{i,R} \in X_i$.

Note that this specification is given in a highly compressed form. It is always the case that the "special" vertices (in this case, those in R^* and the $u_{i,r}$) should be in X. So here the set X' pins down some vertices beyond the special vertices which are to be put in the base of the amalgamation.

The clause $X_i \subseteq Q_i$ was omitted, as it is suggested by the inequality given. But when X_i is not contained in one of the blocks of the original loop, we will have to specify its structure more explicitly.

For the construction of G_ε, one needs to know what is used as an anti-edge and where it is attached. Typically the anti-edge has the form $(K \backslash e)$ with K a clique (a copy of one of the blocks of C) and the attachment is at $(a_{3i}, a_{3(i+1)})$. In such cases one wants to know only the size of K. (We write K' here to distinguish it from the anti-edges involved in G_3.)

In other cases we will need to vary the points of attachment, or possibly use a more complicated anti-edge, so in such cases we supply more explicit information.

Now we can turn our attention to the relevant claims.

Claim 1. The nontrivial maximal cliques come from the blocks of C, the triangles of cut vertices in the loops, the cliques contained in attached anti-edges, and the cliques $\tilde{R}_{3i} = R^* \cup \{a_{3i}, a_{3(i+1)}\}$ when the pair $(a_{3i}, a_{3(i+1)})$ is an edge.

By inspection.

Claim 2. The graphs G_ε are C-free.

Suppose $C \cong C' = (P', Q', R') \subseteq G_\varepsilon$. Then $R' = \hat{R}_i$ for some i.

One excludes the possibility $Q' \cap R' \subseteq R^*$ using the hypothesis $n_2 \geq 5$. Then Q' must be a copy of Q_i, and we suppose $Q' = Q_i$.

The cut vertex u of $P'Q'$ must be either $u_{i,R}$ or the cut vertex of P_iQ_i. As $n_1 \geq 3$ and $P' \cap R' = \emptyset$, we cannot have $P' \subseteq \hat{R}_{i-1}$. So P' must be P_i, but then it meets R'.

Claim 3. Let G be a C-free graph containing G_3, and K a clique of G of order n_3 containing R^* and a_i but disjoint from $X_i \setminus \{a_i\}$. Then $K = \hat{R}_i$.

Write $K = R^* \cup K'$ with $a_i \in K'$ and $|K'| = 3$.

We may suppose

$$K \cap P_iQ_i \subseteq K \cap (X_i \cup \{a_{i+1}\}) = \{a_i, a_{i+1}\}$$

If $|K \cap P_iQ_i| = 1$ we embed C into G for a contradiction, so

$$a_{i+1} \in K'$$

We may suppose

$$K \cap P_{i+1}Q_{i+1} \subseteq K' \cap (X_{i+1} \cup \{a_{i+2}\})$$

If $a_{i+2} \in K'$ we consider $P_{i+2}Q_{i+2}K$ for a contradiction, while if $|K \cap X_{i+1}| = 1$ we consider $P_{i+1}Q_{i+1}K$. So

$$|K \cap X_{i+1}| \geq 2$$

As $a_i \in K$ the anti-edges force $K \cap X_{i+1} = \{a_{i+1}, u_{i+1,R}\}$ and the claim follows.

Claim 4. The set $R^* \cup \{a_0, a_1\}$ controls the set $\{a_i \mid i \in \mathbb{N}\}$ in G_3.

By the previous claim. The set $R^* \cup \{a_{i-1}, a_i\}$ controls the set $\{a_{i+1}, u_{i+1,R}\}$, using the parameter a_{i-1} to distinguish a_{i+1} from $u_{i+1,R}$.

So the claim holds follows by induction and then G_3 is a C-skeleton, as required. $\square$

Lemma 31 (3.6)**.** *Let C be a solid block path of length 3 and type (n_1, n_2, n_3) satisfying*

$$n_2 < n_1 < n_3$$
$$n_2 \geq 5$$

Then there is no countable universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(n_1, n_2; 2)$
G_1	Clique Ext'n	$\hat{R}_i = R_i \cup \{u^Q_{i+1,R}\} \cup R^*$
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Anti-edges at $(a_i, u_{i+1,R})$, $\lvert K\rvert = n_2$
G_ε	Ext'n Family	$\lvert K\rvert = \lvert Q\rvert$

We now omit the enumeration of maximal cliques, which is almost always straightforward. But one must keep in mind the undesirable cliques coming either from the loop structure or from R^* together with an adjacent pair (a_i, a_j).

Claim 1. The graphs G_ε are C-free.

Suppose $C \cong (P', Q', R') \subseteq G_\varepsilon$. As $n_3 > n_1, n_2$ we may suppose $R' = \hat{R}_i$ for some i.

As $|Q' \cap R'| = 1$ and $n_2 \geq 4$ it follows easily that Q' does not lie in a clique containing R^*. So the cut vertex v of $Q'R'$ is not in R^*,

If v is $u_{i+1,R}$, then Q' must be a copy of the clique Q_{i+1}. But we have $|Q_{i+1} \cap \hat{R}_i| = 2$.

So the cut vertex v is a_i or a_{i+1}.

As $n_2 > 3$ we do not have to concern ourselves with triangles. So Q' must lie in a copy of one of the cliques P_{i-1}, Q_i, P_i, or Q_{i+1}. We may suppose that this copy is actually the original clique in G_0.

Q_{i+1} is excluded by overlap with R'. Suppose that Q' is Q_i. Consider the cut vertex u of $P'Q'$. If $u = u_{i+1,R}$ then P' is contained in $\hat{R}_{i-1}$. But P' is disjoint from R', hence from $R^* \cup \{a_i\}$, and then $|P'| \leq 2$, a contradiction. On the other hand if $u \neq u_{i+1,R}$ then P' is forced to be P_i, which meets R'.

Finally, if Q' lies in P_{i-1} or P_i, then P' is contained in Q_{i-1} or Q_i; but $n_1 > n_2$.

Claim 2. Let G be a C-free graph containing G_3. If K is a clique of order n_3 in G containing $R^* \cup \{a_i\}$, but not a_{i-1} or $u_{i,R}$, then $K = \hat{R}_i$.

We take j maximal so that $a_j \in K$. We may suppose

$$K \cap P_jQ_j \subseteq \{a_j, u_{j,R}, a_{j+1}\}$$

But $a_{j+1} \notin K$, and if $|K \cap P_j Q_j| = 1$ we embed C in G, so we conclude

$$a_j, u_{j,R} \in K$$

Now $u_{i,R} \notin K$ so $j > i$. Thus

$$K = \{a_i, a_j, u_{j,R}\} \cup R^*$$

If $j \neq i+1$ then a suitable copy of $P_i Q_i$ together with K embeds C in G. Thus $j = i+1$ and $K = \hat{R}_i$.

This proves the claim.

Now it follows easily that $R^* \cup \{a_0, a_1, u_{1,R}\}$ controls $\{a_i \mid i \in \mathbb{N}\}$ and G_3 is a C-skeleton. □

Lemma 32 (3,7)**.** *Let C be a solid block path of length 3 and type (n_1, n_2, n_3) satisfying*

$$n_1 < n_3 < n_2$$
$$n_1 \geq 3$$

Then there is no countable universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(n_1, n_2; n_3 - 1)$
G_1	Clique Ext'n	$\hat{R}_i = R_i \cup \{u^Q_{i+1,R}\}$
G_2	Amalg'n	Base $\bigcup_i X_i$, $\lvert Q_i \setminus X_i\rvert < n_3 - 1$
G_3	Anti-edges	Edges $(u_{i,R}, u_{i+2,R})$ Anti-edges at $(u_{i,R}, a_{i+2})$, $\lvert K\rvert = \lvert P\rvert$; at $((u_{i,R}, u_{i+1,R})$ and (a_i, x) $(x \in X_{i+1} \setminus \{a_{i+1}, u_{i+1,R}\})$ of type $(QP\backslash e)$
G_ε	Ext'n Family	$\lvert K\rvert = \lvert R\rvert$

Claim 1. The graphs G_ε are C-free.

We suppose $C \cong (P', Q', R') \subseteq G_\varepsilon$. As $n_2 > n_1, n_3$ we may suppose $Q' = Q_i$ for some i.

By the choice of X_i, R' cannot lie in another copy of Q_i in G_3. As $n_3 > 3$, R' must be a copy of either $\hat{R}_{i-1}$ or $\hat{R}_i$ in G_3.

In particular the cut vertex of $P'Q'$ is not a_i. By the choice of X_i P' cannot lie in a another copy of Q_i so P' must be P_i, meeting R' in a_{i+1} or $u_{i,R}$. This proves the claim.

Claim 2. Let G be a C-free graph containing G_3 and let (Q', P') be a copy of (Q, P) in G which is free over a subset Y satisfying the following conditions.

$$a_i \in Y \setminus P', a_{i-1} \notin Y$$
$$|Y \cap Q'| = |X_i|, |Y \cap P'| = 2$$
$$Y \cap P' \cap Q' = \emptyset$$

Then $Y \cap P_i = \{u_{i,R}, a_{i+1}\}$.

We show first that

$$Y \cap Q_i = X_i$$

In the contrary case we can find a clique K of order $n_3 - 1$ in a copy of Q connected to a_i, with K disjoint from P_iQ_i, and embed C in G.

We now consider $Y \cap P_i$.

Taking $P'Q'\hat{R}_{i-1}$ in general position we find easily that

$$u_{i,R} \in Y \cap P'$$

Arguing similarly for $P'Q'\hat{R}_i$ we find that Y meets $\{u_{i+1,R}, a_{i+1}\}$. But $(u_{i,R}, u_{i+1,R})$ is a nonedge, so $u_{i+1,R} \notin Y \cap P'$, and thus $a_{i+1} \in Y \cap P'$. This proves the claim.

Now it follows that $\{a_0, a_1\}$ controls $\{a_i \mid \in \mathbb{N}\}$ and G_3 is a C-skeleton. $\square$

This completes the treatment of all cases of length 3 in Proposition 8.

4. Critical Configurations of Length 4

We take up the treatment of the length 4 cases of Proposition 8, using loop constructions. We adopt the abbreviated notation developed in the previous section.

Lemma 33 (4.1)**.** *Let C be a solid block path of length 4 satisfying*

$$4 \le n_1 = n_2 = n_4 < n_3$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(n_1-1; n_1, n_3, n_1-2)$
G_1	Clique Ext'n	$u^Q_{i+1,P};\ u^Q_{i+1,S}, v^P_{i+1,S}$
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+2,P})$, $(u_{i,S}, u_{i+2,S})$ Anti-edges at (a_i, u) with $u = u_{i+1,S}$ or $v_{i+1,S}$, and $(u_{i,P}, a_{i+2})$, $(u_{i,S}, u_{i+1,P})$, $(u_{i,S}, u_{i+2,S})$ $\lvert K\rvert = \lvert P\rvert$
G_ε	Ext'n Family	$\lvert K\rvert = \lvert P\rvert$

Claim 1. Each graph G_ϵ is C-free.

Suppose $C \cong C' \subseteq G_\epsilon$, $C' = (P', Q', R', S')$. Then R' is a copy of some R_i in G_2, and we may suppose $R' = R_i$. Then Q', S' are Q_i and $\hat{S}_i$ in some order.

In view of the overlap between Q_i or $\hat{S}_i$ and other copies of the same cliques in G_2, as well as the overlap with $\hat{P}_{i-1}, \hat{S}_{i-1}$ and $\hat{P}_{i+1}, Q_{i+1}$, the only cliques available to represent P' are the copies of $\hat{P}_i$; but these meet both Q' and S'.

This proves the claim.

Claim 2. Let $G_3 \subseteq G$ with G C-free, and suppose K is a clique of order n_1 free over a set X satisfying the following conditions.

$$X \cap \{a_i, u_{i,S}, u_{i,P}, v_{i,S}\} = \{a_i, u_{i,S}\}$$
$$|X| = 4$$

Then $X = \{a_i, u_{i,S}, u_{i+1,P}, a_{i+1}\}$.

We may suppose

$$K \cap Q_i R_i \hat{S}_i \subseteq X \cap \{a_i, u_{i,P}, u_{i,S}, u_{i+1,S}, v_{i+1,S}, a_{i+1}\} \subseteq \{a_i, u_{i+1,S}, v_{i+1,S}, a_{i+1}\}$$

As $a_i \in X$ and X is a clique, the vertices $u_{i+1,S}$ and $v_{i+1,S}$ are not in X. If the intersection reduces to a_i we embed C in G, for a contradiction. So

$$a_{i+1} \in X$$

Similarly, considering $X \cap Q_{i+1}R_{i+1}\hat{S}_{i+1}$, X must contain one of the vertices

$$u_{i+1,P}, u_{i+1,S}, u_{i+2,S}, v_{i+2,S}, a_{i+2}$$

As $a_i, a_{i+1} \in X$ and X is a clique, we eliminate all but $u_{i+1,P}$ and a_{i+2}. And if $a_{i+2} \in X$ then we have identified the four vertices of X and we switch to $Q_{i+2}R_{i+2}\hat{S}_{i+2}$ for a final contradiction. So

$$u_{i+1,P} \in X$$

and X has been identified as claimed.

This claim tells us that given the parameters

$$a_i, u_{i,P}, u_{i,S}, v_{i,S}$$

we can identify the pair $\{u_{i+1,P}, a_{i+1}\}$. If in addition $i \geq 1$ then using the parameter $u_{i-1,P}$ we can distinguish $u_{i+1,P}$ from a_{i+1}.

Claim 3. Let $G_3 \subseteq G$ with G C-free, and suppose (Q', R', S') is a solid block path of type (n_1, n_3, n_1) free in G over a subset X satisfying

$$\begin{aligned} X \cap R' &= \emptyset \\ X \cap Q' &= X \cap \{a_{i-1}, u_{i-1,S}, u_{i,P}, v_{i,S}, a_i\} = \{a_i, u_{i,P}, v_{i,S}\} \\ a_{i+1} &\in X \cap S' \\ |X \cap S'| &= 3 \end{aligned}$$

Suppose further that for $v \in (X \cap S') \setminus \{a_{i+1}\}$, the pair (a_i, v) is a nonedge in G. Then $X \cap S' = \{u_{i+1,S}, v_{i+1,S}, a_{i+1}\}$.

We may suppose

$$\begin{aligned} Q'R'S' \cap Q_{i+1} &\subseteq X \cap \{a_{i+1}, u_{i+1,P}, u_{i+1,S}\} \\ &= \{a_{i+1}\} \cup (X \cap S' \cap \{u_{i+1,P}, u_{i+1,S}\}) \end{aligned}$$

This intersection cannot reduce to $\{a_{i+1}\}$. As there is an edge $(a_i, u_{i+1,P})$, we cannot have $u_{i+1,P} \in X \cap S'$. So we find

$$u_{i+1,S} \in X \cap S'$$

Now we may suppose

$$Q'R'S' \cap \hat{P}_{i+1} \subseteq \{a_{i+1}\} \cup (X \cap S' \cap \{v_{i+1,S}, u_{i+2,P}, a_{i+2}\})$$

Thus the third vertex in $X \cap S'$ is one of

$$v_{i+1,S}, u_{i+2,P}, a_{i+2}$$

The case in which $a_{i+2} \in X$ leads to a contradiction by considering $\hat{P}_{i+2}$. And as $X \cap S'$ is a clique, we cannot have $v_{i+2,P} \in X$. So we find

$$u_{i+1,S} \in X \cap S'$$

and $X \cap S'$ has been identified.

Now for $i \geq 1$, beginning with the parameters

$$a_{i-1}, u_{i-1,S}, v_{i-1,P}, v_{i-1,S}, u_{i,S}, u_{i,P}, v_{i,S}, a_i$$

we use Claim 2 to identify the parameters $u_{i+1,P}$ and a_{i+1}, and then use Claim 3 to identify the set $\{u_{i+1,S}, v_{i+1,S}\}$.

The parameter $v_{i-1,S}$ will then serve to distinguish $u_{i+1,S}$ and $v_{i+1,S}$.

Thus we may determine the images of all a_i given the images of the first few parameters of the specified form, and conclude that there is no countable weakly universal C-free graph.

□

Lemma 34 (4.2)**.** *Let C be a solid block path of length 4 and type (n_1, n_2, n_3, n_4), and suppose that*

$$n_1, n_4 < n_2, n_3$$

Then there is no countable weakly universal C-free graph.

Proof. We may suppose $n_2 \leq n_3$.

Some details that don't fit neatly in the table are added as notes.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(n_1, n_2; n_3, n_4)$
G_1	Clique Ext'n	No clique extensions
G_2	Amalg'n	Base $\bigcup_i (X_i \cup Y_i)$, $X_i \subseteq Q_i$, $Y_i \subseteq R_i$ (See below)
G_3	Anti-edges	Edges: Perfect matching between $X_i \cup Y_i$ and $(X_{i+2} \cup Y_{i+2})$ Anti-edges: its complement. $\lvert K\rvert$: see below
G_ε	Ext'n Family	Anti-edges $(K' \backslash e)$; $\lvert K'\rvert = \lvert Q\rvert = \min(\lvert Q\rvert, \lvert R\rvert)$

Notes.

- $X_i \subseteq Q_i$ is minimal so that C does not embed in the free amalgam of two copies of $P_i Q_i$ over $X_i \cup \{a_{i+1}\}$; $Y_i \subseteq R_i$ is chosen similarly.
- Anti-edges $(K \backslash e)$: minimal size such that at least one of the configurations $L_i K L_{i+1}$ or $L_i L_i^* K$ contains a copy of C, where L_i is the i-th loop and L_i^* is another copy of L_i in G_2. (If $u \in X_i'$ and $y \in Y_{i+2}'$, or if $n_2 = n_3$, this means $|K| = \min(n_1, n_4)$. In all cases $|K| = n_1$ or n_4.)

Claim 1. The graphs G_ϵ are C-free.

We consider a copy (P', Q', R', S') of C in G_ϵ. As there are some marginal cases to worry about we separate out the cases $n_2 < n_3$ and $n_2 = n_3$.

Suppose first $n_2 < n_3$. Then we may suppose $R' = R_i$ for some i. We may then suppose Q' is Q_i or Q' embeds in another copy of R_i. In either case the cut vertex of $Q'R'$ is a_i.

Suppose Q' is Q_i. The attached anti-edges at vertices of Q_i have order at most n_1. So P' does not embed in one of these, and must be P_i. Similarly S' is forced to be S_i and then P' meets S', a contradiction.

Suppose Q' embeds into a second copy R_i^* of R_i. Then P' does not embed in the corresponding segment $L_i^* = R_i^* S_i^*$, by choice of Y_i. So P' must embed in an antiedge $(K \backslash e)$ attached to a vertex of R_i^*. As $|K| \leq n_4$, we find $n_1 < n_4$. But then we could embed $P'Q'$ in $R_i^* S_i^* \setminus \{a_{i+1}\}$, which we have already ruled out.

Now suppose $n_2 = n_3$. In this case we may suppose $n_1 \leq n_4$. Then we may suppose that R' is Q_i or R_i for some i; and if $n_1 = n_4$ we may choose notation so that R' is R_i.

Suppose first that R' is Q_i (so $n_1 < n_4$). Then Q' embeds in a copy of Q_i or R_i, and the cut vertex of $Q'R'$ is a_i. So S' must embed in an attached antiedge or in P_i. But as $n_1 < n_4$ both are impossible.

So now we suppose $R' = R_i$. Then we may suppose that Q' is either Q_i or another copy R_i^* of R_i. In either case the cut vertex of $Q'R'$ is a_i. The attached anti-edges at vertices of Y_i have order $\min(n_1, n_4)$, so S' must be S_i.

If $Q' = Q_i$ then P' is forced to be P_i, so meets S', a contradiction. So we suppose that Q' is R_i^*. Then P' cannot embed in an attached antiedge so P' is contained in the corresponding block S_i^*. But this is impossible by the choice of Y_i.

This proves the claim.

Claim 2. Let $G_2 \subseteq G$ with G C-free, and suppose that A is a solid block path of type (n_1, n_2) with blocks (P_A, Q_A), free in G over a subset X satisfying the following.

$$X \cap (\{a_{i-1}, a_i\} \cup X_{i-1} \cup Y_{i-1}) = \{a_i\}$$
$$X \cap (P_A \cap Q_A) = \emptyset$$
$$|X \cap Q_A| = |X_i|$$
$$|X \cap P_A| = 1$$

Suppose further the edges of G give a perfect matching between X_{i-2} and $X \cap Q_A \setminus \{a_i\}$. Then $X \cap Q_A = X_i$ and $X \cap P_A = \{a_{i+1}\}$.

By the perfect matching condition we have $X \cap Q_A \setminus \{a_i\}$ disjoint from Y_i. But we may suppose that

$$A \cap (R_i S_i) \subseteq X \cap (Y_i \cup \{a_{i+1}\}) \subseteq \{a_i, a_{i+1}\}$$

This forces $a_{i+1} \in X$, and then our condition on the edges forces

$$X \cap P_A = \{a_{i+1}\}$$

Now if $X \cap Q_A \neq X_i$, then the intersection $X' = X \cap X_i$ is a proper subset of $X + i$ containing a_i, and therefore C embeds into the free amalgam of two copies of P_iQ_i over $X' \cup \{a_{i+1}\}$. But then A together with P_iQ_i contains such a copy of C, and we have a contradiction.

There is a similar claim allowing for the recognition of Y_i. This gives sufficient rigidity to complete the argument in standard fashion. □

Sometimes our loop constructions degenerate, when the loop is symmetric around the distinguished cut vertex. Then we use only half the loop, since the result after amalgamation is the same. The next case treated is one such situation.

Lemma 35 (4.3)**.** *Let C be a solid block path of length 4 and type (n_1, n_2, n_3, n_4) satisfying*

$$n_1 \geq n_4 > n_2, n_3$$
$$n_2, n_3 \geq 3$$

Suppose there is a countable weakly universal C-free graph. Then

$$n_1 \geq n_3 + n_4 - 1$$

In particular, $n_1 > n_4$.

Proof. Set

$$n_2' = \max(n_2, n_3)$$

We suppose

$$n_1 < n_4 + n_3 - 1 fournth$$

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	Linear chain of solid block paths of type (n_1, n_2')
G_1	Clique Ext'n	None
G_2	Amalg'n	Special vertices a_i, b_i are cut vertices of $Q_{i-1}P_i$, P_iQ_i Base is $\bigcup_i X_i$; $X_i \subseteq P_i$, $2n_1 - \lvert X_i \rvert = n_3 + n_4 - 2$
G_3	Anti-edges	Edges (a_i, B_{i+3}) Anti-edges at (a_i, u) $(u \in X_{i+3} \setminus \{b_{i+3}\})$ $\lvert K \rvert = \min(n_2, n_3)$
G_ε	Ext'n Family	Attach edges or anti-edges at $(b_{2i}, b_{2(i+1)})$, $\lvert K \rvert = \min(n_2, n_3)$

Note that the existence of a suitable set $X_i \subseteq P_i$ requires the condition

$$n_1 \leq n_3 + n_4 - 2$$

Claim 1. The graphs G_ϵ are C-free

If $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$ is an embedding then we may suppose that P' is one of the cliques P_i of order n_1 in G_0. Then Q' may be a copy of one of the cliques Q_i or Q_{i-1}, or contained in another copy P_i^* of P_i.

If Q' is contained in a copy P_i^* of P_i, then Q' meets P' in a_i or b_i and does not contain both. This then forces $Q'R'S'$ to be contained in $P_i'^* \setminus \{b_i\}$. Hence

$$(n_2 + n_3 + n_4 - 2) \leq n_1 - 1$$

which is not the case.

So Q' is a copy of Q_i or Q_{i-1}, and we may suppose that Q' coincides with Q_i or Q_{i-1}; by symmetry we may even suppose that Q' is Q_i. If R' is contained in another copy of Q_i then S' is as well, which is a contradiction. So R' is contained in a copy of P_{i+1} and we may suppose that R' is contained in P_{i+1} itself.

Then S' must be contained in some copy P_{i+1}^* of P_{i+1}. So $|R'S'| \leq |P_{i+1}P_{i+1}^*|$ or

$$n_3 + n_4 - 1 \leq 2n_1 - |X_i|$$

a contradiction.

Claim 2. Let $G_1 \subseteq G$ with G C-free. Let K be a clique of order n_1 free over a set X satisfying

$$X \cap P_i \neq \emptyset$$
$$|X| = |X_i|$$

Then $X = X_i$.

We may take i to be maximal so that X meets P_i. Suppose first that X contains some vertex of P_i other than b_i.

We may suppose that

$$K \cap P_i Q_i P_{i+1} \subseteq X \cap X_i X_{i+1} = X \cap X_i$$

If $X \cap X_i \neq X_i$, and in particular $|X \cap X_i| < |X_i|$, we embed PQR as $P'Q'R'$ in $P_{i+1}Q_{i+1}P_i$ naturally so that R' contains some vertex v_i of $X \cap X_i$ other than b_i, and $|R' \cap X \cap X_i$ is minimized. We may then embed S' into $K \setminus X'$ over v_i disjoint from $R' \setminus \{v_i\}$ to get a copy of C in G, and a contradiction.

So we conclude in this case that $X_i \subseteq X$ and therefore $X_i = X$.

In the remaining case we have

$$X \cap P_i = \{b_i\}$$

Similarly, if our claim does not hold and we take j minimal so that X meets P_j, we conclude $X \cap P_j = \{a_j\}$. In particular $j < i$. In this case we embed C into $P_j K Q_i P_{i+1}$ for a contradiction.

Claim 3. Let $G_1 \subseteq G$ with G C-free. Let (Q', P') be a block path of type (n_2, n_1) free in G over a set X satisfying the following conditions.

$$X \cap Q' = \{b_i\} \cup Q' \cap P'$$
$$|X \cap P'| = |X_i|$$

Then $X \cap P' = X_{i+1}$.

As P' is free over $X \cap P'$ it suffices to show that P' meets P_{i+1}.

Suppose the contrary. We may suppose that

$$Q'P' \cap Q_iP_{i+1} \subseteq X \cap \{b_i\} \cup X_{i+1} = \{b_i\}$$

Then we have an embedding of C into $P_{i+1}Q_{i+1}Q'P'$, and a contradiction.

Now given X_i and a_{i-3} we can determine b_i, and then determine X_{i+1}. This gives sufficient rigidity to complete the argument.

□

Lemma 36 (4.4). *Let C be a solid block path of length 4 satisfying*

$$n_4 > n_1 > n_2 > n_3 \geq 3$$
$$n_2 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(2; n_2, n_3)$
G_1	Clique Ext'n	$u^Q_{i+1,P}, P^*$; $\hat{S}_i = P_{i+1} \cup S^*$
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges at (a_i, a_{i+3}) Anti-edges at (a_i, a_{i+2}) and $(a_i, u_{i+3,P})$ $\lvert K\rvert = \lvert R\rvert$
G_ε	Ext'n Family	$\lvert K\rvert = \lvert Q\rvert$, at $(u_{6i,P}, u_{6(i+1),P})$

Claim 1. The graphs G_ϵ are C-free.

Suppose $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$. Then S' is S_i for some i, so $P' \cap S^* = \emptyset$. Hence $P' = P_j$ for some j.

If Q' is contained in some clique of the form $\hat{P}_k$ for some k, then as $|P' \cap P^*| \leq |P' \cap Q'| = 1$, we get $|Q'| \leq 1 + |\hat{P}_k \setminus P^*| = 4$, a contradiction.

Hence the clique Q' does not meet P^* and is not contained in $P_{j\pm1}$. Furthermore $Q' \neq Q_{j+1}$ so the vertex in $P' \cap Q'$ is not $u_{j+1,P}$. So Q' is a copy of Q_j, and we may suppose $Q' = Q_j$. Now as $R' \cap P' = \emptyset$, we do not have $R' = R_j$. There remains the possibility

$$R' \cap Q' = \{u_{j,P}\}$$

But in this case $R' \subseteq \hat{P}_{j-1}$ and $|R'| \leq |R' \cap (P_{j-1} \setminus Q_j)| + 1 \leq 2$, a contradiction.

This proves the claim.

Claim 2. Suppose that $G_3 \subseteq G$ with G C-free. Let K be a clique of order n_1 containing $\{a_i\} \cup P^*$ and disjoint from $\{u_{i,P}\} \cup S^*$. Then $K = \hat{P}_i$.

Let j be maximal so that K meets $Q_j R_j \hat{S}_j$. We may suppose that

$$K \cap Q_j R_j \hat{S}_j \subseteq K \cap (\{a_j, u_{j,P}, a_{j+1}, a_{j+2}\} \cup S^*) = K \cap \{a_j, u_{j,P}\}$$

As G is C-free it follows that K contains a_j, $u_{j,P}$ and thus $j > i$ and

$$K = \{a_i, a_j, u_{j,P}\} \cup P^*$$

As (a_i, a_{i+2}) carries an antiedge we have $j \neq i+2$.

We may suppose

$$K \cap Q_i R_i \hat{S}_i \subseteq K \cap (\{a_i, u_{i,P}, a_{i+1}, a_{i+2}\} \cup S^*) = K \cap \{a_i, a_{i+1}, a_{i+2}\}$$

and as G is C-free we conclude

$$j \leq i+2$$

Hence $j = i+1$ and $K = \hat{P}_i$.

This claim shows that the images of $a_i, u_{i,P}$, and P^* determine the image of the pair $\{a_{i+1}, u_{i+1,P}\}$. As these vertices can be distinguished over the parameter $u_{i-2,P}$, we have sufficient rigidity to show the nonexistence of a countable weakly universal C-free graph. □

Lemma 37 (4.5)**.** *Let C be a solid block path of length 4 and type (n_1, n_2, n_3, n_4) satisfying*

$$n_4 > n_2 > n_1 > n_3 \geq 3$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(n_1, n_2, n_3; 2)$
G_1	Clique Ext'n	$u^Q_{i+1,S}, S^*$
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(a_i, u_{i+2,S})$ Anti-edgesat (a_i, a_{i+2}) of type (R, Q, P) (see below)
G_ε	Ext'n Family	C_0 again

Note

The antiedges here have the form $(C_0 \backslash e)$ with C_0 a solid block path of type (n_3, n_2, n_1) and with the edge e deleted from the first block (not involving a cut vertex). In the table, it suffices to give the type of C_0 with the block of attachment listed first.

Claim 1. The graphs G_ϵ are C-free.

Suppose $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$ is an embedding as a subgraph.

The clique S' must be $\hat{S}_i$ for some i. Then Q' can only be a copy of one of cliques Q_j for some j. We may suppose $Q' = Q_j$. Then $P' = \hat{P}_j$, so $R' = R_j$. Hence $j = i$ or $i+1$. But these possibilities conflict with $|R' \cap S'| = 1$, $P' \cap S' = \emptyset$. The claim follows.

Claim 2. Let $G_3 \subseteq G$ with G C-free. Suppose that K is a clique of order n_4 containing $S^* \cup \{a_i\|$ but not $u_{i,S}$. Then $K = \hat{S}_{i+1}$.

We may suppose that $K \cap P_iQ_iR_i \subseteq K \cap \{a_i, u_{i,S}, a_{i+1}\} = K \cap \{a_i, a_{i+1}\}$. As G is C-free we find $a_{i+1} \in K$. Then considering $P_{i+1}Q_{i+1}R_{i+1}$ and $P_{i+2}Q_{i+2}R_{i+2}$ similarly, we find $u_{i+1,S} \in K$, $K = \hat{S}_{i+1}$ as claimed.

We now know that the parameters a_i and S^* determine the pair $\{a_{i+1}, u_{i+1,S}\}$ and then the parameter a_{i-1} determines each vertex separately. This gives sufficient rigidity to complete the argument.

□

Lemma 38 (4.6)**.** *Let C be a solid block path of length 4 and type (n_1, n_2, n_3, n_4) satisfying*

$$n_2 \geq n_4 > n_1 > n_3 \geq 3$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(n_1 - 1; n_2, n_3)$
G_1	Clique Ext'n	$u^R_{i+1,P}$
G_2	Amalg'n	$\lvert Q_i \setminus X_i\rvert = n_1 - 2$
G_3	Anti-edges	Anti-edges at (a_i, a_{i_2}), $\lvert K\rvert = \lvert P\rvert$ and at $(u_{i,P}, a_{i+2})$, $\lvert K'\rvert = \lvert Q\rvert$
G_ε	Ext'n Family	$\lvert K\rvert = \lvert P\rvert$

Claim 1. The graphs G_ϵ are C-free.

Let $C_0 = (P, Q, R)$ be a solid block path of type (n_1, n_2, n_3). We will show that the G_ϵ are C_0-free.

Suppose $C_0 \cong C'_0 = (P', Q', R') \subseteq G_\epsilon$ is an embedding as a subgraph. Then Q' must be a copy of a clique Q_i for some $i \in \mathbb{N}$, and we may suppose

$$Q' = Q_i$$

The choice of the sets X_i excludes the possibility that P' could be contained in another copy of Q_i, so P' must be a copy of $\hat{P}_{i-1}$ or $\hat{P}_i$, and in particular $a_i \notin R'$. Thus R' must be R_i, which however meets both candidates for P'.

This proves the claim.

Claim 2. Let $G_3 \subseteq G$ with G C-free. Suppose that K is a clique of order n_1 free in G over a subset X with the following properties.

$$a_i \in X, u_{i,P} \notin X$$
$$|X| = 3$$

Then $X = \{a_i, u_{i+1,P}, a_{i+1}\}$.

We may suppose that

$$K \cap Q_i R_i Q_{i+1} \subseteq X \cap \{a_i, u_{i,P}, a_{i+1}\} = X \cap \{a_i, a_{i+1}\}$$

If the intersection reduces to $\{a_i\}$ then we get an embedding of C into G, and a contradiction. So

$$a_{i+1} \in X$$

We may suppose

$$K \cap Q_{i+1} R_{i+1} Q_{i+2} \subseteq X \cap \{a_{i+1}, u_{i+1,P}, a_{i+2}\}$$

Again, the intersection cannot reduce to $\{a_{i+1}\}$. If $u_{i+1,P} \in X$ we are done. And if $a_{i+2} \in X$ we switch to $KQ_{i+2}R_{i+2}Q_{i+3}$ for a contradiction.

Thus $a_i, u_{i,P}$ determine the pair $\{a_{i+1}, u_{i+1,P}\}$. Using the parameter u_{i-1} we may also distinguish a_{i+1} and $u_{i+1,P}$. Thus we may argue as usual that there is no countable weakly universal C-free graph. $\square$

Lemma 39 (4.7)**.** *Let C be a solid block path of type $(5, n_2, n_3, 5)$ with*

$$n_3 < 5 < n_2$$

Then there is no countable weakly universal C-free graph.

Reverse C here.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(4; n_2, n_4, 4)$
G_1	Clique Ext'n	$u^S_{i+1,P}$, $u^S_{i+1,S}$
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+2,P})$, $(u_{i,S}, u_{i+2,S})$ Anti-edges at all other (u, v) with $u \in \{a_i, u_{i,P}, u_{i,S}\}$ and $v \in \{a_i, u_{i+2,P}, u_{i+2,S}\}$ and at $(a_i, u_{j,S})$ for $j = i+1, i+2$ $\lvert K\rvert = \lvert Q\rvert$
G_ε	Ext'n Family	$\lvert K\rvert = \lvert Q\rvert$

Claim 1. The graphs G_ϵ are C-free.

Let $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$ be an embedding as a subgraph. Then R' is a copy of some R_i and we may suppose $R' = R_i$. Then Q' and S' can only be Q_i and $\hat{S}_i$, in that order. This leaves only copies of $\hat{P}_{i-1}$ and $\hat{S}_{i-1}$ to serve as P', and both have too much overlap with Q'.

Claim 2. Let $G_3 \subseteq G$ with G C-free. Suppose that K is a clique of order 5 free in G over a set X satisfying the following.

$$a_i \in X,\ a_{i-1} \notin X$$
$$|X| = 3$$

Then $X = \{a_i, u_{i+1,P}, a_{i+1}\}$.

We may suppose that

$$K \cap Q_i R_i \hat{S}_i \subseteq X \cap \{a_i, u_{i+1,S}, a_{i+1}\}$$

If the intersection reduces to $\{a_i\}$ then we have an embedding of C into G, and a contradiction.

As $a_i \in X$ and X is a clique, the vertex $u_{i+1,S}$ is not in X. Thus

$$a_{i+1} \in X$$

Now consider $KQ_{i+1}R_{i+1}\hat{S}_{i+1}$ similarly. We conclude that the third vertex in X is one of

$$u_{i+1,P}, u_{i+1,S}, u_{i+2,S}, a_{i+2}$$

Again as $a_i \in X$ and X is a clique, we eliminate $u_{i+1,S}$ and $u_{i+2,S}$. If the third vertex is a_{i+2} we pass to $KQ_{i+2}R_{i+2}\hat{S}_{i+2}$ for a contradiction. So the third vertex of X is $u_{i+1,P}$, as claimed.

The claim shows that a_i determines $\{a_{i+1}, u_{i+1,P}\}$. Then the parameter $u_{i-1,P}$ allows the two vertices to be distinguished.

This gives sufficient rigidity to complete the proof. □

Lemma 40 (4.8)**.** *Let C be a solid block path of length 4 satisfying*

$$n_4 > n_1 > n_3 > n_2 \geq 3$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(n_1 - 1, n_2; n_3, 2)$
G_1	Clique Ext'n	$u^Q_{i+1,P}$; S^*
G_2	Amalg'n	No amalgamation
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+2,P})$ Anti-edges at $(u_{i,P}, a_{i+2})$, $\lvert K\rvert = \lvert Q\rvert$
G_ε	Ext'n Family	$\lvert K\rvert = \lvert Q\rvert$

Claim 1. The graphs G_ϵ are C-free.

Suppose $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$.

Then S' must contain S^* and therefore P' must be a copy of some $\hat{P}_i$. We may suppose

$$P' = \hat{P}_i$$

Notice that as $n_3 \geq 4$, it is not possible for R' to meet S' in a vertex of S^*. This observation will be applied tacitly below (R' functions here mainly as a connecting link between Q' and S', so it is not discussed very explicitly).

If Q' is contained in another copy $\hat{P}_i'$ of $\hat{P}_i$, then as $n_3 > n_2$, R' is contained in $\hat{P}_i'$ as well. So we cannot reach S' in this case.

So Q' is Q_i or Q' is contained in a copy of R_{i+1}, which we may take to be R_{i+1} itself.

If $Q' = Q_i$ then S' can only meet R' in the cut vertex c_{i-1} of $R_{i-1}\hat{S}_{i-1}$, but then $R' \subseteq S_{i-1}$ and $|R'| \leq 2$, a contradiction.

If Q' is contained in R_{i+1} then similarly R' must be contained in $\hat{S}_{i+1}$ and we have a contradiction.

This proves the claim.

Claim 2. Let $G_3 \subseteq G$ with G C-free. Let (R', S') be a block path of type (n_3, n_4) free in G over a subset $X = \{a_i, v\} \cup S^*$ where

$$X \cap R' \cap S' = \emptyset$$
$$X \cap R' = \{a_i\}$$

Then $v = a_{i+1}$.

We may suppose that

$$R'S' \cap P_iQ_i = X \cap \{a_i, u_{i,P}, a_{i+1}\} \cup S^*$$

If the intersection reduces to $\{a_i\}$ then we have an embedding of C in G, and a contradiction.

So it suffices to rule out the possibility

$$v = u_{i,P}$$

Assuming $v = u_{i,P}$, let $S'' = \{a_i, u_{i,P}\} \cup S^*$. Then this is a clique of order n_4. Then $S''R_iS_iP_i$ contains a copy of C, and we have a contradiction.

This proves the claim. Thus the parameter a_{i+1} is determined by S^* and a_i.

Claim 3. Let $G_3 \subseteq G$ with G C-free. Let (Q', P') be a block path of type (n_2, n_1) free in G over a set of the form $X = \{a_i, v, a_{i+1}\}$, with $a_i, v \in Q' \setminus P'$ and $a_{i+1} \in P' \setminus Q'$. Suppose $v \notin S^*$. Then $v = u_{i,P}$.

We may suppose that

$$Q'P' \cap P_{i-1}\hat{S}_i \subseteq X \cap (\{a_i, u_{i,P}, a_{i+1}\} \cup S^*) = X \cap \{a_i, a_{i+1}, u_{i,P}\}$$

If the intersection is $\{a_i, a_{i+1}\}$ then $S_i P' Q' P_{i-1}$ contains a copy of C and we have a contradiction. The claim follows.

Now by a rigidity argument it follows that there is no countable weakly universal C-free graph. □

Lemma 41 (4.9). *Let C be a solid block path of length 4 satisfying*

$$n_1 = n_3 = 4 < n_4 < n_2$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(3, n_2, 4; n_4 - 1)$
G_1	Clique Ext'n	$u^S_{i+1,P}$; $u^R_{i+1,S}$
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+1,P})$, $(u_{i,S}, u_{i+1,S})$ Anti-edges at $(u_{i,P}, v)$ for $v \in \{a_{i+1}, u_{i,S}, u_{i+1,S}, u_{i+2,P}, a_{i+2}\}$ and $(u_{i,S}, a_{i+2})$ $\lvert K\rvert = \lvert S\rvert$
G_ε	Ext'n Family	$\lvert K\rvert = \lvert S\rvert$

Claim 1. The graphs G_ϵ are C-free.

Suppose $C \cong C' = (P', Q', R', S') \subseteq G$ is an embedding as a subgraph. Then Q' is a copy of some Q_i and we may take $Q' = Q_i$. Then P', R' are $\hat{P}_i$, R_i in some order. The overlap with $\hat{S}_{i\pm 1}$ force $\hat{S}'$ to be a copy of $\hat{S}_i$, but this meets P'.

This proves the claim.

Claim 2. Suppose $G_3 \subseteq G$ with G C-free. Let K be a clique of order n_4 free over a subset X satisfying the following.

$$a_i \in X$$
$$a_{i-1}, u_{i-1,S} \notin X$$
$$|X| = 4$$

Then

$$X = \{a_i, u_{i,P}, u_{i+1,S}, a_{i+1}\}$$

We may suppose that

$$K \cap \hat{P}_{i-1} Q_{i-1} R_{i-1} \subseteq X \cap \{a_{i-1}, u_{i-1,S}, u_{i,P}, a_i\} = X \cap \{u_{i,P}, a_i\}$$

If the intersection reduces to $\{a_i\}$ then we embed C in G, for a contradiction. So

$$u_{i,P} \in X$$

We may suppose that

$$K \cap \hat{P}_i Q_i R_i \subseteq X \cap \{a_i, u_{i,S}, u_{i+1,P}, a_{i+1}\}$$

If the intersection reduces to $\{a_i\}$ then we have an embedding of C into G, and a contradiction. As there is no edge $(u_{i-1,S}, x)$ with $x \in X$, the vertex $u_{i,S}$ is not in X. As $a_i \in K$ and K is a clique, the vertex $u_{i+1,S}$ is not in X. So

$$a_{i+1} \in X$$

We may suppose that

$$K \cap \hat{P}_{i+1} Q_{i+1} R_{i+1} \subseteq X \cap \{a_{i+1}, u_{i+1,S}, u_{i+2,P}, a_{i+2}\}$$

The intersection cannot reduce to $\{a_{i+1}\}$.

As $a_i \in K$ and K is a clique, the vertices $u_{i+2,P}, a_{i+2}$ are not in K. So

$$u_{i+1,S} \in X$$

The claim follows.

Thus $a_{i-1}, u_{i-1,P}, u_{i-1,S}, a_i$ determine the set $\{u_{i,P}, u_{i+1,S}, a_{i+1}\}$ and we can distinguish the three vertices.

Now it follows as usual that there is no countable weakly universal C-free graph.

□

Lemma 42 (4.10)**.** *Let C be a solid block path of length 4 satisfying*

$$n_2 = n_4 > n_1 = n_3 = 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(3; n_2, 4, 2)$
G_1	Clique Ext'n	$u^Q_{i+1,P}$; $\hat{S}_i = S_i \cup \{u^P_{i+1,S}\} \cup Q^*_{i+1}$, $Q^*_i = Q_i \setminus (\{a_i, u_{i,P}\} \cup R_i)$
G_2	Amalg'n	Base $\bigcup_i \hat{S}_i$
G_3	Anti-edges	Edges $(u_{i,S}, u_{i+2,S})$ Anti-edges at (u, v) for $u \in Q_i \setminus \{a_i\}$, $v \in \hat{P}_i \setminus \{a_i\}$ (a_i, v) for $v \in (Q_{i+1} \cup \hat{S}_i) \setminus \hat{P}_i$ $(u_{i,S}, c_{i+1})$ with c_i the cut vertex of $R_i S_i$ (u, v): $u \in \hat{S}_i \setminus Q_{i+1}$, $v \in (Q_i \setminus \hat{S}_i) \cup \hat{S}_{i+1}$ $\lvert K\rvert = \lvert R\rvert$
G_ε	Ext'n Family	$\lvert K\rvert = \lvert R\rvert$

Claim 1. The graphs G_ϵ are C-free.

Suppose that $C \cong C' = (P', Q', R', S') \subseteq G$ is an embedding as a subgraph. Then Q' is Q_i or $\hat{S}_i$ for some i, and S' is Q_j or $\hat{S}_j$. As Q', S' are disjoint but connected by an edge we find that either $j = i$ or R' is a copy of $\hat{P}_k$ for some k.

Suppose first that $R' = \hat{P}_k$. Then Q', S' must be Q_k, $\hat{S}_k$ in some order, and there is no possibility for P'.

So $j = i$ and $R' = R_i$. If P' is contained in Q_{i+1} or $\hat{S}_{i-1}$ then $|P'| \leq 1 + |P' \setminus Q'| \leq 3$, a contradiction. But P' cannot be $\hat{P}_{i\pm 1}$ or $\hat{P}_i$, so the claim follows.

Claim 2. Suppose that $G_3 \subseteq G$ with G C-free. Let K be a clique of order n_2 containing $Q_i \cap L_{i-1}$. Then $K = Q_i$

We need to show that the cut vertex b_i of $Q_i R_i$ is in K. We may suppose that

$$K \cap R_i \hat{S}_i \hat{P}_i \subseteq K \cap (\{a_i, u_{i+1,P}, b_i\} \cup \hat{S}_i)$$

and in view of the anti-edges, as $a_i \in K$ and K contains Q_i^* this reduces to $K \cap \{a_i, b_i\}$. As G is C-free, $b_i \in K$.

Claim 3. Suppose that $G_3 \subseteq G$ with G C-free. Let K be a clique of order 4 containing the cut vertex b_i of $Q_i R_i$, disjoint from $L_{i-1} \cup Q_i \setminus \{b_i\}$, and free over a set X of order 2. Suppose further that X contains no vertex adjacent to $u_{i-1,S}$. Then $X = \{b_i, c_i\}$ with c_i the cut vertex of $R_i S_i$.

We may suppose that

$$K \cap L_i L_{i+1} = X \cap L_i L_{i+1}$$

If K meets Q_{i+1} then C embeds via $K Q_{i+1} R_{i+1} \hat{S}_{i+1}$, a contradiction. As $(u_{i-1,S}, u_{i+1,S})$ is an edge, X does not contain $u_{i+1,S}$.

So

$$K \cap Q_i \hat{P}_i \hat{S}_i \subseteq \{b_i, c_i\} \cup (X \cap P_i)$$

But $P_i \subseteq L_{i-1} \cup Q_{i+1}$ so $X \cap P_i = \emptyset$ and

$$K \cap Q_i \hat{P}_i \hat{S}_i \subseteq \{b_i, c_i\}$$

Since G is C-free, $c_i \in X$.

Claim 4. Let K be a clique of order n_2 disjoint from $L_{i-1} \cup Q_i$ and containing the cut vertex c_i of $R_i S_i$. Then $K = \hat{S}_i$.

As $c_i \in K$, $K \cap Q_{i+1} \subseteq \hat{S}_i \cap Q_{i+1} = Q_{i+1}^* \cup \{a_{i+1}\}$.

Suppose first that

$$K \cap Q_{i+1} = \emptyset$$

Then there is $K' \subseteq K$ of order 4 so that $K' \cap \hat{S}_i = \{c_i\}$. Then

$$K' \cap Q_i P_i \hat{S}_i = \{c_i\}$$

and we embed C into G, a contradiction.

Now suppose

$$K \text{ meets } Q_{i+1} \text{ but is not equal to } Q^*_{i+1} \cup \{a_{i+1}\}$$

Then there is $K' \subseteq K$ of order 4 with

$$|K' \cap Q_{i+1}| = 1$$

We may suppose

$$K' \cap Q_{i+1} R_{i+1} \hat{S}_{i+1} \subseteq K' \cap Q_{i+1} \cup K' \cap \hat{S}_{i+1}$$

But $K \cap \hat{S}_{i+1} = \emptyset$, so we embed C in G for a contradiction.

So finally we may assume

$$Q^*_i \cup \{a_{i+1}\} \subseteq K$$

It suffices to show that

$$u_{i+1,S} \in K$$

Now

$$K \cap \hat{P}_{i+1} \hat{S}_{i+1} R_{i+1}$$

must contain some vertex other than a_{i+1}. As $K \cap \hat{S}_{i+1} = \emptyset$ this vertex lies either in $\hat{P}_{i+1}$ or $R_{i+1} \setminus \hat{S}_{i+1}$. But we may suppose $K \cap R_{i+1}$ is contained in the set $\{b_{i+1}, c_{i+1}\}$ of cut vertices of $Q_{i+1} R_{i+1} \hat{S}_{i+1}$ and these have both been ruled out. So apart from $u_{i+1,S}$, the remaining possibilities in $\hat{P}_{i+1}$ are

$$a_{i+2}, u_{i+2,P}$$

and both lead to an embedding of C in G.

The claim follows.

The previous claims show that knowing the amalgamation base up through a_i determines Q_i and $\hat{S}_i$. In particular one can then recognize a_{i+1} and continue inductively.

It then follows as usual that there is no countable weakly universal C-free graph. □

Lemma 43 (4.11)**.** *Let C be a solid block path of length 4 satisfying*

$$n_1 = n_3 = 4 < n_2 < n_4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(3, n_2; 3)$ (length 3)
G_1	Clique Ext'n	$u^Q_{i+1,P}$; $u^Q_{i+1,R}$; $\hat{S}_i = \{a_i, a_{i+2}\} \cup S^*$
G_2	Amalg'n	$\bigcup_i Q^*_i$, $\lvert Q \setminus Q^*_i \rvert = 2$
G_3	Anti-edges	Edges matching $Q^*_i \setminus \{a_i\}$ with $Q^*_{i+2} \setminus \{a_{i+2}\}$ Anti-edges at (u, v) with $u \in Q^*_i$, $v \in Q^*_{i+2}$ unmatched at (a_i, u) with $u \in Q^*_{i+1} \setminus \{a_{i+2}\}$ $\lvert K \rvert = \lvert R \rvert$
G_ε	Ext'n Family	$\lvert K \rvert = \lvert R \rvert$

Claim 1. The graphs G_ϵ are C-free.

Suppose that $C \cong C' = (P', Q', R', S') \subseteq G$ is an embedding as a subgraph. Then S' must be a clique of the form $\{a_j, a_k\} \cup S^*$ with $j < k$. It follows easily that the configuration (P', Q', R') of type $(4, n_2, n)$ is disjoint from S^* and then by inspection we get a contradiction.

Claim 2. Suppose that $G_3 \subseteq G$ with G C-free. Let K be a clique of order n_2 free in G over a subset X with the following properties.

$$\begin{aligned} a_i &\in X \\ X \cap S^* &= \emptyset \\ |X| &= n_2 - 2 \end{aligned}$$

Then $X = Q^*_i$.

Let j be maximal so that $a_j \in X$. We may suppose that

$$\begin{aligned} K \cap Q_j \hat{P}_j S_j &\subseteq X \cap (Q^*_i \cup \{u_{j+1,P}, a_{j+1}, a_{j+2}\} \cup S^*) \\ &= X \cap (Q^*_i \cup \{u_{j+1,P}\}) \end{aligned}$$

Since $a_j \in X$ and X is a clique, the vertex $u_{j+1,P}$ is not in X.

If $X \neq Q^*_j$ then there is $K' \subseteq K$ of order 4 with $K' \cap Q^*_j = \{a_j\}$. Then $K' \cap Q_j \hat{P}_j S_j = \{a_j\}$ we get an embedding of C into G, and a contradiction.

Therefore $X = Q^*_j$, and thus $j = i$.

Claim 3. Suppose that $G_3 \subseteq G$ with G C-free. Let (Q', P') be a block path of type $(n_2, 4)$ free in G over a subset X with the following properties.

$$\begin{aligned} X \cap Q' \cap P' &= \emptyset \\ X \cap Q' &= Q^*_i \\ X \cap S^* &= \emptyset \\ |X \cap P'| &= 2 \end{aligned}$$

Suppose further that there is no edge $(u_{i-1,R}, x)$ with $x \in X \cap P'$, and that there is a unique edge (a_i, x) with $x \in X \cap P'$. Then the unique neighbor of a_i in $X \cap P'$ is a_{i+1}.

We may suppose that

$$\begin{aligned}(Q'P') \cap (R_iS_i) &\subseteq X \cap \{a_i, u_{i+1,R}, a_{i+1}, a_{i+2}\} \cup S^* \\ &= \{a_i\} \cup (X \cap P' \cap \{u_{i+1,R}, a_{i+1}, a_{i+2}\})\end{aligned}$$

If the intersection reduces to $\{a_i\}$ then we have an embedding of C into G, and a contradiction.

As $(u_{i-1,R}, u_{i+1,R})$ is an edge, the vertex $u_{i+1,R}$ is not in $X \cap P'$. So

$$a_{i+1} \in X \cap P'$$

On the other hand there is an edge (a_i, x) with $x \in X \cap P'$, so a_{i+1} is in $X \cap P'$.

This proves the claim.

Now our claims allow us to recover Q_i^* from a_i and S^*, and thus also the identification of the individual vertices in Q_i^* from Q_{i-2}^*, and then to recover a_{i+1}.

It follows as usual that there is no countable weakly universal C-free graph. □

Lemma 44 (4.12)**.** *Let C be a solid block path of length 4 and type $(4,4,4,4)$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(2,4;4,2)$
G_1	Clique Ext'n	$u^Q_{i+1,P}, v^R_{i+1,P}, u^Q_{i+1,S}, v^R_{i+1,S}$
G_2	Amalg'n	No amalgamation
G_3	Anti-edges	Edges: match $U_i = \{u_{i,P}, u_{i,S}, v_{i,P}, v_{i,S}\}$ with U_{i+2} Anti-edges at other (u,v), $u \in U_i$, $v \in U_{i+2}$ and at non-edges of $P_iQ_iR_iS_iP_{i+1}Q_{i+1}P_{i+1}R_{i+1}S_{i+1}$ $\|K\| = 4$
G_ε	Ext'n Family	$\|K\| = 4$

Claim 1. The graphs G_ϵ are C-free.

Suppose $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$.

The construction is completely symmetrical, so we may suppose that $P'$is $\hat{P}_i$ for some i. The overlap with the neighboring loops forces C to go into a single loop, and this is clearly impossible.

This proves the claim.

Claim 2. Suppose $G_3 \subseteq G$ with G C-free. Let K be a clique of order 4 in G containing a_i. Then K is a clique of G_1.

Let us denote by b_i the vertex in $P_i \cap Q_i$, and by c_i the vertex in $R_i \cap S_i$.

Suppose first that K contains the vertex b_i. Then consider $\hat{P}_i K \hat{P}_{i-1} Q_{i-1}$. As K is a clique containing a_i, b_i, the construction of G_3 forces $K \cap \hat{P}_i \hat{P}_{i-1} Q_{i-1}$ to be contained in $\{a_i, b_i, u_{i,P}\}$. If the intersection reduces to $\{a_i, b_i\}$ then we have an embedding of C into G, and a contradiction. So $u_{i,P} \in K$, and similarly $u_{i,S} \in K$, and in this case K is identified as Q_i.

If K contains b_{i-1}, c_{i-1}, or c_i, then K may be identified similarly. So now suppose that none of the these vertices lies in K.

Then K contains at most two of the neighbors of a_i, and these lie in one of the four cliques containing a_i in G_1. However the other two neighbors in the set $\{u_{i,P}, u_{i,S}, v_{i,P}, v_{i,S}\}$ also lie in a clique K' containing a_i, and G_1 contains an extension of KK' to an embedding of C into G. E.g., if K contains $Q_i \setminus \{b_i\}$, then $K \cap R_i = \{a_i\}$ and $K R_i \hat{S}_i \hat{P}_i$ gives an embedding of C into G.

This proves the claim.

Now we may use our perfect matching to work out which clique is which, and which vertex is which, inductively. It follows that there is no countable weakly universal C-free graph. □

Lemma 45 (4.13)**.** *Let C be a solid block path of length 4 and type (n_1, n_2, n_3, n_4) with all $n_i \geq 3$ and with*

$$n_1, n_2, n_3 = 4$$
$$n_4 > 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(2,4;4,2)$
G_1	Clique Ext'n	$u^Q_{i+1,P}, v^R_{i+1,P}$; S^*
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+2,P})$, $(v_{i,P}, v_{i+2,P})$ Anti-edges at (a_i, u) and (v, a_{i+1}) with $u \in \{u_{i+1,P}, v_{i+1,P}\}$, $v \in \{u_{i,P}, v_{i,P}\}$ $\lvert K \rvert = 4$
G_ε	Ext'n Family	$\lvert K \rvert = 4$

Claim 1. The graphs G_ϵ are C-free.

If $C \cong (P', Q', R', S') \subseteq G_\epsilon$ then S' is either a copy of $\hat{S}_i$ for some i, or a copy of $R_i^* = \{a_i, c_i\} \cup S^*$ where c_i is the cut vertex of $R_i S_i$. We may suppose that $S' = \hat{S}_i$ or R_i^*.

Suppose first that $S' = \hat{S}_i$. Then R' is either R_i or a copy of $\hat{P}_i$, Q_{i+1}, R_{i+1}. We may suppose that R' is equal to one of these: R_i, $\hat{P}_i$, Q_{i+1}, R_{i+1}; and the final alternative R_{i+1} is clearly untenable.

The possibility $R' = R_i$ or $\hat{P}_i$ leads to $Q' = Q_i$ and then a contradiction. The possibility $R' = Q_{i+1}$ leads to $Q' = \hat{P}_{i+1}$ and then a contradiction.

Now suppose that $S' = R_i^*$. Then we may suppose R' is Q_i or $\hat{P}_i$, and arrive at essentially the same contradiction.

This proves the claim.

Claim 2. Suppose G is a C-free graph containing G_3 and QP is a block path of type $(4,4)$ in G free over a set X satisfying the following conditions.

$$\begin{aligned} X \cap Q &= \{a_i, u_{i,P}\} \\ v_{i,P} &\notin X \\ X \cap S^* &= \emptyset \\ X \cap Q \cap P &= \emptyset \\ |X \cap Q| &= 3 \end{aligned}$$

Then $X \cap P = \{u_{i+1,P}, v_{i+1,P}, a_{i+1}\}$.

We may suppose that

$$\begin{aligned} QP \cap R_i\hat{S}_i &= X \cap (\{a_i, v_{i,P}, a_{i+1}\} \cup S^*) \\ &= X \cap \{a_i, a_{i+1}\} \end{aligned}$$

It follows that $a_{i+1} \in X \cap P$.

Take j maximal so that $a_j \in X \cap P$. By considering $QPR_j\hat{S}_j$ we find $v_{j,P} \in X$. By considering $PQ_j\hat{P}_j\hat{S}_j$ we find that X meets $\{u_{j,P}, u_{j+1,P}, v_{j+1,P}\}$. As $|X \cap P| = 3$ we conclude that $j = i + 1$. As $a_{i+1} \in X \cap P$ we conclude that $u_{i+2,P}, v_{i+2,P} \notin X$, so X is as stated.

In view of the claim and the pattern of edges and anti-edges in G_3, after fixing some vertices the sequences (a_i), $u_{i,P}$ and $v_{i,P}$ are uniquely determined. By a rigidity argument, there is no countable weakly universal C-free graph. □

Lemma 46 (4.14). *Let C be a solid block path of length 4 satisfying*

$$n_4 > n_1 > n_2 = n_3 = 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(2; 4, 4, 2)$
G_1	Clique Ext'n	$u^Q_{i+1,P}$, P^*; S^*
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+2,P})$ Anti-edges at $(u_{i,P}, a_{i+2})$, $\lvert K\rvert = 4$
G_ε	Ext'n Family	$\lvert K\rvert = 4$

Claim 1. The graphs G_ϵ are C-free.

Suppose $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$. Then S' is a copy of $\hat{S}_j$ for some j, and we may suppose $S' = \hat{S}_j$. Since $P' \cap S^* = \emptyset$, $P' = \hat{P}_i$ for some i. Hence $R' \cap P^* = \emptyset$. Therefore R' is R_j or a copy of Q_{j+1}, and in the latter case we may assume $R' = Q_{j+1}$.

If $R' = R_j$ then $Q' = Q_j$. But then there is no suitable value for i.

Suppose $R' = Q_{j+1}$. The possibility $Q' = R_{j+1}$ is out of the question. We should consider also the possibility that Q' meets R' at $u_{j+1,P}$. Now $|Q' \cap P^*| \leq 1$ so this gives us $|Q'| \leq |P_j \setminus \{a_{j+1}\}| + 1 = 3$, a contradiction.

The claim follows.

Claim 2. Suppose that $G_3 \subseteq G$ with G C-free. Let K be a clique of order n_1 in G with the following properties.

$$a_i \in K, u_{i,P} \notin K$$
$$P^* \subseteq K$$
$$K \cap S^* = \emptyset$$

Then $K = \hat{P}_i$.

We may suppose that

$$\begin{aligned} K \cap Q_i R_i \hat{S}_i &\subseteq K \cap (\{a_i, u_{i,P}, a_{i+1}\} \cup S^*) \\ &= K \cap \{a_i, a_{i+1}\} \end{aligned}$$

The intersection cannot reduce to $\{a_i\}$, so

$$a_{i+1} \in K$$

Now consider $KQ_{i+1}R_{i+1}S_{i+1}$ similarly to conclude that $u_{i+1,P}$ or a_{i+2} belongs to K. If $a_{i+2} \in K$ pass to $Q_{i+2}R_{i+2}S_{i+2}$ for a contradiction.

The claim follows.

So from $a_i, u_{i,P}$ and some fixed parameters we recover $\{a_{i+1}, u_{i+1,P}\}$, and we distinguish these vertices using the parameter $u_{i-1,P}$. By a rigidity argument there is no countable weakly universal C-free graph.

$\square$

Lemma 47 (4.15)**.** *Let C be a block path of length 4 satisfying*

$$n_1 = n_4 = 4$$
$$n_2 \geq 5$$
$$n_3 = 3$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(3, n_2, 3; 3)$
G_1	Clique Ext'n	$u^R_{i+1,P}$, $u^P_{i+1,S}$
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(u_{i,S}, u_{i+2,S})$ Anti-edges at $(a_i, u_{i+1,P})$, $(u_{i,P}, a_j)$, $(j = i+1, i+2)$, $(u_{i,S}, a_j$, $(j = i+2, i+3)$, $(u_{i,S}, u_{i+2,P})$ $\lvert K\rvert = 4$
G_ε	Ext'n Family	$\lvert K\rvert = 4$

Claim 1. The graphs G_ϵ are C-free.

If $C \cong C' = (P', Q', R', S')$ then we may suppose $Q' = Q_i$ for some i, and correspondingly $P' = \hat{P}_i$, $R' = R_i$. There is then no plausible candidate for S'.

Claim 2. Let G be a C-free graph containing G_3, and K a clique of order 4 free over a set X of order 2, with $a_i \in X$ and no neighbor of $u_{i-2,S}$ in X. Then X is $\{a_i, u_{i,P}\}$.

We may suppose that

$$K \cap \hat{P}_i Q_i R_i \subseteq X \cap \{a_i, u_{i,P}, u_{i,S}, a_{i+1}\}$$

If the intersection reduces to $\{a_i\}$ then embed C via $KR_iQ_i\hat{P}_i$; and if $a_{i+1} \in X$ embed C via $K\hat{P}_{i+1}Q_{i+1}R_{i+1}$. As $u_{i,S}$ is a neighbor of $u_{i-2,S}$, this is ruled out. So we are left with $X = \{a_i, u_{i,P}\}$.

Claim 3. Let G be a C-free graph containing G_3, and K a clique of order 4 free over a set X of order 3, with $a_i \in X$ and $a_{i-1}, u_{i,P} \notin X$, and no neighbor of $u_{i-2,S}$ in X. Then $X = \{a_i, u_{i+1,S}, a_{i+1}\}$.

We may suppose that

$$\begin{aligned} K \cap \hat{P}_i Q_i R_i &\subseteq X \cap \{a_i, u_{i,P}, u_{i,S}, a_{i+1}\} \\ &= X \cap \{a_i, a_{i+1}\} \end{aligned}$$

since $u_{i,S}$ is a neighbor of $u_{i-2,S}$.

If the intersection reduces to $\{a_i\}$ then we embed C via $KR_iQ_i\hat{P}_i$. So

$$a_{i+1} \in X$$

Then we consider $X \cap \hat{P}_{i+1}Q_{i+1}R_{i+1}$ and as $a_i \in K$ we find $u_{i+1,P}, a_{i+2} \notin X$ and $u_{i+1,S} \in X$.

The last two claims give sufficient rigidity to show that there is no countable weakly universal C-free graph. □

Lemma 48 (4.16)**.** *Let C be a block path of length 4 satisfying*

$$n_1 = 4$$
$$n_3 = 3$$
$$n_2 \geq n_4 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(3; n_2, 3)$
G_1	Clique Ext'n	$u^R_{i+1,P}$
G_2	Amalg'n	$\bigcup_i (Q^*_i \cup \{u_{i,P}\}$, $\lvert Q_i \setminus Q^*_i\rvert = 2$
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+2,P})$ $\lvert K\rvert = 4$ Anti-edges at (a_i, v) for $v \in Q^*_{i+1} \setminus \{a_{i+1}\}$ $\lvert K'\rvert = 3$
G_ε	Ext'n Family	$\lvert K\rvert = 4$

Claim 1. The graphs G_ϵ are C-free.

We show that G_ϵ contains no solid block path (P', Q', R') of type $(4, n_2, 3)$. Here Q' would have to be a copy of some Q_i, and we may suppose $Q' = Q_i$. Then P', R' meet Q' in a_i and the vertex b_i in $Q_i \cap R_i$. This forces R' to be R_i. Then there is no plausible candidate for P'.

Claim 2. Suppose that $G_3 \subseteq G$ with G C-free. Let K be a clique of order n_2 which is free in G over a subset X with the following properties.

$$a_i \in X$$
$$|X| = |Q^*_i|$$

Suppose there is no edge $(u_{i-2,P}, x)$ with $x \in X$. Then $X = Q^*_i$.

Take j maximal so that K meets Q^*_j. We may suppose that

$$\begin{aligned} K \cap Q_jR_jQ_{j+1} &\subseteq X \cap (Q^*_j \cup \{u_{j,P}\} \cup Q^*_{j+1}) \\ &= X \cap Q^*_j \end{aligned}$$

in view of the conditions on X and the choice of j.

If $X = Q_j^*$ then evidently $j = i$ and the claim is proved.

if $X \neq Q_j^*$ then there is a clique $K' \subseteq K$ of order 4 with $|K' \cap Q_j^*| = 1$, and hence C embeds in $K'Q_jR_jQ_{j+1}$, a contradiction.

This proves the claim.

Claim 3. Suppose that $G_3 \subseteq G$ with G C-free. Let K be a clique of order 4 free in G over a set X with the following properties.

$$X \cap Q_i^* = \{a_i\}$$
$$|X| = 3$$

Suppose that X contains no neighbor of $u_{i-2,P}$. Then $a_{i+1} \in X$.

We may suppose

$$\begin{aligned} K \cap Q_iR_iQ_{i+1} &\subseteq X \cap (Q_i^* \cup \{u_{i,P}\} \cup Q_{i+1}^*) \\ &= X \cap \{a_i, a_{i+1}\} \end{aligned}$$

since X is a clique containing a_i and Q_{i+1}^* contains no neighbor of a_i other than a_{i+1}, and no neighbor of $u_{i-2,P}$.

The intersection cannot reduce to $\{a_i\}$, so $a_{i+1} \in X$.

Now the last claims show that from the parameter a_i (essentially) we can recover Q_i^*, then a_{i+1}, and hence also Q_{i+1}^*. To conclude, we identify $u_{i,P}$.

Claim 4. Suppose that $G_3 \subseteq G$ with G C-free. Let K be a clique of order 4 free in G over a subset X with the following properties.

$$X \cap Q_i^* = \{a_i\}$$
$$X \cap Q_{i+1}^* = \emptyset$$
$$|X| = 3$$

Then $X = \{a_i, u_{i+1,P}, a_{i+1}\}$.

We may suppose that

$$\begin{aligned} K \cap Q_iR_iQ_{i+1} &\subseteq X \cap (Q_i^* \cup \{u_{i,P}\} \cup Q_{i+1}^*) \\ &= X \cap \{a_i, u_{i,P}\} \end{aligned}$$

As the intersection cannot reduce to $\{a_i\}$, the claim follows.

Now it follows that there is no countable universal C-free graph. □

Lemma 49 (4.17). *Let C be a block path of length 4 satisfying*

$$3 \leq n_1, n_3 < n_2 < n_4$$
$$n_1 \neq n_3$$

Then there is no countable weakly universal C-free graph.

Proof. Set

$$m = \min(n_1, n_3)$$
$$n = \max(n_1, n_3)$$

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(n-1; n_2, m)$
G_1	Clique Ext'n	$u^R_{i+1,P}$; $\hat{S}_i = \{a_{i+1}, a_{i+2}\} \cup S^*$
G_2	Amalg'n	Base $\bigcup_i (Q_i^* \cup \{u_{i,P}\})$, $\lvert Q_i \setminus Q_i\rvert = n-2$ if $n = n_1$, and $Q_i^* = \{a_i\}$ otherwise
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+2,P})$ Anti-edges at $(u, u_{i,P})$ for $u \in Q_i^* \setminus \{a_i\}$ and (a_i, a_{i+2}) $\lvert K\rvert = m$
G_ε	Ext'n Family	$\lvert K'\rvert = n$

Claim 1. The graphs G_ϵ are C-free.

Suppose $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$ is an embedding as a subgraph. Then S' has the form

$$\{a_j, a_k\} \cup S^*$$

with (a_j, a_k) an edge.

As $Q' \cap S^* = \emptyset$ we find that Q' is a copy of some Q_i, and we may suppose $Q' = Q_i$. Then P', R' meet Q' in a_i and the vertex $b_i \in Q_i \cap R_i$. The clique meeting Q' in b_i must be R_i, of order m, and equal to P' or Q'. The other clique $K = P'$ or R' meeting Q' in a_i must have order n.

We claim that K cannot be contained in another copy of Q_i. If $n_1 = n$ then the choice of Q_i^* ensures this. If $n_1 = m$ then $K = R'$, and if K were contained in a copy of Q_i, then S' would be forced into the same copy of Q_i, for a contradiction.

As $P' \cap R' = \emptyset$, K is not $\hat{P}_{i-1}$ or $\hat{P}_i$. As $m < n$ K is not contained in R_{i-1}. There is one more possibility to consider: $P' = R_i$ and R' meets S' in a vertex of S^*. In this case

$$|R'| \leq 1 + |R' \setminus S^*| \leq 3$$

But $n > 3$ so this is impossible as well.

The claim is proved.

For the rest of the argument we need to distinguish the cases $n_1 = m$, $n_1 = n$.

Claim 2. Suppose $G_3 \subseteq G$ with G C-free, and $n_1 = m$. Let (Q', R') be a solid block path of type (n_2, m) free in G over a subset X with the following properties.

$$X \cap Q' \cap R' = \emptyset$$
$$X \cap S^* = \emptyset$$
$$X \cap Q' = \{a_i\}$$
$$a_{i-1} \notin X$$
$$|X \cap R'| = 2$$

Suppose also that the elements of $X \cap R'$ are neighbors of a_i. Then $X \cap R' = \{u_{i,P}, a_{i+1}\}$.

We may suppose that

$$\begin{aligned} Q'R' \cap \hat{P}_{i-1}S_{i-2} &\subseteq X \cap (\{a_{i-1}, u_{i,P}, a_i\} \cup S^*) \\ &= X \cap \{u_{i,P}, a_i\} \end{aligned}$$

If the intersection reduces to $\{a_i\}$ then we have an embedding of C into G, and a contradiction. So

$$u_{i,P} \in X \cap R'$$

We may suppose that

$$\begin{aligned} Q'R' \cap \hat{P}_i S_i &\subseteq X \cap (\{a_i, u_{i+1,P}, a_{i+1}, a_{i+2}\} \cup S^*) \\ &= \{a_i\} \cup (X \cap R' \cap \{u_{i+1,P}, a_{i+1}, a_{i+2}\}) \end{aligned}$$

The intersection cannot reduce to $\{a_i\}$. As $u_{i,P} \in X \cap R'$ and $X \cap R'$ is a clique, we cannot have $u_{i+1,P}$ or a_{i+2} in $X \cap R'$. So $a_{i+1} \in X \cap R'$ and the claim is proved.

As we can then distinguish the vertices $u_{i,P}, a_{i+1}$ over the parameter $u_{i-1,P}$, it follows that there is no countable weakly universal C-free graph in this case.

So in what follows we deal with the case $n_1 = n$.

Claim 3. Suppose that $G_3 \subseteq G$ with G C-free, and that $n_1 = n$. Let K be a clique of order n_2 free in G over a subset X with the following properties.

$$a_i \in X$$
$$X \cap S^* = \emptyset$$
$$|X| = |Q_i^*|$$

Then $X = Q_i^*$.

Take j maximal with $a_j \in X$. We may suppose that

$$\begin{aligned} K \cap Q_j R_j S_j &\subseteq X \cap (Q_j^* \cup \{u_{j,P}, a_{j+1}, a_{j+2}\} \cup S^*) \\ &= X \cap (Q_j^* \cup \{u_{j,P}\}) \end{aligned}$$

If $X \neq Q_j^*$ then there is a clique $K' \subseteq K$ of order n so that $|K' \cap Q_j| = 1$. As G is C-free, this forces $u_{j,P} \in K'$ and $|X \cap Q_j^*| = |Q_j^*| - 1 > 1$. As X is a clique meeting $Q_j^* \setminus \{a_j\}$, the vertex $u_{j,P}$ is not in X. This contradiction completes the proof of the claim.

Claim 4. Suppose that $G_3 \subseteq G$ with G C-free, and that $n_1 = n$. Let K be a clique of order n free in G over a subset X with the following properties.

$$\begin{aligned} X \cap Q_i^* &= \{a_i\} \\ a_{i-1} &\notin X \\ X \cap S^* &= \emptyset \\ |X| &= 2 \end{aligned}$$

Suppose that the unique vertex $v \in X \setminus \{a_i\}$ is not a neighbor of $u_{i-2,P}$. Then $X = \{a_i, a_{i+1}\}$.

We may suppose that

$$\begin{aligned} K \cap Q_i R_i S_i &\subseteq X \cap (Q_i^* \cup \{u_{i,P}, a_{i+1}, a_{i+2}\} \cup S^*) \\ &= X \cap \{a_i, u_{i,P}, a_{i+1}, a_{i+2}\} \end{aligned}$$

The intersection cannot reduce to $\{a_i\}$ and $u_{i,P}$ is a neighbor of $u_{i-2,P}$, so X must contain a_{i+1} or a_{i+2}. As $a_i \in X$ and X is a clique, $a_{i+2} \notin X$. So $a_{i+1} \in X$ and the claim is proved.

At this point, given a_i we can identify Q_i^*, and then a_{i+1}.

Claim 5. Suppose that $G_3 \subseteq G$ with G C-free, and that $n_1 = n$ Suppose that (Q', R') is a solid block path of type (n_2, m) free in G over a subset X with the following properties.

$$\begin{aligned} X \cap Q' \cap R' &= \emptyset \\ X \cap Q' &= Q_i^* \\ a_{i+1} &\in X \\ |X \cap R'| &= 2 \end{aligned}$$

Then $X \cap R' = \{u_{i,P}, a_{i+2}\ ]$.

It suffices to consider a suitably chosen copy of $\hat{P}_{i-1} Q' R' S_i$.

From these claims, by a rigidity argument there is no countable weakly universal C-free graph. $\square$

Lemma 50 (4.18)**.** *Let C be a block path of length 4 satisfying*

$$\begin{aligned} n_1 &= 3 \\ n_3 &= 4 < n_2 = n_4 \end{aligned}$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(3, n_2; 3)$
G_1	Clique Ext'n	$u^P_{i+1,R}$; $\hat{S}_i = \{a_{i+1}\} \cup S^*$
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(u_{i,R}, u_{i+2,R})$ Anti-edges at $(u_{i,R}, a_{i+2})$ of type (R, S) and at $(u_{i,R}, u_{i+1,R})$, $\lvert K\rvert = n_2$
G_ε	Ext'n Family	$\lvert K'\rvert = 4$

Claim 1. The graphs G_ϵ are C-free.

Suppose that $C \cong C' = (P', Q', R', S")$ $\subseteq G_\epsilon$ is an embedding as a subgraph. Clearly Q' cannot be a copy of one of the S_i, so Q' is a copy of some Q_i, and we may suppose that $Q' = Q_i$. Then the points of intersection of Q' with P' and R' are a_i and the vertex b_i in $Q_i \cap \hat{P}_i$, so P' must be $\hat{P}_i$ and R' meets Q' in a_i.

Evidently R' is not contained in a copy of S_{i-1} or P_{i-1}, and overlap with P' prevents R' from being a copy of $\hat{R}_{i-1}$ or $\hat{R}_i$. If R' is contained in a copy of Q_i, there is no candidate for S'. So all cases are eliminated and the claim follows.

Claim 2. Suppose that $G_3 \subseteq G$ with G C-free. Let (Q', P') be a solid block path of type $(n_2, 3)$ which is free in G over a set X with the following properties.

$$\begin{aligned} X \cap Q' \cap P' &= \emptyset \\ X \cap Q' &= \{a_i\} \\ a_{i-1} &\notin X \\ |X \cap P'| &= 2 \end{aligned}$$

Then $X \cap P' = \{u_{i,R}, a_{i+1}\}$.

We may suppose that

$$Q'P' \cap \hat{R}_{i-1}Q_{i-1} \subseteq X \cap \{a_{i-1}, u_{i,R}, a_i\}$$

If the intersection reduces to $\{a_i\}$ then we have an embedding of C into G, and a contradiction. So

$$u_{i,R} \in X \cap R'$$

We may suppose that

$$\begin{aligned} Q'P' \cap \hat{R}_i S_i &\subseteq X \cap \{a_i, u_{i+1,R}, a_{i+1}\} \\ &= \{a_i\} \cup (X \cap R' \cap \{u_{i+1,R}, a_{i+1}\}) \end{aligned}$$

If the intersection reduces to $\{a_i\}$ then we have an embedding of C into G, and a contradiction. As $X \cap R'$ is a clique containing $u_{i,R}$, the vertex $u_{i+1,R}$ is not in $X \cap R'$. So $a_{i+1} \in X \cap R'$ and the claim follows.

Now by a rigidity argument there is no countable weakly universal C-free graph.

□

Lemma 51 (4.19)**.** *Let C be a block path of length 4 satisfying*

$$n_1 = 3$$
$$n_3 = 4 < n_4 < n_2$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(3, n_2; 4, n_4)$
G_1	Clique Ext'n	No clique extension
G_2	Amalg'n	Base $\bigcup_i Q_i^*$, $\lvert Q_i^*\rvert = \max(1, n_2 - (n_4 + 1))$
G_3	Anti-edges	No additional edges or anti-edes
G_ε	Ext'n Family	$\lvert K\rvert = 4$

Claim 1. The graphs G_ϵ are C-free.

Suppose that $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$ is an embedding as a subgraph. Then Q' is a copy of some Q_i, and we may suppose $Q' = Q_i$. Then P', R' meet Q' in a_i and the vertex $b_i \in Q_i \cap P_i$. So P' must be P_i and R' meets Q' in a_i.

If R' is contained in another copy $Q^{\#}$ of Q_i then so is S' and we find

$$|R'S'| \leq 1 + |Q_i^{\#} \setminus Q_i^*| = n_4 + 2$$

This is a contradiction. Other possibilities for R' are quickly eliminated.

The claim follows.

For the reconstruction of the sequence (a_i) we follow the logic of the preceding proof.

This suffices to show the nonexistence of a countable weakly universal C-free graph. □

Lemma 52 (4.20)**.** *Let C be a block path of length 4 satisfying*

$$n_1 = 3$$
$$n_3 = n_4 = 4$$
$$n_2 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(3, n_2; 4, 2)$
G_1	Clique Ext'n	$u^Q_{i+1,S}$, $v^Q_{i+1,S}$
) G_2	Amalg'n	Base $\bigcup Q^*_i$, $\lvert Q^*_i\rvert = \max(3, n_2 - 5)$
G_3	Anti-edges	Edges $(u_{i,S}, u_{i+2,S})$ and $(v_{i,S}, v_{i+2,S})$ Anti-edges at (u, v) for non-edges with $u \in Q^*_i$, $v \in Q^*_j$, $j = i + 1, i + 2$ $\lvert K\rvert = 4$ at (a_i, a_{i+1}), $\lvert K'\rvert = 4$
G_ε	Ext'n Family	$\lvert K'\rvert = 4$

Claim 1. The graphs G_ϵ are C-free.

Suppose $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$ is an embedding as a subgraph. Then Q' is a copy of some Q_i and we may suppose $Q' = Q_i$. The possible points of intersection of Q' with P' and R' are a_i, $u_{i,S}$, $v_{i,S}$, and the cut vertex $b_i \in Q_i \cap P_i$.

The case in which the point of intersection is $u_{i,S}$ or $v_{i,S}$ can be ruled out, as the clique K in question would be contained in a copy of $\hat{S}_{i-1}$, leaving $|K| \le |\hat{S}_{i-1} \setminus Q_i| + 1 = 2$, a contradiction.

So P', R' meet Q' in a_i and b_i in some order. The clique meeting Q' in b_i must be P_i, so $P' = P_i$ and R' meets Q' in a_i. We first consider the exotic possibility that R' is contained in a second copy $Q^\#_i$ of Q'. This forces S' into $Q^\#_i$ as well and then $|R'S'| \le 1 + |Q^\#_i \setminus Q_i| = 1 + n_2 - |Q^*_i| \le 1 + 5 = 6$, a contradiction. Thus R' is a copy of R_i, which we may suppose is R_i, and then S' must be $\hat{S}_i$ and we have a contradiction.

This proves the claim.

Claim 2. Suppose that $G_3 \subseteq G$ with G C-free, and that $|Q^*_i| = n_2 - 5$. Let K be a clique of order n_2 free in G over a subset X with the following properties.

$$a_i \in X$$
$$|X| = |Q^*_i|$$

Then $X = Q^*_i$.

We may suppose that

$$\begin{aligned} K \cap (\hat{P}_i Q_i) &\subseteq X \cap (Q^*_i \cup \{a_{i+1}\}) \\ &= X \cap Q^*_i \end{aligned}$$

since X is a clique containing a_i.

If $X \neq Q_i^*$ then we can find $K' \subseteq K$ with $|K'| = 7$ and $|K' \cap Q_i| = 1$. This gives an embedding of C into G, and a contradiction.

Claim 3. Suppose that $G_3 \subseteq G$ with G C-free, and that $|Q_i^*| = 3$. Let K be a clique of order n_2 which is free in G over a set X satisfying the following conditions.

$$a_i \in X$$
$$|X| = 3$$

Then either $X = Q_i^* = \{a_i, u_{i,S}, v_{i,S}\}$ or $X \cap R_{i-1}\hat{S}_{i-1} = \{a_i\}$.

We may suppose that

$$\begin{aligned} K \cap (\hat{R}_{i-1}\hat{S}_{i-1} \cup R_i\hat{S}_i) &\subseteq X \cap (\{a_{i-1}\} \cup Q_i^* \cup Q_{i+1}^*) \\ &= X \cap Q_i^* \end{aligned}$$

since X is a clique containing a_i.

Suppose $X \neq Q_i^*$ but the intersection contains $u_{i,S}$ or $v_{i,S}$. Then we find a clique $P_0 \subseteq \hat{S}_{i-1}$ of order 3 with $|P_0 \cap K| = 1$, where the common vertex is $u_{i.S}$ or V_i'. In this case extend by $R_i\hat{S}_i$ to get an embedding of C.

This proves the claim.

Claim 4. Suppose that $G_3 \subseteq G$ with G C-free. Let (Q', P') be a solid block path of type (n_2, n_1) free in G over a subset X with the following properties.

$$X \cap Q' \cap P' = \emptyset$$
$$a_i \in X \cap Q'$$
$$|X \cap Q'| = |Q_i^*|$$
$$|X \cap P'| = 1$$

Then $X \cap Q' = Q_i^*$ and $X \cap P' = \{a_{i+1}\}$.

As far as $X \cap Q'$ is concerned, this is largely dealt with in the previous two claims, but we need the extension by P' to ensure that $u_{i,S}$ or $v_{i,S}$ gets into $X \cap Q'$.

A similar argument shows that the unique vertex v in $X \cap P'$ lies in Q_{i+1}^*. If it is not a_{i+1} then we embed C into G using $P'Q_{i+1}R_{i+1}\hat{S}_{i+1}$.

From these claims and the edge/antiedge structure on $\bigcup_i Q_i^*$ we have enough rigidity to show that there is no countable weakly universal C-free graph. □

Lemma 53 (4.21). *Let C be a solid block path of length 4 satisfying the following.*

$$n_1 = n_3 = 3$$
$$n_4 > n_2 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(2, n_2, 3; 3)$
G_1	Clique Ext'n	$u^S_{i+1,P}$; $u^R_{i+1,S}$, S^*
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+2,P})$, $(u_{i,S}, u_{i+2,S})$ Anti-edges at (u, v) with $u \in \{a_i, u_{i,P}, u_{i,S}\}$, $v \in \{a_{i+2}, u_{i+2,P}, u_{i+2,S}\}$, not an edge of type C, i.e., $C\backslash e$, e in P
G_ε	Ext'n Family	$\lvert K\rvert = 3$

Claim 1. The graphs G_ϵ are C-free.

Suppose $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$. Then $S' = S_i$ for some i. Since $Q' \cap S^* = \emptyset$ and $|Q'| \geq 5$, Q' is a copy of some Q_j; we suppose $Q' = Q_j$.

If R' meets S' in the vertex v, then v has a neighbor in Q'. Hence $v \notin S^*$, and $v = a_i$ or a_{i+1}. By symmetry we may suppose $v = a_i$. There is no viable candidate for R' other than R_i, so $i = j$ and then P' is forced to meet S', a contradiction.

If $G_3 \subseteq G$ with G C-free, then over the parameters in S^*, from the vertices $a_i, u_{i,P}, u_{i-1,P}, u_{iS}, u_{i-1,S}$ we can determine $u_{i+1,P}$ and a_{i+1} by considering cliques of order n_4 containing $\{a_i\} \cup S^*$ and realizing an appropriate type over the parameters. And then we may determine $u_{i+1,P}$ by considering solid block paths of an appropriate type relative to the same parameters together with a_{i+1} and $u_{i+1,S}$.

It then follows by a rigidity argument that there is no countable weakly universal C-free graph. □

Lemma 54 (4.22)**.** *Let C be a solid block path of length 4 satisfying the following.*

$$n_1 = n_3 = 3$$
$$n_2 > n_4 \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(2, n_2, 3; n_4 - 1)$
G_1	Clique Ext'n	$u^S_{i+1,P}$; $u^R_{i+1,S}$
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(u_{i,S}, u_{i+2,S})$ Anti-edges at (u, v) with $u \in \{a_i, u_{i,P}, u_{i,S}\}$ and $v \in \{a_{i+2}, u_{i+2}, u_{i+2,S}\}$, not an edge $\lvert K\rvert = \lvert S\rvert$
G_ε	Ext'n Family	$\lvert K\rvert = \lvert S\rvert$

Claim 1. The graphs G_ϵ are C-free.

Suppose that $C \cong C' = (P', Q', R', S') \subseteq G_\epsilon$ is an embedding as a subgraph. Then Q' is a copy of some Q_i, and we may suppose that $Q' = Q_i$. So P', R' are $\hat{P}_i$, R_i in some order. By symmetry we may suppose that $R' = R_i$.

Then the only candidates for S' are copies of $\hat{S}_{i-1}$ and $\hat{S}_i$, and one meets R' in two vertices while the other meets P'.

This proves the claim.

Now we may argue as usual that under an embedding of G_3 into a C-free graph G, the sequence (a_i) is determined by a finite amount of data, and thus there is no countable weakly universal C-free graph.

□

Lemma 55 (4.23)**.** *Let C be a solid block path of length 4 and type $(3, 4, 4, n_4)$ with*

$$n_4 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(2; 4, 4, 2)$
G_1	Clique Ext'n	$u^Q_{i+1,P}$, P^*; S^*
G_2	Amalg'n	Base $(\bigcup_i Q_i \setminus R_i) \cup S^*$
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+2,P})$, (v_i, v_{i+2}) where $Q_i \setminus R_i = \{a_i, u_{i,P}, v_i\}$ Anti-edges at (a_i, v_{i+1}) and (u, a_{i+2}), $u \in Q_i \setminus R_i$, $\lvert K\rvert = 4$
G_ε	Ext'n Family	$\lvert K\rvert = 3$

Claim 1. The graphs G_ϵ are C-free.

If $C \cong (P', Q', R'S') \subseteq G_\epsilon$ then S' is either a copy of some $\hat{S}_i$ or of the form $\{a_i, a_j\} \cup S^*$ with (a_i, a_j) an edge.

It then follows that Q', R' are copies of some Q_k, R_k in some order. In order for P' to exist, the order must be $R'Q' = R_k Q_k$, and this forces S' to be a copy of $\hat{S}_k$. So we may suppose $S'R'Q' = \hat{S}_k R_k Q_k$. Then one sees that there is no suitable clique corresponding to P' (noting that any other copy of Q_k contains $Q_k \setminus R_k$).

Claim 2. Suppose G is a C-free graph containing G_3 and K is a clique of order 4 in G free over a set X satisfying the following.

$$a_i \in X$$
$$X \cap S^* = \emptyset$$
$$|X| = 3$$

Then $X = Q_i \setminus R_i$.

Let j be maximal so that $a_j \in X$. We may suppose that

$$\begin{aligned} K \cap Q_j R_j \hat{S}_j &= X \cap ((Q_k \setminus R_k) \cup \{a_{j+1}\} \cup S^*) \\ &= X \cap (Q_k \setminus R_k) \end{aligned}$$

If the intersection is not $Q_k \setminus R_k$ then there is a clique $K' \subseteq K$ of order 3 meeting $Q_j R_j \hat{S}_j$ only in a_j, and C embeds in G. The claim follows.

Claim 3. Suppose G is a C-free graph containing G_3 and K is a clique of order 3 in G such that

$$K \cap Q_i \setminus R_i = \{a_i\}$$
$$K \cap S^* = \emptyset$$

Then $K = \hat{P}_i$.

We may suppose that

$$\begin{aligned} K \cap Q_i R_i \hat{S}_i &= K \cap ((Q_i \setminus R_i) \cup \{a_{i+1}\} \cup S^*) \\ &= K \cap \{a_i, a_{i+1}\} \end{aligned}$$

As G is C-free we have

$$a_{i+1} \in K$$

It then follows easily that K meets Q_{i+1} in at least two points. As $a_i \in K$ it follows that $K \cap Q_{i+1} = \{a_{i+1}, u_{i+1,P}\}$ and $K = \hat{P}_i$.

From the preceding claims and the edge/antiedge structure in G_3 it follows that we can inductively determine the vertices a_i and prove the nonexistence of a countable weakly universal C-free graph by a rigidity argument. □

Lemma 56 (4.24)**.** *Let C be a solid block path of length 4 and type* $(3, 3, 4, n_4)$ *with*

$$n_4 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(2,3;3,2)$
G_1	Clique Ext'n	$u^R_{i+1,P}$; S^*
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+2,P})$ Anti-edges at $(u_{i,P}, a_{i+2})$, $\|K\| = 3$
G_ε	Ext'n Family	$\|K\| = 3$

Claim 1. The graphs G_ϵ are C-free.

Suppose $C \cong (P', Q', R', S') \subseteq G_\epsilon$. We may suppose $S' = \hat{S}_i$ for some i. Then R' must be R_i or a copy of R_{i+1}, and the latter alternative is not viable. So $R' = R_i$.

Then Q' must be a copy of Q_i, and we may suppose $Q' = Q_i$, forcing $P' = \hat{P}_i$, and a contradiction.

Claim 2. Let G be a C-free graph containing G_3 and $(Q', P') \subseteq G$ a solid block path of type $(3,3)$ free in G over a subset X satisfying the following.

$$\begin{aligned} a_i &\in P' \\ X &= \{a_i\} \cup (P' \setminus Q') \\ u_{i,P} &\notin X \\ X \cap S^* &= \emptyset \end{aligned}$$

Then $X = \{a_i, u_{i+1,P}, a_{i+1}\}$.

We may suppose that

$$\begin{aligned} Q'P' \cap R_i\hat{S}_i &= X \cap \{a_i, u_{i,P}, a_{i+1}\} \cup S^* \\ &= X \cap \{a_i, a_{i+1}\} \end{aligned}$$

As the intersection cannot reduce to a_i, we have

$$a_{i+1} \in X$$

Then we consider the intersection with $R_{i+1}\hat{S}_{i+1}$ and conclude rapidly that $X = \{a_i, u_{i+1,P}, a_{i+1}\}$.

Now we easily recover the sequence (a_i) from a finite set of parameters and conclude by a rigidity argument. □

Lemma 57 (4.25)**.** *Let C be a solid block path of length 4 and type $(4,3,4,n_4)$ with*

$$n_4 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	$(2;3,4,2)$
G_1	Clique Ext'n	$u^Q_{i+1,P}$, P^*; S^*
G_2	Amalg'n	Special vertices
G_3	Anti-edges	Edges $(u_{i,P}, u_{i+2,P})$ Anti-edges at (a_i, c_j) $(j = i, i+1)$ with c_i the cut vertex of R_iS_i $\lvert K\rvert = 4$
G_ε	Ext'n Family	$\lvert K'\rvert = 3$

Claim 1. The graphs G_ϵ are C-free.

If $C \cong (P', Q', R', S') \subseteq G_\epsilon$ then S' must be $\hat{S}_i$ for some i. Then R' is a copy of R_i or $\hat{P}_{i+1}$, so we suppose $R' = R_i$, $\hat{P}_i$, or $\hat{P}_{i+1}$.

If $R' = R_i$ then $Q' = Q_i$ and there is no viable candidate for P'.

If $R' = \hat{P}_i$ or $\hat{P}_{i+1}$ then P' must be a copy of some R_j. Then there is no viable candidate for Q'; the main to consider would be $R' = \hat{P}_{i+1}$ and $P' = R_{i+2}$, $Q' = Q_{i+2}$. But we also need to notice that our choice of amalgamation base prevents the possibility that $R' = \hat{P}_i$, $Q' = Q_i$, and P' is a copy of R_i disjoint from $\hat{S}_i$.

Claim 2. Let G be a C-free graph containing G_3 and K a clique of order 4 in G such that

$$\begin{aligned} a_i &\in K \\ P^* &\subseteq K \\ a_{i-1} &\notin K \\ S^* \cap K &= \emptyset \end{aligned}$$

Suppose that $K \setminus P^*$ contains no neighbor of $u_{i-2,P}$. Then $K = \hat{P}_i$.

We may suppose that

$$\begin{aligned} K \cap Q_iR_i\hat{S}_i &\subseteq K \cap \{a_i, u_{i,P}\} \cup \hat{S}_i \\ &= K \cap (\{a_i, u_{i,P}, a_{i+1}\}) \end{aligned}$$

since (a_i, c_i) is an antiedge. This intersection cannot reduce to $\{a_i\}$. As $(u_{i-2,P}, u_{i,P})$ is an edge, we find $a_{i+1} \in K$. Then considering $Q_{i+1}R_{i+1}\hat{S}_{i+1}$ we find $K = \hat{P}_i$.

The claim follows, and the pattern of edges and anti-edges in G_3 allows us to reconstruct the sequence $(u_{i,P})$ from a finite set of parameters. A rigidity argument completes the proof. □

5. Critical configurations of length 5

From this point on, in our loop constructions we generally give only the construction G_0, G_1; that is, we omit the specification of an amalgamation base and the decoration by edges and anti-edges involved in constructing G_3 and then the various G_ε, unless something exceptional is involved. The default includes: amalgamation over the set of special vertices; anti-edges of type $(K\backslash e)$ for some suitable value of $|K|$ (minimal so that K itself would give an embedding of C); attachment of edges/anti-edges according to ε at $(a_{3i}, a_{3(i+1)})$, as amply illustrated in the previous section.

The next two lemmas will cover clause 5.1 and a little more.

Lemma 58 (5.1 with $n_1 > n_2$)**.** *Let C be a solid block path of length 5, satisfying the following.*

$$n_1 > n_2 > n_3 = 3$$
$$n_4 = n_2,\ n_5 = n_1$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n_1 - 1, n_2; n_3, n_2, n_1 - 1)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$, $u^Q_{i+1,R}$

We check that the graphs G_ε are C-free.

Suppose $j : C \cong C' = (P', Q', R', S', T') \subseteq G_\varepsilon$. Then each of P', T' is a copy of some $\hat{P}_i$ or $\hat{T}_i$ and P', T' are disjoint, so we may suppose $P' = \hat{P}_i$ or $\hat{T}_i$, $T' = P_j$ or $\hat{T}_j$, with distinct indices. And we may suppose $i < j$, so that the cut vertex of $P'Q'$ should lie in $Q_{j+1} \cup R_{j+1}$. But this is impossible.

□

The next result covers clause 5.1 for the case $n_2 > n_1$, but in considerably greater generality. The extra generality is not needed elsewhere: one may set $n_3 = 3$ and $n_2 = n_4$ here.

Lemma 59 (5.1 with $n_2 > n_1$)**.** *Let C be a solid block path of length 5 with all block sizes $n_i \geq 3$, satisfying the following.*

$$n_4 \geq n_2 > n_1, n_3$$
$$n_5 = n_1$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n_1 - 1; n_2, n_3, n_4, n_1 - 1)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$, $u^!_{i_1,T}Q$

We check that the graphs G_ε are C-free.

Suppose $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$. Then S' is a copy of some S_i or Q_i (with $n_2 = n_4$ in the second case) and so we may assume that S_i is equal to either S_i or Q_i.

If $S' = S_i$, then R', T' are $R_i, \hat{T}_i$ in some order, with $n_3 = n_5$ if the order is reversed. In any case no clique of order n_2 meets $\hat{T}_i$ in a single vertex, so $R' = R_i$ and then $Q' = Q_i$. There is no viable candidate for P'.

There remains the case $S' = Q_i$, with $n_2 = n_4$. In this case if Q' is a copy of some S_j then we fall into the previous case treated, so we may suppose Q' is Q_j for some j. Then by symmetry we may take $i = j - 1$ and $R' \subseteq \hat{P}_i$. But the inclusion must be proper so $n_3 < n_1$ and P' is not contained in R_i. This leaves no viable candidate for P'.

Thus the graphs G_ε are C-free and the rest goes as usual. □

Lemma 60 (5.1)**.** *Let C be a solid block path of length 5 satisfying the following.*

$$n_1 = n_5,\ n_2 = n_4$$
$$n_1, n_2 > n_3 = 3$$
$$n_1 \neq n_2$$

Then there is no countable weakly universal C-free graph.

Proof.

If $n_1 > n_2$ then Lemma **??** applies. If $n_2 > n_1$ then Lemma 59 applies. □

Lemma 61 (5.2)**.** *Let C be a solid block path of length 5 satisfying the following.*

$$n_5 > n_1 > n_2 > 3$$
$$n_3 = n_4 = 3$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2; n_2, 3)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$, P^*; $S_i = \{a_{i+1}, a_{i+2}, t$ (t fixed) $T_i = T$ fixed, $t \in T$

We check that the graphs G_ε are C-free.

Suppose $C \cong (P', Q', R', S', T') \subseteq G$. Then $T' = T$.

Then $P' = \hat{P}_i$ for some i. The only cliques of order n_2 meeting P' in a single vertex are the copies of Q_i. So we may suppose that $Q' = Q_i$.

Evidently R' is not R_i. The only alternative is that R' meets Q' at $u_{i,P}$. But then $R' \subseteq \hat{P}_{i-1}$ and hence $R' \subseteq P_{i-1}$ with $|R' \cap Q_i| = 1$, so $|R'| \leq 2$, which is impossible.

This proves the claim, and the rest follows as usual. □

Lemma 62 (5.3)**.** *Let C be a solid block path of length 5 satisfying the following.*

$$n_5 > n_2 > n_1 > 3$$
$$n_3 = n_4 = 3$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n_1 - 1; n_2, 3)$
G_1: Clique Ext'n	$u^R_{i+1,P}$, $S_i = \{a_{i+1}, a_{i+2}, t\}$, $T_i = T$; $t \in T$

We check that the graphs G_ε are C-free.

If $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$, then $T' = T$. So Q' must be a copy of some Q_i and we may suppose $Q' = Q_i$. The choice of Q_i^* prevents P' from being inside another copy of Q_i, so P' must be a copy of P_{i-1} or P_i. Then R' must be R_i, but this meets P'.

□

Lemma 63 (5.4)**.** *Let C be a solid block path of length $\ell \geq 4$ with*

$$n_2 > 4$$
$$n_i = 4 \textit{ for } i \neq 2$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2, n_2; 4, 4, 3)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$, $v^R_{i-1,P}$; $u^Q_{i+1,T}$

For the base of the amalgamation we take $\bigcup_i (Q_i^* \cup \{v^R_{i,P}\})$ with $|Q_i \setminus Q_i^*| = 2$.

We check that the graphs G_ε are C-free.

Suppose $C \cong (P', Q', \ldots, Z') \subseteq G_\varepsilon$. Then Q' is a copy of some Q_i, and we may suppose $Q' = Q_i$.

The overlap prevents P' and R' from being a copy of $\hat{P}_{i-1}$ or $\hat{Z}_{i-1}$. The inclusion of Q_i^* in the base prevents them from lying in another copy of Q_i. Thus P' and R' are $\hat{P}_i$ and a copy of Q_i, in some order. If $P' = \hat{P}_i$ and R' is a copy of R_i then C' is trapped in a single loop and we have a contradiction. If the roles are reversed then in view of the overlap of $\hat{P}_i$ with Q_{i+1} and R_{i+1}, C' is still trapped in the same loop.

This proves the claim.

$\square$

Lemma 64 (5.5)**.** *Let C be a block path of length 5 with all block sizes equal to 4. Then there is no countable weakly universal C-free path.*

Proof.

This is very similar to the preceding.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2, 4, 4; 4, 2)$
G_1: Clique Ext'n	$u_{i+1}, P^R, v^S_{i+1,P}; u^R_{i+1,T}, v^S_{i+1,T}$

It is easy to check that G_ε is C-free. It is clear from the construction that any embedding of C into G_ε would involve copies in G_2 of a single loop of G_1, and further that without loss of generality the embedding would be into a single loop, which is impossible.

$\square$

Lemma 65 (5.6)**.** *Let C be a solid block path of length 5 and type $(3, n_2, 4, n_4, 3)$ with*

$$n_2, n_4 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof. We may suppose $n_2 \geq n_4$.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, n_2, 4; n_4, 3)$
G_1: Clique Ext'n	No clique extension

If $C \cong C' \subseteq G_\varepsilon$, then we may suppose either that $Q' = Q_i$ for some i, or that $n_2 = n_4$ and $Q' = S_i$.

In the first case $R' = R_i$ and we may suppose $S' = S_i$, to reach a contradiction. In the second case S' cannot be a second copy of S_i, so must be a copy of Q_i, and we arrive at the first case by a change of notation. $\square$

Lemma 66 (5.7)**.** *Let C be a solid block path of length 5 and type $(3, 4, 4, n_4, 3)$ with*

$$n_4 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, 4; 4, n_4, 3)$
G_1: Clique Ext'n	No clique extension

- Amalgamation base: $\bigcup_i R_i^*$, $|R_i^*| = 2$

With this choice of amalgamation base, it is easy to see that the resulting graphs are C-free. □

Lemma 67 (5.8)**.** *Let C be a solid block path of length 5 of type $(3, 4, 4, 4, 3)$ (switch to 44433). Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, 4, 4; 4, 3)$
G_1: Clique Ext'n	No clique extension

- Amalgamation Base: $\bigcup R_i^*$, $|R_i| = 2$

With this choice of amalgamation base, it is easy to see that the resulting graphs are C-free. □

Lemma 68 (5.9)**.** *Let C be a solid block path of length 5 of type $(4, 4, 4, 3, 3)$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2, 2, 4; 3, 2)$
G_1: Clique Ext'n	$u^Q_{i+1,P}, v^R_{i+1,P}$; Q^*; $u^R_{i+1,T}$

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$. Then R', S', T' can only be a copy of some $R_i\hat{S}_i\hat{T}_i$ or its reversal. As the construction is symmetrical, we may suppose $R'S'T'$ is $R_i\hat{S}_i\hat{T}_i$. The construction rules out the natural possibilities for Q': a copy of $\hat{P}_{i-1}$ or Q_i.

There remains the possibility that Q' is contained in a copy of $\hat{T}_{i-1}$. We suppose $Q' \subseteq \hat{T}_{i-1}$. Then $P' \subseteq \hat{S}_{i-1}$ and $P' \cap S^* = \emptyset$, a contradiction.

□

Lemma 69 (5.10). *Let C be a solid block path of length 5 and type $(n,4,4,3,n)$ with $n \geq 5$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2,4,4;3,n-1)$
G_1: Clique Ext'n	P^*; $u^Q_{i+1,T}, v^Q_{i+1,T}$

For the amalgamation base we take $P^* \cup \bigcup_i T_i$ with $|T \setminus T_i| = \min(5, n-3)$. We check that G_ε is C-free.

Suppose $C \cong C' \subseteq G_\varepsilon$.

We may suppose that P' is $\hat{P}_i$ or $\hat{T}_i$.

If $P = \hat{P}_i$: then we may suppose that $T' = \hat{T}_j$ for some $j \neq i$ and there is no plausible candidate.

$P' = \hat{T}_i$: the choice of amalgamation base ensures Q', R' are not contained in another copy of $\hat{T}_i$. So we may suppose $Q' \subseteq \hat{P}_i$.

It follows that T' is not a copy of any $\hat{P}_j$, so we may suppose $T' = \hat{T}_j$ for some j. It then follows easily that R' is not contained in any $\hat{P}_j$ (including the case $j = i$).

So $R' = Q_i$, $S' \subseteq R_i$. But then $j = i-1$ and $|S' \cap T'| \geq 2$, a contradiction.

□

Lemma 70 (5.11, 6.5). *Let C be a solid block path of of length 5 and type $(4,4,4,3,4)$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

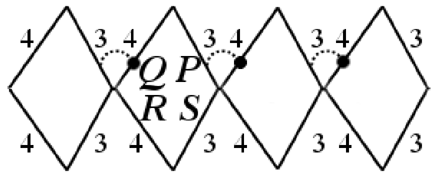
Graph, Type	*Specification*
G_0: Chain	$(3,4;4,3)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$

We check that the graphs G_ε are C-free, and even $(4,4,4,3)$-free.

If $C' = P'Q'R'S' \subseteq G_\varepsilon$ has type $(4,4,4,3)$, then $P'Q'R'$ must lie in amalagmated copies of a single loop L_i and without loss of generality, in a single loop L_i. So the claim is clear.

For the *rigidity argument* we must show that small deformations of this construction would contain a solid block path of type $(4,4,4,3,4,4)$. These embeddings will depend to an unusual degree on multiple loops.

We mention some key configurations. Schematically, the construction is as shown.



If we replace $\hat{P}_i Q_i$ (freely) by another copy $\tilde{P}\tilde{Q}$ which meets Q_{i+1} in just one vertex, then we can embed C into $Q_{i+1}\tilde{P}\tilde{Q}P_{i-1}Q_{i-1}R_{i-1}$.

On the other hand, if we replace $\hat{P}_i Q_i$ by a copy $\tilde{P}\tilde{Q}$ which does not meet Q_{i+1} at all, we embed C into $\tilde{P}\tilde{Q}R_iS_iQ_{i+1}\hat{P}_{i+1}$.

This is the basis of the rigidity argument. □

Lemma 71 (5.12)**.** *Let C be a solid block path of length 5 and type* $(4, 4, 4, 3, n)$ *with*

$$n \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2, 4; 4, 3)$
G_1: Clique Ext'n	$u^Q_{i+1,P}, v^R_{i+1,P}$; $\hat{T}_i = \{a_{i+1}\} \cup T^*$

The *extension process* that gives G_ε at the end involves attachments at the pairs

$$(u_{3i}, u_{3(i+1)})$$

(If we were to add edges (a_i, a_j) we would run into difficulties with T^*.) We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$. We may suppose $T' = T_i$ for some i. Then $P'Q'R'$ is forced by the construction to lie in copies of a single loop L_j, and to be a copy of $\hat{P}_j Q_j R_j$ or its reversal. In particular $i \neq j, j+1$ and it is easy to see that there is no candidate for S'.

□

Lemma 72 (5.13)**.** *Let C be a solid block path of length 5 and type* $(3, 4, 4, 3, n)$ *with*

$$n \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, 4)$
G_1: Clique Ext'n	$\hat{T}_i = \{a_{i+1}\} \cup T^*$

We can also start with a chain of loops of type $(3, 4; 4, 3)$, but after the free amalgamation step this gives us the same result.

We check that the resulting graphs G_ε are C-free. So suppose $C \cong C' \subseteq G_\varepsilon$.

For $n \geq 5$ everything is clear since T' must be one of the $\hat{T}_i$. So we consider the cases $n = 3$ or 4. But in this case one may see easily that there is no embedding of a solid block path of type $(3, 4, 4, 3)$. □

Reverse C to have type **35+433.**

Lemma 73 (5.14)**.** *Let C be a solid block path of length 5 and type $(3, n_2, 4, 3, 3)$ with*

$$n_2 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2, n_2, 4; 3, 2)$
G_1: Clique Ext'n	$u^R_{i+1,P}$; $u^R_{i+1,T}$

It is easy to check that the graphs G_ε are C-free. □

Lemma 74 (5.15)**.** *Let C be a solid block path of length 5 and type $(4, 4, 3, n_4, 3)$ with*

$$n_4 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, 4; 3, n_4, 2)$
G_1: Clique Ext'n	$u^R_{i+1,P}$; $u^Q_{i+1,T}$

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$. Then we may suppose that $R' = R_i$ for some i, and in view of the symmetry of the construction, that $Q' = Q_i$ and $S' = S_i$. There are then no viable candidates for (Q', P'). □

Lemma 75 (5.16)**.** *Let C be a solid block path of length 5 and type $(4, 4, 3, 4, 3)$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2, 3, 4; 4, 3)$
G_1: Clique Ext'n	$u^R_{i+1,P}, v^S_{i+1,P}$

Everything is straightforward. □

Lemma 76 (5.17)**.** *Let C be a solid block path of length 5 and type $(4, 4, 3, 3, n)$ with $4 \leq n \leq 8$. Then there is no countable weakly universal C-free graph.*

Proof. Suppose first

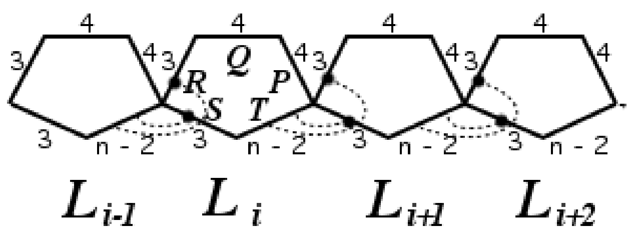
- $n \geq 5$

Then we use the following loop construction.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(4, 3, 3; 3, n - 2)$
G_1: Clique Ext'n	$u^Q_{i+1,T}, v^S_{i+1,T}$

In the amalgamation base, in addition to the special vertices $a_i, u_{i,T}, v_{i,T}$, we add the cut vertex c_i between S_i and T_i.



We check that the graphs G_ε are C-free.

Suppose $C \cong C' \subseteq G_\varepsilon$. We may suppose $T' = \hat{T}_i$ for some i.

Then S' cannot lie in a copy of the next loop L_{i+1}, and the option $S' \subseteq P_i$ quickly yields a contradiction. The presence of the cut vertex c_i in the amalgamation base eliminates the possibility that S' could lie in another copy of $\hat{T}_i$. There remains the possibility

$$S' = S_i$$

.

There are then essentially two possibilities for R': $R' \subseteq \hat{T}_{i-1}$ and $R' = R_i$. Since $|P'Q'R' \setminus S'| = 8$, the first possibility would force $n \geq 10$. Thus we may suppose $R' = R_i$.

Then it is easy to see that the cut vertex of $Q'R'$ must be $u_{i,T}$ and thus Q' (and then P') lies in $\hat{T}_{i-1,T}$ As $|P'Q' \setminus R'| = 6$ this forces $n \geq 9$, a contradiction.

Now suppose

- $n = 4$

We make a simplified loop construction based on block paths of length 2.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, 4)$
G_1: Clique Ext'n	No clique extension

It is easy to see that the resulting graphs are C-free. For the *rigidity argument* we note that if the configuration P_iQ_i is duplicated freely over the base point a_i, then we can find $P_i'Q_i'$ and $P_i''Q_i''$ with the vertices a_{i+1}', a_{i+1}'' corresponding to a_{i+1} identical.

Then the configuration $P_i''P_i'Q_i'Q_iP_i$ gives an embedding of C. This is the essential point in the proof that a_i controls a_{i+1}.

□

Lemma 77 (5.18)**.** *Let C be a solid block path of length 5 and type $(n, 4, 3, 3, n)$ with $n \geq 5$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$n - 3, 4; 3; 3)$
G_1: Clique Ext'n	$u^Q_{i+1,P}, v^Q_{i+1,P}, u^R_{i+1,P}$

We check that G_ε is C-free. Suppose $C \cong C' \subseteq G_\varepsilon$.

We may suppose $P' = \hat{P}_i$. Then Q' is either Q_i, or contained in another copy $\tilde{P}$ of $\hat{P}_i$.

If $Q' = Q_i$, then we may suppose R' is contained in the previous loop L_{i-1} and that $T' \subseteq \hat{P}_{i-2}$. But then S' should be contained in a copy of Q_{i-1} or R_{i-1} and this is ruled out by the overlap.

If $\tilde{P}\tilde{Q}$ is another copy of $\hat{P}_iQ_i$ with $Q' \subseteq \tilde{P}$, then similarly T' should be contained in a copy of $\hat{P}_{i-1}$ and there is no candidate for S'.

For the *rigidity argument,* the first case to consider is a copy $\tilde{P}\tilde{Q}$ of $\hat{P}_iQ_i$ freely joined over a_i. In this case C embeds as $\tilde{P}\tilde{Q}R_iS_i\hat{P}_i$.

The other cases, in which $\tilde{P}\tilde{Q}$ contains both a_i and a_{i+1} but omits one of the additional special vertices, will lead to an embedding involving a suitable copy of $\hat{P}_{i+2}$. □

Lemma 78 (5.19)**.** *Let C be a solid block path of length 5 and type $(4, 4, 3, 4, 4)$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, 4; 3)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$. Then P', Q' are copies of some $\hat{P}_i$, Q_i in some order, and we may suppose $P'Q'$ is $\hat{P}_iQ_i$ or $Q_i\hat{P}_i$. Similarly we may suppose $S'T'$ is $\hat{P}_jQ_j$ or $Q_j\hat{P}_j$. Then $j = i \pm 2$ and there is no candidate for R'.

The rigidity argument involves consideration of both $\hat{P}_iQ_iR_iQ_{i+1}\hat{P}_{i+1}$ and $Q_i\hat{P}_iR_{i+1}Q_{i+2}\hat{P}_{i+2}$.

□

Lemma 79 (5.20)**.** *Let C be a solid block path of length 5 with all block sizes $n_i \geq 3$ and with*

$$n_2 = n_4 = 3$$
$$n_1, n_5 < n_3$$

Then there is no countable weakly universal C-free graph.

Proof. We may suppose $n_1 \leq n_5$, and the details vary depending on whether $n_1 = n_5$ or $n_1 < n_5$.

Case 1. $n_1 = n_5$

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n_1 - 1, 3, n_3, 3; n_5 - 1)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$; $u^S_{i+1,T}$

Case 2. $n_1 < n_5$

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n_1, 3, n_3, 3; n_5 - 1)$
G_1: Clique Ext'n	$u^S_{i+1,T}$

As $n_3 > n_i$ for $i \neq 3$ it is easy to show that all G_ε are C-free. □

Lemma 80 (5.21)**.** *Let C be a solid block path of length 5 with all block sizes $n_i \geq 3$ and with*

$$n_2 = n_4 = 3$$
$$n_3 = n_5 > n_1 \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n_1, 3; n_3, 3, n_3')$, $n_3' = n_3 - (n_1 + 1)$
G_1: Clique Ext'n	$\hat{T}_i = T_i \cup \{u^Q_{i+1,T}\} \cup R^*_{i+1}$, $\lvert R_i \setminus R^*_i \rvert = n_1$ $(a_i \in R^*_i)$

We check that the G_ε are C-free. If $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$ then we may suppose that (R', S', T') is either $(R_i, S_i, \hat{T}_i)$ or its reversal. Our main concern is then the possibility that (P', Q') might embed over a_i or a_{i+1} in $\hat{T}_{i-1}$ or R_{i+1}, and meet R^*_i or R^*_{i+1} in just one vertex. This would then give

$$(n_1 + 2) = |P'Q'| \leq 1 + (n_3 - (n_3 - n_1)) = n_1 + 1$$

The rest is as usual. □

Lemma 81 (5.22)**.** *Let C be a solid block path of length 5 and type* $(3, 3, 4, 3, 4)$. *Then there is no countable weakly universal C-free graph.*

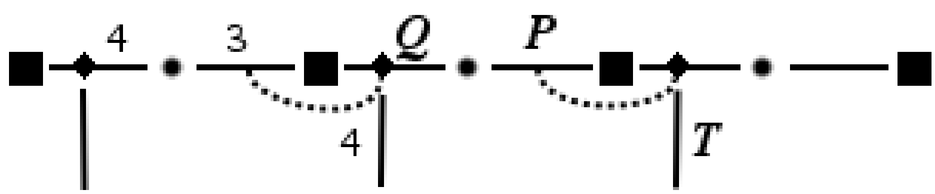
Proof. We deviate somewhat from our usual construction.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(4, 2)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$; $T_i = \{u_{i+1,P} \cup T^*_i$

Thus we begin with a chain of type $(4, 2, 4, 2, \dots)$ extended by edges to an overlapping chain of type $(4, 3, 4, 3, \dots)$. We then attach complete graphs of order 4 freely over the vertices $u_{i,P}$ lying in Q_i.

We amalgamate over the special vertices; we must be careful about the additional copies of $Q_i \hat{P}_i$ which are produced by this.



One must check both the C-freeness and the rigidity.

For the C-freeness, suppose $C \cong C' \subseteq G_\varepsilon$. Then we may suppose T' is Q_i or T_i.

Case 1. $T' = Q_i$

Then R' cannot be a copy of $Q_{i\pm 1}$ in view of the overlap, so we may suppose $R' = T_i$, and then $S' = \hat{P}_i$. There is then no option for Q'.

Case 2. $T' = T_i$

Now S' can be taken to be either $\hat{P}_i$ or a subset of Q_{i+1}.

If S' is $\hat{P}_i$ then R' is Q_i. Since $P'Q'$ cannot be contained in T_{i-1}, Q' must be contained in another copy $\tilde{Q}$ of Q_i, and then P' must be the corresponding copy $\tilde{P}$ of $\hat{P}_i$, which however meets S'.

If S' is a subset of Q_{i+1} containing $u_{i+1,P}$, then there is no option for R'.

Now we consider the *rigidity argument.*

Claim 1. Let $\tilde{Q}\tilde{P}$ be a copy of $Q_i\hat{P}_i$ which is free over a subset X satisfying the following.

$$X \cap \tilde{Q} = \{a_i, u_{i,P}\} \subseteq \tilde{Q} \setminus \tilde{P}$$
$$|X \cap \tilde{P}| = 2$$

Then $X \cap \tilde{P} = \{a_{i+1}, u_{i+1,P}\}$.

We show first that X meets $\{a_{i+1}, u_{i+1,P}\}$. We may suppose that

$$\tilde{Q}\tilde{P} \cap Q_i\hat{P}_iT_i = X \cap \{a_i, u_{i,P}, a_{i+1}, u_{i+1,P}\}$$

So if $X \cap \{a_{i+1}, u_{i+1,P}\} = \emptyset$ then we can embed C in $\tilde{P}\tilde{Q}Q_i\hat{P}_iT_i$.

Now we may suppose

$$\tilde{Q}\tilde{P} \cap Q_{i+1}\hat{P}_{i+1}Q_{i+2} = X \cap \{a_{i+1}, u_{i+1,P}, a_{i+2}u_{i+2,P}\}$$

Now if $|\tilde{Q}\tilde{P} \cap Q_{i+1}\hat{P}_{i+1}Q_{i+2}| = 1$ we get an embedding of C into $\tilde{Q}\tilde{P}Q_{i+1}\hat{P}_{i+1}Q_{i+2}$, and otherwise we get an intersection of $X \cap P$ with Q_{i+2} and we may repeat the argument.

So we find $X \cap Q_i = \{a_{i+1}, u_{i+1,P}\}$. □

Lemma 82 (5.23)**.** *Let C be a solid block path of length 5 with all block sizes $n_i \geq 3$ and with*

$$n_2 = n_4 = 3$$
$$n_1 = n_3 = n_5 \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n_1 - 1, 3; n_1, 2)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$; $u^R_{i+1,S}$; T^*

We check that the graphs G_ε are C-free. If $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$ then it is clear that R' is not a copy of any T_i, and in view of the symmetry of the construction we may suppose $R' = R_i$. Then easily P', T' must be copies of $\hat{P}_i$ and T_i, which intersect.

□

The next should be rewritten to reverse C.

Lemma 83 (5.24)**.** *Let C be a solid block path of length 5 with all block sizes $n_i \geq 3$ and with*

$$n_2 = n_4 = 3$$
$$n_3 = 4$$
$$n_5 > n_1 \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2343; 2)$
G_1: Clique Ext'n	P^*; $u^S_{i+1,T}$, T^*

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$. Then we may suppose that $T =' \hat{T}_i$ for some i, and then P' contains P^* (but is not necessarily a copy of $\hat{P}_i$). It follows that R' is a copy of some R_j, and we may suppose $R' = R_j$.

The possibilities are $j = i$ or $i + 1$, and the latter is blocked by the construction. So we take $R' = R_i$, and $Q' = Q_i$. This quickly leads to a contradiction. □

Lemma 84 (5.25)**.** *Let C be a solid block path of length 5 and type $(4, 3, 4, 3, n)$ with*

$$n \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, 4; 4, 2)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$; $u^R_{i+1,S}$; T^*

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$. Then we may suppose that T' is $\hat{T}_i$ for some i. Furthermore, the cut vertex os $S'T'$ cannot be in T^*, so it is a_i.

It follows that S' must be a copy of $\hat{S}_i$ or Q_i, with R' the corresponding copy of R_i or $\hat{P}_i$, and as the construction is symmetrical we may suppose that $R' = R_i$. Then we arrive quickly at a contradiction.

□

Lemma 85 (5.26)**.** *Let C be a solid block path of length 5 and type $(3,3,4,3,n)$ with*

$$n \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof. We perform a loop construction much like the previous one.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3,3,4,3;2)$
G_1: Clique Ext'n	$u^S_{i+1,T}$, T^*

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$. Then we may suppose that $T' = \hat{P}_i$ for some i. It follows that R' is a copy of some R_j, and we may suppose $R' = R_j$.

The possibilities are $j = i$ or $i+1$, and the latter is blocked by the construction. So we take $R' = R_i$, and $Q' = Q_i$. This quickly leads to a contradiction. □

Lemma 86 (5.27)**.** *Let C be a solid block path of length 5 with all block sizes $n_i \geq 3$ and with*

$$n_2 = n_4 = 3$$
$$n_1 = n_5 > n_3 \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3,3,n_3,3;2)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$, P^*; T^*

We check that the graphs G_ε are C-free. Suppose that $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$.

The cliques of order n_1 in G_ε are copies of the $\hat{P}_i$, the $\hat{T}_i$, and $T^* \cup \{a_i, a_j\}$ when (a_i, a_j) is an edge. So P', T' are $\hat{P}_i$ and one of the cliques containing T^* in some order, and by symmetry we may suppose

$$P' = \hat{P}_i$$
$$T' \supseteq T^*$$

If follows that R' is a copy of R_j for some j and we may suppose

$$R' = R_j$$

Then Q', S' are Q_j, S_j in some order, and as $|Q' \cap \hat{P}_i| = 1$ we find $Q' = Q_j$, $S' = S_j$, $i = j$. Now there is no suitable clique T'.

One point that always deserves some attention is the incompatibility of the graphs G_ε as ε varies. In this case the claim is that if we adjoin a clique of order n_3 at $(a_{3i}, a_{3(i+1)})$ then C embeds in the resulting graph. The embedding would involve cut vertices in P^*, T^* and is thus of the "unexpected" type.

Of course, if there were no such embedding we would simply have adjusted the size of the antiedges correspondingly (to $\min(n_1, n_3 + 4) - 1$). □

Lemma 87 (5.28)**.** *Let C be a solid block path of length 5 and type $(3, n_2, 3, n_4, 3)$ with*

$$n_4 > n_2 \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, n_2; 3, n_4, 2)$
G_1: Clique Ext'n	$u^Q_{i+1,T}$

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$. Then we may suppose that $S' = S_i$ for some i. Thus R' is R_i or $\hat{T}_i$. If R' is $\hat{T}_i$ there is no candidate for Q'. If R' is R_i then Q' is a copy of Q_i and there is no candidate for P'.

□

Lemma 88 (5.29)**.** *Let C be a solid block path of type $(3, n_2, 3, n_2, 3)$ with*

$$n_2 \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2, n_2; 3, n_2, 2)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$; $u^Q_{i+1,T}$

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$. Then each of Q', S' is a copy of one of the cliques Q_i or S_i, and clearly at least one must be a copy of Q_i. We may suppose therefore that $Q' = Q_i$. It is then easy to see that S' cannot be a copy of $Q_{i\pm1}$ or S_{i-1}, and hence $S' = S_i$. We then arrive quickly at a contradiction.

□

6. Remaining critical cases

We deal with the remaining cases of Proposition 8, namely those of length 6 or 7, and those of variable length.

6.1. **Length 6.**

Lemma 89 (6.1)**.** *Let C be a solid block path of length 6 and type* $(3,4,4,4,3,3)$*. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3,4,4;4,3)$
G_1: Clique Ext'n	No clique extension

It is clear that the graphs G_ε are C-free. □

Lemma 90 (6.2)**.** *Let C be a solid block path of length 6 and type* $(4,4,4,3,3,n)$*, with $n \geq 5$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2,4,4;3,3,2)$
G_1: Clique Ext'n	$u^R_{i+1,P}$, $v^S_{i+1,P}$; T^*

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P',Q',R',S',T',U') \subseteq G_\varepsilon$. Then T' contains T^*. It follows that $P'Q'R'$ may be taken to be $\hat{P}_iQ_iR_i$ for some i. The alternatives fo S' are then a copy of S_i or a clique contained in some antiedge K. But both alternatives lead quickly to a contradiction.

We remark that there are some "nonstandard" cliques of order n in this construction but they don't come up in the analysis. They could also be avoided by an adjustment in G_ε.

□

Reverse C here.

Lemma 91 (6.3)**.** *Let C be a solid block path of length 6 with all block sizes $n_i \geq 3$ and with and type* $(4,3,3,4,4,4)$*. Then there is no countable weakly universal C-free graph.*

Proof. We perform a simplified loop construction based on a block path.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	Block path $(3,4)$
G_1: Clique Ext'n	T^*U^*

Here $T_i = \{a_{i+1}\} \cup T^*$ and $U_i = U^*$ is attached to T^*. Attached anti-edges have order 4.

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T', U') \subseteq G_\varepsilon$. Consider $S'T'U'$. We must have $U' = U^*$ and S' is a copy of some P_i, with the cut vertex of $S'T'$ being $\{a_{i+1}\}$. We may suppose S' is P_i, and then R' will be Q_i.

The only plausible candidate for Q' is another copy of Q_i, and then P' is forced to meet S'. □

Lemma 92 (6.4)**.** *Let C be a solid block path of length* 6 *and type* $(4, 4, 4, 3, n, 3)$ *with all $n \geq 5$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2, n, 3; 4, 4, 3)$
G_1: Clique Ext'n	$u^S_{i+1,P}R$; $u^R_{i+1,U}$

Everything is straightforward in this case. □

Lemma 93 (6.5)**.** *Let C be a solid block path of length* 6 *and type* $(4, 4, 4, 3, 4, 4)$ *with all $n \geq 5$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(4, 4, 2)$
G_1: Clique Ext'n	$u^R_{i_1,P}$, $v^R_{i+1,P}$

It is clear the result of amalgamation is C-free. For the *rigidity argument*, we note that if we attach a copy $\tilde{R}\tilde{Q}\tilde{P}$ of $R_iQ_i\hat{P}_i$ freely over a_i, then C embeds in $\tilde{P}\tilde{Q}\tilde{R}R_iQ_i\hat{P}_i$, while if we we attach the copy freely over a set meeting R_{i+1} but not containing all of $a_i, u_{i+1,P}, v_{i+1,P}$ then we extend through R_{i+1} (or else meet R_{i+2} and continue similarly). □

Lemma 94 (6.6)**.** *Let C be a solid block path of length* 6 *and type* $(3, 4, 4, 3, 3, n)$ *with all $n_i \geq 3$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(4, 3)$
G_1: Clique Ext'n	$T_iU_i = \{a_{i+1}\} \cup T^*_iU^*_i$

The passage from G_0 to G_1 comes by attaching solid block paths of type TU, i.e., $(3, n_6)$, with the base point of T_i identified with a_{i+1}.

The proof that the resulting graphs are C-free is straighforward, distinguishing the cases $n \leq 4$ and $n \geq 5$, and for the *rigidity argument* one observes that if a copy $P'Q'$ of P_iQ_i is freely attached over a_i then $P'Q'Q_iP_iT_iU_i$ gives an embedding of C. □

Lemma 95 (6.7)**.** *Let C be a solid block path of length 6 and type $(3,4,4,3,n,3)$ with $n \geq 3$. Then there is no countable weakly universal C-free graph.*

Proof. We perform a simplified loop construction. This goes as follows if $n \neq 4$.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	Block path $(4,3)$
G_1: Clique Ext'n	$T_iU_i = \{a_{i+1}\} \cup T^*U^*$

If $n = 4$, we we omit T^*U^*.

It is easy to see that the graphs G_ε are C-free; distinguishes the cases $n \leq 4$ and $n \geq 5$. □

Lemma 96 (6.8)**.** *Let C be a solid block path of length 6 and type $(4,4,3,3,n,3)$ with*

$$n \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2,4;3,3,n,3)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$; $u^R_{i+1,P}$

We check that the graphs G_ε are C-free. Suppose $C \cong C' \subseteq G_\varepsilon$.

We may suppose that $S'T'U'$ is $S_iT_iU_i$ in some order, and that R' is R_i or R_{i+1}, or is contained in $\hat{P}_i$ or Q_{i+1}.

The natural case $R' = R_i$ leads quickly to a dead end. The case $R' \subseteq R_{i+1}$ is even less tenable.

If $R' \subseteq \hat{P}_i$ then the cut vertex of $R'Q'$ is not a_{i+1}, so Q' is Q_i and there is no viable candidate for P'.

If $R' \subseteq Q_{i+1}$, then $P' = \hat{P}_{i+1}$, and we again reach a dead end. □

Lemma 97 (6.??)**.** *Let C be a solid block path of length 6 and type $(4,4,3,3,3,n)$ with*

$$n \geq 5$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2,4,3;3,3,2)$
G_1: Clique Ext'n	$u^R_{i+1,P}$, $v^S_{i+1,P}$; U^*

Here we need to avoid introducing edges at pairs (a_i, a_j), so the construction of the graphs G_ε will attach edges or anti-edges to pairs $(u_{3i,P}, u_{3(i+1),P})$.

We check that the graphs G_ε are C-free. Suppose $C \cong C' \subseteq G_\varepsilon$.

Then $U' = \hat{U}_i$ for some i, and $P'Q'$ is $\hat{P}_j Q_j$ in some order, for some $j \neq i$.

One sees quickly that the cut vertex of $T'U'$ must lie in U^*, as otherwise the only plausible alternative would be $j = i-1$, and as R' cannot be a copy of R_i there is no way to connect T' and $P'Q'$.

So we suppose $T' = U_k \cup \{u\}$ for some $u \in U^*$ and some k. Then again the only plausible possibility is $j = k-1$, but the overlap with S_i rules this out. □

Requires reversal.

Lemma 98 (6.9)**.** *Let C be a solid block path of length 6 and type $(4,3,4,3,4,4)$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3,3;4,2)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$; $u^R_{i+1,S}$

We will be more explicit than usual, giving the details of the minor edges and antiedges that play a role in the rigidity statement. The main point here is to formulate the rigidity statement precisely, and then to exploit the various near-embeddings of C into our graphs, which are unusually varied.

Claim 1. The graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T', U') \subseteq G_\varepsilon$. Then T', U' must be copies of some $\hat{P}_i$, R_{i+1} in some order. In view of the symmetry of the construction we may take $T' = P_i$, $U' = R_{i+1}$. One then arrives quickly at a contradiction.

The rigidity is also worth checking in some detail.

Note that the hypotheses of the following claim are satisfied when (S', T') is (Q_i, P_i).

Claim 2. Let G be a C-free graph containing G_3 and let (S', T') be a solid block path of type $(3,4)$ in G, free over a subset X satisfying the following conditions.

$$\begin{aligned} X \cap S' \cap T' &= \emptyset \\ X \cap S' &= \{a_i, u_{i,P}\} \\ u_{i,S} &\notin X \\ |X \cap T'| &= 2 \end{aligned}$$

Suppose that no neighbor of $u_{i,P}$ or $u_{i,S}$ belongs to $X \cap T'$. Then $X \cap T' \cap Q_{i+1}\hat{P}_{i+1}$ is either $\{a_{i+1}, u_{i+1,P}\}$ or empty.

We may suppose that

$$S'T' \cap Q_{i+1}\hat{P}_{i+1}R_{i+2} = X \cap \{a_{i+1}, u_{i+1,P}, u_{i+2,P}, a_{i+2}, u_{i+2,S}\}$$

The hypotheses on X eliminate $u_{i+2,P}$ and $u_{i+2,S}$. Suppose now that $a_{i+2} \in X$.

Then we must have either $X \cap \hat{P}_{i+1} = \{a_{i+2}\}$ or $X \cap R_{i+1} = \{a_{i+2}\}$. This gives an embedding of C into G of the type of $\hat{P}_i\hat{S}_iR_iS'T'\hat{P}_{i+1}$ or $P_i\hat{S}_iR_iS'T'R_{i+1}$, and a contradiction.

So we may suppose.

$$S'T' \cap Q_{i+1}\hat{P}_{i+1}R_{i+2} = X \cap \{a_{i+1}, u_{i+1,P}\}$$

Now our claim follows unless this intersection consists of the single vertex a_{i+1} or $u_{i+1,P}$. In this case we embed C as $R_iS'T'Q_{i+1}\hat{P}_{i+1}R_{i+2}$.

Claim 3. Let G be a C-free graph containing G_3 and let (S', T', U') be a solid block path of type $(3, 4, 4)$ in G, free over a subset X such that $S'T'$ and $X \cap S'T'$ satisfy the conditions of the previous claim and in addition

$$\begin{aligned} |X \cap U'| &= 2 \\ T' \cap U' &\subseteq X \\ u_{i,S} &\notin X \end{aligned}$$

Suppose that the vertex in $X \cap U' \setminus T'$ is a neighbor of $u_{i-1,S}$ and the vertex in $X \cap T' \cap U'$ is not a neighbor of $u_{i-1,P}$. Then

$$\begin{aligned} X \cap T' &= \{a_{i+1}, u_{i+1,P}\} \\ T' \cap U' &= \{a_{i+1} \end{aligned}$$

Suppose first that $X \cap Q_{i+1}\hat{P}_{i+1}$ is empty. We may suppose that

$$\begin{aligned} S'T'U' \cap R_i\hat{S}_i &= X \cap \{a_i, u_{i,S}, u_{i+1,S}, a_{i+1}\} \\ &= X \cap \{a_i, u_{i+1,S}, a_{i_1}\} \end{aligned}$$

Now a_{i+1} is not in $X \cap T'$ and a_{i+1} is not a neighbor of $u_{i-1,S}$, so $a_{i+1} \notin X$. On the other hand if $S'T'U' \cap R_i\hat{S}_i$ reduces to the vertex a_i, then we embed C into G as $\hat{P}_i\hat{S}_iR_iS'T'U'$.

So in this case we conclude that $T'U' \cap R_i\hat{S}_i = \{u_{i+1,S}$. Then we embed C into G as $R_i'Q_i\hat{P}_i\hat{S}_iU'T'$ where R_i' is a copy of R_i which does not meet $\hat{S}_i$.

So this rules out the case in which $X \cap Q_{i+1}\hat{P}_{i+1}$ is empty and we conclude

$$X \cap T' = \{a_{i+1}, u_{i+1,P}\}$$

Our assumptions on X then imply that the cut vertex of $T'U'$ is a_{i+1}.

The claim is proved.

Claim 4. Let G be a C-free graph containing G_3 and let (S', T') be a solid block path of type $(4, 3)$ in G which is free over a subset X satisfying the following.

$$\begin{aligned} X \cap \hat{P}_{i-1} &= \{a_i\} \\ X \cap S' &= \{a_i, u_{i,S}\} \\ X \cap S' \cap T' &= \emptyset a_{i+1} & \in X \\ |X \cap T'| &= 2 \end{aligned}$$

Suppose that the vertex in $X \cap U' \setminus \{a_{i+1}\}$ is a neighbor of $u_{i-1,S}$. Then $X \cap T' = \{u_{i+1,S}, a_{i+1}\}$.

We may suppose that

$$S'T' \cap R_{i+1}\hat{S}_{i+1}\hat{P}_{i+1} = X \cap \{a_{i+1}, u_{i+1,S}, u_{i+2,S}, a_{i+2}, u_{i+2,P}\}$$

and by our assumptions on X this intersection reduces to

$$X \cap \{a_{i+1}, u_{i+1,S}\}$$

If $u_{i+1,S} \notin X$ we embed C. So the claim follows.

Application of these claims gives sufficient rigidity to determined the sequence $(a_i, u_{i,P}, u_{i,S})$ inductively.

$\square$

Lemma 99 (6.**??**). *Let C be a solid block path of length 6 and type $(n_1, 3, 3, 3, n_5, 3)$ with*

$$n_1 \neq n_5$$
$$n_1, n_5 \geq 4$$

. Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2,3,3;3,2,2)$
G_1: Clique Ext'n	P^*; T^*; $u^S_{i+1,U}$

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T', U') \subseteq G_\varepsilon$. We may suppose that $P' = \hat{P}_i$ and $T' = \hat{T}_j$ for some i, j. We find that S', U' are S_j, $\hat{U}_j$ in some order so $i \neq j$.

By inspection $i = j + 1$, but this is blocked by the overlap with R_{j+1}. $\square$

Lemma 100 (6.10)**.** *Let C be a solid block path of length 6 and type $(n_1, 3, 3, n_4, 3, n_6)$ with*

$$\begin{aligned} n_1 &> n_4, n_6 \\ n_4, n_6 &\geq 4 \\ n_4 &\neq n_6 \end{aligned}$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2, 3, 3; n_4, 3, n_6)$
G_1: Clique Ext'n	P^*

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T', U') \subseteq G_\varepsilon$. We may suppose that $P'\hat{P}_i$ for some i.

We claim that $S'T'U'$ is a copy of some $S_jT_jU_j$. The choice of amalgamation base rules out the possibilities that $S' = S_j$ and $T'U'$ is contained in another copy of S_j (when $n_4 > n_6$) or $U' = U_j$ and $S'T'$ is contained in another copy of U_j (when $n_4 < n_6$).

In particular the cut vertex of $R'S'$ is a_j. One then arrives quickly at a contradiction. $\square$

Lemma 101 (6.11)**.** *Let C be a solid block path of length 6 and type $(n_1, 3, n_3, 3, 3, n_6)$ with*

$$n_3 > n_6 > n_1 \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n_1, 3, n_3, 3, 3; n_6 - 1)$
G_1: Clique Ext'n	$u^T_{i+1,U}$

If $C \cong C' \subseteq G_\varepsilon$, then we may take $R' = R_i$, $P' = P_i$, $T' = T_i$, and as $n_6 > n_1, 3$ and T_i overlaps with $\hat{U}_{i-1}$, this forces U' to be a copy of $\hat{U}_i$, giving a contradiction. $\square$

Lemma 102 (6.12). *Let C be a solid block path of length 6 and type $(n_1, 3, n_1, 3, 3, n_6)$ with*

$$n_6 > n_1 \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction	
Graph, Type	*Specification*
G_0: Chain	$(n_1 - 1, 3; n_1, 3, 3, 2)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$; U^*

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T', U') \subseteq G_\varepsilon$. Then $U' = \hat{U}_i$ for some i.

Suppose first that the cut vertex v of $T'U'$ is in U^*. Then T' is $U_j \cup \{v\}$ for some j. Then there is no viable candidate for $R'S'$.

So T' must be contained in a copy of $\hat{P}_i$, Q_{i+1}, R_{i+1}, or T_i. The cases $\hat{P}_i$ and R_{i+1} are easily ruled out, so we suppose T' is Q_{i+1} or T_i.

If T' is Q_{i+1} then clearly S' cannot be contained in a copy of $\hat{P}_i$ and so S' is contained in a copy of $\hat{P}_{i+1}$. Now the overlap with Q_{i+2} quickly leads to a contradiction.

If T' is T_i then $R'S'$ is R_iS_{i+1}. Clearly Q' is not contained in another copy of R_i, or in $\hat{P}_{i-1}$, so Q' is a copy of Q_i and then P' meets U', a contradiction. □

Lemma 103 (6.13). *Let C be a solid block path of length 6 and type $(3, 3, n_3, 3, 3, n_6)$ with*

$$n_1 > n_4 \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction	
Graph, Type	*Specification*
G_0: Chain	$(2, 3, n_3; 3, 3, 2)$
G_1: Clique Ext'n	$u^R_{i+1,P}$; U^*

For the amalgamation phase, amalgamate over the base $\bigcup_i (R_i \setminus Q_i) \cup U^*$.

We check that the graphs G_ε are C-free.

Suppose that $C \cong C' \subseteq G_\varepsilon$. We may suppose that $U' = \hat{U}_i$ for some i.

Then we may suppose that R' is R_j for some j. The amalgamation base ensures that one of Q', S' is a copy of Q_j, and the other meets R' at a_j. It is easy to see that Q' cannot meet R' at a_j and so $Q' = Q_j$, $a_i \in S'$. Now one quickly arrives at a contradiction. □

Lemma 104 (6.14). *Let C be a solid block path of length 6 and type $(3,3,n_3,3,3,n_6)$ with*

$$n_3 > n_6 \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2,3,n_3,3,3,;n_6-1)$
G_1: Clique Ext'n	$u^U_{i+1,P}$; $u^T_{i+1,U}$

Everything is straightforward. □

Lemma 105 (6.15). *Let C be a solid block path of length 6 and type $(n_1,3,3,n,3,n)$ with*

$$3 \leq n_1 < n$$

Then there is no countable weakly universal C-free graph.

Proof. If $n_1 \geq 4$ we proceed as follows.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n_1,3,3;n,3,n')$ with $n-n' = \min(n_1+3,n-3)$
G_1: Clique Ext'n	$u^R_{i+1,U}$, U^*

We check that G_ε is C-free. If $C \cong C' \subseteq G_\varepsilon$, we may suppose that $S'T'U'$ is $S_iT_i\hat{U}_i$ in some order. Taking into account the overlap with R_{i+1}, the only plausible alternative is that $P'Q'R'$ lies in a copy of $U_{i\pm1}$, sharing the cut vertex $a_{i\pm1}$ with S', so $|PQR| = n_1+4 \leq n-|U^*| = n-n'$, a contradiction.

For the case $n_1 = 3$ we refer to Lemma **??** below. □

Lemma 106 (6.16). *Let C be a solid block path of length 6 and type $(n,3,n_3,3,3,n)$ with*

$$n_3 > n \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n-1,3,n_3,3;3,n-1)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$; $u^S_{i+1,U}$

We check that the graphs G_ε are C-free. Suppose that $C \cong (P',Q',R',S',T',U') \subseteq G_\varepsilon$. Then we may suppose that $R' = R_i$ for some i.

So Q', S' are Q_i and S_i in some order. In particular P' cannot be a copy of $\hat{P}_{i-1}$ or $\hat{U}_{i-1}$.

So P' is $\hat{P}_i$. Then T' cannot be another copy of S_i, and there is no way to complete the embedding. □

Lemma 107 (6.17)**.** *Let C be a solid block path of length 6 and type $(n, 3, n_3, 3, 3, n)$ with all $n > n_3 \geq 4$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n-2, 3, n_3; 3)$
G_1: Clique Ext'n	$u^R_{i+1,P}, v^S_{i+1,P}$

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T', U') \subseteq G_\varepsilon$. Then we may suppose that $P' = \hat{P}_i$ for some i. The candidates for R' are R_i, a clique contained in a copy of $\hat{P}_{i-1}$ or $\hat{P}_{i+1}$, or a copy of Q_{i+2}. The last two possibilities are not viable since Q' would need to be a copy of S_{i+1}.

If $R' = R_i$ then U' should be a copy of $\hat{P}_{i-3}$, but this is blocked by the overlap with R_{i-2} and S_{i-2}. If R' is contained in $\hat{P}_{i-1}$ then U' should be a copy of $\hat{P}_{i-2}$ but this is blocked similarly.

This proves the claim.

□

Lemma 108 (6.18)**.** *Let C be a solid block path of length 6 and type $(n, 3, n, 3, 3, n)$ with all $n \geq 4$. Then there is no countable weakly universal C-free graph.*

Proof. We perform a simplified loop construction.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	Block path $(3, n)$
G_1: Clique Ext'n	No clique extensions

We use the amalgamation base $\bigcup_i (P_i \setminus Q_i)$.

The graphs G_ε are easily seen to be C^--free where C^- is of type $(n, 3, 3, n)$. This depends on the choice of the amalgamation base.

We comment on the *rigidity argument.*

Claim 1. If G is a C-free graph containing G_3 and P' is a clique of order n containing a_i, then P' contains P^*_i.

We indicate the main point. Assuming the contrary, we find a block path (Q'', P'') of type $(3, 3)$ contained in a copy of $Q_i P_i$ so that $P'' \cap P' = \{a_{i+1}\}$. We must then extend to a copy of C. There are some additional points to deal with along the way (e.g., G_2 should have an antiedge at (a_i, a_{i+2})).

□

Lemma 109 (6.19). *Let C be a solid block path of length 6 and type $(3, n_2, 3, 3, n_5, 3)$ with $n_5 \geq n_2 \geq 4$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2; n_2, 3, 3, n_5, 2)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$; $u^Q_{i+1,U}$

Amalgamation base: $\bigcup_i (Q_i \setminus R_i) \cup \{c_i\}$ with c_i the cut vertex of $R_i S_i$.

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T', U') \subseteq G_\varepsilon$. We may suppose that $T' = T_i$ for some i. Then S', U' are S_i, $\hat{T}_i$ in some order.

If S' is S_i then Q' is Q_i and the overlap with $\hat{P}_i$, $\hat{U}_i$, excludes most candidates for P'. The amalgamation base prevents an embedding of P' into another copy of Q_i. So this case leads quickly to a contradiction.

If S' is $\hat{U}_i$ then the amalgamation base prevents R' from being another copy of $\hat{U}_i$. So the natural candidates for Q' are copies of Q_i or Q_{i+2}. But Q_{i+2} is blocked by the overlap with $\hat{P}_{i+1}$. And $Q' = Q_i$ would force P' to be a copy of R_i, and hence to meet S_i, in view of the amalgamation base.

□

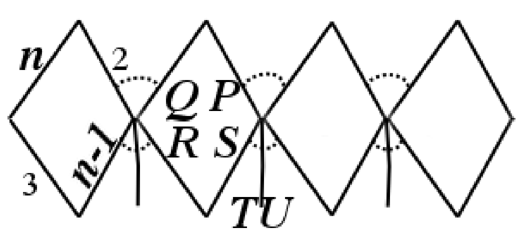
Lemma 110 (6.20, 6.15 with $n_1 = 3$, and 7.4). *Let C be a solid block path of length 6 and type $(3, 3, n, 3, n, 3)$ or $(3, 3, 3, n, 3, n)$ with $n \geq 4$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2, n; 3, n-1)$
G_1: Clique Ext'n	$u^Q_{i+1,P}$; $u^R_{i+1,S}$; $T_i U_i = \{a_{i+1}\} \cup T^*_i U^*_i$

That is, we make the usual loop construction based on a loop of length 4 with overlaps, but attach a solid block path $T_i U_i$ at a_{i+1} as well (this only matters if $n = 4$, for the rigidity argument).



The overlap ensures that an embedding $C \cong C' \subseteq G_\varepsilon$ would take $Q'R'S'$ to a copy of $Q_iR_i\hat{S}_i$, in some order. Thus any solid block path of type $(3, 3, n, 3, n)$ would be of the type of $U_iT_iS_iR_iQ_I$ and will not extend to an embedding of C.

For the *rigidity argument*, if we attach a copy of $\widetilde{QPTU}$ of $QPTU$ freely over $a_i, u_{i,P}$ then $\widetilde{UTPQ}RSTU$ has type $(3, 3, 3, n, 3, n, 3, 3)$, which contains the three possible forbidden configurations.

On the other hand, if we allow $\widetilde{QPTU}$ exactly one point of intersection with Q_{i+1}, then $U_{i+1}T_{i+1}\hat{P}_{i+1}Q_{i+1}\widetilde{PQ}R_i$ has type $(3, 3, 3, n, 3, n, 3)$ which also contains the forbidden configurations. □

Lemma 111 (6.21)**.** *Let C be a solid block path of length 6 and type $(3, 3, 4, 3, n, 3)$ with $n \geq 5$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, 3, 4; 3, n, 2)$
G_1: Clique Ext'n	$u_{i+1,U}$ $:^R$

Everything is straighforward. □

6.2. **Length 7.**

Lemma 112 (7.1)**.** *Let C be a solid block path of length y and type $(3, 4, 4, 3, 3, n, 3)$ with $n \geq 3$. Then there is no countable weakly universal C-free graph.*

Proof. Similar to the proof of 6.7.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(4, 3)$
G_1: Clique Ext'n	Attach a solid block path $T_iU^*V^*$ of type $(3, n_6, 3)$ at a_{i+1}

In the proof that the resulting graphs are C-free, distinguish the cases $n_6 \geq 5$, $n_6 \leq 4$. □

Lemma 113 (7.2)**.** *Let C be a solid block path of length 7 and type $(n, 3, 3, 3, n, 3, n)$ with $n \geq 4$. Then there is no countable weakly universal C-free graph.*

Proof. We perform a loop construction of length 4.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, 3; 3, n-2)$
G_1: Clique Ext'n	$u^Q_{i+1,S}$, $v^R_{i+1,S}$

The graphs G_ε are C^--free, where C^- has type $(n, 3, 3, 3, n)$. We add a few words about the *rigidity argument.*

Claim 1. Let G be a C-free graph containing G_3 and S' a clique of order n free over a set X of order 3. Then for any i, subject to appropriate conditions on edges and antiedges in G_3, and corresponding conditions on X, we have either $a_{i+1}, v_{i+1,S} \in X$, or $X \cap R_{i+1} = \emptyset$.

Otherwise we may suppose $|S' \cap R_{i+1}| = 1$ and we look for an embedding of C into $S' R_{i+1} \hat{S}_{i+1} R_{i+2} \hat{S}_{i+2} R_{i+3} \hat{S}_{i+3}$. We omit the details.

This claim becomes relevant when we consider rigidity for copies of $R_i \hat{S}_i$ containing a_i, and more particularly copies $R' S' R'' S''$ of $R_i \hat{S}_i R_{i+1} \hat{S}_{i+1}$. It brings us down to the case in which $S' \cap R_{i+1} = \emptyset$, and similarly $S' \cap Q_{i+1} = \emptyset$, which then allows us to assume $S' \cap \hat{S}_i = \emptyset$. So one works toward an embedding of C into $R'' S'' R' S' Q_i P_i \hat{S}_i$. □

Lemma 114 (7.??). *Let C be a solid block path of length 7 and type* $(3, n_2, 3, 3, 3, n_6, 3)$ *with*

$$n_6 > n_2 \geq 4$$

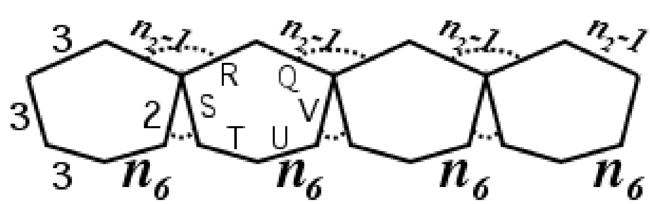
Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n_2 - 1, 3; , 3, 3, 3, n_6, 2)$; $L_i = (Q, R, S, T, U, V)$
G_1: Clique Ext'n	$u^R_{i+1,Q}$; $u^S_{i+1,V}$

Note that we adjust the notation for the blocks to match that associated with C.



We check that G_ε is C_0-free with C_0 of type $(n_2, 3, 3, 3, n_6, 3)$. So suppose $C_0 \cong C_0' \subseteq G_\varepsilon$.

We may suppose $U' = U_i$. Then $T' U' V'$ is $T_i U_i \hat{V}_i$ in some order. Neither possibility leads very far: for example, if $T' = \hat{V}_i$ and S' is contained in $\hat{Q}_i$, we would be looking toward $Q' R' = \hat{Q}_{i-1} R_i$, which is blocked by overlap. □

Clause 7.4 was treated in Lemma 110.

Lemma 115 (7.3). *Let C be a solid block path of length 7 and type $(3,3,n_3,3,3,n_6,3)$ with*

$$n_3, n_6 \geq 4$$
$$n_3 \neq n6$$

Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2,3;n_3,3,3,n_6,2)$
G_1: Clique Ext'n	$u^R_{i+1,P}$; $u^R_{i+1,V}$

For the proof that G_ε is C-free, suppose $C \cong C' \subseteq G_\varepsilon$. We may take $R'S'T'U'$ to be $R_iS_iT_iU_i$ for some i. Overlap of R_i with $\hat{P}_{i-1}$ and $\hat{V}_{i-1}$ keeps the configuration within the same loop and gives a contradiction. □

6.3. **Variable length.** We deal with the final set of critical configurations, those of variable length.

Now our notation for the blocks of C becomes $(P, Q, \ldots, Y, Z)$.

Lemma 116 (V.1). *Let C be a solid block path of length $\ell \geq 4$ with all block sizes $n_i \geq 3$ and with*

$$n_1, n_\ell > n_i \ (1 < i < \ell)$$
$$n_2 = 4$$

If $\ell = 4$, suppose further that $n_1 \neq n_\ell$. Then there is no countable weakly universal C-free graph.

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2,n_2;n_3,\ldots,n_{\ell-1},2)$
G_1: Clique Ext'n	P^*; Z^*

We check that the graphs G_ε are C-free.

Suppose $C \cong C' = (P', Q', \ldots, Z') \subseteq G_\varepsilon$. Then P' and Z' are copies of some $\hat{P}_i$ and $\hat{Z}_j$ and we may suppose they coincide with $\hat{P}_i$, $\hat{Z}_j$ in some order.

Case 1. $P' = \hat{P}_i$ and $T' = \hat{Z}_j$.

As $n_2 = 4$, the cut vertex of $P'Q'$ is not in P^*. Evidently Q' cannot be Q_{i+1}, so we have two possibilities: $Q' = Q_i$ or Q' is a copy of R_{i+1}. In either case we show that the embedding of C is trapped in the corresponding loop L_i or L_{i+1}. This would be obvious, apart from the possibility that the cut vertex of $Y'Z'$ might lie in Z^*, if $n_{\ell-1} = 3$.

This becomes relevant only if Q' embeds into R_{i+1} and eventually X' embeds into Y_{i+1}, Y' embeds into Z_{i+1}. But then as $|Q'| = 4$ it follows that all the block sizes through $n_{\ell-1}$ are at least 4, and we reach a contradiction.

Case 2. $P' = \hat{Z}_j$ *and* $Z' = \hat{P}_i$.

Then $n_1 = n_\ell$, and we now use the hypothesis $\ell \geq 5$. Indeed, a solid block path of type $(n_1, 4, 3, n_1)$ would allow an embedding with the cut vertex of $Y'Z'$ in P^*. But as $\ell \geq 5$ it is easy to see that Q' cannot be a copy of Q_{j+1}. One concludes quickly that Q' is Y_j and in particular $n_{\ell-1} = 4$.

Thus this case is the same as the previous one, after a change in notation.

This concludes the verification that G_ε is C-free □

Lemma 117 (V.2)**.** *Let C be a solid block path of length $\ell \geq 4$ and type $(3, 4, 4, 3^{\ell-3})$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	Block path $(4, 3)$
G_1: Clique Ext'n	No clique extensions

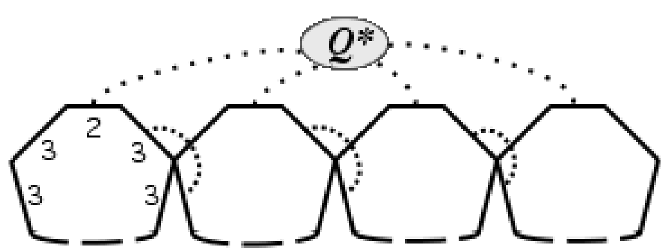
After amalgamation this is the same as a loop construction beginning with $(3, 4; 4, 3)$. The result is C_4-free where C_4 has type $(4, 3, 3, 4)$, but (for the rigidity argument) if one attaches a copy of Q_iP_I freely over a_i then it extends to type $(3, 4, 4, 3, 3, \dots)$ of any length. □

Lemma 118 (V.3)**.** *C be a solid block path of length $\ell \geq 5$ and type $(4, 4, 3^{\ell-3}, 4)$. Then there is no countable weakly universal C-free graph.*

Proof. For $\ell \geq 6$ we proceed as follows.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(3, 2, 3; 3, \dots, 3)$
G_1: Clique Ext'n	$u^S_{i+1,P}$; Q^*



For $\ell = 5$ we adjust this by an additional overlap of $\hat{P}_i$ with R_{i+1}: $\hat{P}_i = P_i \cup \{u^S_{i+1,P}, v^R_{i+1,P}\}$, where now $|P_i| = 2$.

Claim 1. Tthe graphs G_ε are C-free.

Suppose that $C \cong C' \subseteq G_\varepsilon$. Then we may suppose that P', Q' are $\hat{P}_i$, $\hat{Q}_i$ in some order. Then U' will be a copy of $\hat{P}_j$ for some $j \neq i$, and we may suppose $U' = \hat{P}_j$.

Case 1. $P'Q' = \hat{P}_i Q_i$.

Either the cut vertex of $Q'R'$ is in Q^*, or R' is R_i. We consider both possibilities.

- $R' = R_i$

The only plausible alternative would be that $j = i - 2$ and that R' is connected to Z' via a copy of S_{i-1}. But the overlap prevents this.

- The cut vertex v of QR' lies in Q^*.

This is a more plausible alternative. We may suppose R' is $Q_k \cup \{v\}$ for some $k \neq j$. We consider the path from $R' = Q_k \cup \{v\}$ to $Z' = \hat{P}_j$.

This path cannot pass through $Q_{k\pm1}$, so in view of the lengths of the segments involved, we find $j = k - 1$. Again, the lengths of the segments allow only the possibility $S' = R_k$, meaning that $\ell = 5$; but in this case we have an additional overlap to prevent this.

Case 2. $P'Q' = Q_i \hat{P}_i$.

We consider the path from $Q' = \hat{P}_i$ to $Z' = \hat{P}_j$. The case $j = i + 1$ is clearly ruled out, and hence $j < i$. The path from Q' to Z' should then pass through S_{j+1}, and this is ruled out by the overlap.

This proves the claim.

We add a few words about the *rigidity argument* in the case $\ell \geq 6$.

If $R'Q'P'$ is another copy of $R_i \hat{Q}_i \hat{P}_i$ free over a simlar base then we first try embedding C as $P'Q'R'S_i \ldots Z_i \hat{P}_i$. This forces P' to meet $\{a_i, u_{i+1,P}\}$. If the intersection with S_{i+1} contains a unique vertex, then we use the embedding $Q'P'S_{i+1} \ldots Z_{i+1} \hat{P}_{i+1}$. The possibility that P' meets $\hat{P}_{i+1}$ is quickly eliminated.

The case $\ell = 5$ is similar. $\square$

Lemma 119 (Var.4)**.** *Let C be a solid block path of length $\ell \geq 7$ and type $(n, 3^{\ell-2}, n)$ with*

$$n \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof. This construction is based on a loop of type $(n, 3^{\ell-2})$, of length $\ell - 1$. So we will denote the blocks by $(P, Q, \ldots, Y)$.

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n, 3^{\ell-3}; 3)$
G_1: Clique Ext'n	$u^Y_{i+1,P}$

We check that the graphs G_ε are C-free. Suppose $C \cong C' \subseteq G_\varepsilon$.

We may suppose $P' = \hat{P}_i$, $Z' = \hat{P}_j$, with $i < j$. The path from P' to Z' cannot pass through Y_{i+1} and therefore may be supposed to pass through $X_{i+1}, \ldots, Q_{i+1}, \hat{P}_{i+1}$. But as this would place Y' inside $\hat{P}_{i+1}$ and as $\ell \geq 4$ there is no option for Z'.

We may say something about the *rigidity argument.* If a copy $\widetilde{X \ldots P}$ of the segment $X_i \ldots \hat{P}_i$ is freely attached over a_i, then $\widetilde{(P \ldots X)} Y_i \hat{P}_i$ gives an embedding of C. On the other hand if the segment $\widetilde{(X \ldots P)}$ meets Y_{i+1} in a single vertex, then we embed C as $\tilde{P} Y_{i+1} Y_{i+2} \ldots Y_{i+\ell-2} \hat{P}_{i+\ell-2}$. □

Lemma 120 (V.5)**.** *Let C be a solid block path of length $\ell \geq 5$ and type $(n_1, 3^{\ell-2}, n_\ell)$ with*

$$n_\ell > n_1 \geq 4$$

Then there is no countable weakly universal C-free graph.

Proof. Set

$$k = \lfloor \ell/2 \rfloor$$

As in the previous case, we use a chain of loops of length $\ell - 1$, denoting the first and last blocks by P_i and Y_i (eventually, $\hat{P}_i$ and $\hat{Y}_i$) respectively. But for the k-th or $(k+1)$-st block of L_i we use the more explicit notation B_k^I and B_{k+1}^i.

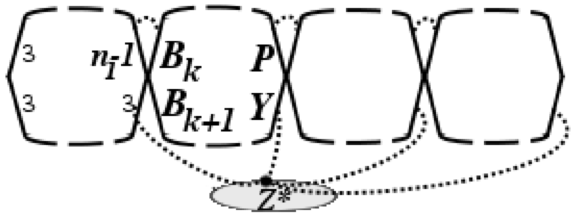
Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(n_1 - 1, n_2, \ldots, n_k; n_{k+1}, \ldots, n_{\ell-1}, 2)$
G_1: Clique Ext'n	$u_{i+1,P}^{B_k^{i+1}}$, $\hat{Y}_i = Y_i \cup \{z\}$; $z \in Z^*$, $\lvert Z^* \rvert = n_\ell$

We have some additional adjustments to make, but first we explain what we have so far.

The basic loop construction is type $(n_1; 3^{\ell-2}, 2)$ with overlap between $\hat{P}_i$ and B_{i+1}^k. We also take a clique Z^* of order n_ℓ with basepoint z, and attach it to each of the cliques Y_i.

$$L_i = (\hat{P}_i, Q_i, \ldots, B_k^i; B_{k+1}^i, \ldots, \hat{Y}_i) \quad (\text{plus } Z^*)$$



Now the *amalgamation base* will be taken to be

$$\bigcup_i P_i^* \cup Z$$

where $|\hat{P}_i \setminus P_i^*| = \min(n_1 - 2, 2(\ell - 2 - k) - 1)$.

In forming G_ε, edges and antiedges are placed at $(u_{3i,P}, u_{3(i+1),P})$ to avoid creating additional cliques of order 3 joined to z. As $\ell \geq 5$, there are no edges (a_i, a_j) in this graph.

Claim. The graphs G_ε are C-free.

Suppose that $C \cong C' \subseteq G_\varepsilon$. We will use the notation $C' = (B_1', \ldots, B_\ell')$.

Then B_ℓ' must be Z, and $B_{\ell-1}$ is a copy of some $\hat{Y}_i$, so we suppose $B_{\ell-1} = \hat{Y}_i$. We may suppose $B_1' = \hat{P}_j$ for some $j \neq i$.

Case 1. $j < i$

As $2k \geq \ell - 1$, we must have $j = i - 1$. In view of the overlap, the path from $\hat{Y}_i$ to $\hat{P}_{i-1}$ cannot pass through B_k^i. So it passes through B_{k+1}^i and $(\ell - 1) - k = \ell - 2$, $k = 1$, a contradiction.

Case 2. $j > i$

The block path from $\hat{Y}_i$ to $\hat{P}_j'$ must be contained in a number of copies of the segments $(B_k, B_{k-1}, \ldots, B_1)$ embedded in various loops. In view of the overlap between $\hat{P}_{i+1}$ and B_k^{i+2}, these all lie in the loop L_{i+1}, so $j = i + 1$.

The segment $B_k^{i+1} \ldots \hat{P}_{i+1}$ is too short, but there is a less obvious path that continues through $\hat{P}_{i+1}$ into a second copy $\tilde{P}$ of $\hat{P}_{i+1}$.

This alternative is blocked by the choice of the amalgamation base: as the path from $\hat{Y}_i$ through Q_{i+1} contains k blocks, there must be $(\ell - 2) - k$ blocks of order 3 contained in $\hat{P}_{i+1}$ and meeting $\tilde{P}$ in $\{a_i\}$ or $u_{i+1,P}$, hence

$$1[2(\ell - 2 - k) \leq 1 + |\hat{P}_i \setminus P_i^*| \leq 1 + 2(\ell - 2 - k) - 1$$

a contradiction. $\square$

The general case.
Reverse C here.

Lemma 121 (Var.??). *Let C be a solid block path of length 6 and type $(n_1, 3, \ldots, 3, n_\ell)$ with*

$$n_\ell > n_1 \geq 4$$
$$n_i = 3 \text{ for } 1 < i < \ell$$

Then there is no countable weakly universal C-free graph.

Proof. By Lemma **??** (applied to the reversal) it suffices to deal with the case

$$\ell = 6$$

Loop Construction

Graph, Type	*Specification*
G_0: Chain	$(2,3,3;3,3,n_\ell - 1)$
G_1: Clique Ext'n	

G_0	Chain of loops of type $(2,3,3;3,3,n_\ell - 1)$
G_1	Extend cliques $\hat{P}_i = P_i \cup P^*$, $\hat{U}_i = U_i \cup \{u_{i+1,U}\}$,
	$u_{i,U} \in S_i$, $\lvert P^*\rvert = n_1 - 2$
G_2	Amalgamation base $\bigcup_i (\hat{U}_i \setminus T_i)$
G_3	Edges, antiedges as usual.
G_ε	Edges, antiedges $(C^- \backslash e)$, C^- of type $(3,3,3,3,n_1)$

at $a_{3i}, a_{3(i+1)}$

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T', U') \subseteq G_\varepsilon$. We may suppose that P' is $\hat{P}_i$ for some i. Then we may also suposose that U' is $\hat{U}_j$ for some $j \neq i$.

If the cut vertex v of $P'Q'$ is in P^* then Q' is either a copy of P_k for some $k \neq i$ or of the form $\{v, a_k, a_{k'}\}$ with $(a_k, a_{k'})$ an edge, and then the cut vertex of $Q'R'$ is may be taken to be a_k. So $j \neq k$.

The possibility $j < k$ is excluded by the overlap with S_{j+1}. If $j > k$ then we must have $j = k+1$ and T' must be contained in a copy of $\hat{U}_j$. But the amalgamation base blocks this.

A similar, though not identical, analysis applies if the cut vertex of $P'Q'$ is a_{i+1}. The remaining possibility is that $Q'R'$ is $Q_i R_i$. This is immediately blocked by the construction.

□

Reverse C here. Instance of V.6. Rewrite for general case?

Lemma 122 (V.6 for length 6)**.** *Let C be a solid block path of length 6 and type $(3,3,n,3,3,3,3)$ with $n \geq 4$. Then there is no countable weakly universal C-free graph.*

Proof. We perform a loop construction as in the proof of Lemma **??**.

G_0 Chain of loops of type $(2, 3, n_3, 3; 2)$

G_1 Extend cliques $\hat{P}_i = P_i \cup \{u_{i+1,P}\}$, $\hat{T}_i = T_i \cup \{u_{i+1,T}\}$,

$u_{i,P} \in Q_i$, $u_{i,T} \in S_i$

G_2 Amalgamation base $\{a_i, u_{i,P}, u_{i,T} \mid i \in \mathbb{N}\}$

G_3 Edges, antiedges as usual.

G_ε Edges, antiedges $(K \backslash e)$, $|K| = 3$ at $a_{3i}, a_{3(i+1)}$

As in the proof of Lemma **??**. these graphs are C^--free where C^- has type $(3, 3, 5, 3, 3)$. □

Length 5 case of V.6

Lemma 123 (V.6 for $\ell = 5$)**.** *Let C be a solid block path of length 5 and type $(3, 3, 4, 3, 3)$. Then there is no countable weakly universal C-free graph.*

Proof.

Loop Construction

Graph	*Type*	*Specification*
G_0	Chain	
G_1	Clique Ext'n	
G_2	Amalg'n	
G_3	Anti-edges	
G_ε	Ext'n Family	

G_0 Chain of loops of type $(2, 3, 4; 3, 2)$

G_1 Extend cliques $\hat{P}_i = P_i \cup \{u_{i+1,P}\}$, $\hat{T}_i = T_i \cup \{u_{i+1,T}\}$,

$u_{i,P}, u_{i,T} \in R_i$

G_2 Amalgamation base $\{a_i, u_{i,P}, u_{i,T} \mid i \in \mathbb{N}\}$

G_3 Edges, antiedges as usual.

G_ε Edges, antiedges $(K \backslash e)$, $|K| = 3$ at $a_{3i}, a_{3(i+1)}$

We check that the graphs G_ε are C-free.

Suppose that $C \cong (P', Q', R', S', T') \subseteq G_\varepsilon$. We may suppose that $R' = R_i$ for some i, and find that the block path lies in copies of a single loop L_i. Taking note that the amalgamation base contains three vertices of R_i, we see easily that this is impossible. □

Proof of Proposition 8. We have dealt with length 1–5 in §§3–5 and with lengths 6 and 7, and variable length, in the present section. □

Proof of Theorem 1. As explained at the end of §2, Propositions 18 and 19 cover all cases, and follow from Proposition 8. □

7. Block paths with trivial blocks

The main obstacle to a general treatment of general solid block paths, where we now allow trivial blocks (i.e. single edges as blocks), is our frequent use of the "overlapping" method of killing unwanted embeddings of the constraint C into our basic construction.

Variations of these constructions will work reasonably well when one has a sufficient number of nontrivial blocks, without requiring that every block be nontrivial. However, the number of exceptional cases will rise correspondingly.

Here we give some further analysis of the general case, showing mainly that our analysis so far is directly relevant: any solid block path C for which there is a countable weakly universal C-free graph must be made up of constituents in our catalog, connected by paths. And we indicate some additional constraints which suggest that the final classification is not so very much more elaborate than the one we have given. As always, we set aside, perhaps for another occasion, the question of the verification that those solid block paths C left in the final catalog actually do belong there; that is, that the corresponding algebraic closure operations are locally finite, and thus there is a canonical countable weakly universal C-free graph with oligomorphic automorphism group. This is probably difficult in most cases. It is also quite possibly false in some instances; for example, the case of type $(n_1, 3, 3, 3, n_5)$ is doubtful.

Definition 124. Let C be a block path. A 3-*component* of C is a maximal segment in which all blocks are nontrivial.

At this point it may be well to introduce, belatedly, a shorter way of referring to constraints C which allow a countable weakly universal C-free graph. We shall call them *Rado constraints* (though they come to us more via Komjáth and Pach than from Rado).

We rephrase Theorem 2 as follows.

Proposition (Theorem 2)**.** *Let C be a solid block path and a Rado constraint. Then the 3-Components of C are Rado constraints.*

This was proved in §1.3.

This already gives us quite a bit of information. Now recall from Lemma 2 that a solid block path of type $(2, n_2, n_3, 2)$ with $n_2, n_3 \geq 3$ is not a Rado constraint. A similar argument should show that no solid block path of type $(2, n_2, \ldots, n_{\ell-1}, 2)$ with all $n_i \geq 3$ for $1 < i < \ell$ is a Rado constraint. This is hard to write down in a very general way so we would prove it by noting first that we may suppose that the pruned block path of type $(n_2, \ldots, n_{\ell-1})$ is a Rado constraint, and then go through the catalog, dealing first with the case $n_2 = 3$, then with the cases in which there is an index i with $n_i = n_{i+1} = 3$, and then with the leftovers, one of which is Lemma 2. We have not checked the details.

This of course suggests a more general conjecture.

Conjecture 8. *Let C be a solid block path which is a Rado constraint and C_0 an internal 3-component of C, i.e., one which does not contain a block leaf. Then C consists of a single block.*

As with the various "forbidden segment" results proved here, this would require dealing with many critical configurations, and apparently would require quite detailed knowledge of the final form of the classification of all block paths which are Rado constraints. So we view this as highly likely, but also challenging.

This would then shift the focus onto the case of solid block paths with no adjacent nontrivial blocks. Here one would like an explicit small bound on the number of blocks which occur.

We mention one easy result in this direction.

Lemma 125. *Let C be a solid block path of type $(n, 2, ..., 2.n)$ and even length ℓ with $n \geq 3$ and $\ell \geq 4n$. Then C is not a Rado constraint.*

We remark that for $\ell \geq 6$ if there is a countable weakly homogeneous C-free graph, then its automorphism group is not oligormorphic. But the transition from the skeleton G_3 to G_ϵ does not work in that case.

Proof. We perform a simplified loop construction. Set $k = (\ell - 2)/2$. We are assuming $n \leq (k+1)/2$.

G_0 Chain of block paths of type $(2, \ldots 2, n)$ of length k

G_1 No clique extensions

G_2 Amalgamation base $\{a_i \mid i \in \mathbb{N}\}$

G_3 No modifications

G_ϵ Edges, antiedges $(K \backslash e)$, $|K| = n$ at a_{6i}, a_{6i+3}

We check that the graphs G_ϵ are C-free.

Suppose that $C \cong (P', Q', \ldots, Q'', P'') \subseteq G_\epsilon$. Then we may suppose that P', P'' are P_i, P_j for some $i < j$.

If the image of C lies in G_3 then $j > i + 1$ since $k < \ell - 2$, but hen since $2k + 1 > \ell - 2$ this does not work either.

On the other hand, if C contains a unique path lying in an antiedge $(K \backslash e)$ connecting a pair of vertices (i, i') with $a_{i'}$ lying between a_i and P'' on C, then clearly $j = i' \pm 1$ and the length of the path from a_i to P'' is at most

$$k + 2(n-1) \leq (k-2) + (k-1) < 2k$$

so this is ruled out, and $|j - i'| > 1$ is clearly impossible.

Finally, if two paths lying in antiedges are involved, then we would require some connection between vertices $a_{i'}$ $a_{j'}$ in G_3 with $|i' - j'| \geq 3$, and this is clearly impossible.

The rest goes as usual. □

This is not a very strong restriction, but it illustrates what remains to us when we cannot make use of "overlap" conditions to control the construction. The next case to consider would be type $(n_1, \ldots, n_\ell)$ with exactly three indices i with $n_i > 2$, no two adjacent (and without loss of generality, $n_1, n_\ell \geq 3$), and primarily under the assumption that n_1, n_2, n_3 are distinct.

But a more substantial issue remains: the reduction of the general classification of Rado constraints to the case of slight extensions of block paths (Conjecture 5). As the analysis should be inductive, it is useful to have an explicit catalog of the target result, but this is only a point of departure for

Appendix
Statement of Theorem 1,
Critical Cases,
Structure of the Proof
Glossary (?)

Statement of Theorem 1

Theorem (1)**.** *Let C be a finite block path with no trivial blocks. Suppose that there is a weakly universal C-free graph. Let ℓ be the number of blocks in C. Then C has one of the following types.*

ℓ	*Form*
(general)	$(3^{\ell-1}, n)$*; or* $(3^{\ell-2}, n, 3)$ *or* $(3^{\ell-2}, 4, 4)$
2	$(4, n)$ *or* $(5, n)$ *with* $n \geq 6$
3	(n_1, m, n_3) *with* $m = 3$ *or* 4
4	$(n_1, 3, 3, n_4)$ *with* $n_4 \geq n_1 + 2$
"	$(3, n, 3, n)$ *with* $n > 4$
"	$(3, 4, 4, 4)$
"	$(3, 4, 3, n)$ $(4, 4, 3, n)$ *with* $n \geq 4$
5	$(4, 4, 3, 3, n)$ *with* $n \geq 9$
"	$(3, n_2, 3, 3, n_5)$ *with* $n_2, n_5 \geq 4$ *and* $\lvert n_2 - n_5 \rvert \geq 2$
"	$(3, 3, n, 3, n)$ *with* $n \geq 5$

Index of Critical Cases

Proposition. *Let C be a solid block path of length ℓ and type $(n_1, \ldots, n_\ell)$. Under any of the following conditions, C is not a Rado constraint.*

Length 2:

Conditions	**Code**	*Conditions*	**Code**
(1) $n_1, n_2 \geq 6$	**6+6+**	*(2)* $n_1 = n_2 = 5$	**55**

Length 3

Conditions	**Code**	*Conditions*	**Code**
(1) $(5, 5, n)$, $n \geq 5$	**55n,n>=5**	*(2)* $(6, 6, 6)$	**666**
(3) (m, n, m), $4 \leq m < n$	**mnm,4<=m<n**	*(4)* (m, n, m), $5 \leq n < m$	**mnm,5<=n<m**
(5) $3 \leq n_1 < n_2 < n_3$, $n_1 \geq 3$, $n_2 \geq 5$	**3<=n1<n2<n3,n2>=5**	*(6)* $5 \leq n_2 < n_1 < n_3$	**5<=n2<n1<n3**
(7) $3 \leq n_1 < n_3 < n_2$	**3<=n1<n3<n2**		

Length 4:

Conditions	**Code**	*Conditions*	**Code**
(1) $n_1 = n_2 = n_4 < n_3$	**4<=n1=n2=n4<n3**	*(2)* $n_1, n_4 < n_2, n_3$	**SmallEnds**
(3) $n_1 \geq n_4 > n_2, n_3 \geq 3$, $n_1 < n_3+n_4-1$	**n1n4>n2n3>=3,n1=n4approx**		
(4) $n_4 > n_1 > n_2 > n_3 \geq 3$	**n4>n1>n2>n3>=3,n2>=5**		
(5) $n_4 > n_2 > n_1 > n_3 \geq 3$	**n4>n2>n1>n3>=3**	*(6)* $n_2 \geq n_4 > n_1 > n_3 \geq 3$	**n2>=n4>n1>n3>=3**
(7) $n_3 < n_1 = n_4 = 5 < n_2$	**n3<n1=n4=5<n2**	*(8)* $n_4 > n_1 > n_3 > n_2 \geq 3$	**n4>n1>n3>n2>=3**
(9) $n_1 = n_3 = 4 < n_4 < n_2$	**n1=n3=4<n4<n2**	*(10)* $n_1 = n_3 = 4 < n_2 = n_4$	**n1=n3=4<n2=n4**
(11) $n_1 = n_3 = 4 < n_2 < n_4$	**n1=n3=4<n2<n4**	*(12)* $(4, 4, 4, 4)$	**4444**
(13) $n_1 = n_2 = n_3 = 4 < n_4$	**4445+**	*(14)* $n_4 > n_1 > n_2 = n_3 \geq 4$	**n4>n1>n2=n3=4**
(15) $n_2 \geq 5$, $n_1 = n_4 = 4$, $n_3 = 3$	**4n34,n>=5**		
(16) $n_2 \geq n_4 \geq 5$, $n_1 = 4$, $n_3 = 3$	**n1=4,n3=3,n2>=n4>=5**		
(17) $n_1, n_3 < n_2 < n_4$, $n_1 \neq n_3$	**n1n3<n2<n4,n1-ne-n3**		
(18) $n_1 = 3$, $n_3 = 4 < n_2 = n_4$	**n1=3,n3=4<n2=n4**	*(19)* $n_1 = 3$, $n_3 = 4 < n_4 < n_2$	**n1=3,n3=4<n4<n2**
(20) $n_2 \geq 5$, $n_1 = 3$, $n_3 = n_4 = 4$	**n1=3,n2>=5,n3=n4=4**	*(21)* $(3, n_2, 3, n_4)$, $n_4 > n_2 \geq 5$	**3n3n',n'>n>=5**
(22) $(3, n_2, 3, n_4)$, $n_2 > n_4 \geq 4$	**3n3n',n>n'>=4**	*(23)* $(3, 4, 4, n)$, $n \geq 5$	**344n,n>=5**
(24) $(3, 3, 4, n)$, $n \geq 5$	**334n,n>=5**	*(25)* $(4, 3, 4, n)$, $n \geq 5$	**434n,n>=5**

Length 5:

Conditions	**Code**
(1) $(n_1, n_2, 3, n_2, n_1)$, $n_1 > n_2 \geq 4$	**nn’3n’n,nn’>=4,n-ne-n’**
(2) $n_3 = n_4 = 3$, $n_5 > n_1 > n_2 > 3$	**n3=n4=3,n5>n1>n2>3**
(3) $n_3 = n_4 = 3$, $n_5 > n_2 > n_1 > 3$	**n3=n4=3,n5>n2>n1>3**
(4) $(4, 4, 4, 5^+, 4)$	**4445+4**
(5) $(4, 4, 4, 4, 4)$	**44444**
(6) $(3, n_2, 4, n_4, 3$, $n_2, n_4 \geq 5$	**35+45+3**
(7) $(3, 4, 4, n_4, 3)$, $n_4 \geq 5$	**3445+3**
(8) $(3, 4, 4, 4, 3)$	**34443**
(9) $(4, 4, 4, 3, 3)$	**44433**
(10) $(n, 4, 4, 3, n)$, $n \geq 5$	**n443n,n>=5**
(11) $(4, 4, 4, 3, 4)$	**44434**
(12) $(4, 4, 4, 3, n)$, $n \geq 5$	**4443n,n>=5**
(13) $(3, 4, 4, 3, n_5)$, $n_5 \geq 5$	**3443n**
(14) $(3, n_2, 4, 3, 3)$, $n_2 \geq 5$	**35+433**
(15) $(4, 4, 3, n, 3)$, $n \geq 5$	**443n3,n>=5**
(16) $(4, 4, 3, 4, 3)$	**44343**
(17) $(4, 4, 3, 3, 5)$, $n_5 \geq 4$	**4433n,4<=n<=8**
(18) $(n_1, 4, 3, 3, n_1)$, $n_1 \geq 5$	**n433n,n>=5**
(19) $(4, 4, 3, 4, 4)$	**44344**
(20) $(n_1, 3, n_3, 3, n_5)$, $n_1, n_5 < n_3$	**n3n’3n”,nn”<n’**
(21) $(n_1, 3, n_3, 3, n_3)$, $n_3 > n_1 \geq 4$	**n3n’3n’,n’>n>=4**
(22) $(3, 3, 4, 3, 4)$	**33434**
(23) $(n, 3, n, 3, n)$ $n \geq 4$	**n3n3n,n>=4**
(24) $(n_1, 3, 4, 3, n_5)$, $n_5 > n_1 \geq 5$	**n343n’,n’>n>=5**
(25) $(4, 3, 4, 3, n)$, $n \geq 5$	**4343n,n>=5**
(26) $(3, 3, 4, 3, n)$, $n \geq 5$	**3343n,n>=5**
(27) $(n, 3, n_3, 3, n)$, $n > n_3 \geq 4$	**n3n’3n,n>n’>=4**
(28) $(3, n_2, 3, n_4, 3)$, $n_4 > n_2 \geq 4$	**3n3n’3,n’>n>=4**
(29) $(3, n, 3, n, 3)$, $n \geq 4$	**3n3n3,n>=4**

Length 6:

Conditions	**Code**
(1) $(3,4,4,4,3,3)$	**344433**
(2) $(4,4,4,3,3,n)$, $n \geq 5$	**44433n,n>=5**
(3) $(4,4,4,3,4,4)$	**444334**
(4) $(4,4,4,3,n_5,4)$, $n_5 \geq 5$	**44435+4**
(5) $(4,4,4,3,4,4)$	**444344**
(6) $(3,4,4,3,3,n)$, $n \geq 3$	**34433n,n>=6**
(7) $(3,4,4,3,n,3)$, $n \geq 3$	**3443n3,n>=3**
(8) $(4,4,3,3,n,3)$, $n \geq 3$	**4433n3,n>=5**
(9) $(4,4,3,4,3,4)$, $n \geq 3$	**434344**
(10) $(n_1,3,n_3,3,3,n_6)$, $n_6 > n_3, n_1$, *and* $n_1, n_3 \geq 4$	**n3n'33m,m>nn'>=4,n-ne-n'**
(11) $(n_1,3,n_3,3,3,n_6)$, $n_3 > n_6 > n_1 \geq 4$	**n3n'33m,n'>m>n>=4**
(12) $(n,3,n,3,3,n_6)$, $n_6 > n \geq 4$	**n3n33n',n'>n>=4**
(13) $(3,3,n_3,3,3,n_6)$, $n_6 > n_3 \geq 4$;	**n3n33n',n'>n>=4**
(14) $(3,3,n_3,3,3,n_6)$, $n_3 > n_6 \geq 4$;	**33n33n',n>n'>=4**
(15) $(n_1,3,3,n,3,n)$, $3 \leq n_1 < n$	**m33n3n,m<n**
(16) $(n,3,n_3,3,3,n)$, $n_3 > n \geq 4$	**n3n'33n,n'>n>=4**
(17) $(n,3,n_3,3,3,n)$, $n > n_3 \geq 4$	**n3n'33n,n>n'>=4**
(18) $(n,3,n,3,3,n)$, $n \geq 4$	**n3n33n,n>=4**
(19) $(3,n_2,3,3,n_5,3)$, $n_5 > n_2 \geq 4$	**3n33n'3,n'>n>=4**
(20) $(3,3,n,3,n,3)$, $n \geq 4$	**33n3n3,n>=4**
(21) $(3,3,4,3,n,3)$, $n \geq 5$	**3343n3**

Length 7:

Conditions	**Code**
(1) $(3,4,4,3,3,n,3)$, $n \geq 3$	**34433n3,n>=3**
(2) $(n,3,3,3,n,3,n)$, $n \geq 4$	**n333n3n,n>=4**
(3) $(3,3,3,n,3,n,3)$	**333n3n3**
(4) $(3,3,n_3,3,3,n_6,3)$, $n_3 \neq n_6$, $n_3 \geq 4$	**33n33n'3,nn'>=4,n-ne-n'**

Variable Length:

Conditions	**Code**
(1) $n_\ell > n_1 > n_i$ $(1 < i < \ell)$, $n_2 = 4$, $\ell \geq 5$	**n1nl>ni,n2=4,l>=4ORn1-ne-nl**
(2) $(3,4,4,3,3,\ldots,3)$, $\ell \geq 4$	**3443*,l>=4**
(3) $(4,4,3,3,\ldots,3,4)$	**443*4,l>=5**
(4) $(n,3,\cdots,3,n)$, $\ell \geq 5$	**n3*n,n>=4,l>=5**
(5) $(n_1,3,\cdots,3,n_\ell)$, $n_\ell > n_1 \geq 4$, $\ell \geq 6$	**n3*n',n'>n>=4,l>=5**
(6) $(3,\ldots,3,n,3,3)$	**3*n33,n>=4,l>=5**

Proof structure

Forbidden (or nearly forbidden) segments

Length	Type	Ingredients	Reference
2	5^+5^+	4.1–8, 5.1–3	Lemmas 10, 12
4	$4^+4^+4^+4^+$	4.9–14; 5.4–5, V.1	Lemma 13
3	$4^+4^+4^+$	5.6–12, 6.1–5 V.1	Lemma 15
2	4^+4^+	5.1,13–19, 6.6–9, V.1–3, 7.1	Lemma 16

Length 1-5

Length	Case	Ingredients	Reference
1		Vacuous	
2		2.1–2	Fact 3
3	$n_1 = n_3$	$n_1 = n_2$: 3.1–2	Proposition 9
		$n_1 \neq n_2$: 3.3–4	"
"	$n_1 < n_3$	3.1, 3.5–7	"
4	$n_1 = n_4$	4.2–3,15	Proposition 14
"	$n_3 \geq 5$	4.8,16–25	"
5.	$n_3 \geq 4$	5.20–27	Proposition 17
"	$n_2 \geq 4$	5.28–29,	"
"	$n_2 = n_3 = n_4 = 3$	V.4, V.5	"

General Case

Case	Ingredients	Reference
$n_1 < n_\ell$	6.10–15, V.4	Proposition 19
$n_1 = n_\ell \geq 4$	6.16–18, 7.2, V.5	"
$n_1 = n_\ell = 3$	6.19–21, 7.3–4, V.6	"

Glossary (perhaps). Terms which might be referenced more explicitly in the text.

- Loop construction
- Clique extension
- Amalgamation base
- Anti-edge (clique type or otherwise - always delete an edge from a block leaf)
- C-skeleton
- C-control

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