6

Definable Groups

We study groups definable in Lie coordinatized structures. We will eventually characterize the groups interpretable in Lie coordinatized structures in terms of their intrinsic model theoretic properties. For the stable theory, see the monograph by Poizat [**PoGS**] and the relevant sections of [**Bu,PiGS**].

6.1 GENERATION AND STABILIZERS

We work with $\mathcal{M}^{\mathrm{eq}}$, and we will consider certain subsets that may meet infinitely many sorts of $\mathcal{M}^{\mathrm{eq}}$. In such cases we adopt the following terminology, reflecting the greater generality of this situation relative to the usual context of model theory.

Definition 6.1.1. *Let $\mathcal{M}$ be a many-sorted structure. A subset S of $\mathcal{M}$ is* locally definable *if its restriction to any sort (equivalently, any finite set of sorts) is definable. In particular, a group is said to be locally definable in $\mathcal{M}$ if its underlying set and its operations are locally definable. When the sorts of $\mathcal{M}$ all have finite rank, a locally definable subset is said to have* finite rank *if its restrictions to each sort have* bounded *rank; in this case, the maximum such rank is called the* rank *of S.*

Remark 6.1.2. *One sort of pathology should be noted here. Suppose that in $\mathcal{M}$, $\operatorname{dcl}(\emptyset)$ meets infinitely many sorts. Let C be a subset of $\operatorname{dcl}(\emptyset)$ meeting each sort in a finite set. Then any group structure whatsoever on C is locally definable.*

As in §5.5 we say that a structure has the type amalgamation property if Proposition 5.1.15 applies.

We have to be unusually careful with our notation for types in the presence of a group operation, distinguishing $\operatorname{tp}(ab)$ (i.e., $\operatorname{tp}(a \cdot b)$) from $\operatorname{tp}(a, b)$; indeed, the two notions will occur in close proximity.

Lemma 6.1.3. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Let G be a locally definable group in $\mathcal{M}^{\mathrm{eq}}$, and S a definable subset closed un-*

der inversion and generic multiplication: for a, b in S independent, $ab \in S$. Then $H = S \cdot S$ is the subgroup of G generated by S and $\operatorname{rk}(H - S) < \operatorname{rk} S$.

Proof. We show first that the product of any three elements a_1, a_2, a_3 of S lies in $S \cdot S$; this shows both that $S \cdot S$ is a subgroup of G, and that $S \subseteq S \cdot S$ (take $a_2 = a_1^{-1}$).

Given a_1, a_2, a_3 we take $u \in S$ independent from a_1, a_2, a_3 and of maximal rank. Let $b_1 = a_1 u$ and $b_2 = u^{-1} a_2$. Then $b_1, b_2 \in S$. Furthermore, b_2, a_3 are independent and thus $b_2 a_3 \in S$. But $a_1 a_2 a_3 = b_1 \cdot b_2 a_3$.

It remains to consider $\operatorname{rk}(H - S)$. Let $a_1 a_2 \in H$ have rank at least $\operatorname{rk}(S)$. Take $u \in S$ of maximal rank independent from a_1, a_2. Then $a_2 u$ belongs to S and is independent from a_1. Thus $b = a_1 a_2 u$ is also in S and

$$\operatorname{rk}(b/u) = \operatorname{rk}(a_1 a_2 / u) = \operatorname{rk}(a_1 a_2) \geq \operatorname{rk} S$$

and thus equality holds, and b and u are independent. We therefore have $a_1 a_2 = b u^{-1} \in S$. Thus $\operatorname{rk}(H - S) < \operatorname{rk} S$.

Lemma 6.1.4. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank, with the type amalgamation property. Let G be a locally 0-definable group of finite rank k in $\mathcal{M}^{\mathrm{eq}}$, and $S \subseteq G$ the locus of a complete type over* $\operatorname{acl}(\emptyset)$*, of rank k. Then $S \cdot S^{-1}$ generates a definable subgroup of G.*

We do not claim that S itself generates a definable subgroup; for example, if S reduces to a single element then the group in question is trivial. On the other hand, the statement of the lemma is equivalent to the claim that S generates a coset of a definable group under the affine group operation $ab^{-1}c$.

Proof. Let $X = \{ab^{-1} : a, b \in S; \operatorname{rk}(a, b) = 2k\}$. Note that for $a, b \in S$ independent of rank k, $\operatorname{rk}(a, ab^{-1}) = \operatorname{rk}(a, b) = 2k$ and thus also a, ab^{-1} are independent of rank k. We claim that the previous lemma applies to X, and that the groups generated by $S \cdot S^{-1}$ and by X coincide. In any case X is closed under inversion. We show now that X is closed generically under the operation ab^{-1}, and hence also under multiplication.

Let $c_1, c_2 \in X$ be independent, $c_i = a_i b_i^{-1}$ with $a_i, b_i \in S$, $\operatorname{rk}(a_i, b_i) = 2k$. We may suppose that (a_1, b_1) is independent from (a_2, b_2) and hence that a_1, a_2, b_1, b_2 is an independent quadruple. We seek d independent from this quadruple satisfying

$$\operatorname{tp}(d/c_1) = \operatorname{tp}(b_1/c_1); \quad \operatorname{tp}(d/c_2) = \operatorname{tp}(b_2/c_2).$$

As S is a complete type over $\operatorname{acl}(\emptyset)$ and b_i is independent from c_i, this is a type amalgamation problem of the sort that can be solved. The type of d now ensures the solvability of the equations

$$c_1 = a_1' d^{-1}; \quad c_2 = a_2' d^{-1}$$

with a_1', a_2' in S. Thus $c_1 c_2^{-1} = a_1' {a_2'}^{-1}$. We claim that this forces $c_1 c_2^{-1}$ into S, with a_1', a_2' as witnesses. Since $a_i' \in \operatorname{dcl}(a_i, b_i, d)$, we have a_1' and a_2' independent over d. Also $\operatorname{rk}(a_i', b_i, d) = \operatorname{rk}(a_i, b_i, d) = 3k$, so a_i and e are independent. Thus a_1' and a_2' are independent. Thus $c_1 c_2^{-1} \in X$.

Now suppose $a, b \in S$. Take $d \in S$ independent from a. Then $ab^{-1} = (ad) \cdot (bd)^{-1} \in X \cdot X$. Thus $S \cdot S^{-1}$ and X generate the same subgroup.

Lemma 6.1.5. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Let G be a locally definable group in $\mathcal{M}^{\mathrm{eq}}$, and S a definable subset generically closed under the ternary operation $ab^{-1}c$ (an affine group law). Then S lies in a coset C of a definable subgroup H of G, with* $\operatorname{rk}(C - S) < \operatorname{rk} S$.

Proof. We consider $X = \{ab^{-1} : a, b \in S \text{ independent}\}$. The condition on S implies that X is generically closed under multiplication and Lemma 6.1.3 applies, so X generates a definable subgroup H with $\operatorname{rk}(H - X) < \operatorname{rk} X$.

We claim that S lies in a single coset C of H. Indeed, if $a, b \in S$ and $c \in S$ is independent from a, b, then $ab^{-1} = (ab^{-1}c)c^{-1} \in H$.

Lastly, we claim that $\operatorname{rk}(C - S) < \operatorname{rk} S$. Let $a_\circ \in S$ and let $h \in H$ be independent from a, of maximal rank. Then $h \in X$ and $h = ab^{-1}$ with $a, b, a_\circ$ independent. Then $ha_\circ = ab^{-1}a_\circ \in S$. Thus $Ha_\circ$ lies in S up to a set of smaller rank.

Definition 6.1.6. *Let $h : G_1 \longrightarrow G_2$ be a map between groups. Then h is an* affine homomorphism *if it respects the operation $ab^{-1}c$.*

Lemma 6.1.7. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Let G, H be locally 0-definable groups in $\mathcal{M}^{\mathrm{eq}}$, S a 0-definable subset of G, and $h : S \longrightarrow H$ a 0-definable function.*

1. *If S is generically closed under the affine group operation $ab^{-1}c$ and h generically respects this operation, then h extends to an affine group homomorphism with domain the coset of a definable subgroup generated by S (under the affine group operation).*
2. *If S is generically closed under the operation ab^{-1} and h generically respects this operation, then h extends to a group homomorphism defined on the subgroup of G generated by S.*

Proof. Consider the graph Γ of the map h as a definable subset of the product group $G \times H$. Then Γ satisfies the hypotheses of Lemma 6.1.5 or Lemma 6.1.3, respectively. Thus in case (1) under the affine group operation Γ lies in a coset $\bar{\Gamma}$ of a definable subgroup of $G \times H$, with $\operatorname{rk}(\bar{\Gamma} - \Gamma) < \operatorname{rk} \Gamma$, and in case (2) Γ lies in a definable subgroup $\bar{\Gamma}$ of

$G \times H$, with $\mathrm{rk}\,(\bar{\Gamma} - \Gamma) < \mathrm{rk}\,\Gamma$. Here $\bar{\Gamma}$ will again be the graph of a function, as otherwise $\bar{\Gamma}$ will contain a translate of Γ disjoint from Γ, violating the rank condition. $\bar{\Gamma}$ is of course the graph of the desired extension of Γ in either case.

In the next lemma the avoidance (or neutralization) of the pathological case referred to at the outset is particularly important.

Lemma 6.1.8. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank, with the type amalgamation property. Let G be a locally definable group in $\mathcal{M}^{\mathrm{eq}}$ of bounded rank which is abelian, of bounded exponent. Then for any definable subset S of G, the subgroup generated by S is definable.*

Proof. We may take everything (locally) 0-definable. We may also suppose that S generates G. Our statement then amounts to the claim that G meets only finitely many sorts of $\mathcal{M}^{\mathrm{eq}}$. The case of rank 0 will play a key role below; in this case we are considering a finitely generated subgroup of a locally finite group, so the group in question is finite and hence definable.

Now let $k = \mathrm{rk}\,G$. Replacing S by a larger set if necessary we may suppose S has rank k. Let $S_\circ$ be the locus of some type of rank k over $\mathrm{acl}\,(\emptyset)$ contained in S. Then under the affine group operation $S_\circ$ generates a coset of a definable subgroup H of rank k (Lemma 6.1.4). Now work in G/H. $S + H$ meets a finite number of sorts, and $k \geq \mathrm{rk}\,(S + H) \geq \mathrm{rk}\,(S/H) + \mathrm{rk}\,(H) \geq \mathrm{rk}\,(S/H) + k$, so S/H is finite and therefore generates a finite subgroup of the locally finite quotient G/H, as noted at the outset.

We now turn to the notion of the stabilizer of a definable set S. Though it is tempting to define this as the group of $g \in G$ such that gS and S agree modulo sets of smaller rank, this tends to define the trivial subgroup and is therefore not useful. Note that most of our underlying geometries do not in any sense have "Morley degree" 1, or even finite degree.

Definition 6.1.9. *Let $\mathcal{M}$ have finite rank, G a definable group in $\mathcal{M}$, and let D, D' be complete types over $\mathrm{acl}\,(\emptyset)$, contained in G, with $\mathrm{rk}\,D = \mathrm{rk}\,D' = r$. Then*

1. $Stab_\circ(D, D') = \{g \in G : \mathrm{rk}\,(Dg \cap D') = r\}$.
2. $Stab_\circ(D) = Stab_\circ(D, D)$ *and* $Stab(D)$ *(the full stabilizer of D) is the subgroup of G generated by* $Stab_\circ(D)$.

Though we claim that $Stab_\circ(D)$ is generically closed under multiplication, it will not in general actually be a subgroup.

Example 6.1.10. *Let* (V, Q) *be an infinite dimensional orthogonal space over a finite field of characteristic 2, with the associated symplectic form degenerate, with a 1-dimensional radical* K *on which* Q *is nonzero. Let* $D = \{x \neq 0 : Q(x) = 0\}$. *Then* $Stab_\circ(D) = V - (K - (0))$ *is not a subgroup.*

Lemma 6.1.11. *Let* $\mathcal{M}$ *be* $\aleph_0$*-categorical of finite rank with the type amalgamation property,* G *a 0-definable group in* $\mathcal{M}^{\mathrm{eq}}$. *Let* D, D', *and* D'' *be complete types over* $\operatorname{acl}(\emptyset)$ *of rank* r *contained in* G. *If* $a \in Stab_\circ(D, D')$ *and* $b \in Stab_\circ(D', D'')$ *are independent, then* $ab \in Stab_\circ(D, D'')$.

Proof. $\operatorname{rk}(Da) \cap D' = r = \operatorname{rk}(D''b^{-1} \cap D')$, so by the corollary to Proposition 5.1.15 we have also $\operatorname{rk}(Da \cap D' \cap D''b^{-1}) = r$, and after multiplication on the right by b we have $\operatorname{rk}(Dab \cap D'') = r$.

Lemma 6.1.12. *Let* $\mathcal{M}$ *be* $\aleph_0$*-categorical of finite rank with the type amalgamation property,* G *a 0-definable group in* $\mathcal{M}^{\mathrm{eq}}$, *and* D *a complete type over* $\operatorname{acl}(\emptyset)$. *Then*

$$Stab(D) = Stab_\circ(D)\, Stab_\circ(D);$$
$$\operatorname{rk}(Stab(D) - Stab_\circ(D)) < \operatorname{rk}(Stab_\circ(D)).$$

Proof. Lemmas 6.1.3 and 6.1.11.

Lemma 6.1.13. *Let* $\mathcal{M}$ *be* $\aleph_0$*-categorical of finite rank with the type amalgamation property,* G *a 0-definable group in* $\mathcal{M}^{\mathrm{eq}}$, *and* D *a complete type over* $\operatorname{acl}(\emptyset)$ *with* $\operatorname{rk} D = \operatorname{rk} G$. *Then* $[G : Stab(D)] < \infty$.

Proof. It suffices to show that $\operatorname{rk}\, Stab_\circ(D) = \operatorname{rk} G$. Let a, b be independent elements of D of rank $r = \operatorname{rk} G$ and $c = a^{-1}b$. Then $\operatorname{rk}(b, c) = 2r$ so $\operatorname{rk}(b/c) = r$, and $b \in D \cap Dc$. Thus $c \in Stab_\circ D$. As c has rank r, we are done.

Lemma 6.1.14. *Let* $\mathcal{M}$ *be* $\aleph_0$*-categorical of finite rank with the type amalgamation property,* G *a 0-definable group in* $\mathcal{M}^{\mathrm{eq}}$, D *a 0-definable subset of* G *with* $\operatorname{rk} D = \operatorname{rk} G$, *and suppose that* G *has no proper 0-definable subgroup of finite index. Then there are pairwise independent* $a_1, a_2, a_3 \in D$ *with* $a_1 a_2 = a_3$.

Proof. We may take D to be the locus of a complete type over $\operatorname{acl}(\emptyset)$. Then by the preceding lemma and our hypothesis $Stab(D) = G$. By Lemma 7 $\operatorname{rk}(D \cap Stab_\circ D) = \operatorname{rk} G$. Pick $a_1 \in D \cap Stab_\circ D$ of rank $\operatorname{rk} G$ and $a_3 \in Da_1 \cap D$ with $\operatorname{rk}(a_3/a_1) = \operatorname{rk} G$. Then set $a_2 = a_3 a_1^{-1}$.

6.2 MODULAR GROUPS

Definition 6.2.1. *Two subgroups H_1, H_2 of a group G are said to be* commensurable *if their intersection has finite index in each. This is an equivalence relation. When G has finite rank, this is equivalent to* $\mathrm{rk}\,(H_1) = \mathrm{rk}\,(H_2) = \mathrm{rk}\,(H_1 \cap H_2)$.

Lemma 6.2.2. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank and modular. Let G be a definable group in $\mathcal{M}$, and H_d a subgroup defined uniformly from the parameter d for d varying over a definable set D. Let $E(d,d')$ hold if and only if H_d and $H_{d'}$ are commensurable. Then the relation E has finitely many equivalence classes.*

Proof. Choose $d \in D$ of maximal rank, $a \in G$ of maximal rank over d, and b in $H_d a$ of maximal rank over a, d. Let $B = \mathrm{acl}\,(b) \cap \mathrm{acl}\,(d,a)$. Let d', a' be conjugate to d, a over b and independent from d, a over b. Then b, d, a, and d', a' are independent over B by modularity and the choice of d', a'. Thus $\mathrm{rk}\,(b/aa'dd') = \mathrm{rk}\,(b/B) = \mathrm{rk}\,(b/ad)$ and $\mathrm{rk}\,(H_d a \cap H_{d'} a') = \mathrm{rk}\,(H_d a)$. Therefore $\mathrm{rk}\,(H_d \cap H_{d'}) = \mathrm{rk}\,(H_d)$, in other words $E(d,d')$ holds. Thus $d/E \in B$.

Furthermore, as $(H_d \cap H_{d'})a'a^{-1}$ is nonempty, $a'a^{-1}$ lies in $H_d H_{d'} = X_d X_{d'}(H_d \cap H_{d'})$ for sets $X_d, X_{d'}$ of coset representatives of the intersection in H_d, $H_{d'}$, respectively. Thus $\mathrm{rk}\,(a'/a,d,d') \le \mathrm{rk}\, H_d$ and hence $\mathrm{rk}\,(a/B) \le \mathrm{rk}\, H_d$. Now we compute $\mathrm{rk}\,(d/E)$:

$$\begin{aligned}
\mathrm{rk}\,(d,a,b) &= \mathrm{rk}\,(d) + \mathrm{rk}\,(a) + \mathrm{rk}\,(b/a,d) = \mathrm{rk}\,(a) + \mathrm{rk}\, G + \mathrm{rk}\, H_d \\
&= \mathrm{rk}\,(b) + \mathrm{rk}\,(a/b) + \mathrm{rk}\,(d/a,b) \\
&\le \mathrm{rk}\, G + \mathrm{rk}\, H_d + \mathrm{rk}\,(d/(d/E))
\end{aligned}$$

showing $\mathrm{rk}\,(d/(d/E)) = \mathrm{rk}\,(d)$ and $\mathrm{rk}\,((d/E)) = 0$, so $d/E \in \mathrm{acl}\,(\emptyset)$.

The next proposition, for which we give a purely model theoretic argument, can be proved in greater generality as a purely group theoretic statement [**Sch,BeLe**]. This was drawn to our attention by Frank Wagner, who has generalized the result even further [**Wa**].

Proposition 6.2.3. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property and modular. Let G be a 0-definable group in $\mathcal{M}$, and H a definable subgroup. Then H is commensurable with a group defined over* $\mathrm{acl}\,(\emptyset)$.

Proof. Let $H = H_d$ have defining parameter $d \in D$, with D a complete type over $\mathrm{acl}\,(\emptyset)$. Let $E(d,d')$ be the equivalence relation: H_d, $H_{d'}$ are commensurable. As this has finitely many classes and D realizes a unique type over $\mathrm{acl}\,(\emptyset)$, all groups H_d $(d \in D)$ are commensurable.

Define $B = \{g \in G :$ For some $d \in D$ independent from g, $g \in H_d\}$. By the corollary to Proposition 5.1.15,

$$\text{For } b_1,\, b_2 \text{ in } B \text{ independent, } b_1 b_2^{-1} \in B.$$

Thus by Lemma 6.1.3, $H = \langle B \rangle$ is a definable subgroup of G with $\mathrm{rk}\,(H - B) < \mathrm{rk}\,H$. Let $h \in H$ be an element of maximal rank. Then $h \in B$. Take $d \in D$ independent from h with $h \in H_d$. Then $\mathrm{rk}\,(h) \leq \mathrm{rk}\,H_d$ and thus $\mathrm{rk}\,H \leq \mathrm{rk}\,H_d$. On the other hand any element of H_d independent from d is in B, so $\mathrm{rk}\,(H \cap H_d) \geq \mathrm{rk}\,H_d$. This shows that H and H_d are commensurable.

Proposition 6.2.4. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property, and modular. Let G be a 0-definable group in $\mathcal{M}$. Then G has a finite normal subgroup N such that G/N contains an abelian subgroup of finite index.*

Proof. Let $Z^* = \{g \in G : [G : C(g)] < \infty\}$. We work mainly in $G^2 = G \times G$. For $a \in G$ let H_a be the subgroup $\{(x, x^a) : x \in G\}$ of G^2. Define $E(a, a')$ as follows: H_a and $H_{a'}$ are commensurable. This is an equivalence relation with finitely many classes. Notice that $E(a, a')$ holds if and only if $Z^*a = Z^*a'$: $E(a, a')$ holds if and only if on a subgroup G_1 of G of finite index we have $x^a = x^{a'}$; that is, $G_1 \leq C(a'a^{-1})$, $a'a^{-1} \in Z^*$.

Thus we have proved that Z^* is of finite index in G and we may replace G by Z^*. Then any element of G has finitely many conjugates and thus for $x, y \in G$, the commutator $[x, y]$ is algebraic over x and over y. In particular for $x, y \in G$ independent, the commutator $[x, y]$ is algebraic over $\emptyset$. On the other hand, every commutator $[x, y]$ can be written as $[x, y']$ with y' independent from x, since $C(x)$ has finite index in G. Thus $N = G'$ is finite, and G/N is abelian.

As this result tends to reduce the study of definable groups to the abelian case, we will generally restrict our attention to abelian groups in the sequel, even when this hypothesis is superfluous.

Lemma 6.2.5. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property, and modular. Let A be a 0-definable abelian group in $\mathcal{M}$, and $D \subseteq A$ the locus of a complete type over $\mathrm{acl}\,(\emptyset)$, S the stabilizer of D in A. Then*

1. $\mathrm{rk}\,S = \mathrm{rk}\,D$.
2. *D is contained in a single coset of S.*
3. *If D' is the locus of another complete type over $\mathrm{acl}\,(\emptyset)$ of the same rank, and if $Stab_\circ(D, D')$ is nonempty, then $Stab_\circ(D, D')$ agrees with a coset of S up to sets of smaller rank, and $Stab(D') = S$.*

4. *If $a, b \in S$ are independent with the same type over* $\operatorname{acl}(\emptyset)$*, then* $a - b \in Stab_\circ(D)$.

Proof. Ad 1. We apply the fundamental rank inequality of Proposition 5.5.4 taking both 0-definable sets to be G, and $G_a = D+a$, relative to the equivalence relation $E(a,b)$: $a - b \in S$. Then for inequivalent elements a, b the intersection $G_a \cap G_b$ has lower rank, so the fundamental rank inequality 5.5.4 applies and yields

$$\operatorname{rk}(A/S) \leq \operatorname{rk} A - \operatorname{rk} D, \text{ hence } \operatorname{rk}(S) \geq \operatorname{rk}(D).$$

The opposite inequality is elementary: if $s \in S$ has maximal rank and $d \in D$ has maximal rank over s, with $d + s \in D$, then $\operatorname{rk}(d + s/d) = \operatorname{rk}(s/d) = \operatorname{rk} S$, so $\operatorname{rk}(S) \leq \operatorname{rk}(D)$. Notice also that $a + S$ meets D in a set of rank $\operatorname{rk} D$.

Ad 2. We have seen that some coset of S meets D in a set of rank $\operatorname{rk} D$. There can be only finitely many such cosets, so they lie in $\operatorname{acl}(\emptyset)$, and as D realizes a single type over $\operatorname{acl}(\emptyset)$, there is only one such coset, and it contains D.

Ad 3. According to Lemma 6.1.11, if $a \in Stab(D)$, $b \in Stab(D, D')$ are independent, then $a + b \in Stab(D, D')$. Thus under the stated hypothesis $Stab(D, D')$ contains most of a coset of S, up to a set of lower rank. Conversely if $a, b \in Stab(D, D')$ are independent then by the same lemma $a - b \in S$, so $Stab(D, D')$ agrees with a single coset of S modulo sets of lower rank. Replacing a by $-a$ we find that $Stab(D', D)$ agrees with a single coset of $Stab(D')$ modulo sets of lower rank and thus S and $Stab(D')$ agree modulo sets of lower rank; as they are groups, they are equal.

Ad 4. By $(1, 2)$ we have $\operatorname{rk} SD = \operatorname{rk} D = \operatorname{rk} Stab_\circ D$. Thus the corollary to Proposition 5.1.15 applies.

Lemma 6.2.6. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property, and modular. Let A be a 0-definable group in $\mathcal{M}^{\mathrm{eq}}$ of rank n. Then there is a sequence of subgroups $(0) = A_\circ \triangleleft A_1 \triangleleft \ldots \triangleleft A_n = A$ with $\operatorname{rk}(A_i/A_{i-1})$=1, and all A_i defined over* $\operatorname{acl}(\emptyset)$.

Proof. We may replace A by a quotient modulo a finite normal subgroup of a subgroup of a finite index, so we may take A abelian. It suffices then to find a subgroup of rank 1 defined over $\operatorname{acl}(\emptyset)$ as we may factor it out and proceed inductively. Let D be the locus in A of a complete rank 1 type over an algebraically closed set. By Lemma 6.2.5 the stabilizer of D in A is a rank 1 subgroup. By Proposition 6.2.3 it is commensurable with a group defined over $\operatorname{acl}(\emptyset)$.

Lemma 6.2.7. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property, G, H 0-definable groups in $\mathcal{M}^{\mathrm{eq}}$, $D \subseteq G$ the locus of a complete type over* $\mathrm{acl}(\emptyset)$ *with* $\mathrm{rk}\, D = \mathrm{rk}\, G$*, $f : D \longrightarrow H$ 0-definable, and suppose that for any independent triple $a_1, a_2, a_3 \in D$ for which $a_1 a_2^{-1} a_3 \in D$, we have $f(a_1 a_2^{-1} a_3) = f(a_1) f(a_2)^{-1} f(a_3)$. Then f extends to a definable affine homomorphism from the coset in G generated by D, to H.*

Proof. Let $r = \mathrm{rk}\, G$. We first define a function $h : Stab(D) \longrightarrow H$. Let $S^* = \{a \in Stab(D) : \mathrm{rk}\,(a) = \mathrm{rk}\, G\}$. Then $S^* \subseteq Stab_\circ(D)$.

If $a \in S^*$ then $a = b_1 b_2^{-1}$ with b_1, b_2 independent elements of rank r in D. We define $h(a) = h(b_1)h(b_2)^{-1}$ and we must check that this is in fact well defined. Suppose also $a = b_3 b_4^{-1}$ with b_3, b_4 independent of rank r in D. Take further b_5, b_6 independent and of rank r, with $a = b_5 b_6^{-1}$, such that $\mathrm{rk}\,(b_5, b_6 / a b_1 b_2 b_3 b_4) = r$. Then b_1, b_2, b_6 and b_3, b_4, b_6 are independent triples with $b_6 b_2^{-1} b_1 = b_6 b_4^{-1} b_3 = b_5$, so applying the affine homomorphism law for f and cancelling $f(b_6)$, we get $f(b_1)b(b_2)^{-1} = f(b_3)f(b_5)^{-1}$ and f is well defined on S^*.

In order to extend h from S^* to $Stab(D)$, we show that part (2) of Lemma 6.1.7 applies. Let $a, b \in S^*$ be independent, and $c = ab^{-1}$. Certainly $c \in S^*$. We have $\mathrm{rk}\,(D \cap Da) = \mathrm{rk}\,(D \cap Db) = \mathrm{rk}\, D$ and thus by the corollary to Proposition 5.1.15 $\mathrm{rk}\,(D \cap Da \cap Db) = r$. Take $d_1 \in D \cap Da \cap Db$ of rank r over a, b, and set $d_2 = d_1 a^{-1}$, $d_3 = d_1 b^{-1}$. Thus $a = d_2^{-1} d_1$, $b = d_3^{-1} d_1$, $c = d_2^{-1} d_3$, with pairs of independent elements of rank r. The resulting formulas $h(a) = f(d_2)^{-1} f(d_1)$ and so forth combine to give $h(c) = h(a)h(b)^{-1}$, as required. Thus we may now take h to be a homomorphism from $S = Stab(D)$ to H.

D is contained in a single left coset C of S. For $b \in D$ we define a map $f_b : C \longrightarrow H$ by $f_b(x) = f(b)h(b^{-1}x)$. This is an affine homomorphism from $C \longrightarrow H$ which agrees with f on elements of D independent from b, using the basic property of f and the definition of h. Our final point will be that f_b is independent of $b \in D$ and therefore gives the desired extension f^* of f to C.

To see that f_b does not depend on b it suffices to prove $f_b = f_{b'}$ for $b, b' \in D$ independent. For any $c \in C$ we have $h(b^{-1}c)h({b'}^{-1}c)^{-1} = h(b^{-1}b') = f(b^{-1})f(b')$ and thus $f_b(c) = f_{b'}(c)$.

Lemma 6.2.8. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property and modular. Let A_1, A_2 be 0-definable abelian groups in $\mathcal{M}^{\mathrm{eq}}$. Suppose that any* $\mathrm{acl}(\emptyset)$*-definable subgroup of $A_1 \times A_2$ is* 0*-definable, and that* $\mathrm{acl}(\emptyset) \cap A_1 = (0)$. *Let C be a finite set with* $\mathrm{acl}(C \cap A_1) \subseteq C$, *and let $a_2 \in A_2$ have maximal rank over C. Then*

1. $\mathrm{acl}(a_2, C) \cap A_1 \subseteq \mathrm{dcl}(a_2, \mathrm{acl}(C))$*;*

2. *If no proper definable subgroup of* A_2 *of finite index is definable over* $\mathrm{acl}(\emptyset)$*, then* $\mathrm{acl}(a_2, C) \cap A_1 = [\,\mathrm{dcl}(a_2) \cap A_1] + [C \cap A_1]$.

Proof. Let $a_1 \in \mathrm{acl}(a_2, C) \cap A_1$. Let D be the locus of (a_1, a_2) over $\mathrm{acl}(C)$, and $S = Stab(D)$ in $A_1 \times A_2$. By Proposition 6.2.3, the group S is commensurable with a group S' defined over $\mathrm{acl}(\emptyset)$; by hypothesis, S' is 0-definable.

By Lemma 6.2.6 D is contained in a coset of S, and hence in a finite union of cosets of $S \cap S'$; as D is the locus of a complete type over an algebraically closed set, D is contained in a single coset X of $S \cap S'$. In particular $S \subseteq D - D \subseteq S'$. Thus X is a coset of S'.

Now $S \cap [A_1 \times (0)]$ is finite, since a_1 is algebraic over a_2, C. Thus $S' \cap [A_1 \times (0)]$ is finite, and since $\mathrm{acl}(\emptyset) \cap A_1 = (0)$, we conclude that $S' \cap [A_1 \times (0)] = 0$. Let $\pi_2 : A_1 \times A_2 \longrightarrow A_2$ be the projection. Thus $S' = S \cap S'$ is the graph of a homomorphism from $\pi_2 S'$ to A_1 and X is the graph of an affine homomorphism $f : \pi_2 X \longrightarrow A_1$. As X is definable over $\mathrm{acl}(C)$, $f(a_2) \in \mathrm{dcl}(a_2, \mathrm{acl}(C))$. This proves the first claim. Under the hypothesis of (2), S' is the domain of a homomorphism $h : A_2 \longrightarrow A_1$ and $f - h$ is a constant $a \in A_1$, so $a \in \mathrm{acl}(C) \cap A_1 = C \cap A_1$ and $a_1 = h(a_2) + a \in [\,\mathrm{dcl}(a_2) \cap A_1] + [C \cap A_1]$ as claimed.

Remark 6.2.9. *With the above hypotheses and notation, the same result can be proved, with the same proof, for the affine space* S_1 *over* A_1*, assuming* $\mathrm{acl}(C) \cap S_1 = \mathrm{dcl}(C) \cap S_1$.

Proposition 6.2.10. *Let* $\mathcal{M}$ *be* $\aleph_0$*-categorical of finite rank with the type amalgamation property, and modular. Let* A *be a 0-definable rank 1 abelian group in* $\mathcal{M}^{\mathrm{eq}}$*. Assume that* $\mathrm{acl}(\emptyset) \cap A = (0)$ *and that* A *has no proper* $\mathrm{acl}(\emptyset)$*-definable subgroup of finite index. Then there is a finite field* F *such that* A *has a definable vector space structure over* F *for which linear dependence coincides with algebraic dependence.*

Proof. Let F be the ring of $\mathrm{acl}(\emptyset)$-definable endomorphisms of A. Our assumptions on A imply that F is a division ring and by $\aleph_0$-categoricity of $\mathcal{M}$, F is finite; thus it is a finite field. Take A now as a vector space over F.

We show by induction on n that any n algebraically dependent elements $a_1, \ldots, a_n$ of A will be linearly dependent. For the inductive step, suppose that $a_1, \ldots, a_{n+1}$ are algebraically dependent and $a_1, \ldots, a_n$ are algebraically independent. Thus $a = a_{n+1} \in \mathrm{acl}(a_1, \ldots, a_n)$ and we wish to express a as an F-linear combination of $a_1, \ldots, a_n$. Let D be the locus of $(a_1, \ldots, a_n, a)$ over $\mathrm{acl}(\emptyset)$, and S its stabilizer. We have $\mathrm{rk}\, S = \mathrm{rk}\, D = n \times \mathrm{rk}\, A = n$, with D contained in a coset of S. Let T be the projection of S onto the first n coordinates. As the projection of D

to these coordinates contains $(a_1, \dots, a_n)$ and is contained in a coset of T, $\operatorname{rk} T = n$. Therefore the kernel of this projection has rank 0 and is finite, and T has finite index in A^n. By our hypotheses on A, the kernel is trivial and $T = A^n$ (consider the intersection of T with the standard copies of A in A^n). In other words S is the graph of a homomorphism $h : A^n \longrightarrow A$, i.e., $h = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i : A \longrightarrow A$ definable over $\operatorname{acl}(\emptyset)$. We claim naturally that $a = \sum \alpha_i a_i$ with $\alpha_i \in F$.

As D is contained in a coset of S, for $(x_1, \dots, x_n, y)$ and $(x'_1, \dots, x'_n, y')$ in D, we get $y - y' = h(\mathbf{x} - \mathbf{x}') = \sum_i \alpha_i(x_i - x'_i)$ and thus $y - \sum_i \alpha_i x_i$ is a constant on D, belonging to $\operatorname{acl}(\emptyset) \cap A = (0)$. This proves our claim.

Lemma 6.2.11. *Let $\mathcal{M}$ be Lie coordinatized, and A an infinite abelian group interpreted in $\mathcal{M}$ without parameters. Suppose that A has no nontrivial $\operatorname{acl}(\emptyset)$-definable proper subgroup, and that $\operatorname{acl}(\emptyset) = \operatorname{dcl}(\emptyset)$. Then A is part of a basic linear geometry in $\mathcal{M}$.*

Proof. By the previous proposition A has a vector space structure over a finite field F such that algebraic dependence coincides with F-linear dependence. Let P be the corresponding projective space. Then P is nonorthogonal to some $\operatorname{acl}(\emptyset)$-definable projective Lie geometry PJ, and there is then a 0-definable bijection between these geometries. Taking a cover of $\mathcal{M}$ if necessary, PJ will be the projectivization of a basic linear geometry J. By Lemma 2.4.7 there is a 0-definable isomorphism of A with J, so A is a basic linear geometry.

6.3 DUALITY

We will be dealing with groups definable in weakly Lie coordinatized structures below. As we make some use of envelopes, we observe that by Lemma 6.2.6 any such group is nonmultidimensional in the sense that it lies in a part of the structure which is coordinatized by a finite number of Lie geometries, each defined over $\mathrm{acl}(\emptyset)$. (More precisely: first adjust the base language temporarily so that the group in question is viewed as defined over $\mathrm{acl}(\emptyset)$.) In particular, only a finite number of quadratic geometries are involved, and after naming the Witt classes by introducing finitely many algebraic parameters, we may work in a Lie coordinatized structure. This being the case, it suffices to state the results in the Lie coordinatized setting; they then apply in the weakly Lie coordinatized setting as well.

Definition 6.3.1. *If $\mathcal{M}$ is a structure, and A a group of prime exponent p interpreted in $\mathcal{M}$, then A^* denotes the group of $\mathcal{M}^{\mathrm{eq}}$-definable homomorphisms from A to a cyclic group of order p (equivalently the set of definable F-linear maps from A to the field F of order p).*

Note that the elements of A^* are almost determined by their kernels, which are definable subgroups of A. However, we do not necessarily have $A^* \subseteq A^{\mathrm{eq}}$; for example, A may be one side of a polar geometry.

The reader should bear in mind that the abelian groups A of this section are not intended to be reminiscent of affine geometries.

Proposition 6.3.2. *Let $\mathcal{M}$ be a Lie coordinatized structure, A a 0-definable group in $\mathcal{M}^{\mathrm{eq}}$ of prime exponent p. Then A^* and the evaluation map $A \times A^* \longrightarrow \mathbb{F}_p$ are 0-definable in $\mathcal{M}^{\mathrm{eq}}$. If A has no nontrivial proper 0-definable subgroups, then either $A^* = (0)$ or the pairing $A \times A^* \longrightarrow \mathbb{F}_p$ is a perfect pairing (the annihilator of each factor in the other is trivial).*

Proof. A^* is a locally definable group. Arrange the sorts of $\mathcal{M}^{\mathrm{eq}}$ in some order and let D_n be the definable subset of A^* consisting of elements which lie in the first n sorts.

Our first claim is that $\mathrm{rk}\, A^*$ is finite, bounded by $\mathrm{rk}\, A$. Fix a definable subset D of A^*, and suppose $\mathrm{rk}\, D > \mathrm{rk}\, A$. We apply Proposition 5.2.2 concerning the sizes of envelopes. Accordingly, the number of elements of D is a polynomial of degree $2\mathrm{rk}\, D$ in the variables used there, and similarly for A. Taking envelopes of large and constant dimension, we deduce that $D \cap E$ eventually is larger than $A \cap E$, while (again for large enough envelopes) $D \cap E \subseteq (A \cap E)^*$; this is a contradiction.

We apply Lemma 6.1.8 and deduce that for any n the subgroup A_n^* generated by D_n is 0-definable. Let K_n be the annihilator in A of

A_n^*. The decreasing chain K_n of 0-definable groups must stabilize with $K_n = K$ constant from some point on. We may factor out K and suppose $K = (0)$ (note in passing that the last part of the proposition will be covered by the argument from this point on).

After these preliminaries we see that $A \times A_n^* \longrightarrow F$ is a perfect pairing for all large n. Therefore, with n, n' fixed, looking at the same situation in large finite envelopes, we find that $A_n^* \cap E = A_{n'}^*$ in such envelopes. Thus A_n^* is independent of n for n large, and $A_n^* = A^*$.

We note that one can form a structure consisting of a set D and a vector space V, with a generic interaction of D with V in which the elements of D act linearly on V. The foregoing proposition will fail for this structure, which is not Lie coordinatizable.

We now mention a variation of somewhat greater generality:

Lemma 6.3.3. *Let $\mathcal{M}$ be a Lie coordinatized structure, A a 0-definable group in $\mathcal{M}^{\mathrm{eq}}$ of finite exponent n, and A^* the definable $\mathbb{Z}/p\mathbb{Z}$-dual of A. Then A^* and the pairing $A \times A^* \longrightarrow \mathbb{Z}/n\mathbb{Z}$ are interpretable in $\mathcal{M}$. Furthermore, any definable subgroup B of A of finite index is an intersection of the kernels of elements of A^*.*

Proof. The definability of A^* is just as before. For the final statement, since A/B has exponent dividing n, it is perfectly paired with its $\mathbb{Z}/n\mathbb{Z}$ dual.

Lemma 6.3.4. *Let $\mathcal{M}$ be a Lie coordinatized structure, A a 0-definable vector space in $\mathcal{M}^{\mathrm{eq}}$ relative to a finite field K of characteristic p. Let A^* be the definable $\mathbb{Z}/p\mathbb{Z}$-dual of A, and Tr the trace from K to the prime field. Then A^* can also be given a K-space structure, and there is then a definable K-bilinear map $\mu : A \times A^* \longrightarrow K$ such that $Tr\,\mu(a, f) = f(a)$ for $(a, f) \in A \times A^*$. This pairing makes A^* the full definable K-linear dual of A.*

Proof. Let A' be the space of all definable K-linear maps of A to K. Let $Tr : A' \longrightarrow A^*$ be defined by $Tr(f)(a) = Tr(f(a))$. If $Tr(f) = 0$ then for $a \in A$ and $\alpha \in K$ we have $Tr(\alpha f(a)) = Tr(f)(\alpha a) = 0$, and thus $f(a) = 0$ by the nondegeneracy of the bilinear form $Tr(xy)$. Thus Tr embeds A' into A^*. Conversely, if $g \in A^*$ then for $a \in A$ the linear map $g_a : K \longrightarrow F$ defined by $g_a(\alpha) = g(\alpha a)$ must have the form $Tr(\gamma_a \alpha) = g(\alpha a)$ for a unique $\gamma_a \in K$. Letting $f(a) = \gamma_a$ we get $Tr(f) = g$, and f is K-linear since $f(\alpha\beta a) = Tr(\beta\gamma_a\alpha)$. Thus Tr identifies the K-linear dual with the F-linear dual. Let μ be the transport to A^* of the natural pairing on $A \times A'$.

Definition 6.3.5. *Let $\mathcal{M}$ be a structure of finite rank, A a group interpretable in $\mathcal{M}$ without parameters.*

1. *Let S, T be definable sets. We write $S \subseteq^* T$ if* $\mathrm{rk}\,(S-T) < \mathrm{rk}\, S$. *For corresponding definable formulas σ, τ we use the notation $\sigma \Longrightarrow^* \tau$.*
2. *If B is a subgroup of A^*, and $a \in A$, then $gtp(a/B)$ denotes the atomic type of a over B in the language containing only the bilinear map $A \times A^* \longrightarrow \mathbb{Z}/n\mathbb{Z}$, with n the exact exponent of A.*
3. *The group A is* settled *if for every algebraically closed parameter set C and $a \in A$ of maximal rank over C, we have* $\mathrm{tp}(a) \cup gtp(a/A^* \cap C) \Longrightarrow^* \mathrm{tp}(a/C)$.
4. *The group A is* 2-ary *if for any algebraically closed parameter set C and any set $\mathbf{b} = b_1, \ldots, b_n$ in A of elements which are independent over C of maximal rank, we have*

$$\bigcup_i \mathrm{tp}(b_i/C) \cup \bigcup_{ij} \mathrm{tp}(b_i b_j/\ \mathrm{acl}\,\emptyset) \Longrightarrow^* \mathrm{tp}(\mathbf{b}/C).$$

Our primary objective in the long run is to show that every group becomes both settled and 2-ary after introducing finitely many constants. The linear part of a quadratic geometry is an example of an unsettled group.

We close this section with a few miscellaneous lemmas.

Lemma 6.3.6. *Let $\mathcal{M}$ be a Lie coordinatizable structure and A, B groups 0-definably interpreted in $\mathcal{M}$ with no proper 0-definable subgroups of finite index. Suppose that B is settled. If a, b, c are independent, with $a \in A$ and $b \in B$ of maximal rank, then*

$$\mathrm{tp}(b/a,\ \mathrm{acl}\,(\emptyset)) \cup \ \mathrm{tp}(b/\ \mathrm{acl}\,(c)) \Longrightarrow^* \ \mathrm{tp}(b/a, c).$$

Proof. As B is settled, taking $C = \ \mathrm{acl}\,(a, c)$ we get

$$\mathrm{tp}(b/\ \mathrm{acl}\,(\emptyset)) \cup gtp(b/\ \mathrm{acl}\,(a, c) \cap B^*) \Longrightarrow^* \ \mathrm{tp}(b/a, c).$$

We will check that

$$(*) \qquad \mathrm{tp}(b/a,\ \mathrm{acl}\,(\emptyset)) \cup \ \mathrm{tp}(b/\ \mathrm{acl}\,(c)) \Longrightarrow gtp(b/\ \mathrm{acl}\,(a, c) \cap B^*)$$

Let $d \in \ \mathrm{acl}\,(a, c) \cap B^*$. We will apply Lemma 6.2.8 with: $A_1 = B^*$; $A_2 = A$; $C = \{c\} \cup [\ \mathrm{acl}\,(c) \cap B^*]$; $a_2 = a$. To do so, we must work over $\mathrm{acl}\,(\emptyset)$, noting that there are no $\mathrm{acl}\,(\emptyset)$-definable proper subgroups of A or B of finite index, and thus, in particular, $\mathrm{acl}\,(\emptyset) \cap B^* = (0)$. Thus by Lemma 6.2.8,

$$d = d_a + d_c \text{ with } d_a \in \ \mathrm{dcl}\,(a,\ \mathrm{acl}\,(\emptyset)) \cap B^* \text{ and } d_c \in \ \mathrm{acl}\,(c) \cap B^*.$$

Thus $(*)$ holds.

Lemma 6.3.7. *Let $\mathcal{M}$ be a Lie coordinatizable structure and suppose that A_i $(1 \le i \le n)$ is a family of groups 0-definable in $\mathcal{M}^{\mathrm{eq}}$, each having no 0-definable subgroups of finite index, and all but the first settled. Let C be algebraically closed and let $a_i, b_i \in A_i$ have maximal rank with $a_1, b_1, \ldots, a_n, b_n$ independent over C. If $\operatorname{tp}(a_i/C) = \operatorname{tp}(b_i/C)$ for all i and $\operatorname{tp}(a_i, a_j/\operatorname{acl}(\emptyset)) = \operatorname{tp}(b_i, b_j/\operatorname{acl}(\emptyset))$ for all i, j, then $\operatorname{tp}(\mathbf{a}/C) = \operatorname{tp}(\mathbf{b}/C)$.*

Proof. We proceed inductively. Thus we may suppose

$$\operatorname{tp}(a_1, \ldots, a_{n-1}/C) = \operatorname{tp}(b_1, \ldots, b_{n-1}/C)$$

and even that $a_i = b_i$ for $i < n$. Let $A = A_1 \times \cdots \times A_{n-1}$ and apply the previous lemma to the pair A, B_n.

Corollary 6.3.8. *Let $\mathcal{M}$ be a Lie coordinatizable structure and A a 0-definable settled group in $\mathcal{M}^{\mathrm{eq}}$ such that $\operatorname{acl}(\emptyset) \cap A^* = (0)$. Then A is 2-ary.*

Note that the property that A is 2-ary will persist over a larger set of parameters, though the hypothesis will not necessarily persist.

Lemma 6.3.9. *Let $\mathcal{M}$ be a Lie coordinatizable structure, A and B 0-definable groups of exponent n with no 0-definable subgroups of finite index. Let $D \subseteq (A \times B)$ be a type over $\operatorname{acl}(\emptyset)$ of maximal rank. Then the following are equivalent:*

1. *For $(a, b) \in D$, a lies in every b-definable subgroup of A of finite index.*

1′. *For $(a, b) \in D$, b lies in every a-definable subgroup of A of finite index.*

2. *For $(a, b) \in D$, there are a_1, a_2, a_3 in A with $a_1 + a_2 = a_3$, all realizing $\operatorname{tp}(a/\operatorname{acl}(b))$, and with a_1, a_2, b independent.*
3. *There are a_1, a_2, a_3 in A and $b \in B$ such that $(a_1, b) \in D$, with $a_1 + a_2 = a_3$, and*

$$\operatorname{tp}(a_2 b/\operatorname{acl}(\emptyset)) = \operatorname{tp}(a_3 b/\operatorname{acl}(\emptyset)).$$

4. *Every $\operatorname{acl}(\emptyset)$-definable bilinear map $A \times B \longrightarrow \mathbb{Z}/n\mathbb{Z}$ vanishes on D.*

Proof. (1) implies (2): Let $(a, b) \in D$, and let A^b be the smallest b-definable subgroup of A of finite index. Then $a \in A^b$. Let D' be the locus of a over $\operatorname{acl}(b)$. Working over $\operatorname{acl}(b)$, Lemma 6.1.4 applies. Thus the stabilizer $Stab(D')$ is a b-definable subgroup of A^b of finite index, and $Stab(D') = A^b$. Let $a_3 = a$. As $\operatorname{rk}[Stab(D') - Stab_\circ(D')] < \operatorname{rk} A^b$, we can find $a_2 \in D' \cap Stab_\circ(D')$ independent from a_3, b, and let $a_1 = a_3 - a_2$.

Evidently (3) is a weakening of (2). We show next that (3) implies (4). Let $f : A \times B \longrightarrow \mathbb{Z}/n\mathbb{Z}$ be $\mathbb{Z}$-bilinear and algebraic over $\operatorname{acl}(\emptyset)$. As D represents a complete type over $\operatorname{acl}(\emptyset)$, f is constant on D; let the value be u. Then $f(a_2, b) = f(a_3, b) = f(a_1, b) + f(a_2, b)$ so $u = f(a_1, b) = 0$.

Since condition (4) is symmetric in A and B it suffices now to show that (4) implies (1). Assume condition (1) fails: $(a, b) \in D$, H is a b-definable subgroup of A of finite index, and $a \notin H$. Fix $f \in A^*$ vanishing on H with $f(a) \neq 0$. Note that $f \in \operatorname{acl}(b)$. Let D^* be the locus of (f, b) over $\operatorname{acl}(\emptyset)$, and S the stabilizer of D^* in $A^* \times A$. As f is algebraic over b, $S \cap [A^* \times (0)]$ is finite, and thus lies in $\operatorname{acl}(\emptyset) \cap A^* = (0)$ by the condition on A. Furthermore $\operatorname{rk} S = \operatorname{rk} B$, and thus S projects onto B and is the graph of a homomorphism $h : B \longrightarrow A^*$. D^* lies in the coset $S + (f, b) = S + (f - h(b), 0)$. Now the representative $f - h(b) \in \operatorname{acl}(\emptyset) \cap A^* = (0)$ so $f = h(b)$. Define $(x, y) = [h(y)](x)$; then $(a, b) = f(a) \neq 0$. Thus (4) fails.

6.4 RANK AND MEASURE

We can attempt to construct a measure on subsets of a group A by taking cosets of a subgroup of index n to have measure $1/n$. Thus we may assign to a set S the infimum of the sums $\sum_i 1/n_i$ corresponding to coverings of S by finitely many such cosets. Our objective here is to show that the "measure zero" sets are those of less than full rank.

Lemma 6.4.1. *Let $\mathcal{M}$ be a Lie coordinatizable structure and A an abelian group of exponent p, 0-definably interpretable in $\mathcal{M}$. Let D be a 0-definable subset of A of full rank, and $a_1^*, \ldots, a_n^* \in A^*$ independent generics. Let $\alpha_1, \ldots, \alpha_n$ be elements of the prime field F_p. Then $\{d \in D : (d, a_i^*) = \alpha_i\}$ has full rank.*

Proof. By induction and the addition of parameters this reduces to the case $n = 1$. If this fails, then for $a_* = a_n$ a generic element of A^*, the complement D' of D contains a coset C_{a^*} of $\ker(a^*)$, modulo a set of smaller rank. We will argue that $\operatorname{rk} D < \operatorname{rk} A$.

Fix m, and let $b_1, \ldots, b_m$ be independent conjugates of a^*. We will consider the cardinality of D and of other definable sets in large envelopes E of $\mathcal{M}$. We have $|C_{a^*}| = q^{-1}|A|$ for some fixed q. The b_i are linearly independent in A^*, so $b_1, \ldots, b_m$ maps A onto F_p^m and the C_{b_i} are statistically independent. Thus the complement of $\bigcup_i C_{b_i}$ has cardinality $(1 - q^{-1})^m|A|$. Now $C_{b_i} \cap D$ has rank less than $\operatorname{rk} A$, so in the limit $|C_{b_i} \cap D|/|A| \longrightarrow 0$ by Lemma 5.2.6. Thus $\limsup_E |D|/|A| \leq (1 - q^{-1})^m$; varying m, $\lim_E |D|/|A| = 0$ and $\operatorname{rk} D < \operatorname{rk} A$ by Lemma 5.2.6.

Lemma 6.4.2. *Let $\mathcal{M}$ be Lie coordinatizable, let A be an abelian group interpreted in $\mathcal{M}$, and let $D \subseteq A$ be definable with $\operatorname{rk} D = \operatorname{rk} A$. Then finitely many translates of D cover A. More specifically, if D is c-definable then one may find $\mathbf{b} = (b_1, \ldots, b_n)$ in A with $A = \bigcup_i (D + b_i)$ and $\mathbf{b}$ independent from c.*

Proof. We may suppose that A is 0-definable, and we proceed by induction on the maximal length of a chain of $\operatorname{acl}(\emptyset)$-definable subgroups of A.

We claim first that the result holds when A is part of a basic linear geometry for $\mathcal{M}$. We leave this essentially to the reader, but as an example, suppose A is an orthogonal space with quadratic form Q and $D = \{x \neq 0 : Q(x) = 0\}$. Let $V \leq A$ be nondegenerate of dimension 5. Then we claim $D + V = A$. Take $v \in A$, and choose w so that $\langle v, w \rangle$ is nondegenerate. Then $V_\circ = \langle v, w \rangle^\perp \cap V$ is a nondegenerate subspace of dimension at least 3, not containing v, and $Q(v) = Q(u)$ for some nonzero $u \in V_\circ$. Then $v - u \in D$.

Now suppose that A has a nontrivial $\operatorname{acl}(\emptyset)$-definable finite subgroup B. Then $\bar{D} = (D+B)/B$ has full rank in A/B and induction applies to $\bar{D}$, A/B. As B is finite this yields the claim in A.

Assume now that A has no nontrivial $\operatorname{acl}(\emptyset)$-definable finite subgroup, and is not part of a basic linear geometry. There is an $\operatorname{acl}(\emptyset)$-definable subgroup A_1 of A which is part of a stably embedded basic linear geometry of $\mathcal{M}$ (Lemma 6.2.11). Let D be c-definable of full rank in A. Pick $b \in A$ of maximal rank over c such that $[b + A_1] \cap D$ is infinite. Then $D - b$ meets A_1 in an infinite set and thus there is a finite subset $F \subseteq A_1$ such that $A_1 \subseteq F + D - b$, and we may take the elements of F to be independent from b, c. Let B be the locus of b over $F \cup \{c\}$. Then B has full rank and for $b' \in B$, $A_1 \subseteq F + D - b'$. Now by induction in A/A_1, for some finite set F', $F' + B + A_1$ covers A. We claim that $F + F' + D = A$.

Let $a \in A$. Then for some $b' \in B$, we have $a \in F' + b' + A_1 \subseteq F' + b' + (F + D - b') = F' + F + D$, as claimed.

Lemma 6.4.3. *Let $\mathcal{M}$ be Lie coordinatizable, let A be an abelian group interpreted 0-definably in $\mathcal{M}$, and suppose A has no proper 0-definable subgroups of finite index. Let $h_i : A \longrightarrow B_i$ for $i = 1, 2$ be definable homomorphisms onto finite 0-definable groups B_1, B_2 and let $h = (h_1, h_2) : A \longrightarrow B_1 \times B_2$ be the induced map. If h_1, h_2 are independent then h is surjective.*

Proof. Let the range of h be $C \leq B_1 \times B_2$ and let $C_1 = C \cap [B_1 \times (0)]$, $C_2 = C \cap [(0) \times B_2]$. C can be interpreted as the graph of an isomorphism between B_1/C_1 and B_2/C_2. Let $g_i : A \longrightarrow B_i/C_i$ be the map induced by h_i. Then $g_i \in \operatorname{acl}(h_i)$ and g_1 and g_2 differ only by an automorphism of the range. Thus $g_i \in \operatorname{acl}(h_1) \cap \operatorname{acl}(h_2) = \operatorname{acl}(\emptyset)$ and thus by assumption $B_1 = C_1$, $B_2 = C_2$, and h is surjective.

Lemma 6.4.4. *Let $\mathcal{M}$ be Lie coordinatizable, let A be an abelian group interpreted 0-definably in $\mathcal{M}$, let A^0 be the smallest 0-definable subgroup of finite index, and let $D \subseteq A$ be 0-definable with $\operatorname{rk} D = \operatorname{rk} A$. Assume that D lies in a single coset C of A^0 and let $h : A \longrightarrow B$ be a definable homomorphism into a finite group B. Then for any $b \in h[C]$, D meets $h^{-1}[b]$ in a set of full rank.*

Proof. If h is algebraic over $\emptyset$ then h is constant on C and there is nothing to prove. Suppose, therefore, that $h \notin \operatorname{acl}(\emptyset)$.

Using the previous lemma, the proof of Lemma 6.4.1 can be repeated (for the case $n = 1$), using independent conjugates of h. Alternatively, the following argument can be given which does not make use of finite approximations but again makes use of an infinite family of independent conjugates of h.

Let $\nu(h) = \nu_D(h) = |\{c \in h[C] : D \cap h^{-1}[c] \text{ has full rank }\}|/|h[C]|$. We claim $\nu(h) = 1$. For $h' = (h_1, h_2)$ induced by two homomorphisms, if ${h'}^{-1}[(c_1, c_2)] \cap D$ has full rank, then the same applies to $h_i^{-1}[c_i]$ and thus by the previous lemma, if h_1 and h_2 are independent then we get $\nu(h') \leq \nu(h_1)\nu(h_2)$. Thus if $\nu(h) < 1$, then by taking enough independent conjugates h_i of h we can construct a homomorphism f with finite image for which $\nu(f)$ is arbitrarily small. But a finite number m of translates $D + a_i$ cover C, and $\nu_D = \nu_{D+a}$ for each translate. Hence $1 = \nu_C(f) \leq m\nu_D(f)$, and we have a lower bound on ν_D, a contradiction.

Lemma 6.4.5. *Let $\mathcal{M}$ be Lie coordinatizable, let A be an abelian group interpreted 0-definably in $\mathcal{M}$, and let D be the locus of a complete type over* $\operatorname{acl}(\emptyset)$ *of maximal rank. Then there are independent $a, a' \in D$ such that $a - a'$ lies in every a-definable subgroup of A of finite index.*

Proof. Take $a \in D$. Let A^a be the smallest a-definable subgroup of A of finite index. We consider the canonical homomorphism $h : A \longrightarrow A/A^a$. The previous lemma applies and shows that $(A^a + a) \cap D$ has full rank. It suffices to take a' in the intersection, of maximal rank.

6.5 THE SEMI-DUAL COVER

It is remarkable that duality can be used to reduce many aspects of the treatment of affine covers to the treatment of finite covers. (Affine covers are covers with affine fibers in the sense of §4.5, corresponding, for us, to stages in a Lie coordinatization in which affine geometries are involved.)

Suppose that $\pi : \mathcal{N} \longrightarrow A$ is a cover with affine fibers $N_a = \pi^{-1}[a]$, affine over A. (Some might prefer to call A "V" here, but as in the previous section we tend to call our abelian groups A for the present.) Then the affine dual N_a^* is a finite cover of the linear dual A^*. Let $\mathcal{N}^*$ be the corresponding cover; then it seems that $\mathcal{N}^*$ should contain the same information as $\mathcal{N}$. We show below that a group structure on $\mathcal{N}$ corresponds to what we call a "bilinear group structure" on $\mathcal{N}^*$. This approach will lead to our sharpest result on groups, the "finite basis theorem" for definability in definable groups. Cf. the work of Ahlbrandt and Ziegler in [**AZ2**].

On the other hand, this method does not appear to apply to iterated covers, as a cover of $\mathcal{N}$ does not appear to correspond to a cover of $\mathcal{N}^*$, and thus the use of affine covers cannot be eliminated systematically.

Definition 6.5.1. *Let A_1, A_2 be abelian groups. A* bilinear cover *of A_1, A_2 is a surjective map $\pi = (\pi_1, \pi_2) : L \longrightarrow A_1 \times A_2$, where L is a structure with two partial binary operations $q_1, q_2 : L \times L \longrightarrow L$, with the following properties:*

- BL1. *q_i is defined on $\cup_{a \in A_i}[\pi_i^{-1}[a] \times \pi_i^{-1}[a]]$, and gives an abelian group operation on each subset $L[a] = \pi_i^{-1}[a]$.*
- BL2. *For $i, i' = 1, 2$ in either order, $\pi_{i'}$ is a group homomorphism on each group $(L[a]; q_i)$ for $a \in A_i$.*
- BL3. *Given elements $a_{ij} \in A_i$ for $i = 1, 2$, $j = 1, 2$, and $c_{ij} \in \pi^{-1}(a_{1i}, a_{2j})$, we have*

$$q_2(q_1(c_{11}, c_{12}), q_1(c_{21}, c_{22})) = q_1(q_2(c_{11}, c_{21}), q_2(c_{12}, c_{22})).$$

(In (BL3), note that the result of the calculation on either side lies in $\pi^{-1}(a_{11} + a_{12}, a_{21} + a_{22})$.)

Such covers will normally occur interpreted within some $\mathcal{M}^{\mathrm{eq}}$, in which case L and all the associated structure is taken to be interpretable in $\mathcal{M}$. Generally, q_1 and q_2 will be given the more suggestive notations "$+^1$, $+^2$," or just "$+$" if no ambiguity results. The same applies to iterated sums $\sum^1$, $\sum^2$, or $\sum$. We will also write $L(a_1, a_2)$ for $\pi^{-1}[(a_1, a_2)]$.

Lemma 6.5.2. *Let $\pi : L \longrightarrow A_1 \times A_2$ be a bilinear cover relative to the operations q_1 and q_2. Then*

1. *q_1 and q_2 agree on $L(0,0)$. Let this group be denoted $(A,+)$.*
2. *If $0_1, 0_2$ are the identity elements of A_1 and A_2 respectively, then there are canonical identifications $L(0_1) \simeq A \times A_2$ and $L(0_2) \simeq A_1 \times A$.*
3. *Each set $L(a_1,a_2)$ is naturally an affine space over $L(0,a_2)$ and $L(a_1,0)$, giving two A-affine structures on $L(a_1,a_2)$ which coincide.*

Proof. *Ad* 1. Let $A = L(0,0)$ as a set. Let e_1, e_2 be the 0-element of A with respect to q_1 and q_2 respectively. With all a_{ij} equal to 0 (in A_1 or A_2, as the case may be) and with c_{ij} equal to e_1 in (BL3), and setting $e' = q_2(e_1,e_1)$, condition (BL3) can be written as $e' = q_1(e',e')$. Hence we have $q_2(e_1,e_1) = e' = e_1$, and this implies $e_1 = e_2$.

Then with $c_{12} = c_{21} = e_1$ we get $q_2 = q_1$ on A. We note in passing that with $c_{11} = c_{22} = e_1$ we would also get the commutative law (or laws) on A, which in any case we have assumed.

Ad 2. We now consider the structure of $L(0_1)$. By (BL2) we have $q_2[L(a_1,a_2), L(a_1',a_2)] \subseteq L(a_1+a_1',a_2)]$ and, in particular, $L(0_1,a_2)$ is a subgroup of $L(a_2)$ for $a_2 \in A_2$. Let its identity element be denoted $z(a_2)$. We will show that $z : A_2 \longrightarrow L(0_1)$ is a homomorphism. Let $a, a' \in A_2$ and let $z = q_1(z(a), z(a'))$. Applying (BL3), we get $q_2(z,z) = q_1(q_2(z(a),z(a)), q_2(z(a'),z(a'))) = q_1(z(a),z(a')) = z$ and thus $z = z(a+a')$. Thus z is a homomorphism. By definition $\pi_2 z$ is the identity and as the kernel of π_2 on $L(0_1)$ is the group A, we get a direct product decomposition $(L(0_1),q_1) \simeq A_2 \times A$. This identification respects π_2; that is, $L(0_1,a)$ corresponds to $(a_2) \times A$ with q_2 acting on $L(0_1,a)$ as on A.

A similar analysis applies on the other side.

Ad 3. According to (BL2) under q_2 $L(0,a_2)$ acts on $L(a_1,a_2)$ for any $a_1 \in A_1$, making the latter an affine space over the former. After identifying A with $L(0_1,a_2)$ and $L(a_1,0_2)$ we get two affine actions of A on $L(a_1,a_2)$. These can be compared as follows. Let $x \in A$, $y \in L(a_1,a_2)$, and let z_1 be the identity element of $L(a_1)$, z_2 the identity element of $L(a_2)$. The identification of A with $L(0_1,a_2)$ takes x to $q_1(x,z_2)$; the other identification takes x to $q_2(x,z_1)$. For the action of A via $L(0_1,a_2)$ on $L(a_1,a_2)$ we get $q_2(q_1(x,z_2),y) = q_2(q_1(x,z_2), q_1(z_1,y)) = q_1(q_2(x,z_1), q_2(z_2,y)) = q_1(q_2(x,z_1),y)$, which is the action of A via $L(a_1,0_2)$.

Lemma 6.5.3. *Let L be a bilinear cover of $A_1 \times A_2$. Let $a_i \in A_1$, $a_j' \in A_2$, and let $x_{ij} \in L(a_i,a_j')$, r_i, s_j integer coefficients. Then $\sum_i^2 r_i \sum_j^1 s_j x_{ij} = \sum_j^1 s_j \sum_i^2 r_i x_{ij}$ and, in particular, if $r_i = s_j = 1$ then the order of summation can be reversed.*

Proof. We first deal with the case in which $r_i = s_j = 1$, proceeding by induction on the numbers m, n of indices i and j respectively, beginning with $m = n = 2$, which is (BL3). Case $(m, n+1)$ is easily derived from cases (m, n) and $(m, 2)$ as in the usual proofs of basic properties of sums, and case $(m+1, n)$ follows similarly from (m, n) and $(2, 2)$, so from the basic case $m = 2$, $n = 2$ we can first get case $(m, 2)$ for any m and then (m, n) for any m, n.

The general case of integer coefficients follows by simply expanding out the definitions from the case of coefficients ± 1. So consider now the case in which the r_i are ± 1, but keep the $s_j = 1$. Splitting the set I of indices i into I^+ and I^- according to the sign of r_i, our claim is

$$\sum_{I^+}^{2} \left(\sum_{j}^{1} x_{ij} \right) -^{2} \sum_{I^-}^{2} \left(\sum_{j}^{1} x_{ij} \right) = \sum_{j}^{1} \sum_{i}^{2} r_i x_{ij}.$$

Moving the negative term from left to right and applying the positive case twice, with a little care, the claim falls out. The case of $r_i, s_j = \pm 1$ then follows by repeating the argument.

Lemma 6.5.4. *Let $\mathcal{M}$ be a structure, and*

$$0 \longrightarrow A_1 \longrightarrow B \longrightarrow A_2 \longrightarrow 0$$

be an exact sequence of abelian groups with A_1, A_2 of prime exponent p, and assume this sequence is interpreted in $\mathcal{M}$. For $a \in A_2$ let B_a be the preimage in B of a, a coset of A_1, and let B_a^* be the set of definable affine homomorphisms from B_a to the field F of p elements. Let $L = \{(a, f) : a \in A_2, f \in B_a^*\}$, take $\pi_1 : L \longrightarrow A_2$ natural, and let $\pi_2 : L \longrightarrow A_1^*$ be defined by $\pi_2 f \in A_1^*$ the linear map associated to f, i.e. $f(x+y) - f(y)$ as a function of x. Then L is a cover of $A_2 \times A_1^*$ with respect to the operations q_1, q_2 described as follows. The operation q_1 acts by addition in the second coordinate. The operation q_2 also acts by addition but in a somewhat more delicate sense: if $\pi_2(a, f) = \pi_2(a', f')$ then f and f' are affine translates of the same linear map $f_\circ$, and we set $q_2((a, f), (a', f')) = (a + a', f + f')$ where $f + f'$ is the function g on $B_{a+a'}$ defined by $g(b + b') = f(b) + f'(b')$ for $b \in B_a$, $b' \in B_{a'}$.

Proof. One checks in the first place that q_2 is well defined: for $a_1 \in A_1$, $f(b + a_1) + f'(b' - a_1) = f(b) + f_\circ(a_1) + f'(b') - f_\circ(a_1) = f(b) + f'(b')$.

The verification of the axioms is straightforward. Axiom (BL3) concerns the situation $a, a' \in A_2$, $f_1, f_2 \in B_a^*$, $f_1', f_2' \in B_{a'}^*$, f_1 and f_2 induce the same linear map, and f_1' and f_2' induce the same linear map. The result of applying the appropriate combinations of q_1 and q_2 in either order is $(a + a', (f_1 + f_2) + (f_1' + f_2'))$ with the sum on the right involving $B_{a+a'} = B_a + B_{a'}$.

The cover associated to an exact sequence as described above will be called a *semi-dual* cover since it involves two groups, one of which is a dual group. Notice that the "structure group" $L(0,0)$ for the semi-dual cover associated with such an exact sequence is the set of constant maps from A_1 to F, which we identify with F. If $\mathcal{M}$ is Lie coordinatized then the cover obtained is definable since the dual group is definable.

Now we present a construction in the reverse direction.

Lemma 6.5.5. *Let $\mathcal{M}$ be a structure, A_1 and A_2 abelian groups interpreted in $\mathcal{M}$, and L a bilinear cover of $A_2 \times A_1$ interpreted in $\mathcal{M}$. Set $F = L(0,0)$, and let B be the set*

$$\begin{aligned}\{(a,f):\quad & a \in A_2,\ f : L(a) \longrightarrow F \text{ definable},\\ & f \text{ is the identity on } L(a,0) \text{ identified with } L(0,0)\}.\end{aligned}$$

Then B is a group with respect to the operation $(a,f)+(a',f') = (a+a',f'')$ with $f''(q_2(x,x')) = f(x)+f(x')$ for $x \in L(a)$, $x' \in L(a')$ and $\pi_2(x) = \pi_2(x')$, and there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}(A_1,F) \longrightarrow B \longrightarrow A_2 \longrightarrow 0$$

where Hom *is the group of definable homomorphisms.*

Proof. Wherever one sees an expression $q_2(x,x')$ it should be assumed that $\pi_2(x) = \pi_2(x')$, both in the above and in the proof following.

We check first that the operation $+$ on B is well defined. Let $x, y \in L(a)$, $x', y' \in L(a')$, with $q_2(x,x') = q_2(y,y')$. We may write $y = q_1(x,\alpha)$, $y' = q_1(y,\beta)$, with $\alpha \in L(a,0)$, $\beta \in L(a',0)$ (or $\alpha, \beta \in L(0,0)$ after appropriate identifications). The relation $q_2(x,x') = q_2(y,y')$ after application of (BL3) becomes $q_1(\alpha,\beta) = 0$ or $\alpha+\beta = 0$ in $L(0,0)$. Thus $f(y)+f'(y') = f(x)+\alpha+f'(x')+\beta = f(x)+f'(x')$ as needed.

The operation $+$ is clearly commutative and associative, and one can easily construct inverses. Thus we have a group B, and a projection from B to A_2. The kernel is $\{(0,f) : 0 \in A_2,\ f : L(0) \longrightarrow F,\ f$ is the identity on $F\}$. But $L(0)$ can be identified with $A_1 \times F$ and thus this kernel can be identified with the definable homomorphism group $\mathrm{Hom}(A_1,F)$.

Definition 6.5.6. *An abelian group A of prime exponent interpreted in a Lie coordinatized structure will be called* reflexive *if the natural map $A \longrightarrow A^{**}$ is an isomorphism.*

Lemma 6.5.7. *Let $\mathcal{M}$ be a Lie coordinatizable structure, A an abelian group interpreted in $\mathcal{M}$. Then the following are equivalent:*

1. *A is reflexive.*

2. *The natural map* $A \longrightarrow A^{**}$ *is injective.*
3. A *is definably isomorphic to a dual group* B^*.

Proof. (2) implies (1): As in the proof of Proposition 6.3.2, using finite approximations to compare cardinalities, we get $|A^{**}| \le |A^*| \le |A|$.

Evidently (3) implies (2) and (1) implies (3).

Lemma 6.5.8. *Let* $\mathcal{M}$ *be a Lie coordinatized structure, and* A_1, A_2 *abelian groups interpreted in* $\mathcal{M}$ *of prime exponent* p, *with* A_1 *reflexive. Let* F *be the field of order* p. *Then there is a natural correspondence between interpretable exact sequences* $0 \longrightarrow A_1 \longrightarrow B \longrightarrow A_2 \longrightarrow 0$ *and definable bilinear covers* L *of* $A_2 \times A_1^*$ *with structure group* $L(0,0) = F$, *up to the natural notions of isomorphism.*

Proof. This is largely contained in Lemmas 6.5.4 and 6.5.5, bearing in mind that the groups A_1, A_2 of Lemma 6.5.5 are A_1^* and A_2 in our present notation. It is also necessary to trace through the claim that these two correspondences reverse one another up to canonical isomorphism, a point which we leave to the reader.

The next proposition (after a preparatory lemma) states essentially that definable sections of bilinear covers are locally affinely bilinear, uniformly in a parameter: on a complete type, they respect the bilinear structure, up to translation. It would be interesting to get a global analysis. The proof requires that one of the groups be settled, a hypothesis which will eventually be seen to hold generally over an appropriate set of parameters; but the proof of the latter result requires the present one.

Notation 6.5.9

1. For $D \subseteq A \times B$, $s : A \times B \longrightarrow C$, *and* $a \in A$, *we write* D_a *for* $\{b \in B : (a,b) \in D\}$ *and* $s_a : D_a \longrightarrow C$ *for the map induced by* s.

2. For A *an* $\aleph_0$*-categorical group,* c *a parameter or finite set of parameters, let* A^c *be the smallest* c*-definable subgroup of* A *of finite index. This will be called the* principal component *of* A *over* c.

Notice the law

$$(A_1 \times A_2)^c = A_1^c \times A_2^c$$

and hence $(A^n)^c = (A^c)^n$.

Lemma 6.5.10. *Let* $\mathcal{M}$ *be Lie coordinatizable,* A *and* B *abelian groups and* $\pi : L \longrightarrow A \times B$ *a bilinear cover, all 0-definably interpreted in* $\mathcal{M}$, *with structure group* $F = L(0,0)$. *Let* $f : A' \longrightarrow A$ *be a generically surjective 0-definable map,* $D \subseteq A' \times B$ *the locus of a complete type over* $\operatorname{acl}(\emptyset)$ *of maximal rank, and* $s : D \longrightarrow L$ *a 0-definable section relative to* f, *i.e.* $s(a', b) \in L(fa', b)$ *on* D. *Assume*

1. *The group* B *is settled.*

2. *A and B have no 0-definable proper subgroups of finite index.*
3. $\operatorname{acl}(a') \cap B^* = \operatorname{dcl}(a') \cap B^*$ *for $a' \in A'$.*
4. *For $(a', b) \in D$, b lies in $B^{a'}$, the principal component of B over a'.*

Then for any $a' \in A'$, the map $s_{a'} : D_{a'} \longrightarrow L(fa')$ is affine; that is, it is induced by an affine map.

Proof. We may work over $\operatorname{acl}(\emptyset)$. As B is settled it follows from (3) that $D_{a'}$ is the locus of a complete type over $\operatorname{acl}(a')$.

Let D^* be

$$\{(a', b_1, b_2, b_3, b_4) : \begin{array}{l} \text{the first four coordinates are independent,} \\ \text{all } (a', b_i) \text{ lie in } D, \text{ and } b_4 = b_1 - b_2 + b_3\}. \end{array}$$

By Lemma 6.2.7 it suffices to check the relation

$$s(a', b_4) = s(a', b_1) - s(a', b_2) + s(a', b_3)$$

on D^*.

Fix (a', b_1, b_2, b_3, b_4) in D^*. We claim that there are elements a'_1, a'_2, a'_3 in A' such that

(i) $fa'_3 = fa'_1 + fa'_2$;
(ii) $\operatorname{tp}(a'_1 a'_2 a'_3 b_i)$ does not depend on $i = 1, \ldots, 4$;
(iii) $\operatorname{tp}(a'_i, \mathbf{b}) = \operatorname{tp}(a', \mathbf{b})$ for $i = 1, 2, 3$.

Granted this, we complete the computation as follows. Set $\alpha_i = \sum_j^1 (-1)^j s(a'_i, b_j)$, and $\beta_j = \sum_i^2 (-1)^i s(a'_i, b_j)$. By Lemma 6.5.3 we have $\sum_i^2 (-1)^i \alpha_i = \sum_j^1 (-1)^j \beta_j$. As $\sum_j (-1)^j b_j = 0$, $\alpha_i \in L(a_i, 0_B)$. Let $\theta : L(0_B) \longrightarrow A \times F$ be the canonical isomorphism: $\theta(x) = (a, \theta_2(x))$ for $x \in L(a, 0_B)$. Since we are working over $\operatorname{acl}(\emptyset)$, $\theta_2 : L(a, 0_b) \longrightarrow F$ is constant on the α_i, by condition (iii). Set $\alpha = \theta_2(\alpha_i)$. Thus

$$\theta_2 \left(\sum_i^2 (-1)^i \alpha_i \right) = (0, -\alpha).$$

Similarly, $\beta_j \in L(0_A, b_j)$ and under the isomorphism $\psi : L(0_A) \longrightarrow B \times F$ we get $\psi(\beta_j) = (b_j, \beta)$ for a fixed β, and thus

$$\psi \left(\sum_j^1 (-1)_j^\beta \right) = (0, 0).$$

But ψ and θ agree on $L(0,0)$, so the last two computations yield $\alpha = 0$, $\theta(\alpha_i) = 0$ in $L(0_B)$ and hence also in $L(a_i)$; that is, $\sum_j^1 s(a_i', b_j) = 0$, as required.

It remains to choose the elements a_1, a_2, a_3. Let $a = fa'$. Each b_i is in $B^{a'}$, and hence $(b_1, b_2, b_3) \in (B^3)^a$. By Lemma 6.3.9 there are a_1, a_2, a_3 in A with $a_1 + a_2 = a_3$ such that a_1, a_2, b_1, b_2, b_3 are independent and all a_i realize $\mathrm{tp}(a/\operatorname{acl}(b_1, b_2, b_3))$. Again by Lemma 6.3.9, each a_i lies in $A^{\mathbf{b}}$ for $\mathbf{b} = (b_1, b_2, b_3)$ and thus $(a_1, a_2) \in (A^2)^{\mathbf{b}}$, and again by Lemma 6.3.9 $\mathbf{b} \in (B^3)^{a_1, a_2}$. As $a_3 \in \operatorname{dcl}(a_1, a_2)$ we conclude

$$b_i \in (B^3)^{\mathbf{a}}$$

with $\mathbf{a} = (a_1, a_2, a_3)$, for $i = 1, 2, 3$, but also for $i = 4$, as $(B^3)^{\mathbf{a}}$ is a group.

Choose elements $a_i' \in A'$ above a_i for $i = 1, 2, 3$ satisfying $\mathrm{tp}(a_i') = \mathrm{tp}(a')$. These are not yet the desired elements. Choose $\mathbf{b}' = (b_1', b_2', b_3') \in (B^3)^{a_1', a_2', a_3'}$ with $\mathrm{tp}\mathbf{b}' = \mathrm{tp}\mathbf{b}$ and $\mathrm{rk}\,\mathbf{b}' = 3\mathrm{rk}\,B$. This is possible by Lemma 6.4.4 applied to B^3.

As B is settled $\mathrm{tp}(b_i'/a_1a_2a_3) = \mathrm{tp}(b_i/a_1a_2a_3)$. By the corollary to Lemma 6.3.7 B is 2-ary and thus $\mathrm{tp}(b_1'b_2'b_3'/a_1a_2a_3) = \mathrm{tp}(b_1b_2b_3/a_1a_2a_3)$. Applying an automorphism, we may suppose $b_i' = b_i$ for $i = 1, 2, 3$; this gives new values of a_i'. Condition (i) is satisfied, and as B is settled and $b_i \in B^{a_1', a_2', a_3'}$ for all i, condition (ii) is also satisfied. Finally, as B is settled and 2-ary we get condition (iii) as well.

The next proposition is the preceding lemma with its fourth hypothesis deleted.

Proposition 6.5.11. *Let $\mathcal{M}$ be Lie coordinatizable, A and B abelian groups and $\pi : L \longrightarrow A \times B$ a bilinear cover, all 0-definably interpreted in $\mathcal{M}$, with structure group $F = L(0,0)$. Let $f : A' \longrightarrow A$ be a generically surjective 0-definable map, $D \subseteq A' \times B$ the locus of a complete type over $\operatorname{acl}(\emptyset)$ of maximal rank, and $s : D \longrightarrow L$ a 0-definable section relative to f, i.e. $s(a', b) \in L(fa', b)$ on D. Assume*

1. *The group B is settled.*
2. *A and B have no 0-definable proper subgroups of finite index.*
3. *$\operatorname{acl}(a') \cap B^* = \operatorname{dcl}(a') \cap B^*$ for $a' \in A'$.*

Then for any $a' \in A'$, the map $s_{a'} : D_{a'} \longrightarrow L(fa')$ is affine, that is, is induced by an affine map.

Proof. As in the previous argument we work over $\operatorname{acl}(\emptyset)$.

In the notation of the preceding proof, our claim is the following: for (a', b_1, b_2, b_3, b_4) in D^*, we have $\sum_j (-1)^j s(a', b_j) = 0$. We claim first that there is $b \in D_{a'}$, independent from a', b_1, b_2, b_3, b_4, with $b - b_i \in B^{a',b}$ for $i = 1, 2, 3, 4$. As the principal component is a group, it suffices to deal with b_i for $i \leq 3$. As $D_{a'}$ is the locus of a complete type over the algebraically closed set $\operatorname{acl}(a')$, by the type amalgamation property it will suffice to deal with a single b_i. This case is covered by Lemma 6.4.5.

Now let a'' enumerate $[\operatorname{acl}(a', b) \cap B^*] \cup \{a', b\}$ and let f_1, f_2 be definable functions picking out a', b, respectively, from a''. Let A'' be the locus of a'' and let f'' be $f \circ f_1$. Let D' be the locus of $(a'', b_i - b)$. As B is settled, this set does not depend on i. Define $s' : D' \longrightarrow L$ by $s'(x, u) = s(f_1 x, u + f_2 x) - s(f_1 x, f_2 x)$ with the subtraction performed in $L(f_1 x)$. Then in the context of A'', D', s', hypothesis (3) again holds, and hypothesis (4) of the preceding lemma is achieved. Thus $s'_{a''}$ is affine. Furthermore, each $b_i - b$ lies in $D'_{a''}$, so we get

$$\begin{aligned} 0 = \quad \textstyle\sum_i (-1)^i s'(a'', b_i - b) \quad &= \sum_i (-1)^i [s(a', b_i) - s(a', b)] \\ &= \sum_i (-1)^i s(a', b_i) \end{aligned}$$

as claimed.

6.6 THE FINITE BASIS PROPERTY

Our objective in the present section is to pin down definability in groups rather thoroughly, as follows.

Proposition 6.6.1 (Finite Basis Property). *Let $\mathcal{M}$ be Lie coordinatizable and A an abelian group interpreted in $\mathcal{M}$. Then there is a finite collection of definable subsets D_i of A such that every definable subset of A is a boolean combination of the sets D_i, cosets of definable subgroups of A of finite index, and sets of rank less than* $\operatorname{rk}(A)$.

The proof will occupy most of this section.

Lemma 6.6.2. *Let $\mathcal{M}$ be Lie coordinatizable and A an abelian group interpreted in $\mathcal{M}$. The following are equivalent:*

1. *A is settled over $\emptyset$; i.e., we have*

   $$(*) \qquad \operatorname{tp}(a/\emptyset) \cup gtp(a/C \cap A^*) \Longrightarrow^* \operatorname{tp}(a/C)$$

   for $a \in A$ of maximal rank over the algebraically closed set C.
2. *For every finite set $C_\circ$ there is an algebraically closed set C containing $C_\circ$ such that for $a \in A$ of maximal rank over C the relation $(*)$ holds.*
3. *Every definable subset of A is a boolean combination of 0-definable sets, cosets of definable subgroups of finite index, and sets of rank less than* $\operatorname{rk} A$.

Proof. (2) is a weakening of (1), of course, and it implies (3), taking $C_\circ$ to be a defining set of parameters for the given definable set. Thus we are concerned only with the implication from (3) to (1).

Suppose on the contrary the implication

$$(*) \qquad \operatorname{tp}(a/\emptyset) \cup gtp(a/C \cap A^*) \Longrightarrow^* \operatorname{tp}(a/C)$$

fails to hold generically over some algebraically closed set C, which we may take to be finitely generated. Take a type p over C of full rank other than $\operatorname{tp}(a/C)$, compatible with the data in $(*)$. Let D be the locus of p. Now D lies in a single coset X of the principal component A^C. By (3), the type $\operatorname{tp}(a/C)$ contains the intersection of some definable coset with $\operatorname{tp}(a/\emptyset)$ up to a set of smaller rank; that is, there is a definable homomorphism h from A to a finite group, and a value c of h, such that $\operatorname{tp}(a/\emptyset) \cup \{h(x) = c\} \Longrightarrow^* \operatorname{tp}(a/C)$. Hence $\operatorname{rk}(D \cap h^{-1}[c]) < \operatorname{rk} A$, contradicting Lemma 6.4.4.

Thus Proposition 6.6.1 is equivalent to the statement that every group becomes settled over some finite set.

Lemma 6.6.3. *Let $\mathcal{M}$ be a Lie coordinatizable structure, and let $A_1, \ldots, A_n$ be settled groups 0-definably interpreted in $\mathcal{M}$, with no proper 0-definable subgroups of finite index. Then the product $A = \prod_i A_i$ is settled over* $\operatorname{acl}(\emptyset)$.

Proof. We may assume $n = 2$ and $\operatorname{acl}(\emptyset) = \operatorname{dcl}(\emptyset)$. Let C be algebraically closed, and $a = (a_1, a_2) \in A = A_1 \times A_2$ of maximal rank over C. Note that $A^* = A_1^* \times A_2^*$ and $C \cap A^* = (C \cap A_1^*) \times (C \cap A_2^*)$. Our claim is

$$\operatorname{tp}(a_1, a_2/\emptyset) \cup gtp(a_1/C \cap A_1^*) \cup gtp(a_2/C \cap A_2^*) \Longrightarrow^* \operatorname{tp}(a/C).$$

We have $\operatorname{tp}(a_2/\emptyset) \cup gtp(a_2/C \cap A_2^*) \Longrightarrow^* \operatorname{tp}(a_2/C)$, so to conclude it will suffice to show that $\operatorname{tp}(a_1/a_2) \cup gtp(a_1/C \cap A_1^*) \Longrightarrow^* \operatorname{tp}(a_1/a_2C)$, which is not quite what we have assumed. As A_1 is settled we have, in fact,

$$\operatorname{tp}(a_1/\emptyset) \cup gtp(a_1/\operatorname{acl}(a_2, C) \cap A_1^*) \Longrightarrow^* \operatorname{tp}(a_1/a_2C)$$

so it remains to understand $gtp(a_1/\operatorname{acl}(a_2, C) \cap A_1^*)$.

We apply Lemma 6.2.8 to A_1^* and A_2. Thus $\operatorname{acl}(a_2, C) \cap A_1^* = [\operatorname{dcl}(a_2) \cap A_1^*] + [C \cap A_1^*]$. As $gtp(a_1/\operatorname{dcl}(a_2) \cap A_1^*)$ is determined by $\operatorname{tp}(a_1/a_2)$, we are done.

Definition 6.6.4. *Let A be an abelian group interpreted in a Lie coordinatizable structure $\mathcal{M}$. A definable subset Q of A will be called* tame *if every definable subset of Q is the intersection with Q of a boolean combination of cosets of definable subgroups of finite index, and sets of lower rank. This notion is of interest only when* $\operatorname{rk} Q = \operatorname{rk} A$.

Lemma 6.6.5. *Let $\mathcal{M}$ be a Lie coordinatizable structure, and let A be an abelian group interpreted in $\mathcal{M}$.*

1. *If A contains a definable tame subset Q of full rank, then A is settled over some finite set.*
2. *If A contains a settled definable subgroup B of finite index then A is settled over some finite set.*

Proof. Ad 1. By Lemma 6.4.2 A can be covered by finitely many translates of Q. It suffices to work over a set of parameters C containing defining parameters for Q together with sufficiently many parameters of translation to cover A.

Ad 2. This is a special case of the first part, taking Q to be the locus of a 1-type over $\emptyset$ of full rank in B.

Lemma 6.6.6. *Let $\mathcal{M}$ be a Lie coordinatizable structure, and let A be an abelian group interpreted in $\mathcal{M}$. If A contains a finite subgroup*

$A_\circ$ for which the quotient $A/A_\circ$ is settled over a finite set, then A is settled over a finite set.

Proof. Let $A/A_\circ$ be settled over $C_\circ$. Take $a \in A$ of maximal rank over $C_\circ$ and let $\bar{a} = a + A_\circ$ viewed as an element of the quotient group. Then a is algebraic over $\bar{a}$. Take C containing $C_\circ$, independent from a, with the multiplicity of $\mathrm{tp}(a/\bar{a}, C)$ minimized. Let q be the type of a over C. We claim that the locus Q of q is tame, in other words that for C' containing C and independent from a over C, we have

$$q \cup \mathit{gtp}(a/\ \mathrm{acl}\,(C') \cap A^*) \Longrightarrow^* \ \mathrm{tp}(a/C').$$

In any case our choice of C ensures that

$$(*) \qquad \mathrm{tp}(a/\bar{a}, C) \Longrightarrow \ \mathrm{tp}(a/C').$$

Let q' be $\mathrm{tp}(\bar{a}/C)$. As the quotient group is settled,

$$q' \cup \mathit{gtp}(\bar{a}/(A/A_\circ)^*) \Longrightarrow^* \ \mathrm{tp}(\bar{a}/C).$$

Now $(A/A_\circ)^*$ may be identified with a definable subset of A^* and thus in view of $(*)$, $q \cup \mathit{gtp}(a/A^* \cap\ \mathrm{acl}\,(C')) \Longrightarrow^* \ \mathrm{tp}(a/C')$. Thus Q is tame and A is settled over some finite set.

Lemma 6.6.7. *Let $\mathcal{M}$ be a Lie coordinatizable structure, and let A be an abelian group interpreted in $\mathcal{M}$, A_1 a rank 1 $\mathrm{acl}\,(\emptyset)$-definable subgroup of A, and suppose $\mathrm{acl}\,(\emptyset) \cap A^* = (0)$ (i.e. A has no 0-definable subgroup of finite index), and $\mathrm{acl}\,(\emptyset) \cap A_1 = (0)$. Suppose a is an element of A of full rank over $\emptyset$, with $a \in\ \mathrm{acl}\,(a/A_1, c)$ for some c independent from a/A_1 (an element of A/A_1). Then there is an $\mathrm{acl}\,(\emptyset)$-definable subgroup A_2 with $A = A_1 \oplus A_2$.*

Proof. Let Q be the locus of a over $\mathrm{acl}\,(c)$. With $n = \mathrm{rk}\, A$, the hypotheses give $\mathrm{rk}\,(a/c) = n - 1$. Let $S = \mathit{Stab}(Q)$. Then S is a subgroup of A of rank $n - 1$ (Lemma 6.2.5), and Q lies in a single coset of S. We claim that $S \cap A_1$ is finite.

If $S \cap A_1$ is infinite, let $b \in S \cap A_1$ have rank 1 over $\emptyset$. By Lemma 6.2.5, part (4), we may take $b \in \mathit{Stab}_\circ Q$. Then there is $a' \in Q$ of rank $n - 1$ over b, c such that $a'' = a' - b \in Q$. Thus $\mathrm{tp}(a''/c) = \ \mathrm{tp}(a/c)$ and $a'' \in\ \mathrm{acl}\,(a''/A_1, c)$; that is, $a' - b \in\ \mathrm{acl}\,(a'/A_1, c)$ and hence $b \in\ \mathrm{acl}\,(a', c)$. This contradicts the independence of a', b over c.

Now by Proposition 6.2.3 there is an $\mathrm{acl}\,(\emptyset)$-definable subgroup A_2 commensurable with S. It follows easily that $A_1 \cap A_2 = 0$ and $A_1 \oplus A_2$ is a definable subgroup of A of finite index defined over $\mathrm{acl}\,(\emptyset)$, and thus $A_1 \oplus A_2 = A$.

Lemma 6.6.8. *Let $\mathcal{M}$ be a Lie coordinatizable structure, let*

$$(0) \longrightarrow A_1 \longrightarrow B \longrightarrow A_2 \longrightarrow (0)$$

be an exact sequence interpreted in $\mathcal{M}$, with A_2 settled over $\emptyset$, and let $\pi : L \longrightarrow A_2 \times A_1^*$ be the corresponding bilinear cover. Assume $\operatorname{acl}(\emptyset) \cap A_1 = (0)$ and $\operatorname{acl}(\emptyset) \cap A_2^* = (0)$. Let C be algebraically closed, and let D be a complete type over C in A_2 of maximal rank. Let $a^* \in C \cap A_1^*$ be generic in A_1^* over $\emptyset$, and suppose $g : D \longrightarrow L(a^*)$ is a C-definable section, that is: $g(a) \in L(a, g_2(a))$ for some function g_2; here we use the standard representation of the bilinear cover L, and, in particular, $g_2(a)$ induces a^* on A_1. Then there is a C-definable homomorphism j from A_2 to a finite group, such that for any $b \in B$ with $b/A_1 \in D$, the quantity

$$[g_2(b/A_1)](b)$$

is determined by $j(b)$.

Proof. We apply Proposition 6.5.11 with the groups A_1^* and A_2 here playing the role of the A of B from that proposition. For A' we take the locus of C (as an enumerated set) over $\emptyset$ and for the D of Proposition 6.5.11 we take the locus of (C, d) with d a realization of the type D from the present Lemma. The function f picks out the element corresponding to our a^* in any realization of the type of the sequence C. In particular, in the notation of Proposition 6.5.11, our present C is a typical element a'. Now applying Proposition 6.5.11, the section g is affine. In other words, if A_2' is the principal component A_2^C, then A_2' is a C-definable subgroup of finite index in A_2, and there is a C-definable homomorphism $h : A_2' \longrightarrow L(a^*)$ such that for $d, d' \in D$ we have $g(d) - g(d') = h(d - d')$. We may write $h(a) = (h_1(a), h_2(a))$ and as g is a section we find $h_1(a) = a$.

Let $B' = \{b \in B : b/A_1 \in A_2'\}$. Define a map $j_\circ$ from B' to the prime field F by $j_\circ(b) = [h_2(b/A_1)](a)$. We will show that $j_\circ$ is a homomorphism.

As h is a homomorphism, $j_\circ(b + b')$ is the second component of $h(b/A_1) + h(b'/A_1)$, evaluated at $b + b'$; by the definition of the operation q_2 on L, this is $h_2(b/A_1)(b) + h_2(b'/A_1)(b') = j_\circ(b) + j_\circ(b')$.

Thus $j_\circ$ is a homomorphism. Let B'' be its kernel, and let j be the canonical homomorphism from B to B/B''. We claim that this j works. Suppose $b_1, b_2 \in B$, $b_i/A_1 \in D$, and $j(b_1) = j(b_2)$. Then $b_1 - b_2 \in B''$ and $j_\circ(b_1 - b_2) = 0$, so $g_2(b_1/A_1)(b_1) = g_2(b_2/A_1)(b_2)$ is determined by the value of j.

Lemma 6.6.9. *Let $\mathcal{M}$ be a Lie coordinatizable structure, let A be 0-definably interpretable in $\mathcal{M}$, A_1 a definable subgroup, and suppose* *that A_1 is settled. Suppose there is a 0-definable type of full rank in A with locus Q such that for any C and any $a \in Q$ with a/A_1 of maximal rank over C,*

$$(*) \qquad \operatorname{tp}(a/(a/A_1)) \cup gtp(a/\operatorname{acl}(C) \cap A^*) \implies \operatorname{tp}(a/(a/A_1), C).$$

Then Q is tame in A, and hence A is settled over some finite set.

Proof. Let $\bar{a} = a/A_1$, and let $q = \operatorname{tp}(\bar{a}/C)$. Then

$$q \cup gtp(\bar{a}/\operatorname{acl}(C) \cap (A/A_1)^*) \implies^* \operatorname{tp}(\bar{a}/C).$$

As $(A/A_1)^*$ can be identified with a definable subset of A^*, this together with $(*)$ yields

$$\operatorname{tp}(a/C) \cup gtp(a/\operatorname{acl}(C) \cap A^*) \implies^* \operatorname{tp}(a/C).$$

Thus Q is tame.

The following lemma is critical.

Lemma 6.6.10. *Let $\mathcal{M}$ be a Lie coordinatizable structure, let A be 0-definably interpretable in $\mathcal{M}$, with $\operatorname{acl}(\emptyset) \cap A^* = (0)$, and let A_1 be a 0-definable subgroup of A which is part of a stably embedded linear geometry J in $\mathcal{M}$, not of quadratic type. Assume that A/A_1 is settled and that there is no $\operatorname{acl}(\emptyset)$-definable complement to A_1 in A. Then A is settled over some finite set.*

Proof. We will arrive at the situation of the previous lemma, relative to some finite set of auxiliary parameters $C_\circ$ (so the sets C of the previous lemma should contain $C_\circ$). We work over $\operatorname{acl}(\emptyset)$.

Let $\bar{A} = A/A_1$. Fix an element $a \in A$ of maximal rank, and let $\bar{a} = a/A_1$. Let $S = a + A_1$ viewed as an affine space over A_1. Let $S^{*\circ}$ be the prime field affine dual defined in §2.3. Call a set C *basal* if C is algebraically closed and independent from a. Then we claim

$$\text{For } C \text{ basal, } a \text{ is not in } \operatorname{acl}(\bar{a}, C, J).$$

Otherwise, take $a \in \operatorname{acl}(\bar{a}, C, d_1, \ldots, d_k)$ with $d_i \in J$ and k minimal. Then the sequence

$$\bar{a}, C, d_1, \ldots, d_k$$

is independent. We apply Lemma 6.6.7, noting that $\operatorname{acl}(\emptyset) \cap A_1 = (0)$ by our hypothesis. Then Lemma 6.6.7 produces a complement to A_1 in A, a contradiction. Also, by Lemma 6.2.8 $\operatorname{acl}(\bar{a}, C) \cap J = \operatorname{dcl}(\bar{a}, C) \cap J$. Now Lemma 2.3.17 applies, giving

$$\operatorname{tp}(a/\bar{a}, \operatorname{dcl}(\bar{a}, C) \cap S^{*\circ}) \implies \operatorname{tp}(a/\bar{a}, C).$$

Let $T(C)$ be $\operatorname{dcl}(C) \cap S^{*\circ}$. We need to examine $T(C)$ more closely for basal C. For $f \in A_1^*$ let $S^{*\circ}(f)$ be the set of elements of $S^{*\circ}$ lying above f; this is an affine space over the prime field $F_\circ$, of dimension 1. Let $A_1^*(C) = \operatorname{acl}(C) \cap A_1^*$. Let $T_1(C) = \operatorname{dcl}(C, \bar{a}) \cap \bigcup\{S^{*\circ}(f) : f \in A_1^*(C)\}$. We claim that for some basal C, for all C' containing C, we have

$$(*) \qquad T(C') = T(C) + T_1(C')$$

and hence $T(C') \subseteq \operatorname{dcl}(\bar{a}, T(C), T_1(C'))$.

Let $\beta(C) = \{x \in A_1^*(\bar{a}) : \text{for some } y \in A_1^*(C),\ S^{*\circ}(x+y) \cap T(C) \neq \emptyset\}$. Choose C basal with $\beta(C)$ maximal. Let $C' \supseteq C$ be basal, $t \in T(C')$. Then $t \in S^{*\circ}(x+y)$ for some $x \in A_1^*(\bar{a})$, $y \in A_1^*(C')$. So $t \in \beta(C') - \beta(C)$. Thus there is $y' \in A_1^*(C)$ and $t' \in T(C) \cap S^{*\circ}(x+y')$. Then $t - t' \in T(C') \cap S^{*\circ}(y - y') \subseteq T_1(C')$ and as $t = t' + (t - t')$, our claim is proved.

Using quantifier elimination in $(J, S, S^{*\circ})$, the claim gives

$$\operatorname{tp}(a/\bar{a}, T(C)) \cup \operatorname{tp}(a/\bar{a}, T_1(C')) \Longrightarrow \operatorname{tp}(a/\bar{a}, T(C')).$$

Now in order to show

$$\operatorname{tp}(a/C') \cup gtp(a/\operatorname{acl}(C') \cap A^*) \Longrightarrow^* \operatorname{tp}(a/C')$$

it will suffice to check that

$$(**) \qquad \operatorname{tp}(a/\bar{a}) \cup gtp(a/C' \cap A^*) \Longrightarrow \operatorname{tp}(a/\bar{a}, T_1(C')).$$

We fix C' and let $\pi : L \longrightarrow \bar{A} \times A_1^*$ be the semi-dual cover corresponding to $(0) \longrightarrow A_1 \longrightarrow A \longrightarrow \bar{A} \longrightarrow (0)$. Let D' be the locus of $\bar{a}$ over C'. If $t \in T_1(C')$, then $(\bar{a}, t) \in L$; let $a^* = \pi_2(\bar{a}, t)$ be the induced element of A^*. Then $a^* \in C' \cap A_1^*$. As $t \in \operatorname{dcl}(\bar{a}, C')$ we may write $(\bar{a}, t) = g(\bar{a}) = (\bar{a}, g_2(\bar{a}))$, where $g : D' \longrightarrow L(a^*)$ is a C'-definable section. By Lemma 6.6.8 there is a C'-definable homomorphism j onto a finite group whose values determine $g_2(\bar{u})(u)$ for $u \in A$, $\bar{u} \in D'$. By definition $gtp(a/C' \cap A^*)$ determines the value of $j(a)$ and hence of $t(a)$. Claim $(**)$ follows.

Proof of Proposition 6.6.1 We proceed by induction on the length of a maximal chain of $\operatorname{acl}(\emptyset)$-definable subgroups. We may work over $\operatorname{acl}(\emptyset)$. If A contains a finite subgroup defined over $\operatorname{acl}(\emptyset)$ we may apply induction and Lemma 6.6.6. Accordingly we may suppose $\operatorname{acl}(\emptyset) \cap A = (0)$. Similarly we may suppose $\operatorname{acl}(\emptyset) \cap A^* = (0)$, using Lemma 6.6.5, part (2).

Now A contains an $\operatorname{acl}(\emptyset)$-definable rank 1 subgroup A_1 which is part of a basic linear geometry J (Lemmas 6.2.6, 6.2.11). If A_1 has an $\operatorname{acl}(\emptyset)$-definable complement A_2 then we may assume both A_1 and A_2 are settled, and then $A = A_1 \oplus A_2$ is settled. Accordingly we may suppose that A_1 is not complemented. Now by induction A/A_1 is settled over some set C and after enlarging C if necessary, we may assume that the associated linear geometry is not quadratic (adding an element of the quadratic space Q, if needed). Now the previous lemma applies.

The following is another version of the finite basis property.

Proposition 6.6.11

Let $\mathcal{M}$ be Lie coordinatizable and A an abelian group interpreted in $\mathcal{M}$. Then there is a finite collection D_i of definable subsets of A, such that every definable subset of A is a boolean combination of translates of the D_i together with cosets of definable subgroups.

Proof. We proceed by induction on $\operatorname{rk}(A)$. Let D_i be a finite list of definable sets including all the definable sets associated correspondingly to all $\operatorname{acl}(\emptyset)$-definable subgroups of smaller rank. In addition let C be a finite set over which A is settled, and assume that all C-definable sets occur as well in the list (D_i). We claim this suffices.

As A is settled over C, it will suffice to consider definable subsets D of A of rank less than $\operatorname{rk} A$. Such a set lies in the union of a finite number of cosets of $\operatorname{acl}(\emptyset)$-definable subgroups of A of rank less than $\operatorname{rk}(A)$, by Lemma 6.2.5 and Proposition 6.2.3. We may therefore assume that D lies in one such coset, and since our problem is invariant under translation, we may even assume D lies in an $\operatorname{acl}(\emptyset)$-definable subgroup of smaller rank, and conclude.