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Introduction

1.1 THE SUBJECT

In the present monograph we develop a structure theory for a class of finite structures whose description lies on the border between model theory and group theory. Model theoretically, we study large finite structures for a fixed finite language, with a bounded number of 4-types. In group theoretic terms, we study all sufficiently large finite permutation groups which have a bounded number of orbits on 4-tuples and which are k-closed for a fixed value of k. The primitive case is analyzed in [KLM; cf. Mp2]. The treatment of the general case involves application of model theoretic ideas along lines pioneered by Lachlan.

We show that such structures fall into finitely many classes naturally parametrized by "dimensions" in the sense of Lachlan, which approximate finitely many infinite limit structures (a version of Lachlan's theory of shrinking and stretching), and we prove uniform finite axiomatizability modulo appropriate axioms of infinity (quasifinite axiomatizability). We also deal with issues of effectivity. At our level of generality, the proofs involve the extension of the methods of stability theory—geometries, orthogonality, modularity, definable groups—to this somewhat unstable context. Our treatment is relatively self-contained, although knowledge of the model theoretic background provides considerable motivation for the results and their proofs. The reader who is more interested in the statement of precise results than in the model theoretic background will find them in the next section.

On the model theoretic side, this work has two sources. Lachlan worked out the theory originally in the context of stable structures which are homogeneous for a finite relational language [La], emphasizing the parametrization by numerical invariants. Zilber, on the other hand, investigated totally categorical structures and developed a theory of finite approximations called "envelopes," in his work on the problems of finite axiomatizability. The class of $\aleph_0$-categorical, $\aleph_0$-stable structures provides a broad model theoretic context to which both aspects of the theory are relevant. The theory was worked out at this level in [CHL], including the appropriate theory of envelopes. These were used in particular to show that the corresponding theories are not finitely axiomatizable, by Zilber's method. The basic tool used in [CHL], in accordance with Shelah's general approach to stability theory and geometrical refinements due to Zilber, was a "coordinatization" of an arbitrary structure in the class by a tree of standard coordinate geometries (affine or projective over finite fields, or degenerate. Other classical geometries involving quadratic forms were conspicuous only by their absence at this point.

The more delicate issue of finite axiomatizability modulo appropriate "axioms of infinity," which is closely connected with other finiteness problems as well as problems of effectivity, took some time to resolve. In [AZ1] Ahlbrandt and Ziegler isolated the relevant combinatorial property of the coordinatizing geometries, which we refer to here as "geometrical finiteness," and used it to prove quasifinite axiomatizability in the case of a single coordinatizing geometry. The case of $\aleph_0$-stable, $\aleph_0$-categorical structures in general was treated in [HrTC].

The class of *smoothly approximable* structures was introduced by Lachlan as a natural generalization of the class of $\aleph_0$-categorical $\aleph_0$-stable structures, in essence taking the theory of envelopes as a definition. Smoothly approximable structures are $\aleph_0$-categorical structures which can be well approximated by finite structures in a sense to be given precisely in §2.1. One of the achievements of the structure theory for $\aleph_0$-categorical $\aleph_0$-stable theories was the proof that they are smoothly approximable in Lachlan's sense. While this was useful model theoretically, Lachlan's point was that in dealing with the model theory of large finite structures, one should also look at the reverse direction, from smooth approximability to the structure theory. We show here, confirming this not very explicitly formulated conjecture of Lachlan, that the bulk of the structure theory applies to smoothly approximable structures, or even, as stated at the outset, to sufficiently large finite structures with a fixed finite language, having a bounded number of 4-types.

Lachlan's project was launched by Kantor, Liebeck, and Macpherson in [KLM] with the classification of the primitive smoothly approximable structures in terms of various more or less classical geometries (the least classical being the "quadratic" geometry in characteristic 2, described in §2.1.2). These turn up in projective, linear, and affine flavors, and in the affine case there are some additional nonprimitive structures that play no role in [KLM] but will be needed here ("affine duality," §2.3). Bearing in mind that any $\aleph_0$-categorical structure can be analyzed to some degree in terms of its primitive sections, the results of [KLM] furnish a rough coordinatization theorem for smoothly approximable structures. This must be massaged a bit to give the sort of coordinatization that has been exploited previously in an ω-stable context. We will refer to a structure as “Lie coordinatizable” if it is bi-interpretable with a structure which has a nice coordinatization of the type introduced below. Lie coordinatizability will prove to be equivalent to smooth approximability, in one direction largely because of [KLM], and in the other by the analog of Zilber’s theory of envelopes in this context. One tends to work with Lie coordinatizability as the basic technical notion in the subject. The analysis in [KLM] was in fact carried out for primitive structures with a bound on the number of orbits on 5-tuples, and in [Mp2] it was indicated how the proof may be modified so as to work with a bound on 4-tuples. (Using only [KLM], we would also be forced to state everything done here with 5 in place of 4.)

In model theory, techniques for going from a good description of primitive pieces to meaningful statements about imprimitive structures generally fall under the heading of “geometrical stability theory,” whose roots lie in early work of Zilber on $\aleph_1$-categorical theories, much developed subsequently. Though the present theory lies slightly outside stability theory (it can find a home in the more recent developments relating to simple theories), geometrical stability theory provided a very useful template [Bu, PiGS].

Before entering into greater detail regarding the present work, we make some comments on the Galois correspondence between structures and permutation groups implicit in the above, and on its limitations.

Let X be a finite set. There is then a Galois correspondence between subgroups of the symmetric group $Sym(X)$ on X, and model theoretic structures with universe X, associating to a permutation group the invariant relations, and to a structure its automorphism group. This correspondence extends to $\aleph_0$-categorical structures ([AZ1, Introduction], [CaO]).

When we consider infinite families of finite structures in general, or a passage to an infinite limit, this correspondence is not well behaved. For instance, the automorphism group of a large finite random graph of order n (with constant and nontrivial edge probability) is trivial with probability approaching 1 as n goes to infinity, while the natural model theoretic limit is the random countable graph, which has many automorphisms.

It was shown in [CHL], building on work of Zilber for totally categorical structures, that structures which are both $\aleph_0$-categorical and $\aleph_0$-stable can be approximated by finite structures simultaneously in both categories. Lachlan emphasized the importance of this property, which will be defined precisely in §2.1, and proposed that the class of structures with this property, the *smoothly approximable structures*, should be amenable to a strong structure theory, appropriately generalizing [CHL]. Moreover, Lachlan suggested that the direction of the analysis can be reversed, from the finite to the infinite: one could classify the large finite structures that appear to be "smooth approximations" to an infinite limit, or in other words, classify the families of finite structures which appear to be Cauchy sequences both as structures and as permutation groups. This line of thought was suggested by Lachlan's work on stable finitely homogeneous structures [La], much of which predates the work in [CHL], and provided an additional ideological framework for that paper.

In the context of stable finitely homogeneous structures this analysis in terms of families parametrized by dimensions was carried out in [KL] (cf. [CL, La]), but was not known to go through even in the totally categorical case. Harrington pointed out that this reversal would follow immediately from compactness if one were able to work systematically within an elementary framework [Ha]. This idea is implemented here: we will replace the original class of "smoothly approximable structures" by an elementary class, a priori larger. Part of our effort then goes into developing the structure theory for the ostensibly broader class.

From the point of view of permutation group theory, it is natural to begin the analysis with the case of finite primitive structures. This was carried out using group theoretic methods in [KLM], and we rely on that analysis. However, there are model theoretic issues which are not immediately resolved by such a classification, even for primitive structures. For instance, if some finite graphs G_n are assumed to be primitive, and to have a uniformly bounded number of 4-types, our theory shows that an ultraproduct G^* of the G_n is bi-interpretable with a Grassmannian structure, which does not appear to follow from [KLM] by direct considerations. The point here is that if G_n is "the same as" a Grassmannian structure in the category of permutation groups, then it is bi-interpretable with such a structure on the model theoretic side. To deal with families, one must deal (at least implicitly) with the uniformity of such interpretations; see §8.3, and the sections on reducts. It is noteworthy that our proof in this case actually passes through the theory for imprimitive structures: any nonuniform interpretation of a Grassmannian structure on G_n gives rise to a certain structure on G^*, a reduct of the structure which would be obtained from a uniform interpretation, and one argues that finite approximations (on the model theoretic side) to G^* would have too many automorphisms. In other words, we can obtain results on uniformity (and hence effectivity) by ensuring that the class for which we have a structure theory is closed under reducts. This turns out to be a very delicate point, and perhaps the connection with effectivity explains why it should be delicate.

1.2 RESULTS

A rapid but thorough summary of this theory was sketched in [HrBa], with occasional inaccuracies. For ease of reference we now repeat the main results of the theory as presented there, making use of a considerable amount of specialized terminology which will be reintroduced in the present work. The various finiteness conditions referred to are all given in Definition 2.1.1.

Theorem 1 (Structure Theory)

Let $\mathcal{M}$ be a Lie coordinatizable structure. Then $\mathcal{M}$ can be presented in a finite language. Assuming $\mathcal{M}$ is so presented, there are finitely many definable dimension invariants for $\mathcal{M}$ which are infinite, up to equivalence of such invariants. If C is a set of representatives for such definable dimension invariants, then there is a sentence $\varphi = \varphi_{\mathcal{M}}$ with the following properties:

1. *Every model of φ in which the definable dimension invariants of C are well-defined is determined up to isomorphism by these invariants.*
2. *Any sufficiently large reasonable sequence of dimension invariants is realized by some model of φ.*
3. *The models of φ for which the definable dimension invariants of C are well-defined embed homogeneously into $\mathcal{M}$ and these embeddings are unique up to an automorphism of $\mathcal{M}$.*

There are a considerable number of terms occurring here which will be defined later. Readers familiar with "shrinking" and "stretching" in the sense of Lachlan should recognize the situation. Definable dimension invariants are simply the dimensions of coordinatizing geometries which occur in families of geometries of constant dimension; when the appropriate dimensions are not constant within each family, the corresponding invariants are no longer well-defined. A dimension invariant is reasonable if its parity is compatible with the type of the geometry under consideration; in particular, infinite values are always reasonable.

The statements of the next two theorems are slight deformations of the versions given in [HrBa]. We include more clauses here, and we use definitions which vary slightly from those used in [HrBa].

Theorem 2 (Characterizations)

The following conditions on a model $\mathcal{M}$ are equivalent:

1. $\mathcal{M}$ *is smoothly approximable.*
2. $\mathcal{M}$ *is weakly approximable.*
3. $\mathcal{M}$ *is strongly quasifinite.*

4. *$\mathcal{M}$ is strongly 4-quasifinite.*
5. *$\mathcal{M}$ is Lie coordinatizable.*
6. *The theory of $\mathcal{M}$ has a model $\mathcal{M}^*$ in a nonstandard universe whose size is an infinite nonstandard integer, and for which the number of internal n-types $s^*_n(\mathcal{M}^*)$ satisfies*

$$s^*_n(\mathcal{M}^*) \leq c^{n^2}$$

for some finite c, and in which internal n-types and n-types coincide. (Here n varies over standard natural numbers.)

The class characterized above is not closed under reducts. For the closure under reducts we have:

Theorem 3 (Reducts)
The following conditions on a model $\mathcal{M}$ are equivalent:

1. *$\mathcal{M}$ has a smoothly approximable expansion.*
2. *$\mathcal{M}$ has a weakly approximable expansion.*
3. *$\mathcal{M}$ is quasifinite.*
4. *$\mathcal{M}$ is 4-quasifinite.*
5. *$\mathcal{M}$ is weakly Lie coordinatizable*
6. *The theory of $\mathcal{M}$ has a model $\mathcal{M}^*$ in a nonstandard universe whose size is an infinite nonstandard integer, and for which the number of internal n-types $s^*_n(\mathcal{M}^*)$ satisfies:*

$$s^*_n(\mathcal{M}^*) \leq c^{n^2}$$

for some finite c. (Here n varies over standard natural numbers.)

On the other hand, once the class is closed under reducts it is closed under interpretation, hence:

Theorem 4 (Interpretations)
The closure of the class of Lie coordinatizable structures under interpretation is the class of weakly Lie coordinatizable structures.

An earlier claim that the class of Lie coordinatizable structures is closed under interpretations was refuted by an example of David Evans which will be given below.

Theorem 5 (Decidability)
For any k and any finite language, the theory of finite structures with at most k 4-types is decidable, uniformly in k. The same applies in an extended language with dimension comparison quantifiers and Witt defect quantifiers. Thus one can decide effectively whether a sentence in such a language has a finite model with a given number of 4-types.

This is a distant relation of a family of theorems in permutation group theory giving explicit classifications of primitive permutation groups with very few 2-types. Dimension comparison quantifiers do not allow us to quantify over the dimensions of spaces, but they allow us to compare the dimensions of any two geometries. Witt defect quantifiers are more technical (§2.1, Definition 2.1.1).

Theorem 6 (Finite structures)

Let L be a finite language and k a natural number. Then the class of finite L-structures having at most k 4-types can be divided into families $\mathcal{F}_1, \ldots, \mathcal{F}_n$ for some effectively computable n such that

1. *Each family $\mathcal{F}_i$ is finitely axiomatizable in a language with dimension comparison and Witt defect quantifiers.*
2. *Each family $\mathcal{F}_i$ is associated with a single countable Lie coordinatizable structure $\mathcal{M}_i$. The family $\mathcal{F}_i$ is the class of "envelopes" of $\mathcal{M}_i$, which are the structures described in Theorem 1, parametrized by freely varying definable dimension invariants (above a certain minimal bound, with appropriate parity constraints).*
3. *For $\mathcal{M}$, $\mathcal{N}$ in $\mathcal{F}_i$, if the dimension invariants satisfy $d(\mathcal{M}) \leq d(\mathcal{N})$ then there is a homogeneous embedding of $\mathcal{M}$ in $\mathcal{N}$, unique up to an automorphism of $\mathcal{N}$.*
4. *Membership in each of the families $\mathcal{F}_i$ (and in particular, in their union) can be determined in polynomial time, and the dimension invariants can be computed in polynomial time. Thus the isomorphism problem in the class of finite structures with a bounded number of types can be solved in polynomial time.*
5. *The cardinality of an envelope of dimension d is an exponential polynomial in d; specifically, a polynomial in exponentials of the entries of d (with bases roughly the sizes of the base fields involved). The structure $N_i(d)$ which is the member of $\mathcal{F}_i$ of specified dimensions d can be constructed in time which is polynomial in its cardinality.*

Theorem 7 (Model Theoretic Analysis)

The weakly Lie coordinatizable structures $\mathcal{M}$ are characterized by the following nine model theoretic properties:

- LC1. *$\aleph_0$-categoricity.*
- LC2. *Pseudofiniteness.*
- LC3. *Finite rank.*
- LC4. *Independent type amalgamation.*
- LC5. *Modularity in $\mathcal{M}^{\mathrm{eq}}$.*
- LC6. *The finite basis property in groups.*
- LC7. *General position of large 0-definable sets.*

LC8. *$\mathcal{M}$ does not interpret the generic bipartite graph.*

LC9. *For every vector space V interpreted in $\mathcal{M}$, the definable dual V^* (the set of all definable linear maps on V) is interpreted in $\mathcal{M}$.*

Some of these notions were first introduced in [HrBa], sometimes using different terminology. In particular, the rank function is not a standard rank function, the finite basis property in groups (or "linearity") reduces to local modularity in the stable case, and the general position (or "rank/measure") property is an additional group theoretic property that arises in the unstable case, when groups tend to have many definable subgroups of finite index. The eighth condition is peculiarly different from the ninth. This is a corrected version of Theorem 6 of [HrBa].

David Evans made several contributions to the theory given here, notably the observation that the orientation of quadratic geometries is essential, and bears on the problem of reducts. The detection of all such points is critical. Evans also gave a treatment of weak elimination of imaginaries in linear geometries, in [EvSI].

We will say a few words about the development of this material, using technical notions explained fully in the text. The first author on reading [KLM] understood that one could extract stably embedded geometries from the analysis of primitive smoothly approximable structures given there, and that the group theory gives a decent orthogonality theory (but the orthogonality theory given here will be based more on geometry than on group theory). These ingredients seemed at first to be enough to reproduce the Ahlbrandt–Ziegler analysis, after the routine verification that the necessary geometrical finiteness principle follows from Higman's lemma; all of this follows the lead of [AZ1], along the lines developed in [HrTC]. An attempt to implement this strategy failed, in part because at this stage there was no hint of "affine duality."

The second author then produced affine duality and gave a complete proof of quasifinite axiomatizability, introducing some further modifications of the basic strategy, notably canonical projectives and a closer analysis of the affine case. The theme in all of this is that one should worry even more about the interactions of affine geometries than one does in the stable case. This can perhaps be explained by the following heuristic. Only the projective geometries are actually coordinatizing geometries; the linear and affine geometries are introduced to analyze definable group structures, in keeping with the general philosophy that structures are built from basic 1-dimensional pieces, algebraic closure, and definable groups. Here higher dimensional groups are not needed largely because of the analog of 1-basedness, referred to below as the finite basis property. The developments that go beyond what is needed for quasifinite axiomatizability are all due to the second author. The extension of a considerable body of geometric stability theory to this context is essential to further developements. The high points of these developments, as far as applications are concerned, are the analysis of *reducts* and its applications to issues of *effectivity*. It may be noted also that the remarkable quadratic geometries have been known for some time, and play an essential role in [KLM], in particular. In our view they add considerably to the appeal of the theory.

The treatment of reducts requires a considerably more elaborate transference of techniques of stability theory to this unstable setting than would be required for the quasifinite axiomatizability alone. This would not be indispensable for the treatment of structures already equipped with a Lie coordinatization; but to apply these results to classes which are closed under interpretation requires the ability to recognize an appropriate coordinatization, starting from global properties of the structure; thus one must find the model theoretic content of the property of coordinatizability by the geometries on hand.

Our subject has also been illuminated by recent developments in connection with Shelah's "simple theories," and is likely to be further illuminated by that theory.

Various versions of this material, less fully worked out, have been in circulation for a considerable period of time (beginning with notes written in Spring 1990) and have motivated some of the work in simple theories. In particular, versions of sections 5.1 [KiP], 5.4, and 6.1 [PiGr] have been obtained in that very general context; all of this rests on the theoretical foundation provided by the original paper of Shelah [ShS] and subsequent work by Kim [Ki].

Some comments on the relationship of this theory to Shelah's "simple theories" are in order. Evidently a central preoccupation of the present work is the extension of methods of stability theory to an unstable context. Stability theory is a multilayered edifice. The first layer consists of a theory of rank and the related combinatorial behavior of definable sets. The next layer includes the theory of orthogonality, regular types, and modularity, and was initially believed to be entirely dependent on the foundational layer in its precise form. One of the key conclusions of the present work is that is possible to recover the second "geometric model theory" layer over an unstable base. Because we have $\aleph_0$-categoricity and finiteness of the rank, our basic rank theory becomes as simple as possible; nonetheless, almost all of the "second-level" phenomena connected with simplicity appear in our context with their full complexity—the main exception being the Lascar group. It was perhaps this combination of circumstances that facilitated a very successful generalization of the "geometric theory" to the simple context, once the first layer was brought into an adequate state by Kim's thesis [KiTh].

As far as the present work is concerned, the development of a sufficiently general theory was often due to necessity rather than insight. For example, if we—or the creator of the finite simple groups—had been able to exclude from consideration the orthogonal geometries in characteristic 2, we would have had a considerably simpler theory of generics in groups, with $Stab = Stab_\circ$ (cf. §6.1, Definition 6.1.9, and the Example following). Such a simplified theory would have been much less readily generalizable to the simple context; in addition, under the same hypothesis, this simplified theory would have largely obviated the need for the theory of the semi-dual cover.

A number of features of the theory exposed here have been generalized with gratifying success to the context of simple theories, but some have not. On the positive side, one has first of all the theorem which we originally called *the independence theorem*. This name has become standard in the literature, although in the present manuscript it was eventually renamed "the type amalgamation property." In any case this is still a misnomer, as this amalgamation involves a triple over a base rather than a pair. Compare the following "homological" description. Let $I(n)$ be the space of n-types, over some fixed base, of *independent* n-tuples (whose elements are themselves finite sequences of elements). We have "projection" maps $\pi_i : I(n) \to I(n-1)$ obtained by deletion of one coordinate. The uniqueness of forking in stability theory is the statement that the induced map $I(2) \to I(1)^2$ is *injective*. We replace this by an *exactness* property, characterizing the image of $I(3)$ in $I(2)^3$ by minimal coherence conditions.

The first proof found for this theorem consisted of inspection in the 1-dimensional case, followed by an induction on rank. In the course of related work, an abstract proof was found, assuming finite simplicity rank and definability of the rank. This proof was later generalized by Kim and Pillay, and together with their realization of the relevance of the Lascar group, it became the central pillar of simplicity theory. In §5.1 we retain the original clumsy inductive proof. This may be of use in situations where simplicity is not known in advance.

The main point in any case is not the proof of this theorem but the realization that the uniqueness of nonforking extensions, which seemed characteristic of stability theory and essential to its fabric, can be replaced "densely often" with an appropriate *existential* statement.

The definition of *modularity* could largely be taken over from the stable case. A new idea was required (cf. §5.4) to produce enough geometric imaginaries for proof of the local–global principle; this idea survives in the contemporary treatment of canonical bases in simple theories. The consequences of modularity for groups are not as decisive in general as in the stable case, even generically, so we had to consider stronger variants.

The recognition theorems in rank one which use these properties serve to situate the basic geometries model theoretically to a degree. One would like to see these theorems generalized, as Zilber's characterizations of modular groups were extended from the totally categorical to the strongly minimal case.

The strong presence of duality is also a new feature as far as the model theory is concerned. Initially it arose as a particular instance of instability, which we sought to circumscribe and neutralize as much as possible. At the outset duals must be recognized in order to render the basic geometries stably embedded; the dual space of a finite vector space is also a prime example of a nonuniform interpretation. Eventually duality also emerged as a positive tool, useful for certain purposes even in contexts where stability is initially assumed: see §6.5, on the semi-dual cover, and also the treatment of second-order quantifiers in Chapter 8, dealing with effectivity. It seems possible that linear duality, like modularity, has some significance in general model theoretic frameworks, but at this time our situation remains isolated, awaiting further illumination.

The proof of Theorem 2 will be largely complete by the end of §3.5 (see the discussion in §3.5 for more on this). The final section (§8.4) contains some retrospective remarks on the structure of our development.

Various versions of this paper have benefited from remarks by a variety of model theorists. We thank particularly Ambar Chowdhury, David Evans, Bradd Hart, Dugald Macpherson, Anand Pillay, and Frank Wagner for their remarks. The first author also thanks Amaal for diverting correspondence during the preparation of the final version. We thank Virginia Dunn for a lively discussion of colons—and many other things—and Amélie Cherlin for editorial assistance.

2

Basic Notions

2.1 FINITENESS PROPERTIES

We discuss at length the various finiteness properties to be considered here.

We will make use of nonstandard terminology as a convenient way of dealing with "large" integers; see [FJ, Chapter 13] (in particular, the examples treated therein, in §13.5) for a full presentation of this method. The method is based on the idea of replacing the standard model of set theory in which one normally works by a proper elementary extension, the "enlargement," in which there are "new" (hence, infinite) integers. Since the extension is elementary, all notions of set theory continue to have meaning, and (more or less) their usual properties. In particular, for any set S occurring in the enlargement, there is an associated collection of "all" subsets of S in the sense of the enlargement; this will not actually contain all subsets of S in general, and those which are in fact present in the enlargement are called "internal" (the others could be called "external," but we do not use them). The word "internal" is used in other related ways: we may call an internal set which is finite in the sense of the enlargement either "internally finite," or "nonstandardly finite." A subset of an internally finite set need not be internal, but if it is, it will be internally finite. Again, we refer to the presentation by Fried and Jarden [FJ] for the essential foundational material.

2.1.1 Quasifiniteness, weak or smooth approximability

Definition 2.1.1. *Let $\mathcal{M}$ be a structure.*

1. *$\mathcal{M}$ is* $\aleph_0$-categorical*, or* oligomorphic*, if for each n $\mathcal{M}$ has finitely many n-types.*
2. *$\mathcal{M}$ is* pseudofinite *if it is a model of the theory of all finite structures (in the same language).*
3. *$\mathcal{M}$ is* k-quasifinite *if in a nonstandard extension of the set theoretical universe it is elementarily equivalent to an internally finite model with finitely many internal k-types.*

4. $\mathcal{M}$ *is* quasifinite *if in a nonstandard extension of the set theoretical universe it is elementarily equivalent (in the original language L) to an internally finite L^*-structure with a finite number of internal k-types, for all k.*
5. *A finite substructure $\mathcal{N}$ of $\mathcal{M}$ is* k-homogeneous *in $\mathcal{M}$ if all 0-definable relations on $\mathcal{M}$ induce 0-definable relations on $\mathcal{N}$, and for every pair of k-tuples $\mathbf{a}$, $\mathbf{b}$ in $\mathcal{N}$, $\mathbf{a}$ and $\mathbf{b}$ have the same type in $\mathcal{N}$ if and only if they have the same type in $\mathcal{M}$.*
6. *A structure $\mathcal{M}$ is* weakly approximable *by finite structures if it is $\aleph_0$-categorical, and every finite subset X of $\mathcal{M}$ is contained in a finite substructure $\mathcal{N}$ which is $|X|$-homogeneous in $\mathcal{M}$.*
7. *A structure $\mathcal{M}$ is* smoothly approximable *by finite structures if it is $\aleph_0$-categorical, and every finite subset X of $\mathcal{M}$ is contained in a finite substructure $\mathcal{N}$ which is $|\mathcal{N}|$-homogeneous in $\mathcal{M}$.*
8. $\mathcal{M}$ *is* strongly k-quasifinite *if in a nonstandard extension of the set theoretical universe it is elementarily equivalent to an internally finite model with finitely many internal k-types, which coincide with the k-types.*
9. $\mathcal{M}$ *is* strongly quasifinite *if in a nonstandard extension of the set theoretical universe it is elementarily equivalent (in the original language L) to an internally finite L^*-structure with a finite number of internal k-types, which coincide with the k-types, for all k.*

Remarks 2.1.2

We use freely the usual characterizations of $\aleph_0$-categoricity. Pseudofiniteness is also commonly referred to as the *finite model property.* Quasifiniteness strengthens pseudofiniteness (which is perhaps etymologically incorrect), as one sees by expressing pseudofiniteness in nonstandard terms. It also implies $\aleph_0$-categoricity, since the condition on internal k-types is equivalent to a similar condition on internal formulas with k free variables, and this includes the standard formulas. Decoding the nonstandard formulation yields:

3′. A structure $\mathcal{M}$ is k-quasifinite if and only if there is a finite number N such that for an arbitrary sentence φ true in $\mathcal{M}$, there is a finite structure $\mathcal{N}$ satisfying φ in which there are at most N formulas in k free variables.

4′. A structure $\mathcal{M}$ is quasifinite if and only if there is a function $\nu : \mathbb{N} \to \mathbb{N}$ such that for any n and an arbitrary sentence φ true in $\mathcal{M}$, there is a finite structure $\mathcal{N}$ satisfying φ in which there are at most $\nu(k)$ formulas in k free variables for $k \leq n$.

For strong quasifiniteness one specifies the formulas rather than the number of formulas.

Note that a weakly approximable structure $\mathcal{M}$ is strongly quasifinite, using the formulas which define k-types in a finite k-homogeneous substructure.

One gets an equivalent notion by bounding types rather than formulas, or equivalently, by bounding the number of orbits of the automorphism group of $\mathcal{N}$ on k-tuples. This concept would seem to be the most natural one from a purely permutation group theoretic standpoint. The definition of (strong) quasifiniteness implies (strong) k-quasifiniteness for all k, but the converse is not immediate. As noted in §1, Theorem 2, we will show that (strong) 4-quasifiniteness (or using only [KLM]: (strong) 5-quasifiniteness) already implies (strong) quasifiniteness, so in particular this converse does hold. One might have the impression that $\aleph_0$-categorical pseudofinite structures are strongly quasifinite in general, which is very far from the case. The generic graph seems to be the canonical counterexample; it is not quasifinite. The point is that while one might reasonably expect the property: "*every formula in k variables is equivalent to one in a specified finite set of formulas in k variables*" to be first order, it is not, in general.

As defined here all of these notions are invariant under elementary equivalence. When $\mathcal{M}$ is countable, weak and smooth approximability can be expressed somewhat more concretely in the form that $\mathcal{M}$ is a union of a countable chain of finite substructures $\mathcal{M}_i$ such that $\mathcal{M}_i$ is i-homogeneous (in the weak case), or $|\mathcal{M}_i|$-homogeneous (in the smooth case), respectively.

Digression 2.1.3

It is generally assumed that there is *not* going to be a coherent structure theory for $\aleph_0$-*categorical pseudofinite structures in a finite language*, though there is no solid evidence for this. One complication is that it seems to be quite hard in practice to determine whether a given finitely homogeneous structure is pseudofinite. For finitely homogeneous structures, pseudofiniteness holds in the stable case [La], fails in cases involving nondegenerate partial orders, and is obscure in most other cases, apart from those amenable to probabilistic analysis. The test case would be whether the generic triangle-free graph is pseudofinite.

2.1.2 Geometries

We have described most of the finiteness notions occurring in the statement of Theorem 2, with the exception of the technical notion of coordinatizability by Lie geometries. This notion in its most useful form involves some detailed properties of specific geometries. The relevant collection of geometries was given almost completely in [KLM], with the exception of what we call *affine duality*, which was not needed there. In addition a certain coordinatization theorem was proved there, which requires a further laying on of hands before it acquires the form most useful for a model theoretic analysis. We will now present the relevant geometries, which we give first in their linear forms, and then in projective and affine versions. It should be borne in mind that geometries are understood to be structures in the model theoretic sense, and not simply lattices or combinatorial geometries.

Definition 2.1.4. *A* weak linear geometry *is a structure of one of the following six types, and a* linear geometry *is an expansion of a weak one by the introduction of a set of algebraic constants in* $\mathcal{M}^{\mathrm{eq}}$.

1. *A degenerate space: a pure set, with equality alone.*
   We tend to ignore this case, as our claims are trivial in this context. One may perhaps pretend that it is a vector space over a field of order 1, and that linear dependence over a set is membership; in this case it equals its projectivization and has no affine version.
2. *A pure vector space:* (V, K)*, with* K *a finite field and* V *a* K*-vector space, with the usual algebraic structure.*
   Scalar multiplication is treated as a map from $K \times V$ to V rather than as a set of unary operators. This allows the Galois group of K to act on the structure.
3. *A polar space:* $(V \cup W, K, L; \beta)$*, where* K *is a finite field,* L *a* K*-line (1-dimensional* K*-space),* V *and* W *are* K*-spaces, and there is a nondegenerate bilinear pairing* $\beta : V \times W \to L$.
   We write $V \cup W$ rather than V, W because we treat $V \cup W$ as a set on which there is an equivalence relation with two classes, thereby preserving the symmetry between V and W. In particular, the domain of β is actually $(V \times W) \cup (W \times V)$, and β is symmetric.
4. *An inner product space:* (V, K, L, β) *where* K *is a finite field,* L *a* K*-line,* $\beta : V \times V \to L$ *a nondegenerate sesquilinear form with respect to a fixed automorphism* σ *with* $\sigma^2 = 1$*, and either* σ *is trivial and* β *is symplectic, or* σ *is nontrivial and* β *is hermitian with respect to* σ.
   (The symmetric case is included in the following class.)
5. *An orthogonal space:* (V, K, L, q) *where* K *is a finite field,* L *a* K*-line, and* q *a quadratic form on* V *with values in* L*, whose associated bilinear form is nondegenerate.*
   This point of view allows a treatment independent of the characteristic.
6. *A quadratic geometry:* $(V, Q, K; \beta_V, +_Q, -_Q, \beta_Q, \omega)$*, where* K *is a finite field of characteristic 2,* V *is a* K*-vector space,* β_V *is a nondegenerate symplectic bilinear form on* V*,* Q *is a set of quadratic*

forms q on V for which the associated bilinear form $q(v+w)+q(v)+q(w)$ is β_V, chosen so that V acts regularly on Q by translation, with β_Q, $+_Q$, $-_Q$ giving the interaction between Q and V, and ω specifying the Witt defect [CoAt], which is fairly obscure in the infinite dimensional case.

There is, evidently, a considerable amount to be elucidated here.

In the first place, there are always quadratic forms q for which the associated bilinear form $q(v+w)+q(v)+q(w)$ is the given symplectic form β_V, and any two of them differ by a quadratic form which is additive; this is just the square of a K-linear map. The full linear dual V^* acts regularly by $q \mapsto q + \lambda^2$ ($q \in Q$, $\lambda \in V^*$) on this set of quadratic forms, and via the identification of V with a subspace of V^*, coming from the given symplectic inner product β_V, we get a semiregular action of V on this space of quadratic forms. Q will be one of the V-orbits. We take $\beta_Q : Q \times V \to K$ to be the evaluation map $\beta_Q(q,v) = q(v)$, while $+_Q : V \times Q \to Q$ is the regular action of V on Q and $-_Q : Q \times Q \to V$ the corresponding "subtraction" map; both of these are definable from β_Q, e.g.: $v +_Q q = q + \lambda_v^2$ where λ_v is the linear form $\beta_V(v,\cdot)$. The map $\omega(q)$ is not definable from β_Q. In the finite ($2n$) dimensional case it will give the Witt defect $\pm$ of q, which is the difference between n and the dimension of a maximal totally q-isotropic subspace; this is either 0 or 1. In the infinite dimensional case we require a different description. For $q_1, q_2 \in Q$, $\sqrt{q_1+q_2}$ is a linear function of the form λ_v for a unique $v \in V$. Identifying v and λ_v, we may write $q(\sqrt{q_1+q_2}) \in K$; furthermore, we find $q_1(\sqrt{q_1+q_2}) = q_2(\sqrt{q_1+q_2})$, which translates to $(v,v)=0$. We will write $[q_1,q_2]$ for $q_1(\sqrt{q_1+q_2})$. For $q_1,q_2,q_3 \in Q$ if $v = \sqrt{q_1+q_2}$, $w = \sqrt{q_1+q_3}$, and $\alpha = (v,w)$ we find $[q_1,q_2] + [q_1,q_3] + [q_2,q_3] = \tau(\alpha)$ with $\tau(x) = x^2 + x$ the Artin–Schreier polynomial. Hence the relation $[q_1,q_2] \in \tau[K]$ is an equivalence relation with two classes. ω has the effect of naming these classes as unary predicates. We will construe ω as a function from Q to $\{0,1\} \subseteq K$. In particular, the Witt defect is taken modulo 2, which is quite convenient since it is then additive with respect to orthogonal sums.

Remarks 2.1.5

1. In the case of polar geometries we may write $W = V^*$ and $V = W^*$, informally, but as we are dealing with infinite dimensional spaces this should not be taken too literally. One can give this a precise sense if one associates with each of V and W the corresponding weak topology on its companion, making each the continuous dual of the other.

2. We use K-lines L rather than K itself in order to allow certain permutations of the language as automorphisms. The point is that if f is a bilinear form or a quadratic form and α is a scalar, then αf is another form of the same type with the same automorphism group. It will be convenient to view two structures with the same underlying set whose forms differ by a scalar as isomorphic. If α is a square they are isomorphic via multiplication by $\sqrt{\alpha}$, but in our formalism the identity map on the space extends to an isomorphism by allowing α to act on L. The same effect would be achieved by replacing the L-valued form f by the set of K-valued forms $\{\alpha f : \alpha \in K^\times\}$ and allowing scalars to act on the set of forms.

3. We can view a geometry as having as its underlying set a vector space in most cases, or a pair of spaces in duality in the polar case, or the set (V, Q) in the quadratic case, with the additional structure encoded in $\mathcal{M}^{\mathrm{eq}}$.

Definition 2.1.6.

1. *An* unoriented weak *linear geometry is defined as one of the six types of geometry listed above, with the proviso that in the sixth case we omit the Witt defect function ω.*
2. *A* basic linear *geometry is a linear geometry in which the elements of K and L are named, and in the polar case the two spaces V and W are named (or, equivalently, treated as unary predicates).*

Definition 2.1.7.

1. *A* projective geometry *is the structure obtained from a linear geometry by factoring out the equivalence relation defined by $acl(x) = acl(y)$, with algebraic closure understood in the model theoretic sense.*
2. *A* semiprojective geometry *is the structure obtained from a basic linear geometry by factoring out the relation $x^Z = y^Z$, where Z is the center of the automorphism group, that is, the set of scalars respecting any additional structure present. For example, in the symplectic case, the symplectic scalars are ± 1.*

After we check quantifier elimination in basic linear geometries, it will be clear that this algebraic closure operation is just linear span (in the sense appropriate to each case) and that our projective geometries are indeed projective geometries in the nonquadratic case; in the polar case we will have two projective spaces (PV, PV^*) with a notion of perpendicularity.

Definition 2.1.8. *If V is a definable vector space and A is a definable set, then A is an* affine *V-space if V acts definably and regularly on A.*

If J is a linear geometry and V is its underlying vector space (or one of the two underlying vector spaces in the polar case) then an affine geometry *(J, A) is a structure in which J carries its given structure and A carries the action of V, with no further structure.*

We will deal subsequently with the model theoretic properties of linear, affine, and projective geometries, but first we will deal with the notion of coordinatization that enters into the statement of Theorem 2 from Chapter 1

2.1.3 Coordinatization

Definition 2.1.9. *Let $\mathcal{M} \subseteq \mathcal{N}$ be structures with $\mathcal{M}$ definable in $\mathcal{N}$, and let $a \in \mathcal{N}^{\mathrm{eq}}$ represent the set $\mathcal{M}$ (its so-called canonical parameter).*

1. *$\mathcal{M}$ is* canonically embedded *in $\mathcal{N}$ if the* 0*-definable relations of $\mathcal{M}$ are the relations on $\mathcal{M}$ which are a-definable in the sense of $\mathcal{N}$.*
2. *$\mathcal{M}$ is* stably embedded *in $\mathcal{N}$ if every $\mathcal{N}$-definable relation on $\mathcal{M}$ is $\mathcal{M}$-definable, uniformly. The uniformity can be expressed either by requiring that the form of the definition over $\mathcal{M}$ be determined by the form of the definition over $\mathcal{N}$, or by requiring that the same condition apply to all elementary extensions of the pair $(\mathcal{M}, \mathcal{N})$.*
3. *$\mathcal{M}$ is* fully embedded *in $\mathcal{N}$ if it is both canonically and stably embedded in $\mathcal{N}$.*

Definition 2.1.10. *A structure $\mathcal{M}$ is* coordinatized by Lie geometries *if it carries a tree structure of finite height with a unique, 0-definable root, such that the following coordinatization and orientation properties hold.*

1. *(Coordinatization) For each $a \in \mathcal{M}$ above the root either a is algebraic over its immediate predecessor in the tree ordering, or there exists $b < a$ and a b-definable projective geometry J_b fully embedded in $\mathcal{M}$ such that either*
   (i) *$a \in J_b$; or*
   (ii) *there is c in $\mathcal{M}$ with $b < c < a$, and a c-definable affine or quadratic geometry (J_c, A_c) with vector part J_c, such that $a \in A_c$ and the projectivization of J_c is J_b. (Note that the projectivization of a symplectic geometry in characteristic* 2 *may have both quadratic and affine geometries attached to it in this way.)*
2. *(Orientation) If $a, b \in \mathcal{M}$ have the same type and are associated with coordinatizing quadratic geometries J_a, J_b in $\mathcal{M}$, then there*

is no definable orientation-reversing isomorphism of J_a and J_b as unoriented weak linear geometries; in other words, if a definable map between them preserves everything other than ω, then it also preserves ω.

Example 2.1.11
Let A be the infinite direct sum of copies of $(\mathbb{Z}/p^2\mathbb{Z})$ with p a fixed prime. One coordinatizes this by placing 0 at the root, as a finite set, then putting the projectivization of

$$A[p] = \{a \in A : pa = 0\}$$

above it, and $A[p] \setminus \{0\}$ itself above that (covering each projective point by the corresponding finite set of points above it); finally, one adds $A \setminus A[p]$; above each $a \in A[p] \setminus \{0\}$ one has the affine space $A_a = \{x \in A : px = a\}$. This gives a tree of height 4, with layers of the form: finite, projective, finite, affine, respectively.

We also use the briefer expression *Lie coordinatized* with the same meaning. However, we make a rather sharp distinction between the existence of a coordinatization, as defined above, and coordinatizability in the following more general sense.

Definition 2.1.12. *The structure $\mathcal{M}$ is* Lie coordinatizable *if it is bi-interpretable with a structure having finitely many 1-types which is coordinatized by Lie geometries.*

At this point the notions involved in Theorem 2 of Chapter 1 have all been defined. In Theorems 3 and 4 we also use the notion of *weak Lie coordinatizability*, which involves a notion of Lie coordinatization in which the orientation condition is suppressed.

2.2 RANK

2.2.1 The rank function

Definition 2.2.1. *Let $D \subseteq \mathcal{M}$ be definable. A* rank function *(with finite values, or the symbol ∞ if undefined) is determined by the following conditions:*

1. *$rk\, D > 0$ if and only if D is infinite.*
2. *$rk\, D \geq n+1$ if and only if there are definable D_1, D_2, π, f with $\pi : D_1 \to D$, $f : D_1 \to D_2$ such that*
   (i) *$rk\, \pi^{-1}(d) = 0$ for $d \in D$;*
   (ii) *$rk\, D_2 > 0$;*
   (iii) *$rk\, f^{-1}(d) \geq n$ for $d \in D_2$.*

If we are not in the $\aleph_0$-categorical case then these definitions should take place in a saturated model, and variations are possible using type-definable sets. We work in the $\aleph_0$-categorical setting. We write $rk(a/B)$ for the rank of the type of a over B, which is the minimum of $rk\, D$ for $a \in D$, D B-definable. In practice B is finite and the type reduces to the *locus* of a over B, which is the smallest B-definable set containing a.

Our definition of rank can be applied either to $\mathcal{M}$ or to $\mathcal{M}^{\mathrm{eq}}$, and the latter is the more useful convention in the long run. When the distinction is significant, in connection with specific structures $\mathcal{M}$, we will refer to rk computed in $\mathcal{M}$ as *pre-rank*, and the rank computed in $\mathcal{M}^{\mathrm{eq}}$ as *rank*.

Lemma 2.2.2

1. *$rk\, D = 0$ if and only if D is finite.*

$1'$. *$rk(a/B) = 0$ if and only if $a \in acl\, B$.*

2. *$rk(D_1 \cup D_2) = \max(rk\, D_1, rk\, D_2)$.*

$2'$. *(Extension property) If D is B-definable, then there is a complete type over B containing D and having the same rank.*

$2''$. *If $B_1 \subseteq B_2$ then $rk(a/B_2) \leq rk(a/B_1)$.*

Proof. (1,2) are straightforward and ($1'$, $2'$) are direct consequences. ($2''$) corresponds to the law: if $D_1 \subseteq D_2$, then $rk\, D_1 \leq rk\, D_2$, which is included in (2).

Lemma 2.2.3. *Let $\mathcal{M}$ be $\aleph_0$-categorical. Then the following are equivalent for $a, b \in \mathcal{M}$:*

1. *$rk(a/b) \geq n+1$.*
2. *There are a', c with $a' \in acl(abc) - acl(bc)$, and $rk(a/a'bc) \geq n$.*

Proof. Let D be the locus of a over b.

(1) $\Longrightarrow$ (2). Suppose that $\pi : D_1 \to D$ has finite fibers, $f : D_1 \to D_2$ has fibers of rank at least n, and D_2 is infinite, with D_1, D_2, π, and f c-definable. Take $a' \in D_2 - acl(bc)$, and $a_1 \in f^{-1}(a')$ with $rk(a_1/a'bc) \geq n$ (using the Extension Property). Set $a_0 = \pi a_1$. Then we have $a' \in acl(a_0bc) - acl(bc)$, and as $rk(a_1/a'bc) \geq n$ we find $rk(a_0/a'bc) \geq n$. Furthermore, as $tp(a_0/b) = tp(a/b)$ we can replace a_0 by a, replacing a', c by other elements.

(2) $\Longrightarrow$ (1). Let a', c have the stated properties. Let D_1 be

$$\{(x, y) : tp(xy/bc) = tp(aa'/bc)\}$$

and let $\pi : D_1 \to D$, $f : D_1 \to D_2$ be the projections of D_1 onto the first and second coordinates, respectively. Then $f^{-1}(a')$ contains (a, a') and $rk(a/a'bc) \geq n$, so easily $f^{-1}(a')$ has rank at least n and hence the same applies to all fibers of f. It follows easily that D_1, D_2, f, π have the required properties for (1).

Lemma 2.2.4. *Let $\mathcal{M}$ be $\aleph_0$-categorical. If $rk(a/bc)$ and $rk(b/c)$ are finite, then $rk(ab/c)$ is finite and*

$$rk(ab/c) = rk(a/bc) + rk(b/c).$$

Proof. We use induction on $n = rk(a/bc) + rk(b/c)$, and the criterion of Lemma 2.2.3.

We show first that $rk(ab/c) \leq n$. Let d, e satisfy: $e \in acl(abcd) - acl(cd)$. We will show that $rk(ab/cde) < n$. We have either $e \in acl(abcd) - acl(bcd)$ or $e \in acl(bcd) - acl(cd)$ and correspondingly either $rk(a/bcde) < rk(a/bc)$ or $rk(b/cde) < rk(b/c)$. In either case induction applies to give $rk(ab/cde) < n$.

Now we show that $rk(ab/c) \geq n$. If $rk(b/c) = 0$ we observe that

$$rk(ab/c) \geq rk(a/c) \geq rk(a/bc) = n.$$

Assume $rk(b/c) > 0$ and take b', d with $b' \in acl(bcd) - acl(cd)$, and $rk(b/b'cd) = rk(b/c) - 1$. Using the Extension Property we may suppose also that $rk(a/bb'cd) = rk(a/bc)$. By induction we find $rk(ab/b'cd) = n - 1$ and hence $rk(ab/c) \geq n$.

Corollary 2.2.5. *If $rk\, D = 1$, then acl defines a pregeometry on D, that is, a closure property of finite character with the exchange property.*

Definition 2.2.6. *We say that a and b are* independent *over C if*

$$rk(ab/C) = rk(a/C) + rk(b/C);$$

equivalently, $rk(a/bC) = rk(a/C)$.

Lemma 2.2.7. *The independence relation has the following properties:*

1. *Symmetry: If a and b are independent over C, then the same applies to b and a over C.*
2. *If a is algebraic over bC, then a is independent from b over C if and only if it is algebraic over C.*
3. *The following are equivalent:*
   (i) *a and bc are independent over E;*
   (ii) *a and b are independent over Ec, and a and c are independent over E.*

Proof. Each of these statements is clear on the basis of at least one of the criteria given in Definition 2.2.6.

This theory is relevant to our geometries, as they all have rank 1. This will be verified below.

2.2.2 Geometries

Lemma 2.2.8. *If J is a basic linear geometry then it has elimination of quantifiers.*

Proof. One checks that any suitably normalized atomic type is realized. In other words, using the basic universal axioms appropriate in each case, one shows that any existential formula in one variable can be reduced to a standard form, which is either visibly inconsistent or always realized. As we are dealing with basic geometries, the base field has been incorporated into the language, and we deal with structures whose underlying universe is of one of four types: degenerate, a vector space, a polar pair of spaces, or a quadratic pair (V, Q); these carry, variously, linear, bilinear, and quadratic structure. We may ignore the degenerate case and we defer the case of a quadratic geometry to the end. By taking the relevant bilinear or quadratic form to be identically zero in cases where it is not present, and expanding the domain of the type to a subspace B (or pair of subspaces in the polar case) which is nondegenerate whenever that notion is meaningful (this includes the polar case), we may assume the type to be realized has the following form:

$$x \notin B \tag{1}$$

$$\beta(x, b) = \lambda(b) \tag{2}$$

$$q(x) = \alpha \tag{3}$$

The justification for (1) is that the excluded case is trivial, and the point of (3) is that any remaining conditions on q can be expressed in terms of the associated bilinear form in (2). Furthermore, the condition (2) is satisfied by an element of B, either because it is vacuous, or because B is nondegenerate, and after translation by such an element we get a similar system with $(1,3)$ as above and (2) replaced by

$$(2') \qquad x \in B^{\perp}.$$

There is then nothing more to check unless q is nondegenerate. In this case one needs to know that q takes on all possible values in the orthogonal complement of any finite dimensional space. The following argument applies without looking at the classification of quadratic forms on finite dimensional spaces.

Let $K_0 \subseteq K$ be the set of values α such that q takes on the value α in the orthogonal complement to any finite dimensional space. Easily K_0 contains a nonzero element, is closed under multiplication by squares, and is closed under addition as $q(x+y) = q(x) + q(y)$ when x, y are orthogonal with respect to the associated bilinear form. It follows that $K_0 = K$.

Returning to the quadratic case, if the domain B of the type meets the set Q, then this is covered by the orthogonal case. Otherwise, we first add to the domain an element q of Q (we will have occasion later, in the treatment of imaginary elements, to revert to this point); the quantifier-free type of the extension is determined by the action of q on B, and the ω-invariant, both of which may be specified arbitrarily.

Corollary 2.2.9. *The definable linear functions on the vector space V in a linear geometry are those afforded either by the inner product (if one is given, or is derivable from a quadratic form), or by the dual in the polar case.*

Proof. One checks that a definable subspace of finite codimension contains the kernel of a finite set of linear forms encoded directly in the structure (via a bilinear form, or polarity). Then any linear form whose kernel contains the kernels of these forms is expressible as a linear combination of them.

Lemma 2.2.10. *The linear, affine, and projective geometries are all of pre-rank* 1.

Proof. It suffices to handle the basic linear case, and we can reduce the quadratic geometries to the orthogonal case. By quantifier elimination, algebraic closure is then linear span in the appropriate sense, which in the polar case takes place in two disjoint vector spaces. Thus the computation of rank is unaffected by the fact that the vector space structure may have been enriched.

In the next section we discuss weak elimination of imaginaries, and one may then replace "pre-rank" by "rank" in the preceding.

Corollary 2.2.11. *If $\mathcal{M}$ is Lie coordinatizable, then $\mathcal{M}$ has finite rank, at most the height of the coordinatizing tree.*

Corollary 2.2.12. *If J is a linear, projective, or affine geometry, and a, b are finite sequences with $acl(a) \cap acl(b) = C$, then a and b are independent over C.*

Proof. Note that by definition the affine geometries include the linear model as a component. In the linear nonquadratic case we have noticed that the algebraic closure is the linear span; the analogous statement holds in the projective case. So in the linear and projective cases this is essentially a statement about linear algebra.

The affine and quadratic cases are similar: they may be expressed in the form (J_l, A), where J_l is linear and J_l (or in the polar case, part of J_l) acts regularly on the additional set A. A subspace is either an ordinary subspace of J_l (which may be polar) or a pair (B_l, B_a), where B_l is linear and B_a is an affine copy of B_l (with the usual modification in the polar case). If $acl(a) \cap acl(b)$ contains an affine (or quadratic) point then we are still essentially in the linear case; otherwise, we are working with affine dimension, which is 1 greater than the corresponding linear dimension. In this case it is important that $acl(a)$ and $acl(b)$ have a linear part determined by their affine parts (this should be rephrased slightly in the polar case, but the facts are the same).

2.2.3 A Digression

The remainder of this section is devoted to additional remarks on rank notions which are far removed from our main topic.

Definition 2.2.13. *Let $\mathcal{M}$ be $\aleph_0$-categorical. Then ranks rk_α valued in $\mathbb{N} \cup \{\infty\}$ are defined as follows:*

1. *$rk_0(D) = 0$ if D is finite, and is ∞ otherwise.*
2. *$rk_\alpha(D) > 0$ if and only if $rk_\beta(D) = \infty$ for $\beta < \alpha$.*
3. *$rk_\alpha(D) \geq n + 1$ if and only if there are $\pi : D_1 \to D$, $f : D_1 \to D_2$ definable, with*
   (i) *$rk_\alpha(\pi^{-1}(d)) = 0$ for $d \in D$;*
   (ii) *$rk_\alpha(f^{-1}(d)) \geq n$ for $d \in D_2$;*
   (iii) *$rk_\alpha(D_2) > 0$.*

Remarks 2.2.14

1. In the superstable case working in saturated models with type-definable sets, for D complete and α arbitrary there is a 0-definable

quotient D' with $rk_\alpha D'$ finite and maximal. Writing $rk'_\alpha(D)$ for $rk_\alpha(D')$ we will have $U(D) = \sum_\alpha \omega^\alpha \, rk'_\alpha(D)$.

2. The ranks rk_α are additive and sets of α-rank 1 carry a geometry.

Definition 2.2.15. *If $0 < rk_\alpha \mathcal{M} < \infty$, we call α the* tier *of $\mathcal{M}$. According to the definition of rk_α, there is at most one tier for $\mathcal{M}$.*

Lemma 2.2.16. *There are pseudofinite $\aleph_0$-categorical structures of arbitrarily large countable tier, as well as structures of the same type with no tier.*

Proof. We deal first with countable tier. We have examples for $\alpha = 0$. In all other cases we proceed inductively, writing $\alpha = \sup(\beta_n + 1)$. We take countable pseudofinite $\aleph_0$-categorical structures D_n of tier β_n with $rk_{\beta_n}(D_n) \geq n(n+1)$ (replace D_n by a power if necessary) and encode them into D^{eq} for a new set D as follows.

We take initially a language L^* with sorts $D, D_1, D_2, \ldots$, whose restriction to D_n is the language of D_n. We also add generic maps $f_n : [D]^{n+1} \to D_n$; here the notation $[D]^i$ refers to unordered sets. The axioms are the axioms of D_n, relativized to that set, together with the following:

$$(*) \qquad \begin{array}{l} \text{For } t \in [D]^n \text{ and any } h_i : [t]^i \to D_i \text{ there is} \\ a \in D \text{ for which } f_i(s \cup \{a\}) = h_i(s) \text{ for } s \in [t]^i. \end{array}$$

This theory has D-quantifier elimination and is complete, consistent, and $\aleph_0$-categorical when interpreted as a theory of D, with D_n encoded in $\mathcal{M}^{\mathrm{eq}}$. For the finite model property, we begin with finite approximations to D_i for $i \leq N$, and we let D be large finite, f_n random; most choices satisfy $(*)$. As D^{n+1} maps onto D_n definably, we find $rk_{\beta_n} D \geq n$. Thus $rk_\alpha D \geq 1$; one can show $rk_\alpha D = 1$ and the tier is exactly α.

To get no tier we use sorts D_n and functions $f_n : D_n^n \to D_{n+1}^{n^2}$, satisfying the analog of $(*)$. Then $rk_\alpha(D_n) \geq n \, rk_\alpha D_{n+1}$ for all n and easily $rk_\alpha D_n = \infty$ for all n and α. We view this structure as encoded in D_1^{eq}.

2.3 IMAGINARY ELEMENTS

Definition 2.3.1. *$\mathcal{M}$ has* weak elimination of imaginaries *if for all $a \in \mathcal{M}^{\mathrm{eq}}$, we have $a \in dcl(acl(a) \cap \mathcal{M})$.*

Lemma 2.3.2. *If D is 0-definable in $\mathcal{M}$ and $D(a) = acl(a) \cap D$ for $a \in \mathcal{M}^{\mathrm{eq}}$, then the following are equivalent:*

1. *D is stably embedded in $\mathcal{M}$ and admits weak elimination of imaginaries.*
2. *For $a \in \mathcal{M}^{\mathrm{eq}}$, $tp(a/D(a))$ implies $tp(a/D)$.*

Proof. (1) $\implies$ (2). Let $\varphi(x, y)$ be a formula with x a single variable (of the same sort as a). The relation $\varphi(a, y)$ defined on D is D-definable and hence has a canonical parameter d_0 in D^{eq}; note that $d_0 \in acl(a)$. By weak elimination of imaginaries there is $B \subseteq D(d_0) \subseteq D(a)$ such that $d_0 \in dcl\, B$ and hence $\varphi(a, y)$ is B-definable: $\varphi(a, y) \iff \varphi^*(b, y)$, with b in B. This last fact is part of $tp(a/B)$ and determines the φ-type of a over D. Thus (2) holds.

(2) $\implies$ (1). If $a \in D^{\mathrm{eq}}$, then $a \in dcl(D)$, and hence by (2) we have $a \in dcl(D(a))$, as required for weak elimination of imaginaries.

Now suppose that $\varphi(x, a)$ is a formula implying $x \in D$, where x is a string of free variables. Let $A = D(a)$. If $tp(b/A) = tp(a/A)$, then $\varphi(x, a)$ and $\varphi(x, b)$ are equivalent, by (2). Thus $\varphi(x, a)$ is D-definable.

Lemma 2.3.3. *Let J be a linear, projective, or affine geometry. Let $a \in J^{\mathrm{eq}}$, and $A = acl(a) \cap J$. Then $acl(a) = acl(A)$.*

Proof. We may take J basic. Write $a = f(b)$ with b in J and f 0-definable. Take b' independent from b over $acl(a)$ in the sense of §2.2.1, with $tp(b'/\, acl(a)) = tp(b/\, acl(a))$.

We claim that b and b' are independent over A. We have $a = f(b) = f(b')$ and thus $A \subseteq acl(b) \cap acl(b') \cap J \subseteq acl(a) \cap J = A$. Thus this reduces to Corollary 2.2.12.

Our two independence statements may be written out as follows:

$$rk(b'/Aab) = rk(b'/Aa); \quad rk(b'/Ab) = rk(b'/A).$$

Since $rk(b'/Aab) = rk(b'/Ab)$ and $rk(b'/Aa) = rk(b'a/A) - rk(a/A) = rk(b'/A) - rk(a/A)$, on comparing the two equations we find $rk(a/A) = 0$, and $a \in acl(A)$, as claimed.

Corollary 2.3.4. *Let P be a projective geometry stably embedded in $\mathcal{M}$, A a subset of $\mathcal{M}$, and P_A the geometry obtained by taking acl relative to A as the closure operation. Then P_A is modular, i.e.,*

$$rk(ab) = rk(a) + rk(b) - rk(a \cap b)$$

for finite algebraically closed a, b.

Proof. By stable embedding and the preceding lemma we may replace A by $acl(A) \cap P$.

Lemma 2.3.5. *Let J be a basic linear geometry. Then J has weak elimination of imaginaries.*

Proof. By the preceding lemma it suffices to prove the following: if $A \subseteq J$ is algebraically closed, $a \in J^{\mathrm{eq}}$, and $a \in acl(A)$, then $a \in dcl(A)$.

We write $a = f(b)$ with f A-definable and $b = (b_1, \ldots, b_n)$, and we minimize n. Assuming $a \notin dcl(A)$, we have $n \geq 1$. Working over $A \cup \{b_1, \ldots, b_{n-1}\}$ we may suppose $n = 1$ and $b = b_n$. Let $D \subseteq J$ be the locus of b over A; of course, $b \notin A$. We examine the dependence of f on the element of D chosen.

Let $I = \{(x, y) \in D^2 : \langle xA\rangle \cap \langle yA\rangle = A\}$. The corner brackets are another notation for algebraic closure in J, intended to suggest linear span. For $(x, y) \in I$ the type of xy over A is determined by the inner product $\beta(x, y)$, with β nondegenerate or trivial, and possibly derived from a quadratic form; or else in the quadratic case, if $D \subseteq Q$, by $[x, y] = x(\sqrt{x + y})$. We will write $x \cdot y$ for the corresponding function in each case. So for some subset X of the field K we have

$$\text{For } (x, y) \in I,\ f(x) = f(y) \text{ if and only if } x \cdot y \in X.$$

Let $X_0 = K$ when we are dealing with a bilinear form, and $X_0 = \tau[K]$ with $\tau(x) = x^2 + x$ in the quadratic case with $D \subseteq Q$. Then in any case $X \subseteq X_0$ and it suffices to show that $X = X_0$, as then f is constant on independent pairs, and hence constant on D.

To see that $X = X_0$ it suffices to check that for $\alpha_{12}, \alpha_{13}, \alpha_{23} \in X_0$ there are x_1, x_2, x_3 independent over A for which $x_i \cdot x_j = \alpha_{ij}$ for $1 \leq i < j \leq 3$, as we then take $\alpha_{12} = \alpha_{23} \in X$ and $\alpha_{13} \in X_0$ arbitrary to conclude $X = X_0$. This is essentially a special case of the statement from which quantifier elimination was derived, though this was slightly obscured in the quadratic case by the suppression of some details.

We leave this calculation to the reader, but note that in the quadratic case, if the three elements x_1, x_2, x_3 are quadratic forms, we may write them as $q + \lambda_v^2, q, q + \lambda_w^2$, respectively, and find that the "target" values α_{ij} satisfy: $\alpha_{12} = q(v)$; $\alpha_{23} = q(w)$; and

$$\alpha_{13} = (q + \lambda_v^2)(v + w) = \alpha_{12} + \alpha_{13} + \tau((v, w)).$$

Corollary 2.3.6. *Let J be a basic semiprojective geometry. Then J has weak elimination of imaginaries.*

Proof. Let $a \in J^{\mathrm{eq}}$, let V be the vector space model covering J, and let $A = acl(a) \cap V$. Then $a \in dcl(A)$. Let $\mathbf{a}$ be a sequence of elements of

J over which a is definable, and let $B = acl(a) \cap J$. The orbit of $\mathbf{a}$ in J over B is the same as its orbit over A, so $a \in dcl(B)$.

Remark 2.3.7 Projective geometries J need not have weak elimination of imaginaries, since the semiprojective geometry lies in J^{eq}.

Definition 2.3.8. *Let V be a vector space and A an affine V-space, with A and V definable in a structure $\mathcal{M}$. Let K be the base field.*

1. *A* K-affine *map $\lambda : A \to K$ is a map satisfying*

$$\lambda(\sum_i \alpha_i a_i) = \sum_i \alpha_i \lambda(a_i)$$

for scalars α_i with $\sum_i \alpha_i = 1$ (in which case the left side makes sense; linear operations make sense in A relative to a base point in A, and affine sums are independent of the basepoint).

2. *A^* is the set of $\mathcal{M}$-definable K-affine maps on A.*

Lemma 2.3.9. *In the notation of the previous definition, there is an exact sequence*

$$(0) \to K \to A^* \to V^* \to (0)$$

where V^* is the definable dual of V (consisting of all definable linear functionals).

Proof. K represents the set of constant functions. The map from A^* to V^* is defined as follows. For $\lambda \in A^*$ and $v \in V$, let $\lambda'(v) = \lambda(a + v) - \lambda(a)$, which is independent of the base point a. This is surjective since V^* lifts to A^* by choosing a base point in A.

Remarks 2.3.10

1. In this exact sequence it is possible that $A^* = K$ and $V^* = (0)$; indeed, this must occur in the stable case. V^* is described by the corollary to quantifier elimination in §2.2.1; in particular, V^* is definable.

2. Note that A^* is coded in $(V, V^*, A)^{\mathrm{eq}}$. The algebraic closure of an element of A^* in (V, V^*, A) will be the line in V^* generated by the corresponding linear map. For this reason we do not have weak elimination of imaginaries in (V, V^*, A). Note also that V^* is normally not mentioned explicitly, as it is identified with V when there is a nondegenerate bilinear map (assuming the situation is stably embedded).

3. We do have weak elimination of imaginaries in (V, V^*, A^*), as in the the proof of Lemma 2.3.12 below, but this is not stably embedded in (V, V^*, A, A^*), as a base point in A gives a definable splitting of A^*—that is, a hyperplane complementary to the line of constants.

4. V^* is definable over A^*, so even in the polar case it is not necessary to include it in the geometry when A^* is present.

Lemma 2.3.11. *Let J be a basic, nonquadratic, linear geometry, and (J, A) a corresponding affine geometry. Then (J, A, A^*) admits quantifier elimination in its natural language.*

Proof. We take as the language the previous language for J, predicates for A and A^*, addition and subtraction maps $V \times A \to A$ and $A \times A \to V$, an evaluation map $A \times A^* \to K$, a K-vector space structure on A^*, distinguished elements of A^* corresponding to the constant functions, the canonical map $A^* \to V^*$ if V^* is present in some form, or an evaluation map $A^* \times V \to K$ if V^* is left to be encoded by A^*. As in the linear case we verify the realizability of suitably normalized atomic types. Since we can enlarge the domain of the types we can take a base point in A, identify A with V, and identify A^* with $K \oplus V^*$, putting us into the linear case.

Lemma 2.3.12. *Let J be a basic nonquadratic linear geometry and let (J, A) be a corresponding basic affine geometry. Then (J, A, A^*) has weak elimination of imaginaries.*

Proof. As we have Lemma 2.3.3 for the affine case, and A^* is algebraic over V^*, we just have to check that the proof of Lemma 2.3.5 also goes through. As in that proof, our claim is that if $B \subseteq (J, A, A^*)$ is algebraically closed and $f : (J, A, A^*) \to (J, A, A^*)^{\mathrm{eq}}$ is B-definable, then f is constant on each 1-type D over B.

We consider $I = \{(x, y) \in D^2 : \langle xB\rangle \cap \langle yB\rangle = B\}$, where the span is the algebraic closure in (J, A, A^*). (This includes the constant line in A^*.) We claim that f is constant along pairs in I; this will suffice. When $D \subseteq J$ it is convenient to view V^* as included in J, which is automatically the case except in the polar geometries. Then in dealing with D we may dispense with A and A^* and we are in the situation we treated previously. There remain the possibilities that $D \subseteq A$ or $D \subseteq A^*$.

Suppose that $D \subseteq A$. If B meets A, then we can replace D by a type realized in J. Suppose therefore that $B \cap A = \emptyset$. The type of D includes the values of affine maps on D and gives no further information about the type of a pair in I. Since the linear maps in B are covered by affine maps, this means that the only relevant part of B is $B \cap V$, and furthermore for $(x, y) \in D^2$, $x - y$ is orthogonal to $B \cap V$. Thus the type of such a pair, if it is not already determined, depends on the value of $Q(x - y)$ for a nondegenerate quadratic form Q. To repeat the previous argument we need independent elements v, w lying in $B^\perp$ with $Q(v)$, $Q(w)$, and $Q(v + w)$ taking on arbitrary values. This we have.

Now suppose $D \subseteq A^*$. If B meets A, then A^* becomes identified with $K \oplus V^*$ and we return to the linear case. If $B \cap A = \emptyset$, then for $(x, y) \in I$

the type of the pair over B is determined by the type of the image in V^*, and we again return to the linear case.

We now consider the relationship between the linear dual and the dual over the prime field. It turns out that the distinction is unimportant in the linear case but of some significance in the affine case.

Definition 2.3.13. *Let V be a vector space over the finite field F with prime field $F_\circ$, and A the corresponding affine space. We write $V^{*\circ}$ and $A^{*\circ}$ for the linear and affine dual with respect to the $F_\circ$-structure.*

Lemma 2.3.14. *There is a 0-definable group isomorphism τ between V^* and $V^{*\circ}$ given by $\tau f = Tr \circ f$, and a 0-definable surjection $\tau^A : A^* \to A^{*\circ}$ given similarly by the trace $Tr : F \to F_\circ$, with kernel the set of constant maps of trace 0.*

Proof. In the linear case, the two spaces have the same dimension over $F_\circ$. We check that the kernel of τ is trivial. Assume $\tau f = 0$. Then for any $v \in V$ and $\alpha \in F$, $\tau f(\alpha v) = Tr(\alpha f(v)) = 0$. As the trace form is nondegenerate on F, this means $f(v) = 0$.

In the affine case the difference in dimensions is the dimension of $F/F_\circ$ corresponding to the difference in the space of constant maps. As τ^A induces τ its kernel is contained in the space of constant maps.

We record the degree of elimination of imaginary elements afforded by $A^{*\circ}$.

Lemma 2.3.15. *Let (V, A) be a basic affine geometry, not of quadratic type. Let $C \subseteq (V, A, A^*)^{\mathrm{eq}}$ be definably closed and locally finite, that is, finite in each sort.*

If $acl(C) \cap (V \cup A) \subseteq C$, then $C = dcl(C \cap (V \cup A \cup A^{*\circ}))$.

Proof. Let $V_C = V \cap C$, $A_C = A \cap C$, $A^*_C = A^* \cap acl(C)$. By weak elimination of imaginaries $C = dcl(acl(C) \cap (V \cup A \cup A^*)) = dcl(V_C \cup A_C \cup A^*_C)$. As V^* is identified with a quotient of $A^{*\circ}$ it will suffice to check that

$$Mult(A^*_C/C) = Mult(A^*_C/C \cap A^{*\circ}, C \cap V^*).$$

Let $v^*_1, \ldots, v^*_d$ be a basis for $C \cap V^*$ and let a^*_i be a lifting of v^*_i to A^*. The element a^*_i is chosen from an affine line over the base field F. We have for each i

$$Mult(a^*_i/a^*_1, \ldots, a^*_{i-1}, C) \leq Mult(a^*_i/a^*_1, \ldots, a^*_{i-1}, V^* \cap C, A^{*\circ} \cap C)$$

and it suffices to show equality.

Let $K = Aut(a^*_i + F/a^*_1, \ldots, a^*_{i-1}, C)$, a subgroup of $(F, +)$. Let L be the space of K-invariant affine maps over $F_\circ$ on $a^*_i + F$. We have $Aut(a^*_i + F/L) = K$, since a translation $x \to x + \alpha$ on $a^*_i + F$ leaves

L invariant if and only if the linear maps induced by L annihilate α, and these are just the $F_\circ$-linear maps annihilating K. Accordingly, for $A_i^* = \langle a_1^*, \ldots, a_i^* \rangle$ we have

$$Aut(A_i^*/a_1^*, \ldots, a_{i-1}^*, C) = Aut(A_i^*/v_i, L).$$

Now $L \subseteq dcl(a_1^*, \ldots, a_{i-1}^*, C) \cap (a_1^* + F)^{*\circ}$, and we need

$$L \subseteq dcl(a_1^*, \ldots, a_{i-1}^*, C) \cap A^{*\circ}.$$

For $f \in L$ inducing the linear map $\bar{f}$ and $a^* \in a_i^* + F$ define $f_{a^*} \in A^{*\circ}$ by $f_{a^*}(a) = \bar{f}((a, a^*)) - f(a^*)$. This does not depend on the choice of a^*: $f_{a^*+\alpha}(a) = \bar{f}((a, a^*) + \alpha) - [f(a^*) + \bar{f}(\alpha)] = f_{a^*}(a)$. Thus f defines $f' = f_{a^*}$ and the converse is obvious.

Lemma 2.3.16. *Let V be part of a stably embedded basic linear geometry J with base field F. Let A be an affine space over V. Assume A and V are 0-definable. Then there is a 0-definable space, which we will denote FA, such that FA contains V as a subspace of codimension 1, and A as a coset of V. The space FA is unique up to canonical definable isomorphism.*

Proof. We let FA be $F \times A \times V$ modulo the equivalence relation defined by: $(\alpha, a, v) \sim (\alpha', a', v')$ if and only if $\alpha = \alpha'$, $\alpha(a - a') = v - v'$. Equivalence classes will be denoted in terms of their representatives as $\alpha a + v$. The scalar multiplication will be defined by

$$\beta(\alpha a + v) = (\beta\alpha)a + \beta v.$$

This is clearly well defined.

To define addition on FA, note that for any $a_\circ \in A$ the elements of FA are uniquely representable in the form $\alpha a_\circ + v$. Thus we may set

$$(\alpha a_\circ + v) + (\alpha' a_\circ + v') = (\alpha + \alpha')a_\circ + (v + v').$$

This definition is immediately seen to be independent of the choice of $a_\circ$. Thus the construction is 0-definable. One checks the vector space axioms. Evidently V sits as a subspace of codimension 1 and A as a coset.

Verification of the uniqueness statement is straightforward.

Lemma 2.3.17. *Let V be a nonquadratic basic linear geometry, possibly with distinguished elements, forming part of a geometry J with field of scalars F which is stably embedded in $\mathcal{M}$. Let A be a C-definable affine space over V. Then*

1. *$FA \cup J$ and $FA \cup (FA)^* \cup J$ are stably embedded.*

2. *Suppose A is not in $acl(J, C)$ and let $C_\circ$ be*

$$[acl(C) \cap J] \cup [acl(C) \cap A] \cup [acl(C) \cap A^{*\circ}].$$

Then $(J, FA, FA^{*\circ}, C_\circ)$ with its intrinsic geometric structure is fully embedded in $\mathcal{M}$ over $C \cup C_\circ$.

Proof. Ad 1. For $a \in A$ we have $FA \subseteq dcl(a, V)$. Furthermore, $(FA)^* \subseteq dcl(a, V^*)$ since $f \in (FA)^*$ is determined by its restriction to V and its value at a. Thus this is immediate.

Ad 2. Let $\mathcal{N}$ be $(J, FA, FA^{*\circ}, C_\circ)$ with its intrinsic geometric structure, and let $\mathcal{N}'$ be $\mathcal{N}$ with its full induced structure. As $\mathcal{N}$ is stably embedded, any 0-definable set D in $\mathcal{N}'$ is definable in $\mathcal{N}$ with parameters. We claim that D is 0-definable in $\mathcal{N}$.

Let d be the canonical parameter for D in $\mathcal{N}$, and $d' = [acl(d) \cap (A \cup J)] \cup [dcl(d) \cap A^{*\circ}]$. By Lemma 2.3.15 $d \in dcl(d')$ in $\mathcal{N}$. In $\mathcal{N}'$ by assumption $d' \in dcl(\emptyset)$, and thus $d' \in C_\circ$. Thus D is 0-definable in $\mathcal{N}$.

Lemma 2.3.18. *Let V be a nonquadratic basic linear geometry, possibly with distinguished elements, forming part of a geometry J with field of scalars F which is stably embedded in $\mathcal{M}$. Let A be a C-definable affine space over V. Let $C' \supseteq C$ with $acl(C') \cap (J \cup A^{*\circ}) \subseteq C'$ and $acl(C', J) \cap A = \emptyset$. Then for $a \in A$, $tp(a/C' \cap A^{*\circ})$ implies $tp(a/C')$.*

Proof. We may take $C = \emptyset$. By the preceding lemma $A \cup A^* \cup J$ is fully embedded in $\mathcal{M}$ over the parameters $C_\circ = C' \cap (J \cup A^{*\circ})$. Thus

$$tp(a/\, dcl(C') \cap (A, A^*, J)^{\mathrm{eq}}) \Longrightarrow tp(a/C').$$

By Lemma 2.3.15

$$dcl(C') \cap (A, A^*, J)^{\mathrm{eq}} \subseteq dcl(C_\circ).$$

By quantifier elimination, $tp(v/C' \cap A^{*\circ})$ determines $tp(v/C' \cap A^{*\circ}, a)$ for $v \in J$, so $tp(a/C' \cap A^{*\circ})$ determines $tp(a/C' \cap A^{*\circ}, J)$. The claim follows.

Lemma 2.3.19. *A Lie coordinatizable structure is $\aleph_0$-categorical.*

Proof. It suffices to treat the case of a structure $\mathcal{M}$ equipped with a Lie coordinatization. The argument is inductive, using Lemmas 2.3.5 and 2.3.12 with Lemma 2.3.2, and some control of the algebraic closure. Let $\mathcal{N}_h$ be the part of $\mathcal{M}$ coordinatized by the tree up to height h, let $\mathcal{N}$ be $\mathcal{N}_h$ together with finitely many coordinatizing geometries at height $h + 1$, and let J be a further coordinatizing geometry at any level, with defining parameter a. Our claim is:

J realizes finitely many types over any finite subset of $\mathcal{N} \cup \{a\}$.

In the main case, J is itself at height $h+1$ and thus a is already in $\mathcal{N}_h$. However, with J fixed, we proceed inductively on h and on the number of components at level $h+1$, beginning with $\mathcal{N}$ empty.

Given this result, one can then get the uniform bounds required for $\aleph_0$-categoricity by one more induction over the tree (by height alone).

It will be convenient to assume that the geometries involved are basic, and are either finite, linear, or affine; that is, projective geometries should be replaced by their linear covers. This cannot be done definably. Since the expanded version of $\mathcal{M}$ interprets $\mathcal{M}$ and has essentially the same coordinatizing tree, this implies the stated result for $\mathcal{M}$.

Since the case in which J is finite is trivial, we need deal only with the linear and affine cases, to which Lemmas 2.3.5 and 2.3.12 apply, and may be combined with Lemma 2.3.2. This reduces the problem to the following: for $A \subseteq \mathcal{N}$ finite, show that $acl(Aa) \cap J$ is finite.

Suppose on the contrary that $acl(Aa) \cap J$ contains arbitrarily large finite-dimensional Aa-definable subspaces V of J. Fix such an Aa-definable subspace V of J. $\mathcal{N}$ is B-definable for some set of parameters B lying in $\mathcal{N}_h$, and by induction $acl(Ba) \cap J$ is finite. As A' varies over the set of realizations in $\mathcal{N}$ of the type of A over Ba, the corresponding A'-definable subspace V' varies over the realizations of the type of V over $acl(Ba) \cap J$. Let n_1 be the number of types of sets AA' as A' varies in this manner, and let n_2 be the number of types of the corresponding sets VV'. Now n_1 is bounded, by induction hypothesis, since $B \subseteq \mathcal{N}_h$ and $A \subseteq \mathcal{N}$; $\mathcal{N}$ can be thought of as obtained by appending one geometry J' to a structure $\mathcal{N}'$ with $\mathcal{N}'_h = \mathcal{N}_h$ and with one fewer component at height $h+1$. We have arrived at the following: n_1 is bounded, and as the dimension of V increases, n_2 is unbounded; but $n_1 \geq n_2$. This contradiction yields a bound on the dimension of V and hence on the size of $acl(Aa) \cap J$.

2.4 ORTHOGONALITY

Definition 2.4.1

1. A normal geometry *is a structure J with the following properties (uniformly—in every elementary extension):*

(i) *$acl(a) = a$ for $a \in J$.*

(ii) *Exchange: if $a \in acl(Ba') - acl(B)$, then $a' \in acl(Ba)$.*

(iii) *If $a \in J^{\mathrm{eq}}$, then $acl(a) = acl(B)$ for some $B \subseteq J$.*

(iv) *For $J_0 \subseteq J$ 0-definable and nonempty, if $a, a' \in J$ and $tp(a/J_0) = tp(a'/J_0)$ then $a = a'$.*

2. A normal geometry is reduced *if it satisfies the further condition:*

(v) *$acl(\emptyset) = dcl(\emptyset)$ in J^{eq}.*

This distinction is illustrated by Example 2.4.5.

Lemma 2.4.2. *Projective geometries in our sense are normal geometries. The basic projective geometries are normal and reduced.*

Proof. Note that we include the polar and quadratic cases.

Conditions (i) and (ii) are the usual geometric properties in most cases. In the polar and quadratic case this includes the fact that the various parts of the geometry do not interact pointwise, e.g., for $q \in Q$ in the quadratic case, $acl(q) \cap V = \emptyset$. This can be computed in the basic linear model using quantifier elimination. We remark also that (i) requires $acl(\emptyset) = \emptyset$, which is not so much true as a matter of how the structure is viewed; for this purpose one takes a model in which objects such as the field K are encoded in $\mathcal{M}^{\mathrm{eq}}$ (or in the language). Condition (iii) was verified in §2.3. For (iv), note that apart from the polar and quadratic cases, if there are nontrivial 0-definable subsets they are determined by the set of values of a quadratic form or a hermitian form on the line representing a projective point. If a and a' have the same type over J_0, then they lift to points in the linear space having the same type over the preimage of J_0. But these sets generate the whole vector space.

In the polar case it may happen (e.g., in the basic case) that the two vector spaces involved are 0-definable. However, the type of a linear form over a vector space determines the linear form. Similarly in the quadratic case, the type of a quadratic form over its domain determines the form, and conversely the type of a vector over Q determines the vector as an element of the dual space, and hence determines the vector.

For (v) in the basic case, apply weak elimination of imaginaries for the associated basic semiprojective geometry to an element of $acl(\emptyset)$.

The following is a modified form of Lemma 1 of [HrBa].

Lemma 2.4.3. *Let J_1, J_2 be normal geometries, fully embedded and* 0-*definable in a structure $\mathcal{M}$. Then one of the following occurs:*

1. J_1, J_2 *are* orthogonal*: every 0-definable relation on $J_1 \cup J_2$ is a boolean combination of sets of the form $R_1 \times R_2$ with R_i an $acl(\emptyset)$-definable relation on J_i; or*
2. J_1, J_2 *are* 0-linked*: there is a 0-definable bijection between J_1 and J_2.*

Proof. If (1) fails then there is a 0-definable relation $R \subseteq J_1^{n_1} \times J_2^{n_2}$ for some n_1, n_2 which is not a finite union of $acl(\emptyset)$-definable rectangles $A_1 \times A_2$ ($A_i \subseteq J_i^{n_i}$ $acl(\emptyset)$-definable). It follows by compactness that we have $b_1 \in J_1^{n_1}$ such that $R(b_1) = \{b_2 \in J_2^{n_2} : R(b_1, b_2)\}$ is not $acl(\emptyset)$-definable. Our first claim is

$$(*) \qquad \begin{array}{l} \text{If } b_1 \in J_1^{n_1},\ R \subseteq J_1^{n_1} \times J_2^{n_2} \text{ is 0-definable,} \\ \text{and } R(b_1) \text{ is not } acl(\emptyset)\text{-definable, then } acl(b_1) \text{ meets } J_2. \end{array}$$

By stable embedding, $R(b_1)$ is J_2-definable. Let $c_2 \in J_2^{\mathrm{eq}}$ be its canonical parameter. Then by assumption c_2 is not algebraic over $\emptyset$, and then by (iii) we conclude that $acl(c_2)$ meets J_2, and $(*)$ follows.

Now take $a_2 \in acl(b_1) \cap J_2$ and let $S(a_2)$ be the locus of b_1 over a_2. As a_2 is algebraic over $S(a_2)$, $S(a_2)$ is not definable over $acl(\emptyset)$. Thus by another application of $(*)$, $acl(a_2)$ meets J_1. Let $a_1 \in acl(a_2) \cap J_1$. By the argument just given, we can also find $a_2' \in acl(a_1) \cap J_2$. But then $a_2' \in acl(a_2) \cap J_2 = \{a_2\}$ and thus $acl(a_1) = acl(a_2)$, and furthermore we have shown that in this relation a_1 determines a_2 (and of course, conversely). Thus $dcl(a_1) = dcl(a_2)$ and we have a 0-definable bijection f between two 0-definable sets $D_1 \subseteq J_1$ and $D_2 \subseteq J_2$. By (iv) and compactness each element a_1 of J_1 is determined by some a_1-definable subset of D_1, and hence (using f) by some a_1-definable subset $T(a_1) \subseteq D_2$. Therefore this set $T(a_1)$ is not definable over $acl(\emptyset)$, and by $(*)$ and the subsequent argument a_1 belongs to the domain of some 0-definable bijection between parts of J_1 and J_2. By compactness J_1 and J_2 are 0-linked.

Remark 2.4.4

Under the hypotheses of the preceding lemma, if J_1 and J_2 are reduced, then the first alternative can be strengthened as follows:

1′. J_1, J_2 *are* strictly orthogonal*: every 0-definable relation on $J_1 \cup J_2$ is a boolean combination of sets of the form $R_1 \times R_2$ with R_i a* 0-*definable relation on J_i.*

This holds since $dcl(\emptyset)$-definable sets are 0-definable.

Example 2.4.5. *J_1, J_2 carry equivalence relations E_1, E_2 with two infinite classes and no other structure. Then these are normal geometries, but not reduced. In $J_1 \times J_2$ we may add a bijection between J_1/E_1 and J_2/E_2. This would fall under the orthogonal case, but not the strictly orthogonal case.*

Lemma 2.4.6. *Let J_1 and J_2 be basic linear geometries canonically embedded in the structure $\mathcal{M}$. Suppose that in $\mathcal{M}$ there is a 0-definable bijection $f : PJ_1 \leftrightarrow PJ_2$ between their projectivizations. Then there is a 0-definable bijection $\hat{f} : J_1 \leftrightarrow J_2$ which is an isomorphism of unoriented weak geometries, and which induces f.*

Proof. Without loss of generality we may take the universe to be $J_1 \cup J_2$. Recall that in the basic linear geometries any bilinear or quadratic forms involved may be taken to be K-valued, and $acl(\emptyset) = dcl(\emptyset)$.

J_i consists either of a single vector space, a pair of spaces in duality, or a quadratic geometry (V, Q) and correspondingly PJ_i consists either of the corresponding projective model, two projective spaces, or the pair (PV, Q). The given f preserves algebraic closure, which is the span in the projective sense (except in Q) and hence f is covered by a map $\hat{f}$ which is linear on each vector space in J_i (relative to an isomorphism of the base fields) and which agrees with f on Q in the quadratic case. At this point we will identify the base fields, writing $K = K_1 = K_2$. There are finitely many such maps $\hat{f}$, and the set of them is implicitly definable, so by Beth's theorem they are definable over $acl(\emptyset) = dcl(\emptyset)$, or in other words, are 0-definable.

Fix one such $\hat{f}$. The type of $\hat{f}(a)$ is determined by the type of a, for a a finite string of elements. When a quadratic form is present we may recognize the totally isotropic spaces as those on which only one nontrivial 1-type is realized; in the polar case a totally isotropic space consists of a pair of orthogonal spaces, and one nontrivial 1-type is realized in each factor. It follows that $\hat{f}$ preserves orthogonality. Furthermore, if quadratic or skew quadratic forms Q_1, Q_2 are present (given, or derived from a hermitian form), then there is a function F for which $Q_2(\hat{f}(x)) = F(Q_1(x))$, where $F : K_0 \to K_0$ with $K_0 = K$ except in the hermitian case, where it is the fixed field of an automorphism σ of order 2. The function F is additive (consider orthogonal pairs) and linear with respect to elements of K^2 or in the hermitian case, K_0. In any case, it follows that F is linear on K_0 and is given by multiplication by an element α of K_0; in other words, $Q_2 = \alpha Q_1$. This sort of shift is allowed by a weak isomorphism, so our claim follows except in the polar, symplectic, and quadratic cases, to which we now turn.

In the polar and symplectic cases the 1-type structure is trivial and we have a function $F : K \to K$ for which $\beta_2(\hat{f}v, \hat{f}w) = F(\beta_1(v, w))$, where β_i gives either a duality between two spaces, or a symplectic structure. This map is visibly linear, so we are in the situation considered previously.

Now we consider the quadratic case. On the symplectic part we have $\beta_{V_2} = \alpha\beta_{V_1}$ for some α. Considering pairs (v,q) in $V \times Q$ we find that $\hat{f}q(\hat{f}v) = F'(q(v))$ for some function F' which similarly must be multiplication by a constant (for example, by considering the effect of replacing v by a scalar multiple); as the form associated to $\hat{f}q$ is $\alpha \cdot \beta_{V_1}$, we find $\hat{f}q = \alpha q$. This leads to the particularly unsatisfactory conclusion that the actions of V_1 and V_2 on Q_1 and Q_2 are related by

$$\hat{f}q +_2 \hat{f}v^2 = \hat{f}(q +_1 (\alpha^{1/2}v)^2).$$

We can, however, adjust $\hat{f}$ by taking $\hat{f}'(v) = \alpha^{1/2}v$ and then we find that the inner product, action of Q, and translation by V all agree in the two models.

Lemma 2.4.7. *Let J_1 and J_2 be basic quadratic or polar linear geometries canonically embedded in the pseudofinite structure $\mathcal{M}$. Suppose that in $\mathcal{M}$ there is a 0-definable bijection $f : PJ_1 \leftrightarrow PJ_2$ between the projectivizations of V_1 and V_2 (V_i is one of the two factors of J_i, in the polar case, and the vector part, in the quadratic case). Then there is a 0-definable bijection $\hat{f} : J_1 \leftrightarrow J_2$ which is an isomorphism of weak geometries, and which induces f.*

Proof. By the preceding lemma, f lifts to the linear part of J_1, J_2 covering PV_i. It remains to be seen that the linear or quadratic forms on V_2 which correspond to elements of J_1, transported by $\hat{f}$, are realized by elements of J_2. In finite approximations to $\mathcal{M}$, *all* such maps are realized, and in particular, all definable ones are realized in $\mathcal{M}$ by elements of J_2. If $\hat{f}$ is chosen to preserve the symplectic structure (exactly) in the quadratic case, then all structure will be preserved by the induced map.

Lemma 2.4.8. *Let $\mathcal{M}$ be a structure, D, I definable subsets, and $\{A_i : i \in I\}$ a collection of uniformly i-definable subsets of $\mathcal{M}$. Assume that $acl(\emptyset) = dcl(\emptyset)$ in D^{eq}, and that D is orthogonal to I and is orthogonal to each A_i over i. Then D is orthogonal to $I \cup \bigcup_i A_i$ (and hence strictly orthogonal to the same set).*

Proof. It will be convenient to use the term "relation between A and B" for a subset of $A^m \times B^n$ with m, n arbitrary. We have to analyze a relation between D and $I \cup \bigcup_i A_i$, and it suffices to consider the part relating D to $\bigcup_i A_i$. Fix i. Then R gives a relation between D and A_i involving a finite number of $acl(i)$-definable subsets of D^m for some m. These belong to a finite i-definable boolean algebra of subsets of D^m, which by strict orthogonality is 0-definable over D, and may be taken to be independent of i by dividing I into 0-definable sets. The elements of this boolean algebra belong to $acl(\emptyset)$ in D^{eq} and hence are 0-definable. The relation with A_i can be expressed in terms of them, and I may be broken up further into finitely many 0-definable pieces on which the definition is constant.

Definition 2.4.9. *The* localization P/A *of a projective geometry P over a finite set A is the geometry obtained from the associated linear geometry L as follows. Let $L_0 = acl(A) \cap L$, and projectivize $L_0^{\perp}/\,rad(L_0)$. If the vector space $L_0^{\perp}/\,rad(L_0)$ supports a quadratic geometry Q_0 then add that geometry to the localization as well. (The radical $rad(L_0)$ is $L_0 \cap L_0^{\perp}$; in the quadratic case $L_0^{\perp}$ has a quadratic part which is taken to consist of quadratic forms which vanish on $rad(L_0)$; in the orthogonal case in characteristic 2 we may also have to add a quadratic part—see the following remark.)*

Remark 2.4.10

The previous definition uses the convention that inner products are 0 *where undefined. In the linear case one therefore works with L/L_0. In the polar case L_0 consists of two spaces and the orthogonal spaces "switch sides." In the quadratic case $Q/Q \cap L_0^{\perp}$ is a space of quadratic forms on the correct space $(L \cap L_0^{\perp})/\,rad(L_0)$. (It would not be well-defined, however, as a space of forms on $L/\,rad(L_0)$.) Finally, one unusual phenomenon occurs in localizing orthogonal geometries in characteristic* 2*. Let q be a quadratic form associated to a non-degenerate symplectic form on V, and for simplicity let $A = \{v\}$ be a single nonzero vector of V. If $q(v) = 0$, then the form q descends to $\bar{L} = v^{\perp}/\langle v\rangle$; otherwise, for each nonsingular* 2*-space H containing $\langle v\rangle$ in L, the restriction of q to $H^{\perp}$ induces a quadratic form on $\bar{L}$, and as H varies the collection $\bar{Q}$ enlarges $\bar{L}$ to a quadratic geometry $(\bar{L}, \bar{Q})$.*

If P is a basic projective geometry, then this geometry is again a basic projective geometry, since the base field is named. In most cases it gives a geometry of the same type we began with. We could also define the *full localization* by taking P modulo the equivalence relation $acl(xA) = acl(yA)$ with all induced structure. The nontrivial atoms of the full localization are either components of our localization or affine spaces over its linear part.

Lemma 2.4.11. *Let P, Q be basic projective geometries defined and orthogonal over the set A and fully embedded over A in $\mathcal{M}$. Then their localizations are orthogonal over any set B over which they are defined.*

Proof. We may suppose that $A \subseteq B$ or $B \subseteq A$, with the proviso in the latter case that we allow P, Q to be localizations of geometries defined over B.

If $A \subseteq B$ and $P/B, Q/B$ are nonorthogonal, then they have a B-definable bijection which is unique and hence defined over $A \cup (acl(B) \cap (P \cup Q))$ (which serves to define the localizations). But over A this gives a relation on $P \cup Q$ which violates the orthogonality.

If $B \subseteq A$ and $P = \hat{P}/A$, $Q = \hat{Q}/A$ with $\hat{P}, \hat{Q}$ basic B-definable projective geometries, then nonorthogonality over B gives a B-definable bijection between $\hat{P}$ and $\hat{Q}$ which induces an A-definable bijection between the localizations.

2.5 CANONICAL PROJECTIVE GEOMETRIES

Throughout this section we work in a Lie coordinatized structure $\mathcal{M}$.

Definition 2.5.1. *Let $J = J_b$ be a b-definable weak projective Lie geometry in the structure $\mathcal{M}$. Then J is a* canonical projective *geometry if*

1. *J is fully embedded over b; and*
2. *If $tp(b') = tp(b)$ and $b' \neq b$, then J_b and $J_{b'}$ are orthogonal.*

A terminological note: there is no connection between the use of the term "canonical" in connection with canonical embeddings, and canonical projectives. In the case of embeddings the term refers to the so-called "canonical language," which has not been introduced explicitly here, while in the latter case it refers to the canonicity condition (2).

Lemma 2.5.2. *Let P_b be a b-definable projective geometry fully embedded in a Lie coordinatizable structure $\mathcal{M}$. Then there is a canonical projective geometry in $\mathcal{M}^{\mathrm{eq}}$ nonorthogonal to P_b over a finite set.*

Proof. We may assume P_b is basic, and since it lives in $\mathcal{M}^{\mathrm{eq}}$, we may replace $\mathcal{M}$ by a bi-interpretable structure and suppose that $\mathcal{M}$ is coordinatized by Lie geometries. If P_b is orthogonal to each of the coordinatizing geometries over their defining parameters, then repeated use of Lemma 2.4.8 shows that P_b is orthogonal to the ambient model $\mathcal{M}$, and hence to itself, which is not the case (the equality relation refutes this).

So we may suppose that P_b is one of the coordinatizing geometries, and that b is the parameter associated with P_b in the coordinatization of $\mathcal{M}$, so that it represents a branch $(b_1, \ldots, b_n)$ (or $b_0, \ldots, b_n$ with b_0 the 0-definable root) of the tree structure on $\mathcal{M}$ associated with a sequence of geometries (and finite algebraically closed sets) in $\mathcal{M}$, with $b_n = b$. Minimize n subject to nonorthogonality to the original geometry, so that for each geometry of the form J_{b_i}, with $i < n$, the associated projective geometry is orthogonal to P_b.

Consider the conjugates $P_{b'}$ of P_b. If $P_b, P_{b'}$ are nonorthogonal over a finite set, then the appropriate localization of $P_{b'}$ is orthogonal to the coordinatizing geometries Q for b over any set over which $P_{b'}$ and Q are defined. It follows by induction that $P_{b'} \cap acl(b', b_1, \ldots, b_i) = \emptyset$ for all $i \leq n$; notice that the induction step is vacuous when b_i is algebraic over its predecessor. For $i = n$ we have $acl(b, b') \cap P_{b'} = \emptyset$ and similarly $acl(b, b') \cap P_b = \emptyset$. Thus the nonorthogonality gives a unique (b, b')-definable bijection between P_b and $P_{b'}$, preserving the unoriented weak structure, and also, by an explicit hypothesis, preserving the Witt defect in the quadratic case.

Nonorthogonality of the associated geometries defines an equivalence relation on the conjugates of b and for any pair b', b'' of equivalent conjugates we have a canonical (b', b'')-definable isomorphism $\iota_{b',b''}$ between the geometries as weak geometries. Let b_1, b_2, b_3 be three conjugates of b for which the corresponding geometries are nonorthogonal. Using the orthogonality as above we may show that $acl(b_1, b_2, b_3) \cap P_{b_3} = \emptyset$ and hence the unique $acl(b_1, b_2, b_3)$-definable bijection between P_{b_1} and P_{b_3} agrees with the composition of the canonical bijections $P_{b_1} \leftrightarrow P_{b_2}$ and $P_{b_2} \leftrightarrow P_{b_3}$. So these identifications cohere and we can attach to an equivalence class c of conjugates of b a single weak projective geometry Q_c^w canonically identified with the given weak projective geometries. The geometry we want is the basic projective geometry Q_c associated with Q_c^w. We still must check that it is in fact canonical. This follows since the conjugates of c distinct from c are the classes of conjugates of b inequivalent to b.

Lemma 2.5.3. *Let P_b be a b-definable projective geometry fully embedded in a Lie coordinatizable structure, and let J_c be a canonical projective geometry nonorthogonal to P_b with canonical parameter c. Then $c \in dcl(b)$ and $P_b \subseteq dcl(b, J_c)$.*

Proof. For the first point, if c' is a conjugate of c over b, then $P_{c'}$ is nonorthogonal to P_b and hence to P_c; so $c = c'$. Thus $c \in dcl(b)$. There is a (b, c)-definable bijection between the localizations of P_b and J_c, and the localization of P_b over $\{b, c\}$ is P_b since $c \in dcl(b)$ (or for that matter since $c \in acl(b)$). Thus this bijection induces a function from J_c onto P_b.

Lemma 2.5.4. *Let P_b, $P_{b'}$ be nonorthogonal b-definable and b'-definable canonical projective geometries, not assumed to be conjugate. Then $dcl(b) = dcl(b')$ and there is a unique (b, b')-definable bijection between them, which is an isomorphism of weak, unoriented geometries.*

Proof. The first point follows from Lemma 2.5.3 and allows us to construe (b, b') as either b or b'. The rest is in Lemmas 2.4.3 and 2.4.6.

We will discuss the issue of orientation further.

Lemma 2.5.5. *Let P_b be a canonical projective quadratic geometry. There is a coordinatizing quadratic geometry J_c and a definable unoriented weak isomorphism of P_b with J_c. We may choose c so that if we orient P_c according to this isomorphism, the orientation is independent of the choice of c within its type over b.*

Proof. Let J_c be a coordinatizing geometry not orthogonal to P and minimized in the sense that c is as low in the tree structure on $\mathcal{M}$ as possible. Then by the previous lemma $b \in dcl(c)$ and by the minimization, as in the proof of Lemma 2.5.2, $acl(c) \cap P = \emptyset$. Thus the nonorthogonality gives a definable weak unoriented isomorphism. Conjugates of c over b, or for that matter conjugates of c over the empty set for which the corresponding geometry is nonorthogonal to P_b, have compatible orientations by the orientation condition in the definition of Lie coordinatization.

For a discussion of orientation the following terminology is convenient.

Definition 2.5.6

1. *A* standard system of geometries *is a 0-definable function $J : A \to \mathcal{M}^{\mathrm{eq}}$ whose domain A is a complete type over $\emptyset$ and whose range is a family of canonical projective geometries.*
2. *Two standard systems of geometries are* equivalent *if they contain a pair of nonorthogonal geometries. In this case there is a 0-definable identification between the systems, since nonorthogonality gives us a 1-1 correspondence between the domains, and the nonorthogonal pairs have canonical identifications.*

Lemma 2.5.7. *In a Lie coordinatized structure the quadratic geometries can be assigned compatible orientations, in the sense that in nonorthogonal geometries the orientations are identified by the canonical weak unoriented isomorphism between appropriate localizations. This can be done 0-definably.*

Proof. We first orient the standard systems made up of projective quadratic geometries. Here we just choose one representative of each equivalence class of such systems, and use the given orientations.

With this as a frame of reference we orient an arbitrary quadratic geometry P_b. There is a unique canonical projective quadratic geometry J_c oriented in the first step and nonorthogonal to P_b, and we have $c \in dcl(b)$. There is a canonical isomorphism between P_b and the localization of J_c at $A = acl(b) \cap J_c$. (By Lemma 2.5.5 it provides a well-defined identification of orientations.) It will be convenient to look at the linear quadratic geometry (V, Q) associated with J_c, and at $B = acl(b) \cap (V, Q)$, which carries the same information as A (as far as J_c is concerned).

B does not meet Q, as this would result in the localization of J_c at B, and hence also P_b, being orthogonal rather than quadratic. Let B_0 be a linear complement to $rad\, B$ in B. We can localize at B in two steps: first with respect to B_0, then with respect to $rad\, B$. At the first step the set Q is unchanged, but we modify the Witt defect as follows: $\omega^{B_0}(q{\restriction}B_0^\perp) = \omega(q) + \omega(q{\restriction}B_0)$. Here, on the right, ω is in one case the orientation function chosen already on J_c, and in the other the ordinary Witt defect for a form on an finite and even dimensional space (B_0 carries a nondegenerate symplectic form and is therefore even dimensional). At the second localization, by $rad\, B$, the linear part is replaced by a subspace of finite codimension and the radical is factored out; the space Q is also reduced to the set of forms vanishing on $rad\, B$. As this does not alter the Witt defect of such forms in the finite dimensional case, we let $\omega^B = \omega^{B_0} \restriction Q \cap B^\perp$.

One must check the consistency of such conventions, but this reduces to their correctness in the finite dimensional case, using common localizations.

The initial orientations on the coordinate geometries will not necessarily agree with the ones given here; according to the orientation condition, on a given level of the coordinatization tree, within a given nonorthogonality class, they will be completely correct or completely incorrect. We may change the orientations of the coordinate geometries to agree with our canonical assignment, and nothing is lost.

Example 2.5.8

It is appropriate to return to the canonical unoriented example at this point. Take an unoriented quadratic geometry, and let $\mathcal{M}$ consist of two copies of this geometry, with an identification, and with both possible orientations. To orient this geometry one must name an element of $acl(\emptyset)$.

There are two canonical projectives in this example, with each of the two possible orientations. Our canonical orientation procedure is not available. We can, however, pick an orientation on one of the canonical projective quadratic geometries and extend this orientation to the rest of the structure. Since the orientation is in $acl(\emptyset) - dcl(\emptyset)$, this produces a slightly enriched structure.

If the example is put higher up the coordinatization tree of a structure, it forces us to break the symmetry between elements which are not algebraic over $\emptyset$.

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