Small representations of SL_2 in the finite Morley rank category

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In this article we consider representations of SL_2 which are interpretable in finite Morley rank theories, meaning that inside a universe of finite Morley rank we shall study the following definable objects: a group G isomorphic to SL_2, an abelian group V, and an action of G on V; V is thus a definable G-module on which G acts definably. Our goal will be to identify V with a standard G-module, under an assumption on its Morley rank. (A word on this notion will be said shortly, after we have stated the results.)

It will be convenient to work with a faithful representation, possibly replacing SL_2 by the quotient PSL_2, and we shall write $G \simeq (\mathrm{P})\mathrm{SL}_2$ to cover both cases.

Theorem. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a faithful action of G on V for which V is G-minimal. Assume $\operatorname{rk} V \leq 3 \operatorname{rk} \mathbb{K}$. Then V bears a structure of $\mathbb{K}$-vector space such that:*

- *either $V \simeq \mathbb{K}^2$ is the natural module for $G \simeq \mathrm{SL}_2(\mathbb{K})$, or*
- *$V \simeq \mathbb{K}^3$ is the irreducible 3-dimensional representation of $G \simeq \mathrm{PSL}_2(\mathbb{K})$ with $\operatorname{char} \mathbb{K} \neq 2$.*

The characteristic 0 case essentially reduces to a theorem of Loveys and Wagner (Fact 1.2 below), or the following consequence of it:

Lemma 1.4. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a quasi-simple algebraic group G over $\mathbb{K}$, a torsion-free abelian group V, and a faithful action of G on V for which V is G-minimal. Then $V \rtimes G$ is algebraic.*

In earlier versions of this article we relied on the following proposition, which the reader will now find in an appendix (the notion of unipotence there is not quite the algebraic one).

Proposition. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$ of characteristic p, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a non-trivial action of G on V. Then for v generic in V, $C_G^\circ(v)$ is toral or unipotent (possibly trivial).*

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Our theorem involves the Morley rank of a structure; the reader should bear in mind that this is an abstract analog of the Zariski dimension, which can be axiomatized by some natural properties [4]. The Morley rank is however not necessarily related to any geometry or topology, being a purely model-theoretic notion. Yet in general if a field $\mathbb{K}$ has Morley rank k and V is an algebraic variety of Zariski-dimension d over $\mathbb{K}$, then its Morley rank is dk. The rank hypothesis in the Theorem would thus amount, if the configuration were known to be algebraic, to assuming that $\dim V \leq \dim G$; but of course the possibility for a field to have a finite Morley rank $k > 1$ makes algebraic geometry less general than our context. More precisely, model theorists have constructed what they call "red fields" [1]: fields of finite Morley rank with a definable subgroup of the additive group. These exist only in positive characteristic but horribly complicate matters, as our proof will confirm.

We work in a ranked universe as in [4]. Indeed, the semi-direct product $V \rtimes G$ is a ranked group in the sense of Borovik and Poizat [9, Corollaire 2.14 and Théorème 2.15]. We shall not go too deeply into purely model-theoretic arguments but will merely use the natural, intuitive properties of Morley rank as a notion of dimension.

Let us now say a word about the proof of the Theorem. As we have mentioned, there is no geometry a priori on $V \rtimes G$, and our efforts will be devoted to retrieving a suitable vector space structure on V which arises from the action of G. Model-theoretically speaking, the main tool is Zilber's so-called Field Theorem (Fact 1.9 below), which enables one to find an (algebraically closed) field inside a solvable, non-nilpotent, infinite group of finite Morley rank. A major difficulty is that the action of an algebraic torus of G will not induce a vector space structure on all of V. And even if such a good structure exists, this does not mean that G itself is linear on V. The 2-dimensional case relies on a theorem by Timmesfeld (Fact 1.1 below); as for dimension 3, we extend the field action manually and some curious computations will, in the end, prove linearity of G. Once we have G acting linearly on V, we can apply the classification of linear representations given in [3, Théorème 10.3] to adjust the linear structure so that action becomes algebraic. On the other hand, the detailed analysis leading to the linearity contains enough information to arrive at the same conclusion directly.

Now that we have said what the present paper is, let us say what it is not: it does not relate directly to the classification project for simple groups of finite Morley rank, although some rudimentary aspects of representation theory have been used there, via the amalgam method.

We heartily thank Borovik for directing our attention to the final section of [3], and to the referee for an elegant simplification of our original, at times clumsy analysis.

1 Preparatory Remarks

The proof of our Theorem is in §2; for the moment, we gather and make observations of a more general nature.

In a sense, the starting point of the work was the following characterization of the natural SL_2-module due to Timmesfeld. An action satisfying the second assumption is usually called *quadratic* (since unipotent elements act quadratically).

Fact 1.1 ([12, Chapter I, Theorem 3.4])**.** *Let $\mathbb{K}$ be a field and $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$. Let V be a faithful G-module. Suppose the following:*

(i). $C_V(G) = 0$

(ii). $[U, U, V] = 1$*, where U is a maximal algebraic unipotent subgroup of G.*

Let $0 \neq v \in C_V(U)$ and $W = \langle v^G \rangle$. Then there exists a field action of $\mathbb{K}$ on W such that W is the natural G-module. In particular $G \simeq \mathrm{SL}_2(\mathbb{K})$.

We shall say that an algebraic group is simple if it is simple, group-wise speaking. If the group is perfect, has finite center, and the quotient is simple infinite, we call it quasi-simple.

We shall use the non-standard notation $(+)$ to denote quasi-direct sum, i.e. the sum of two subgroups (of a fixed abelian group) which have a finite, possibly non-trivial, intersection.

In §1.1 we shall apply some model theory to linearize actions on a torsion-free module; the rest of the paper deals with the positive characteristic setting. In §1.2 we recast some classical remarks on actions of finite Morley rank, notably Zilber's Field Theorem. This leads us to §1.3, where we give a general three-fields argument for theories of finite Morley rank. Eventually, a closer analysis of the action of tori will be made in §1.4.

1.1 Algebraicity in characteristic 0

We first deal with actions on torsion-free groups, simply using a general result of Loveys and Wagner. Let us specialize [7] to our context.

Given a group K acting on a connected group of finite Morley rank H, H is said to be *K-minimal* if no non-trivial definable connected proper subgroup of H is K-invariant.

Fact 1.2 (special case of [7, Theorem 4])**.** *In a universe of finite Morley rank, consider the following definable objects: an abelian, torsion-free group A, an infinite group S, and a faithful action of S on A for which A is S-minimal. Then there is a subgroup $A_1 \leq A$ and a field $\mathbb{K}$ such that $A_1 \simeq \mathbb{K}_+$ definably, $A \simeq \mathbb{K}_+^n$, and S embeds into $\mathrm{GL}_n(\mathbb{K})$ for some n.*

The claim that $A \simeq \mathbb{K}_+^n$ is not in the actual statement of [7, Theorem 4], but obvious from its proof. We shall also need the following result.

Fact 1.3 ([8, Theorem 1.4(a)]). *In a universe of finite Morley rank, consider the following definable objects: a field* $\mathbb{K}$*, and a subgroup* $H \leq \mathrm{GL}_n(\mathbb{K})$ *such that* $H/Z(H)$ *is infinite and simple. If* H *is irreducible on* $\mathbb{K}^n$*,* $\operatorname{char}\mathbb{K} = 0$ *and some Borel subgroup of* H *is non-abelian, then* $H = Z(H) \cdot E$ *for some algebraic group* $E \leq H$*.*

As a consequence, if $Z(H)$ is finite then H is Zariski-closed.

Lemma 1.4. *In a universe of finite Morley rank, consider the following definable objects: a field* $\mathbb{K}$*, a quasi-simple algebraic group* G *over* $\mathbb{K}$*, a torsion-free abelian group* V*, and a faithful action of* G *on* V *for which* V *is* G*-minimal. Then* $V \rtimes G$ *is algebraic with respect to some* $\mathbb{K}$*-vector space structure on* V*.*

Proof. By Fact 1.2, there is a field structure $\mathbb{L}$ and an $\mathbb{L}$-vector space structure on V such that $G \hookrightarrow \mathrm{GL}(V)$ definably. By Fact 1.3 the image $\hat{G}$ of G in $\mathrm{GL}(V)$ is an algebraic subgroup of $\mathrm{GL}(V)$. By [3] (or [11]) the isomorphism $G \to \hat{G}$ is the composition of an algebraic isomorphism with an isomorphism induced by a field isomorphism $\alpha : \mathbb{K} \simeq \mathbb{L}$. Using α to identify $\mathbb{K}$ with $\mathbb{L}$, we may suppose that G is an algebraic subgroup of $\mathrm{GL}(V)$, and at the same time we may view the $\mathbb{L}$-vector space structure on V as a $\mathbb{K}$-vector space structure. □

As a consequence, our theorem is virtually trivial *in characteristic* 0 (in characteristic p, matters will be more difficult.)

Proof of our Theorem in characteristic 0. Let V and G as in the statement of our main result (see the introduction), and assume that V is torsion-free. Then by Lemma 1.4, $V \rtimes G$ is algebraic; $\dim_{\mathbb{K}} V$ is 2 or 3, and as irreducible algebraic representations of (P)SL_2 are well-known, the theorem is proved. □

Before we move on and for the sake of pure digression, let us also mention a simplification of an existing result allowed by Lemma 1.4.

Fact 1.5 ([6, Theorem A in char. 0]). *Let* G *be a connected, non-solvable group of finite Morley rank acting definably and faithfully on a* torsion-free *connected abelian group* V *of Morley rank* 2*. Then there is an algebraically closed field* $\mathbb{K}$ *of Morley rank* 1 *and characteristic* 0 *such that* $V \simeq \mathbb{K}^2_+$*, and* G *is isomorphic to* $\mathrm{GL}_2(\mathbb{K})$ *or* $\mathrm{SL}_2(\mathbb{K})$ *in its natural action.*

Proof. V is clearly G-minimal. By Fact 1.2, there is an interpretable field structure $\mathbb{K}$ such that $G \hookrightarrow \mathrm{GL}_n(\mathbb{K})$ with $V \simeq \mathbb{K}^n_+$. Clearly the dimension must be 2, making the rank of the field 1. So there is a field $\mathbb{K}$ of rank 1 such that $V \simeq \mathbb{K}^2_+$ and $G \hookrightarrow \mathrm{GL}_2(\mathbb{K})$. But definable subgroups of $\mathrm{GL}_2(\mathbb{K})$, especially over a field of rank 1, are known: [10, Theorem 5] together with connectedness and non-solvability of G this forces either $G \simeq \mathrm{GL}(V)$ or $G \simeq \mathrm{SL}(V)$. □

1.2 Nilpotent and solvable actions

We start with an abstract version of a famous theorem of Malcev.

Fact 1.6 ([9, Théorème 3.18])**.** *Let G be a connected, solvable group of finite Morley rank acting definably and faithfully on a definable, abelian group A. If a definable subgroup $B \leq A$ is G- or G'-minimal, then B is centralized by G'.*

Lemma 1.7. *In a universe of finite Morley rank, consider the following definable objects: a reductive algebraic group G, a nilpotent group V, and an action of G on V. Let U be a unipotent subgroup of G. Then $V \rtimes U$ is nilpotent.*

Proof. We may assume that U is a maximal unipotent subgroup. In this case, and by reductivity of G, U is the commutator subgroup of the Borel subgroup $B = N_G(U)$ [2, top of p. 65]. Now consider $H = V \rtimes B$ and write $F^\circ(H) = V \rtimes K$ with $K \leq B$. The quotient $H/F^\circ(H) \simeq B/K$ is abelian by [4, Theorem 9.21], so $U = B' \leq K$. $\square$

Lemma 1.8. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a quasi-simple algebraic group G over $\mathbb{K}$, an abelian group V, and a non-trivial action of G for which V is G-minimal. Then V has the same characteristic as $\mathbb{K}$.*

Proof. Let p denote the characteristic of $\mathbb{K}$. Fix a maximal unipotent subgroup U of G. By Lemma 1.7, $V \rtimes U$ is nilpotent. If $p = 0$ and V is torsion or if $p \neq 0$ and $pV = V$, then Nesin's structure theorem for nilpotent groups [4, Theorem 6.8] yields $[V, U] = 0$. As conjugates of U generate G, the action is trivial, a contradiction. $\square$

Now comes Zilber's celebrated Field Theorem.

Fact 1.9 ([9, Théorème 3.7])**.** *Let A be a definable abelian group with an infinite abelian group of automorphisms M definable inside a structure of finite Morley rank. If A is M-minimal, then there is an infinite definable field $\mathbb{K}$ and a definable $\mathbb{K}$-vector space structure of dimension 1 on A such that M acts $\mathbb{K}$-linearly, i.e. $A \simeq \mathbb{K}_+$ and $M \hookrightarrow \mathbb{K}^\times$ definably.*

The reader should keep in mind that if $\mathbb{K}$ is a field of finite Morley rank, then any infinite subgroup of $\mathbb{K}^\times$ additively generates $\mathbb{K}$. Now another word on fields of finite Morley rank.

Lemma 1.10. *In a universe of finite Morley rank, let A, T be definable, abelian, infinite groups such that A is T-minimal and the action is faithful. Let K be a definable group normalizing A and T. Then K centralizes T.*

Proof. We let K act on $\operatorname{End} A$ by $s^\varphi(a) := (s(a^{\varphi^{-1}}))^\varphi$. By assumption, K normalizes the image of T in $\operatorname{End} A$, which additively generates a definable algebraically closed field. As there are no definable groups of automorphisms of a field of finite Morley rank [4, Theorem 8.3], K acts trivially on T. $\square$

1.3 A three fields configuration

The following lemma will appear at a crucial moment in the proof of our main theorem, when dealing with the Cartan subalgebra of the adjoint representation of (P)SL_2.

Lemma 1.11. *In a universe of finite Morley rank, consider the following definable objects: three infinite fields $\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3$, a connected group T acting on the underlying additive groups, and a map $B : \mathbb{K}_1 \times \mathbb{K}_2 \to \mathbb{K}_3$.*

Suppose that for each $i = 1, 2, 3$, $T/C_T(\mathbb{K}_i)$ acts on $(\mathbb{K}_i, +)$ as an infinite subgroup of $\mathbb{K}_i^\times$. Suppose further that $C_T^\circ(\mathbb{K}_1)$ is non-trivial in its action on $(\mathbb{K}_2, +)$. If B is bi-additive and globally T-covariant (in the sense that $B(k_1^t, k_2^t) = B(k_1, k_2)^t$), then either B is identically 0 or gives rise to a definable isomorphism $\mathbb{K}_1 \simeq \mathbb{K}_3$.

Proof. For the sake of clarity we shall write $k_1 \otimes k_2$ for $B(k_1, k_2)$. Moreover, we shall drop field multiplication operations. Last but not least, the action of t on k_i will be denoted by $t \cdot k_i$; as $T/C_T(\mathbb{K}_i)$ acts as a subgroup of $\mathbb{K}_i^\times$, one has $t \cdot (k_i k_i') = (t \cdot k_i)k_i'$, which allows simply writing $t \cdot k_i k_i'$.

Let $T_1 = C_T^\circ(K_1)$ and Θ be its image in $\mathbb{K}_2^\times$; by assumption, $\Theta \neq 1$. It follows that Θ additively generates $\mathbb{K}_2$.

First suppose that there exist $(k_1, k_2) \in K_1 \times K_2$ both non-zero such that $k_1 \otimes k_2 = 0$. By T_1-covariance and right additivity, it follows that $k_1 \otimes \mathbb{K}_2 = 0$. Now by T-covariance and left additivity, $\mathbb{K}_1 \otimes \mathbb{K}_2 = 0$: B is identically zero.

We may therefore suppose that for any $(k_1, k_2) \in \mathbb{K}_1 \times \mathbb{K}_2$ both non-zero, $k_1 \otimes k_2 \neq 0$. So any $k_2 \in \mathbb{K}_2 \setminus \{0\}$ induces a function $f_{k_2} : \mathbb{K}_1 \to \mathbb{K}_3$ given by

$$f_{k_2}(k_1) = (k_1 \otimes k_2)/(1 \otimes k_2)$$

We claim that this function does not depend on the choice of $k_2 \neq 0$. Let $k_2' \in \mathbb{K}_2$ be non-zero. As Θ additively generates $\mathbb{K}_2$, there are finitely many $t_i \in T_1$ such that $k_2' = \sum_i t_i \cdot k_2$. Let $k_1 \in \mathbb{K}_1$. Then by T_1-covariance,

$$\begin{aligned} k_1 \otimes k_2' &= \textstyle\sum_i [t_i \cdot (k_1 \otimes k_2)] = \sum_i [t_i \cdot (1 \otimes k_2) f_{k_2}(k_1)] \\ &= f_{k_2}(k_1) \textstyle\sum_i [t_i \cdot (1 \otimes k_2)] = f_{k_2}(k_1)(1 \otimes k_2') \end{aligned}$$

Since $k_2' \neq 0$, $1 \otimes k_2' \neq 0$, and dividing one finds $f_{k_2'}(k_1) = f_{k_2}(k_1)$, as desired.

So let $f : \mathbb{K}_1 \to \mathbb{K}_3$ be this function. Clearly $f(k_1) = f_1(k_1) = (k_1 \otimes 1)/(1 \otimes 1)$ is additive; we now show that it is multiplicative.

As the image of T in $\mathbb{K}_1^\times$ is by assumption non-trivial, it additively generates $\mathbb{K}_1$. It therefore suffices to show that f is multiplicative on (the image of) T. We shall denote by $\bar{t}$ the elements induced by t in $\mathbb{K}_1^\times$ and in $\mathbb{K}_2^\times$; in context, there is no risk of confusion. Let $s, t \in T$. Then

$$\begin{array}{rcl} f(\bar{s}\bar{t}) & = & (\bar{s}\bar{t} \otimes 1)/(1 \otimes 1) \\ & = & t \cdot (\bar{s} \otimes \bar{t}^{-1})/(1 \otimes 1) \\ & = & t \cdot \left[(\bar{s} \otimes \bar{t}^{-1})/(1 \otimes \bar{t}^{-1})\right] \left[(1 \otimes \bar{t}^{-1})/(1 \otimes 1)\right] \\ & = & t \cdot \left[f_{\bar{t}^{-1}}(\bar{s})(1 \otimes \bar{t}^{-1})/(1 \otimes 1)\right] \\ & = & f(\bar{s}) \left[t \cdot (1 \otimes \bar{t}^{-1})/(1 \otimes 1)\right] \\ & = & f(\bar{s}) \left[(\bar{t} \otimes 1)/(1 \otimes 1)\right] \\ & = & f(\bar{s}) f(\bar{t}) \end{array}$$

So the function $f : \mathbb{K}_1 \to \mathbb{K}_3$ is a non-zero definable ring homomorphism between two infinite definable fields of finite Morley rank. It follows that it is a definable isomorphism. □

1.4 Around tori

We return to abelian-by-abelian situations, trying to capture the behavior of semi-simple elements. The logician's approach to this topic relies on the following notion [5]. A *good torus* is a definable, abelian, divisible group with no torsion-free definable section; the latter condition being equivalent to: every definable subgroup is the definable hull of its torsion subgroup. We shall call a subgroup or an element of a group of finite Morley rank *toral* if it is contained in a good torus.

The following theorem of Wagner states that in finite Morley rank, the multiplicative group of a field of characteristic p is a good torus.

Fact 1.12 ([13, Corollary 9])**.** *Let $\mathbb{K}$ be a field of finite Morley rank of characteristic $p > 0$. Then $\mathbb{K}^\times$ has no torsion-free definable section.*

Lemma 1.13. *In a universe of finite Morley rank, consider the following definable objects: two infinite, abelian groups K and H, and a faithful action of K on H for which H is K-minimal. Suppose that H has exponent p and that K contains a non-trivial q-torus for each $q \neq p$. Then* $\operatorname{rk} H = \operatorname{rk} K$.

Proof. By Zilber's Field Theorem, there is a field structure $\mathbb{L}$ such that $H \simeq \mathbb{L}_+$ and $K \hookrightarrow \mathbb{L}^\times$. In particular, $\operatorname{char} \mathbb{L} = p$. Now $\mathbb{L}^\times/K$ is torsion-free, so by Wagner's Theorem, K cannot be proper in $\mathbb{L}^\times$. Hence $\operatorname{rk} K = \operatorname{rk} \mathbb{L} = \operatorname{rk} H$. □

Recall that $\operatorname{Tor} G$ stands for the set of torsion elements of a group G.

Lemma 1.14. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$ of characteristic p, a subgroup $\Theta \leq \mathbb{K}^\times$, a connected abelian group V, and an action of Θ on V. Then there is $\theta \in \operatorname{Tor} \Theta$ such that $C_V(\Theta) = C_V(\theta)$ and $[V, \Theta] = [V, \theta]$.*

Proof. By Wagner's Theorem (Fact 1.12), $\Theta = d(\operatorname{Tor} \Theta)$. By the descending chain condition on centralizers, $C_V(\Theta) = C_V(\operatorname{Tor} \Theta) = C_V(\theta_1, \ldots, \theta_n)$ for torsion elements, and we take a generator θ_0 of the finite cyclic group $\langle \theta_1, \ldots, \theta_n \rangle$: one has $C_V(\Theta) = C_V(\theta_0)$, and this holds true of any root of θ_0.

Now the group $[V, \operatorname{Tor}\Theta]$ is definable and connected (a consequence of Zilber's indecomposability theorem [4, Theorem 5.26]), so

$$\Sigma = \{t \in \Theta : [V, t] \leq [V, \operatorname{Tor}\Theta]\}$$

is a definable subgroup of Θ containing $\operatorname{Tor}\Theta$. Again, as $\Theta = d(\operatorname{Tor}\Theta)$, it follows that $\Sigma = \Theta$, that is $[V, \Theta] = [V, \operatorname{Tor}\Theta]$. We turn to the lattice of definable, connected groups $\{[V, t] : t \in \operatorname{Tor}\Theta\}$: if t_1 is a root of t_2, then $[V, t_1] \geq [V, t_2]$. So by the ascending chain condition, there is $\theta \in \operatorname{Tor}\Theta$ such that $[V, \theta] = [V, \operatorname{Tor}\Theta] = [V, \Theta]$. We may assume that θ is a root of θ_0, and we are done. $\square$

And now for a little bit of cohomology.

Fact 1.15. *Let A be a connected, abelian group of finite Morley rank of bounded exponent and α a definable automorphism of finite order coprime to the exponent of A. Then $A = C_A(\alpha) \oplus [A, \alpha]$. Moreover, if $A_0 < A$ is a definable, connected, α-invariant subgroup, then $[A, \alpha] \cap A_0 = [A_0, \alpha]$.*

Proof. Let ad_α and Tr_α be the adjoint and trace maps, that is:

$$\operatorname{ad}_\alpha(x) = x^\alpha - x \qquad \text{and} \qquad \operatorname{Tr}_\alpha(x) = x + \cdots + x^{\alpha^{n-1}}$$

where n is the order of α. It is easily seen, as A has no n-torsion, that $\ker \operatorname{ad}_\alpha \cap \ker \operatorname{Tr}_\alpha = 0$. In particular, $\operatorname{rk} A \geq \operatorname{rk}(\ker \operatorname{ad}_\alpha) + \operatorname{rk}(\ker \operatorname{Tr}_\alpha)$. Moreover, $\operatorname{im} \operatorname{ad}_\alpha \leq \ker \operatorname{Tr}_\alpha$ and $\operatorname{im} \operatorname{Tr}_\alpha \leq \ker \operatorname{ad}_\alpha$. It follows therefore that $\operatorname{rk} A \geq \operatorname{rk}(\ker \operatorname{ad}_\alpha) + \operatorname{rk}(\ker \operatorname{Tr}_\alpha) \geq \operatorname{rk}(\ker \operatorname{ad}_\alpha) + \operatorname{rk}(\operatorname{im} \operatorname{ad}_\alpha) = \operatorname{rk} A$, so $\operatorname{im} \operatorname{ad}_\alpha = \ker \operatorname{Tr}_\alpha$. Hence $A = \ker \operatorname{ad}_\alpha \oplus \ker \operatorname{Tr}_\alpha = \ker \operatorname{ad}_\alpha \oplus \operatorname{im} \operatorname{ad}_\alpha = C_A(\alpha) \oplus [A, \alpha]$.

Let $a_0 \in A_0$; then $a_0 \in \operatorname{ad}_\alpha(A_0)$ iff $\operatorname{Tr}_\alpha(a_0) = 0$ iff $a_0 \in \operatorname{ad}_\alpha(A)$. $\square$

Lemma 1.16. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$ of characteristic p, a subgroup T of $\mathbb{K}^\times$, a connected abelian p-group A, and an action of T on A. Then $A = C_A(T) \oplus [A, T]$. Let $A_0 < A$ be a definable, connected, T-invariant subgroup. Then $C_A(T)$ covers $C_{A/A_0}(T)$ and $C_T(A) = C_T(A_0, A/A_0)$.*

Proof. We may apply Lemma 1.14 and find a torsion element $t_0 \in T$ such that $C_A(T) = C_A(t_0)$ and $[A, T] = [A, t_0]$. We use Fact 1.15 and deduce that $A = C_A(T) \oplus [A, T]$.

If $x \in A$ maps to an element in $C_{A/A_0}(t_0)$, then denoting the canonical projection by π one has $\pi \circ \operatorname{ad}_{t_0}(x) = \operatorname{ad}_{t_0} \circ \pi(x) = 0$. Hence $\operatorname{ad}_{t_0}(x) \in A_0$ and by Fact 1.15 there is $x_0 \in A_0$ such that $\operatorname{ad}_{t_0}(x) = \operatorname{ad}_{t_0}(x_0)$, whence $x \in x_0 + \ker \operatorname{ad}_{t_0}$, and $\ker \operatorname{ad}_{t_0} = C_A(t_0)$.

Let $\Theta = C_T(A_0, A/A_0)$; Θ is definable. Then $C_A(\Theta)$ covers $C_{A/A_0}(\Theta) = A/A_0$; it follows that $A = C_A(\Theta) + A_0 \leq C_A(\Theta)$. $\square$

2 Proof of the Theorem

We now attack our main result.

Theorem. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a faithful action of G on V for which V is G-minimal. Assume $\operatorname{rk} V \leq 3 \operatorname{rk} \mathbb{K}$. Then V bears a structure of $\mathbb{K}$-vector space such that:*

- *either $V \simeq \mathbb{K}^2$ is the natural module for $G \simeq \mathrm{SL}_2(\mathbb{K})$, or*
- *$V \simeq \mathbb{K}^3$ is the irreducible 3-dimensional representation of $G \simeq \mathrm{PSL}_2(\mathbb{K})$ with $\operatorname{char} \mathbb{K} \neq 2$.*

Let us begin with something completely different: a piece of notation and an observation regarding $(\mathrm{P})\mathrm{SL}_2$.

Notation 1. Let $G \simeq (\mathrm{P})\mathrm{SL}_2$. Fix a Borel subgroup B of G and let $U = B'$ be its unipotent radical. Let T be an algebraic torus such that $B = U \rtimes T$. Let i be the involution in T, and $\zeta \in N_G(T)$ a 2-element inverting T (the order of ζ depends on the isomorphism type of G).

Fact 2.1. *A definable, connected subgroup of $(\mathrm{P})\mathrm{SL}_2$ is toral, has only unipotent elements, or contains a maximal unipotent subgroup of $(\mathrm{P})\mathrm{SL}_2$.*

Proof. Let K be a definable, connected subgroup. We may assume that K is proper; as K is then solvable (see for instance [10, Théorème 4]), up to conjugacy $K \leq B$. Let $U_1 = U \cap K$; if K is not toral, then $U_1 \neq 1$. If some elements in K are not unipotent, that is if $K > U_1$, then we may split $K = U_1 \rtimes T_1$ for some non-trivial, connected toral subgroup; so fixing $u \in U_1^\#$ one has $K \geq \langle u^K \rangle \geq \langle u^{T_1} \rangle = U$, as observed after Fact 1.9. □

The time has now come to start the proof.

Notation 2. In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a non-trivial action of G on V for which V is G-minimal. Let $k = \operatorname{rk} \mathbb{K}$ and assume $\operatorname{rk} V \leq 3k$.

First of all one may assume that the action does not satisfy all the assumptions of Fact 1.1, as otherwise $\operatorname{rk} V = 2k$; in particular, the action is not quadratic or $C_V(G) \neq 0$. As observed after Lemma 1.4, we may suppose that V is not torsion-free. It is then easily seen that V has prime exponent p, and $\mathbb{K}$ must have characteristic p as well by Lemma 1.8.

Our goal is to show that $G \simeq \mathrm{PSL}_2$ acts on $V \simeq \mathbb{K}^3$ in the usual irreducible way (in characteristic $\neq 2$). The proof will involve studying various subgroups of V, defining a field action piecewise, and eventually proving its linearity. On our way we shall prove $p \neq 2$.

2.1 Structure of the module

A word on terminology: if K is a group acting on a definable, connected, abelian group V, we shall call V a K-*module*. In particular, K-submodules are by definition definable and connected.

Step 3. V has a T-submodule $X_0 \neq 0$ such that $C_T^\circ(X_0) \neq 1$.

Proof. Suppose $C_V^\circ(T) = 0$; by Lemma 1.16, $C_V(T) = 0$ as well, whence $C_V(G) = 0$, so the action is not quadratic. Let $V_1 \leq V_2 \leq V$ be B-submodules, with V_1 and V_2/V_1 B-minimal. Notice that by Malcev's Theorem (Fact 1.6), both V_1 and V_2/V_1 are even T-minimal. Notice further that $V_2 < V$, as otherwise the action is quadratic, a contradiction.

If $\operatorname{rk} V_1 \neq k$ then by Lemma 1.13, the action of T on V_1 can't have a finite kernel: $T_1 = C_T^\circ(V_1)$ must be infinite, and taking $C_V^\circ(T_1) \geq V_1$ we are done. So we may assume $\operatorname{rk} V_1 = k$. Suppose $\operatorname{rk} V_2/V_1 \neq k$. As V_2/V_1 is T-minimal, the group $T_2 = C_T^\circ(V_2/V_1)$ is non-trivial by Lemma 1.13. Since T_2 is a good torus, $C_{V_2}(T_2)$ covers V_2/V_1 by Lemma 1.16, so $C_{V_2}^\circ(T_2)$ is non-trivial; in particular $C_V^\circ(T_2) \neq 1$: we are done.

So we may suppose $\operatorname{rk} V_1 = \operatorname{rk} V_2/V_1 = k$, and in particular $\operatorname{rk} V_2 = 2k$. Let $W_2 = (V_2 \cap V_2^\zeta)^\circ$; clearly $\operatorname{rk} W_2 \geq k$. If $(V_1 \cap W_2)^\circ \neq 0$, then by T-minimality of V_1, one has $V_1 \leq W_2$. By T-minimality of V_2/V_1, one finds that W_2 is either V_1 or V_2, a contradiction as neither is ζ-invariant since they are B-invariant and proper.

Therefore $(V_1 \cap W_2)^\circ = 0$, and in particular $V_2 = V_1(+)W_2$; whence W_2 is T-minimal, and ζ-invariant. As ζ inverts T, Lemma 1.10 then forces T to centralize W_2: we are done. □

Notation 4. Let $\Theta = C_T^\circ(X_0)$ and $X = C_V(\Theta)$. Let $M = [V, \Theta]$ and $Y = [X, U]$.

By Lemma 1.16, $V = M \oplus X$ and each is a non-trivial, $T \cdot \langle\zeta\rangle$-invariant submodule. By the indecomposability theorem, Y is definable and connected; it is U-invariant and non-trivial since otherwise X is $\langle U, \zeta\rangle = G$-invariant.

Step 5. $\operatorname{rk} M \geq 2k$ and $\operatorname{rk} X \leq k$.

Proof. We claim that for x generic in X, $C_G^\circ(x)$ is toral. Otherwise, as $C_G^\circ(x)$ contains $\Theta \leq T$, it contains either U or U^ζ by Fact 2.1; we may assume that for x generic in X, U centralizes x. Thus U centralizes X. As the latter is ζ-invariant, it follows that $G = \langle U, U^\zeta\rangle$ centralizes X, a contradiction.

Hence, the centralizer in G of the generic element of X is toral. Let $x \in X$ be generic, and suppose that $g \in G$ is such that $x \in X^g$. Then $\langle\Theta, \Theta^g\rangle \leq C_G^\circ(x)$ which is toral, so $C_G^\circ(\langle\Theta, \Theta^g\rangle)$ is an algebraic torus, which can be only $C_G^\circ(\Theta) = T$, and only T^g for a similar reason. Hence $g \in N_G(T) = T \cdot \langle\zeta\rangle = N_G(X)$. So X is generically disjoint from its distinct conjugates; it follows that

$$\operatorname{rk} X^G = 2k + \operatorname{rk} X \leq \operatorname{rk} V = \operatorname{rk} M + \operatorname{rk} X$$

Hence $\operatorname{rk} M \geq 2k$, and then $\operatorname{rk} X \leq k$. □

Observe that in particular $\operatorname{rk} V > 2k$.

Step 6. $T = \Theta$ centralizes X.

Proof.

Claim. $[X, T] \leq C_V(U)$.

Proof: Suppose on the contrary that $C_{[X,T]}(U) < [X, T]$. Let $A = A_0/C_{[X,T]}(U)$ be a T-minimal submodule of $[X, T]/C_{[X,T]}(U) \neq 0$. By Lemma 1.16, $C_A(T) = 0$; so by Zilber's Field Theorem there is a field structure $\mathbb{L}_1$ such that $A \simeq (\mathbb{L}_1, +)$ and T induces an infinite subgroup of $\mathbb{L}_1^\times$.

By construction $A_0 \not\leq C_V(U)$; let N be a B-minimal quotient of $[A_0, U] \neq 0$. Let $x \in A_0$ be such that the map $\varphi = \pi \circ \operatorname{ad}_x : U \to N$ is non-trivial, where $\pi : [A_0, U] \to N$ is the canonical projection. By Malcev's Theorem (Fact 1.6), U centralizes N, so N is T-minimal and φ is a morphism. If there is $u \in U^\#$ such that $\varphi(u) = 0$, then since $\langle u^\Theta \rangle = U$ by Zilber's Field Theorem one finds $\ker \varphi = U$: a contradiction. So φ is injective, and $\operatorname{rk} N \geq \operatorname{rk} U = k$. Let $u \in U^\#$ and $s \in \Theta^\#$. Then $u^s \neq u$, and by Θ-covariance and injectivity of φ, $\varphi(u)^s = \varphi(u^s) \neq \varphi(u)$; in particular $\varphi(u) \notin C_N(\Theta)$, and $C_N(\Theta) < N$. By T-minimality of N, $C_N(\Theta)$ is finite, so there is another field structure $\mathbb{L}_3$ such that $N \simeq (\mathbb{L}_3, +)$ and Θ induces an infinite subgroup of $\mathbb{L}_3^\times$.

Since Θ centralizes A (as a section of X) and U centralizes N, we may apply Lemma 1.11 with $B : A \times U \to N$ defined by $B(a, u) = \pi([a_0, u])$, where $a_0 \in A_0$ lies above $a \in A = A_0/C_{[X,T]}(U)$. We get $\mathbb{L}_1 \simeq \mathbb{L}_3$, whence $\operatorname{rk} A = \operatorname{rk} N \geq k$; in particular $\operatorname{rk} X = k$ and $A = X$ is T-minimal. Since ζ normalizes X, Lemma 1.10 implies that T centralizes X, a contradiction. $\lozenge$

We now finish the proof of Step 6. $[X, T]$ is a ζ-invariant submodule of $C_V(U)$. But $C_V(U) \cap C_V(U)^\zeta \leq C_V(G)$ which is finite, hence $[X, T] = 0$. $\square$

Step 7. $Y \leq M$; $V = Y \oplus X \oplus Y^\zeta$; $\operatorname{rk} X = \operatorname{rk} Y = k$; U centralizes Y, $(X + Y)/Y$, and $V/(X + Y)$.

Proof. Fix $x_0 \in X$ and $u_0 \in U^\#$. Let $m \in M$ and $x \in X$ be such that $[x_0, u_0] = m + x$. Since $U = u_0^T \cup \{0\}$, we have:

$$\{[x_0, u] : u \in U\} = \{[x_0, u_0]^t : t \in T\} \cup \{0\} = \{m^t + x : t \in T\} \cup \{0\} \subseteq M + \langle x \rangle$$

But by Zilber's indecomposability theorem, $[x_0, U]$ is connected, so $[x_0, U] \leq M$: hence $Y \leq M$.

By Lemmas 1.13 and 1.16, the rank of any submodule of M is a multiple of k: going back to Step 5 one sees that $\operatorname{rk} M = 2k$ and $\operatorname{rk} X = k$. Now Y is not ζ-invariant since it would otherwise be $G = \langle U, \zeta \rangle$-invariant; on the other hand M is ζ-invariant, so $Y < M$. It follows that Y has rank k as well. In particular Y is T-minimal, and B-minimal, so by Fact 1.6, U centralizes Y. Moreover $Y \cap Y^\zeta$ is finite, whence T-central, so $Y \cap Y^\zeta \leq C_M(T) = 0$ by Lemma 1.16. One thus has $M = Y \oplus Y^\zeta$ and $V = Y \oplus X \oplus Y^\zeta$.

U centralizes $(X + Y)/Y$ by construction. Since $V/(X + Y) \simeq Y^\zeta$ is T-minimal, by Fact 1.6 again, U centralizes $V/(X + Y)$ as well. $\square$

Step 8. The characteristic is not 2; $G \simeq \mathrm{PSL}_2$ and ζ (now of order 2) inverts X.

Proof. Suppose $p = 2$. For any $u \in U^\times$ consider the map $\varphi : V \to V$ given by commutation with u. Since $p = 2$ one finds $\operatorname{im} \varphi \leq \ker \varphi$. By Step 7, $\operatorname{im} \varphi \leq X + Y$; as $U = u^T \cup \{0\}$ and T centralizes X (Step 6), $C_X(u) = C_X(U)$. It follows that $\operatorname{im} \varphi \leq C_{X+Y}(u) = C_X(u) + Y = C_{X+Y}(U)$. Hence $[V, U] \leq C_V(U)$, and Fact 1.1 applied to the action of G on $V/C_V(G)$ yields $\operatorname{rk} V = 2k$, a contradiction.

Hence $p \neq 2$. As T centralizes X, the involution $i \in T$ cannot invert X. It follows that $G \simeq \mathrm{PSL}_2$. In particular ζ has order 2.

Now since Y is T-minimal, i must either invert or centralize it. If i centralizes Y, then it centralizes $M = Y \oplus Y^\zeta$ and X: so i centralizes V, a contradiction. Hence i inverts Y, and also Y^ζ: it follows that i inverts M. So ζ which is conjugate to i must also invert a module of rank $2k$. Let us write $M = M^{+\zeta} \oplus M^{-\zeta}$ under the action of ζ. Then Y is disjoint from both, showing that both have rank k. It follows that ζ must invert X. □

Step 9. $C_V(G) = 0$, $C_X(U) = 0$, and $C_V(U) = Y$.

Proof. By Step 8, $C_V(G) \leq C_V(T) \cap C_V(\zeta) = C_X(\zeta) = 0$. Since ζ inverts X, it normalizes $C_X(U)$ which is $G = \langle U, \zeta\rangle$-invariant, whence finite, whence by connectedness in $C_V(G) = 0$. So $C_X(U) = 0$.

We already know that $Y \leq C_V(U)$ (Step 7). If there is $x + y^\zeta \in C_V(U)$ with $x \in X$, $y \in Y$, then $y \neq 0$ as $C_X(U) = 0$. Hence $y \notin X$, that is T does not centralize y: using Zilber's Field Theorem, $Y = y^T \cup \{0\}$. Now U centralizes $x + y^\zeta$, so using T, U centralizes $x + Y^\zeta$; U then also centralizes $Y^\zeta \leq C_V(U, U^\zeta)$: a contradiction. □

2.2 Linearity

The second part of the proof is of so different a nature that if the reader wishes to take a break, he may now. We shall start afresh with the following knowledge.

- ζ has order 2 (Step 8)
- $Y = C_V^\circ(U) = [X, U]$ is B-minimal (Notation 4 and Steps 7 and 9)
- $V = Y \oplus X \oplus Y^\zeta$; $\operatorname{rk} X = \operatorname{rk} Y = k$ (Notation 4 and Step 7)
- $X = C_V(T)$ is inverted by ζ (Steps 6 and 8)

We now work towards understanding the scalar action on X.

Step 10. Let $x \in X$, $t \in T$, $u \in U^\#$. Then there is a unique $x' \in X$ such that $[x', u] = [x, u]^t = [x, t \cdot u]$; x' depends on x and t, but not on u.

Proof. Fix $u_1 \in U^\#$ and consider the definable morphism from X to Y which maps x to $[x, u_1]$. This is injective, as the kernel lies in $C_X(u_1) = C_X(T, u_1) \leq C_X(U) = 0$. By equality of ranks, the map is a bijection. Now suppose another $u_2 \in U^\#$ is given, and we have elements x'_1, x'_2 such that $[x'_i, u_i] = [x, u_i]^t$. Then there is $\tau \in T$ such that $u_2 = u_1^\tau$, and it follows that:

$$\begin{array}{rclclclcl} [x'_2, u_2] & = & [x, u_2]^t & = & [x, u_1^\tau]^t & = & [x, u_1]^{\tau t} & & \\ & = & [x, u_1]^{t\tau} & = & [x'_1, u_1]^\tau & = & [x'_1, u_1^\tau] & = & [x'_1, u_2] \end{array}$$

whence $x'_1 = x'_2$, as claimed. □

We can finally impose a linear structure on V. This is done piecewise using the decomposition $V = Y \oplus X \oplus Y^\zeta$. By our hypotheses, $C_Y(T) = (0)$ and Y is T-invariant. Let $\mathbb{L}$ be the subring of $\mathrm{End}(Y)$ generated by the image of T. As Y is T-minimal, $\mathbb{L}$ is a field (Fact 1.9) and $Y \simeq (\mathbb{L}, +)$

Notation 11.

- On Y, $\mathbb{L}$ acts as a subring of $\mathrm{End}(Y)$.
- On Y^ζ, we let $k \cdot y^\zeta = (k \cdot y)^\zeta$.
- On X, we let $k \cdot x$ be the unique $x' \in X$ such that $[x', u] = k \cdot [x, u]$ (Step 10; this does not depend on the choice of u).

We shall check that G acts linearly. We do it piecewise; notice that when we claim that U acts linearly on X, we mean that the operation induced by elements of U from X *to* V is linear, without claiming anything about invariance under the action.

Step 12. *$T \cdot \langle \zeta \rangle$ acts linearly on V. U acts linearly on $Y \oplus X$.*

Proof. By construction, T is linear on Y and Y^ζ. It is linear on X, as it acts trivially! By construction, ζ is linear on $Y \oplus Y^\zeta$. As it inverts X, it is also linear on X. So $T \cdot \langle \zeta \rangle$ is linear on V.

As U acts trivially on Y, it is linear on Y. It remains to see that U is linear on X. Let $u \in U$, $x \in X$, and $k \in \mathbb{K}$. By definition of the action on X, one has $[k \cdot x, u] = k \cdot [x, u]$, and therefore:

$$k \cdot x^u - k \cdot x = k \cdot [x, u] = [k \cdot x, u] = (k \cdot x)^u - k \cdot x$$

Linearity follows. □

It remains to prove that U is linear on Y^ζ. As T is, and since T acts transitively on $U^\#$, it suffices to exhibit one non-trivial element of U which is linear on Y^ζ.

Notation 13 (Bryant Park element)**.** Let $w = \zeta$ (it is an involution, after all). Let $u \in U$ be such that (wu) has order 3.

Such an element exists (this may be viewed as a special case of the Steinberg relations). We shall prove that this particular u is linear on Y^ζ.

FUGA

Step 14. For any $y \in Y$, there is a unique $x \in X$ such that $y^{wu} = y + x + y^w$.

Proof. A priori, one has

$$y^{wu} = y_1 + x + y_2^w$$

for elements $y_1, y_2 \in Y$ and $x \in X$. But U centralizes Y, $(X+Y)/Y$, and $V/(X+Y)$ by Step 7. So $y_2 = y$. We push further, using the fact that w inverts X (Step 8).

$$\begin{array}{rcl} y^{(wu)^2} & = & y_1^{wu} + x^{wu} + y^{wwu} \\ & = & y_1^{wu} - x^u + y \end{array}$$

and

$$y = y^{(wu)^3} = y_1^{wuwu} - x^{uwu} + y^{wu}$$

whence applying u^{-1},

$$y = y_1^{wuw} - x^{uw} + y^w$$

Now U^w centralizes Y^w, $(X+Y^w)/Y^w$, and $V/(X+Y^w)$ (Step 7), so $[y_1, u^w] \in X + Y^w$. It follows that y_1 is the projection on Y of y_1^{wuw}. On the other hand, $x^u \in X + Y$, so $x^{uw} \in X + Y^w$. Taking projections on Y modulo $X + Y^w$, one has $y_1 = y$. □

Step 15. Let $y \in Y$ and $x \in X$ be as in Step 14. Then $[x, u] = 2y$.

Proof. By definition,

$$y^{wu} = y + x + y^w$$

Let us iterate:

$$\begin{array}{rcl} y^{(wu)^2} & = & y^{wu} + x^{wu} + y^{wwu} \\ & = & (y + x + y^w) - x^u + y \\ & = & 2y + x - x^u + y^w \end{array}$$

and

$$\begin{array}{rcl} y^{(wu)^3} & = & 2y^{wu} + x^{wu} - x^{uwu} + y^{wwu} \\ & = & 2(y + x + y^w) - x^u - x^{uwu} + y \\ & = & 3y + 2x - x^u - x^{uwu} + 2y^w \end{array}$$

As wu has order three, one has:

$$2y + 2x - x^u - x^{uwu} + 2y^w = 0$$

Now u centralizes $(Y+X)/Y$, so there is $y_1 \in Y$ such that $x^u = x + y_1$. Let x_1 be associated to y_1 by Step 14: one has $y_1^{wu} = y_1 + x_1 + y_1^w$. Hence

$$\begin{array}{rcl} x^{uwu} & = & x^{wu} + y_1^{wu} \\ & = & -x^u + (y_1 + x_1 + y_1^w) \\ & = & -x - y_1 + y_1 + x_1 + y_1^w \\ & = & x_1 - x + y_1^w \end{array}$$

It follows that

$$2y + 2x - (x + y_1) - (x_1 - x + y_1^w) + 2y^w = 0,$$

and projecting onto Y modulo $X + Y^w$,

$$y_1 = 2y$$

so that $[x, u] = y_1 = 2y$. □

Notation 16. For $y \in Y$, let $x(y)$ be the element x given by Step 14.

Step 17. The function $x(y)$ is $\mathbb{L}$-linear.

Proof. Let $k \in \mathbb{L}$. Then

$$[x(k \cdot y), u] = 2(k \cdot y) = k \cdot (2y) = k \cdot [x(y), u] = [k \cdot x(y), u]$$

And we are done. □

Step 18. u is linear on Y^w.

Proof. Let $y \in Y$ and $k \in \mathbb{L}$; let $y_2 = k \cdot y$, and $x_2 = x(y_2)$. Then

$$(k \cdot y^w)^u = y_2^{wu} = y_2 + x_2 + y_2^w = k \cdot y + x_2 + k \cdot y^w$$

On the other hand,

$$k \cdot y^{wu} = k \cdot (y + x + y^w) = k \cdot y + k \cdot x + k \cdot y^w$$

As x is $\mathbb{L}$-linear, both expressions are equal: u is linear on Y^w. □

It follows that $G = \langle T, \zeta, u \rangle$ is $\mathbb{L}$-linear on V.

We may now finish the proof. We have a definable embedding of G into $\mathrm{GL}(V)$ with V 3-dimensional over $\mathbb{L}$. We may view this as a homomorphism from $\mathrm{PSL}_2(\mathbb{K})$ into $\mathrm{GL}(V)$, with the image of $\mathrm{PSL}_2(\mathbb{K})$ acting irreducibly on V. Let $\hat{G}$ be the Zariski closure in $\mathrm{GL}(V)$ of the image of $\mathrm{PSL}_2(\mathbb{K})$, also acting irreducibly on V. As V has dimension 3, the group $\hat{G}$ will be a simple algebraic group, so by Theorem A of [3] (or [11, Theorem 1.3]) the homomorphism $\mathrm{PSL}_2(\mathbb{K}) \to \hat{G}$ has the form $h \circ \varphi^\circ$ with φ an embedding of $\mathbb{K}$ into $\mathbb{L}$ and h a rational homomorphism defined on ${}^\varphi\,\mathrm{PSL}_2(\mathbb{K})$. We now return to G and ${}^\varphi G$, keeping the same notation for φ and h.

Since the composition $h \circ \varphi$ is definable, $\varphi[\mathbb{K}]$ is definable, and is therefore $\mathbb{L}$. As V with its $\mathbb{L}$-structure is a rational representation of ${}^\varphi G$, $V \rtimes {}^\varphi G$ is algebraic, and pulling back via φ^{-1}, we get a $\mathbb{K}$-structure making $V \rtimes G$ algebraic.

We note that in a more general setting, the Zariski closure $\hat{G}$ would be semisimple rather than simple, and there would be several associated maps h_i and φ_i, with the representation V being a tensor product; this is the case discussed in detail in [3, §10], [11, §6].

3 Appendix: actions of (P)SL2 and centralizers

In this appendix we give one further result on the structure of a generic stabilizer in a representation of $(\mathrm{P})\mathrm{SL}_2(\mathbb{K})$ of finite Morley rank in positive characteristic, whose proof is a variation on Step 5 of our main argument. Recall that a connected definable subgroup of a group of finite Morley rank is toral if it is included in a maximal torus, and p-unipotent if it is a nilpotent p-group of bounded exponent.

Proposition. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$ of characteristic p, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a non-trivial action of G on V. Then for v generic in V, $C_G^\circ(v)$ is toral or unipotent (possibly trivial).*

Proof. We first show that we may assume $C_V(G) = 0$. Assume the result holds when $C_V(G) = 0$ and let V be as in the statement. Let $V_0 = C_V(G) < V$. Since G is perfect, one has $C_{V/V_0}(G) = 0$, and the action of G on V/V_0 is non-trivial. By assumption, the result holds for V/V_0. Now let $v \in V$ be generic. Then $\bar{v} \in V/V_0$ is generic too, and in particular $C_G^\circ(\bar{v})$ is either toral or unipotent. As $C_G^\circ(v) \leq C_G^\circ(\bar{v})$, we are done.

So from now on we suppose $C_V(G) = 0$. In Notation 1 we had fixed a maximal unipotent subgroup $U \leq G$, $B = N_G(U)$ its normalizer, T an algebraic torus such that $B = U \rtimes T$, and a 2-element ζ inverting T.

Let $v \in V$ be generic. $C_G^\circ(v)$ is proper in $(\mathrm{P})\mathrm{SL}_2$, hence solvable [10, Théorème 4]; up to conjugacy, $C_G^\circ(v) \leq B$. Assume that $C_G^\circ(v)$ is neither unipotent nor toral. Then by Fact 2.1, $C_G^\circ(v)$ contains U.

So $C_G^\circ(v) = U \rtimes T_v$ for some non-trivial $T_v \leq T$. The family $\{T_v : v \in V, U \leq C_G^\circ(v) \leq B\}$ of subgroups of T is uniformly definable; as $\mathbb{K}$ has positive characteristic, $T \simeq \mathbb{K}^\times$ is a good torus, and the family is finite [5, Rigidity II]. It follows that there is a *common* $T_0 \leq T$ such that generically, $C_G^\circ(v)$ is conjugate to $U \rtimes T_0$.

Now let $V_1 = C_V(U)$. Clearly V_1 is infinite, taking a B-minimal subgroup of V and applying Malcev's Theorem (Fact 1.6). As any two distinct conjugates of U generate G and $C_V(G) = 0$, V_1 must be disjoint from V_1^g for $g \notin B$. It follows that $N_G(V_1) = B$ and that V_1 is disjoint from its distinct conjugates. One therefore has

$$\operatorname{rk} V_1^G = \operatorname{rk} V_1 + \operatorname{rk} G - \operatorname{rk} B = \operatorname{rk} V_1 + \operatorname{rk} \mathbb{K}.$$

By assumption, the generic element of V is centralized by a conjugate of $U \rtimes T_0$. Thus V_1^G is generic in V. But furthermore, for v generic in V_1, $C_G^\circ(v)$ is a conjugate of $U \rtimes T_0$ containing U; conjugacy is therefore obtained by an element of $N_G(U) = B$. As $B' = U$, $U \rtimes T_0$ is normal in B; hence $C_G^\circ(v) = U \rtimes T_0$. This means that T_0 centralizes a generic subset X of V_1; as $X + X = V_1$ it follows that $V_1 = C_V(U \rtimes T_0)$.

Let $W = V_1 \oplus V_1^\zeta$ and $\check{W} = W \setminus (V_1 \cup V_1^\zeta)$. The generic element of W is in $\check{W}$. Let $v \in \check{W}$. Clearly $T_0 \leq C_G^\circ(v)$. Moreover, if $C_G^\circ(v)$ is not toral, then it must meet a unipotent subgroup which can only be either U or U^ζ as $1 \neq T_0 \leq C_G^\circ(v)$. In that case, $C_G^\circ(v)$ contains either U or U^ζ by Fact 2.1, against the definition of $\check{W}$. This means that for $v \in \check{W}$, one has $T_0 \leq C_G^\circ(v) \leq T$. In particular, $\check{W}^G$ is *not* generic in V.

It follows that $W < V$. As V_1^G is generic in V, W cannot be G-invariant. Therefore $T \cdot \langle\zeta\rangle \leq N_G(W) < G$, and equality follows from maximality of $T \cdot \langle\zeta\rangle$. As $T \cdot \langle\zeta\rangle$ also normalizes $V_1 \cup V_1^\zeta$, one sees that $N_G(\check{W}) = T \cdot \langle\zeta\rangle$.

Let $w \in \check{W}$. Assume that $w \in \check{W}^g$ for some $g \in G$. Then $C_G^\circ(v)$ is a non-trivial connected subgroup of T, so $C_G(C_G^\circ(v)) = T = T^g$, and $g \in N_G(T) = T \cdot \langle\zeta\rangle = N(\check{W})$. This implies that

$$\operatorname{rk} \check{W}^G = \operatorname{rk} \check{W} + \operatorname{rk} G - \operatorname{rk} T = 2 \operatorname{rk} V_1 + 2 \operatorname{rk} \mathbb{K} = 2 \operatorname{rk} V_1^G.$$

But V_1^G is already generic in V which is infinite: this is a contradiction. $\square$

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