L^*-GROUPS OF ODD TYPE WITH RESTRICTED 2-TORAL ACTIONS II. PRÜFER RANK 2 AND 2-RANK AT LEAST 3: COMPONENT ANALYSIS

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Abstract. We begin the analysis of connected simple K^*-groups of finite Morley rank and odd type having Prüfer 2-rank 2 and 2-rank at least 3. More generally, we consider certain simple L^*-groups of odd type; degenerate type simple sections are allowed, but their definable automorphism groups are restricted. The present paper analyzes algebraic components in centralizers of involutions, isolating the expected configurations involving components that woujld be encountered in groups of type PSp_4 or G_2, namely $\mathrm{SL}_2 * \mathrm{SL}_2$ and possibly PSL_2. The recognition problem (via verification of BN-pair axioms) will be discussed subsequently. At that point one exotic configuration appears in the case of G_2 which can be eliminated in the context of finite simple group theory but which has not been eliminated in the finite Morley rank context.

In the general L^* context the so-called *uniqueness case* presents additional difficulties, but in the present case, as 2-rank is assumed to be greater than Prüfer rank, the uniqueness case is eliminated by prior work.

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Contents

1. Introduction

1.1. The Algebraicity Conjecture(s). The Algebraicity Conjecture for simple groups of finite Morley rank states that connected simple groups of finite Morley rank are simple algebraic groups over algebraically closed fields.

The Sylow 2-subgroup in a group G of finite Morley rank has a subgroup of finite index of the form

$$U * T$$

with U definable, connected, of finite exponent, and T a divisible abelian 2-group. The group G is said to have even, odd, mixed or degenerate type according as T is trivial, or U is trivial, or neither is trivial, or both are, respectively. Odd type includes algebraic groups over algebraically closed fields of any characteristic other than 2, including characteristic 0. The classification by type is independent of the choice of Sylow 2-subgroup, as they are conjugate.

This four-way division into types gives us four versions of the Algebraicity Conjecture which may appear to be inextricably interlinked, but this is misleading. The even and mixed type cases have been resolved, independently of the other two types.

Even & Mixed Type Theorem ([ABC08]). There are no connected simple groups of finite Morley and mixed type. Those of even type are algebraic; more precisely, they are Chevalley groups over algebraically closed fields of characteristic 2.

The case of degenerate type represents both the most doubtful and the most difficult portion of the conjecture. However, a mix of methods from finite group theory (both the theory of finite simple groups, and black box group theory) with model theoretic ideas of a more geometric character suffices to prove the following, which we will find useful here, even though our focus will be on groups of odd type.

Degenerate Type Theorem ([BBC07]). A connected group of finite Morley rank and degenerate type has trivial Sylow 2-subgroup.

In other words, any connected group of finite Morley rank which contains an involution has an infinite Sylow 2-subgroup.

The case of odd type can be approached by methods closely parallel to methods used in finite simple group theory, but does lead back to difficult problems in degenerate type. One of our concerns has been to separate the difficulties which may be viewed as inherited from the degenerate case from those specific to the analysis in odd type.

We are concerned accordingly with the following.

Odd Type Algebraicity Conjecture. A connected simple group of finite Morley rank of odd type is a Chevalley group over an algebraically closed field of characteristic other than 2.

If one combines this conjecture with the known results it can also be put in the following form.

Non-Degenerate Type Algebraicity Conjecture. A connected simple group of finite Morley rank which contains an involution is a Chevalley group over an algebraically closed field.

1.2. **Inductive strategies.** The analysis of groups of finite Morley rank is inductive, or bottom-up, but really divides into three somewhat independent layers—thin, quasi-thin, and generic type (i.e., tiny, small, and typical)—each with their own particular techniques. In terms of the Algebraicity Conjecture these layers should correspond to Lie ranks 1, 2, or above.

Whichever layer one considers, one typically assumes that the group under consideration is a minimal counterexample to the Algebraicity Conjecture. That is, one works inductively. But it is desirable to restrict this induction hypothesis in a way which clarifies what is actually required for the inductive argument, and which disentangles the various portions of the Algebraicity Conjecture as far as is possible. It is this approach which led to an unconditional proof of the Even and Mixed Type conjectures without first dealing with the caae of degenerate type.

In the odd type case, a similar approach throws into relief what the salient problems are for groups of degenerate type, and possibly some other groups of Prüfer rank 1, with regard to the Algebraicity Conjecture for groups with involutions.

Building on a sequence of results of increasing generality, the following was shown in [BC22a].

High Prüfer Rank Theorem. Let G be a simple group of finite Morley rank of odd type with Prüfer 2-rank at least three. Then one of the following applies.

(1) G is a Chevalley group over an algebraically closed field of characteristic other than 2.

(2) G has a proper definable strongly embedded subgroup.

(3) G has a definable simple section of odd type which is nonalgebraic and has Prüfer rank at most 2.

(4) G has a definable simple section of degenerate type on which some definable section of G of odd type acts faithfully as a group of automorphisms.

Note that in the last case, the section which acts faithfully can be supposed to be the definable hull of a nontrivial 2-torus.

In the K^* case, where all proper definable connected simple sections are assumed to be algebraic, it is known that one can eliminate the second alternative, and for that step it would suffice to have Prüfer rank at least 2.

To put this in an inductive setting, we assume that proper definable connected simple sections *with involutions* are algebraic, which eliminates the third alternative, and that any definable automorphism groups of connected simple sections of degenerate type are themselves of degenerate type, which eliminates the last alternative; but we allow simple definable sections of degenerate type. The High Prüfer Rank Theorem gives either an identification or a configuration of "uniqueness type" that calls for further analysis. One knows in this uniqueness case that Prüfer 2-rank and ordinary 2-rank agree, which is already somewhat pathological.

We continue here, and in subsequent papers, to analyze the situation in Prüfer 2-rank 2 and 2-rank at least 3, under similar inductive hypotheses. The restriction on automorphism groups of simple sections of degenerate type is denoted NTA_2, which is intended to suggest the phrase "no 2-toral automorphisms." A similar restriction on infinite elementary abelian 2-groups acting faithfully on degenerate type groups is known as Altınel's lemma and was the starting point for the successful analysis of groups of even and mixed type—where it is a lemma, rather than an assumption.

1.3. **The target theorem.** We aim ultimately at the following result, in which the terms "L^* group" and "NTA_2" refer to our inductive hypotheses on simple sections of odd type, or on automorphism groups of simple sections of degenerate type, respectively (Definition 2.1). Since the proof is not given in this paper we treat this as a conjecture here.

Conjecture 1. Let G be a connected simple L^* group of finite Morley rank of odd type satisfying the condition NTA_2, with Prüfer 2-rank 2 and

$$m_2(G) \geq 3.$$

Then either G is a simple Chevalley group over an algebraically closed field (PSp_4 or G_2), or G has 2-rank 3 and involves a configuration known from finite simple group theory, associated with groups of type

G_2 in characteristic 3, but with the wrong Borel subgroup (normalizer of a Sylow 3-subgroup),

We intend to give our analysis in three parts. The first part, *component analysis*, will be given here. This part is lengthy and troublesome, and involves a number of special cases which do not arise in nature and must be eliminated. We were able to bypass this kind of detailed analysis in the case of higher Prüfer rank by very general considerations.

With the analysis of the present paper in hand, one can prove a recognition theorem for PSp_4, corresponding to the case of Prüfer rank 2 and 2-rank at least 4 [BC22b]. In the case of Prüfer rank 2 and 2-rank 3 the target group is G_2. Here one gets either the desired identification or a rather specific configuration encountered also in the finite case, involving a base field of characteristic 3 [BC22c]. In the finite case the corresponding configuration is eliminated by character theoretic arguments.

The method of proof of the analog of Conjecture 1 in the case of higher Prüfer rank (where however there is also an unresolved case involving strongly embedding, when the Sylow 2-subgroup is a 2-torus) is to aim at a form of the Curtis-Tits theorem, involving generation by root SL_2-subgroups. That approach requires having some control of Lie rank 2 subgroups a priori and is not appropriate here.

In Prüfer rank 2 we aim at a direct construction of a BN-pair, once the necessary component analysis is in hand, and then apply results of Kramer, Tent, and van Maldeghem to identify the group. To verify that the expected (B, N)-pair has the desired properties, we first verify that the Weyl group has the expected structure, and then examine its action on root groups, reaching a qualitative approximation to the Chevalley commutator formula holds for positive roots, after which the appropriate properties follow.

As far as the pathological configuration arising in the G_2 analysis is concerned, while number of finite group theoretic arguments based on character theory have been successfully replaced in the context of groups of finite Morley rank by structural analyses, using for example the fact that any definable infinite field will be algebraically closed, the particular configuration that arises appears to be challenging, even in the setting of K^* groups.

Conjecture 1 leaves the case of Prüfer rank 2 and 2-rank 2 entirely open. Conjecturally of course this should correspond to groups of type PSL_3 but this is not part of what we are aiming at.

1.4. **Component analysis.** As in finite group theory the analysis in groups of sufficiently high 2-rank begins with an examination of the groups $E(C_G(i))$ for involutions i, which in our context shifts to the subgroup $E_{alg}(C_G(i))$, which is the product of the algebraic components. In favorable cases this analysis is handled largely by signalizer functor theory.

Along the way, some delicate points arise which call for the use of unipotence theory in the finite Morley rank context. The general theory was already given in [BC22a] at a level of generality that allows for some applications in our current setting (Prüfer rank 2, 2-rank at least 3). Up to a point this treatment is fairly uniform, but in Prüfer rank at least 3 there comes a point at which one can study the interaction of subgroups of type $(\mathrm{P})\mathrm{SL}_2$ by reduction to the case of Prüfer rank 2.

In Prüfer rank 2 we cannot escape a close examination of the various pathological cases which may arise in theory. We aim at the following, where part of the analysis involves a subgroup $E_{\mathcal{E}}$ of E_{alg} which is particularly well-behaved.

Theorem 1.1. *Let G be a connected simple L^* group of finite Morley rank of odd type satisfying the condition* NTA_2*, with Prüfer* 2*-rank* 2 *and*

$$m_2(G) \geq 3.$$

Then there are at most two conjugacy classes of involutions, and one of the following applies.

(1) *There are two conjugacy classes of involutions.*

Then the 2*-rank of* G *is* 4*; one conjugacy class of involutions satisfies* $E_{\mathcal{E}}(C_G(i)) = \mathrm{PSL}_2$*, and the other satisfies* $E_{\mathcal{E}}(C_G(i)) = \mathrm{SL}_2 *_2 \mathrm{SL}_2$*, with the same base field in all components; the two components of* $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ *are conjugate, and the Sylow* 2*-subgroup is as in* PSp_4*.*

(2) *There is one conjugacy class of involutions, and these satisfy*

$$E_{alg}(C_G(i)/OFC_G(i)) = \mathrm{SL}_2 *_2 \mathrm{SL}_2$$

(not necessarily over the same base field).

More precisely, if L *is a component of* $C_G(i)$*, then* L *is of type* SL_2*,* $C_G(i) = L *_2 C_G(L)$*, and* $E_{alg}(C_G(L)/OFC_G(L))$ *is of type* SL_2*.*

Furthermore, $C_G(i)$ *is connected and contains a Sylow* 2*-subgroup of* G*, isomorphic to that of* $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ *(in characteristic other than* 2*).*

In the finite case, there are three configurations involving an involution for which $E(C(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$, treated in [FW69, Fon70]: the configuration in which the components are conjugate, corresponding to PSp_4, the configuration where they are normal in the centralizer of the involution but the base fields are isomorphic, corresponding to G_2, and a third configuration where the base fields are different, corresponding to the twisted group ${}^3D_4$. In Theorem 1.1 our first case lies squarely on the road to PSp_4 while our second case points in the general direction of G_2, though at this point in the analysis one branch which has properties reminiscent of both PSp_4 and G_2 remains to be eliminated, while the other branch allows for possibilities reminiscent of ${}^3D_4$.

We carry the analysis a little further (by much the same kind of argumentation) to describe the configurations more precisely.

Theorem 1.2. *Let G be a connected simple L^* group of finite Morley rank of odd type satisfying the condition* NTA_2*, with Prüfer* 2*-rank* 2 *and*

$$m_2(G) \geq 3.$$

Then there are at most two conjugacy classes of involutions, and one of the following applies.

(1) There are two conjugacy classes of involutions.

Then the 2*-rank of G is* 4*; and the Sylow* 2*-subgroup is as in* PSp_4.

One conjugacy class of involutions satisfies

$$C_G^\circ(i) \simeq \mathrm{PSL}_2(k) \times k^\times,$$

and the other satisfies

$$C_G(i) \simeq \mathrm{SL}_2(k) *_2 \mathrm{SL}_2(k),$$

with the two components of $\mathrm{SL}_2(k) *_2 \mathrm{SL}_2(k)$ *conjugate (and all three base fields the same in the sense that they are definably isomorphic).*

In this case, for i an SL_2*-involution, i will be the only* SL_2*-involution in* $C_G^\circ(i)$.

Furthermore the following are equivalent for involutions t.

(a) t is a PSL_2*-involution.*

(b) t lies in a component of a PSL_2-involution.

(c) t lies in a subgroup of G of type PSL_2.

For t a PSL_2-involution, and $L_t = E_{alg}(C_G(t))$, the involutions of $C_G^\circ(t)$ are those of $L\langle t\rangle$. Those in $L_t \cup \{t\}$ are PSL_2-involutions, and the rest are SL_2-involutions.

(2) *There is one conjugacy class of involutions, and these satisfy*

$$C_G(i) = \mathrm{SL}_2(k_1) *_2 \mathrm{SL}_2(k_2)$$

where the base fields k_1, k_2 have the same characterstic. Furthermore, in characteristic zero, we have

$$\bar{\mathrm{r}}_0(k_1^\times) = \bar{\mathrm{r}}_0(k_2^\times)$$

in the sense of characteristic zero unipotence theory (§2.3).

Furthermore, $C_G(i)$ is connected and contains a Sylow 2-subgroup of G, isomorphic to that of $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ (in characteristic other than 2).

2. General background

Here we collect useful material on a variety of topics. We do not go into the history or the origins of the material, but we aim to give reasonably accessible sources for the material in the form it is applied here.

2.1. L-groups, D-groups, NTA_2.

Definition 2.1. Let G be a group G of finite Morley rank.

1. If G is of odd type, then it is an *L-group* in the odd type sense if every definable simple section of odd type is a Chevalley group over an algebraically closed field of characteristic other than 2, and it is an *L^*-group* in the odd type sense if the same applies to every proper definable simple section of odd type.

2. G satisfies the condition NTA_2 if every definable section of G which acts faithfully on a definable simple section of G of degenerate type is itself of degenerate type.

3. The group G is a *D-group* if every definable connected simple section has degenerate type (and hence contains no involutions), and G is a *D^*-group* if the same applies to its proper definable connected simple sections.

Notation 2.2. If G is a group of finite Morley rank, we will write

$$O(G)$$

for the largest definable normal *connected* subgroup without involutions.

There are two points to note here: the connectedness requirement, and the absence of any requirement of solvability. On the other hand, we write $\sigma(G)$ and $F(G)$ for the solvable radical and Fitting subgroup of G, and then $\sigma^\circ(G)$, $F^\circ(G)$ for the respective connected components. We also write $O^\sigma(G)$ for $O\sigma(G)$.

Fact 2.3 ([BC08, Lemma 1.15]). *The Sylow* 2*-subgroup of a connected* D*-group* G *of finite Morley rank is connected.*

Fact 2.4 ([BC22a, Lemma 2.29]). *Let* G *be a connected group of finite Morley rank. Then*

$$[G, \sigma^\circ(G)] \le F^\circ(G).$$

Definition 2.5. Let H be a group of finite Morley rank.

Then $U(H)$ denotes the largest connected definable nilpotent normal subgroup of H whose torsion subgroup has bounded exponent.

In particular we will have $U(H) = B * \prod_r U_{0,r}(U(H))$ with B the torsion subgroup of $U(H)$.

Fact 2.6 ([BC22a, Proposition 3.10]). *Let* H *be a connected* L*-group of finite Morley rank and odd type satisfying the condition* NTA_2. *Suppose that*

$$UF(H) \le Z(H).$$

Then

$$H = E_{alg}(H) * K \text{ where } K \text{ is connected with } K/Z^\circ(K) \text{ of degenerate type.}$$

In particular, the Sylow 2*-subgroup of* K *is central in* H.

Fact 2.7 ([BC22a, Lemma 3.11]). *Let* H *be a connected* L*-group of finite Morley rank and odd type satisfying* NTA_2 *and let* $\bar{H} = H/OF(H)$.

Then $OF(\bar{H}) \le Z(\bar{H})$. *Hence*

$$\bar{H} = E_{alg}(\bar{H}) * \bar{K} \text{ where } \bar{K} \text{ is connected and } \bar{K}/Z^\circ(\bar{K}) \text{ has degenerate type.}$$

Fact 2.8 ([BC22a, Lemma 3.12]). *Let* H *be a connected* D*-group of finite Morley rank and odd type satisfying* NTA_2. *Then* $H/UF(H)$ *has a unique, central, Sylow* 2*-subgroup.*

2.2. **Torsion.**

Fact 2.9 ([BN94, Ex. 11 p. 93; Ex. 13c p. 72])**.** *Let H be a definable normal subgroup of G. If $x \in G$ is an element such that $\bar{x} \in G/H$ is a p-element for some prime p, then xH contains a p-element. Furthermore, if H and G/H are $p^\perp$-groups, then G is a $p^\perp$-group.*

Definition 2.10. For π a set of primes, a *π-torus* is a divisible abelian π-group. We write Π for the set of all primes, so that a Π-torus is a maximal divisible abelian torsion group.

Fact 2.11 ([Che05])**.** *Let G be a group of finite Morley rank. Then any two maximal Π-tori of G are conjugate.*

For the most part one applies the following with $p = 2$ and with the group of odd type, to conclude that all involutions lie in a 2-torus.

Fact 2.12 ([BC09, Theorem 3])**.** *Let G be a connected group of finite Morley rank and odd type, π a set of primes. Then any π-element a in G such that $C_G^\circ(a)$ contains no non-trivial π-unipotent subgroup lies in some π-torus of G.*

Fact 2.13 ([AB08, Theorem 1])**.** *If G is a connected group of finite Morley rank and T is a p-torus of G, then $C_G(T)$ is connected.*

Fact 2.14 ([ABC08, Thm. I.6.4])**.** *Let G be a group of finite Morley rank, S the connected component of a Sylow 2-subgroup with maximal 2-torus T. Then $N(T)$ controls fusion in S.*

We are interested in the case of odd type: so here, $S = T$.

Definition 2.15. A *good torus* in a group of finite Morley rank is a connected definable divisible abelian subgroup such that every definable subgroup is the definable hull of its torsion subgroup.

A *decent torus* is the definable hull of a Π-torus.

Fact 2.16 ([Wag03], cf. [ABC08, Prop. I.4.20])**.** *If k is a field of finite Morley rank and non-zero characteristic, then the multiplicative group $k^\times$ is a good torus.*

From the general theory connected with this notion we quote the following.

Fact 2.17 ([ABC08, Cor. I.4.22]). *Let G be a group of finite Morley rank, and T a connected definable subgroup of a finite product of good tori. Then T is a good torus.*

It is also clear that a definable quotient of a good torus is a good torus.

Fact 2.18 ([Fré06b, Lemma 3.1]). *Let G be a group of finite Morley rank, N a definable normal subgroup of G, and T a maximal decent torus of G. Then TN/N is a maximal decent torus of G/N, and every maximal decent torus of G/N has this form.*

Fact 2.19 ([ABC08, Lemma 8.3]). *Let G be a connected solvable group of finite Morley rank. Then $G/F^\circ(G)$ is divisible abelian.*

The following may be checked most simply in the Tate module associated with a 2-torus.

Fact 2.20 ([BC08, Lemma 1.6]). *Let T be a nontrivial 2-torus, and let i be an involution acting on T . Then either i inverts T, or $C_T(i)$ is infinite.*

Definition 2.21. There are various useful definitions of *Weyl group.* Here we use the one used in [BC09]: $W = N_G(T)/C_G(T)$ with T a maximal Π-torus (bearing in mind Fact 2.13).

Fact 2.22 ([BC09, Theorem 5]). *Let G be a connected group of finite Morley rank. Suppose the Weyl group is nontrivial and has odd order, with p the smallest prime divisor of its order. Then G contains a p-unipotent subgroup.*

2.3. **Unipotence theory.** We use the notation $U_{0,r}(H)$ for the subgroup generated by $(0, r)$-unipotent subgroups in the sense introduced in [Bur04b]. The theory is reviewed in some detail in [BC22a, §2.4]. Here we give a bit of the intuition and recall some useful properties.

In the first place, we have the classical theory of p-unipotent subgroups of a group H of finite Morley rank, for p a prime. These are by definition connected definable solvable p-subgroups. Any such are in fact nilpotent. One defines $U_p(H)$ as the subgroup generated by the p-unipotent subgroups. If H is solvable then $U_p(H)$ is nilpotent and hence lies in $F^\circ(H)$. On the other hand if H is a quasi-simple algebraic group then $U_p(H) = 1$ unless the characteristic of the base field is p, in which case $U_p(H) = H$.

The theory of p-unipotent subgroups will be useful here, but we require an extension to the "prime" 0. In this case, rather than a notion of 0-unipotence, we will have a graded notion of $(0, r)$-unipotence for $r \geq 0$, and a corresponding subgroup $U_{0,r}(H)$. While these notions are not directly comparable, the general sense is that for larger r the $(0, r)$-unipotent subgroups become "more" unipotent. In particular, we attach particular importance to the parameter $\bar{\mathrm{r}}_0(H)$ defined as the largest value r for which $U_{0,r}(G) > 1$ (and $\bar{\mathrm{r}}_0(H) = 0$ if there is no such r).

One subtlety not encountered for ordinary primes p is that in principle a $U_{0,r}$-unipotent subgroup may contain a $U_{0,s}$-unipotent subgroup with $s \neq r$; when this does not occur, the $U_{0,r}$-unipotent subgroup in question is called *homogeneous*.

We have the following formal properties.

Fact 2.23 ([Bur09, Lemma 2.11])**.** *Let $f : G :\to H$ be a definable homomorphism between two groups of finite Morley rank. Then*

(1) (Push-forward) $f[U_{(0,r)}(G)] \leq U_{(0,r)}(H)$ is a $U_{(0,r)}$ -group.

(2) (Pull-back) If $U_{(0,r)}(H) \leq f[G]$ then $f[U_{(0,r)}(G)] = U_{(0,r)}(H)$.

More substantively, we have useful analogs of facts about p-unipotence.

Fact 2.24 ([Bur09, Theorem 2.16]). *Let G be a connected solvable group of finite Morley rank, for which $U_{0,r}(G)$ is nontrivial for some r. Then $U_{0,\bar{r}_0(G)}(G) \le F(G)$.*

Fact 2.25 ([Bur04a, Thm. 2.31]; [Bur06, Cor. 3.6]; cf. [BN94, Thm. 6.8, 6.9]). *Let Q be a nilpotent group of finite Morley rank. Then $Q = B * D$ is a central product of definable characteristic subgroups B, $D \le Q$ where B is connected of bounded exponent and D is divisible.*

Let T be the torsion part of D. Then we have decompositions of D and B into central products as follows.

$$\begin{aligned} B &= U_2(G) \times U_3(G) \times U_5(G) \times \cdots \\ D &= d(T) * U_{0,1}(G) * U_{0,2}(G) * \cdots \end{aligned}$$

with T a Π-torus.

We will only be interested in groups of odd or degenerate type, so in practice the term $U_2(Q)$ wii be trivial and $B \le O(Q)$. On the other hand it is not so clear what the intersections of the various factors in the decomposition of D will be, as the factors need not be homogeneous.

Fact 2.26 ([Bur06, Corollary 4.6]). *Let $G = HT$ be a group of finite Morley rank, with H and T definable and nilpotent, and $H \lhd G$. Suppose that T is a $U_{0,r}$-group for some $r \ge \bar{r}_0(H)$. Then G is nilpotent.*

In particular, if $r > \bar{r}_0(H)$ then T centralizes H.

(For the final statement, apply Fact 2.25.)

We have noted that a failure of homogeneity may cause complications. In that regard the following is helpful.

Fact 2.27 ([Fré06a, Thm. 4.11]). *Let G be a connected group of finite Morley rank acting definably on a nilpotent group H with $H = U_{0,r}(H)$. Then $[G, H]$ is a definable homogeneous $U_{0,r}$-group.*

What follows is intended to be helpful in some extreme cases in the study of L-groups.

Definition 2.28. We say that a group has *abelian Borels* if all of its definable connected solvable subgroups are abelian.

Lemma 2.29. *Let G be a connected group of finite Morley rank with abelian Borels.*

Then the following hold.

(1) The Borel subgroups of G are conjugate.

(2) G is generated by its definable connected solvable subgroups.

(3) $\sigma^\circ(G) = Z^\circ(G)$.

Proof.
Ad 1. We argue by induction on the rank of G. If $\sigma(G)$ is infinite we may factor it out and conclude at once, while if $\sigma(G)$ is finite it is contained in the center and we may in any case factor it out and suppose that $\sigma(G) = 1$, and in particular $Z(G) = 1$.

By rank computations, if B_1, B_2 are Borel subgroups then the union of their conjugates is generic in G. Therefore we may suppose B_1 meets B_2 nontrivially. Taking $a \neq 1$ in the intersection, we may replace G by $C_G^\circ(a)/(B_1 \cap B_2)$, and conclude by induction on rank.
Ad 2,3.

Let H be the subgroup of G generated by its definable connected solvable subgroups. If B is a definable connected solvable subgroup of G then so is $\sigma^\circ(G)B$ and hence B centralizes $\sigma^\circ(G)$. It follows that

$$\sigma^\circ(G) = Z^\circ(H)$$

Furthermore, by a Frattini argument, with B a Borel subgroup of G, we have $G = HN^\circ(B)$. If $G > H$ then $N^\circ(B) > B$ and hence there is an infinite abelian subgroup $\bar{A}$ of $N^\circ(B)/B$. But then the preimage A is definable connected solvable subgroup of G and we arrive at a contradiction.

So $G = H$ and both (2) and (3) follow,

□

Lemma 2.30. *Let G be a connected group of finite Morley rank with abelian Borels.*

Suppose that G has a definable faithful action on a torsion-free abelian group V. Then the following hold.

(1) $U_p(G) = 1$ for all primes p.

(2) Every element of G belongs to a Borel subgroup.

(3) If G is of degenerate type then every Borel subgroup of G is self-normalizing.

(4) The inverse image of $E(G/\sigma(G))$ in G is the central product $\sigma(G) * E(G)$. In particular, if G is nonabelian then $E(G) > 1$.

Proof.

Ad 1. $U_p(G) = 1$ for all primes p.

Suppose that $U \leq G$ is p-unipotent. Then U commutes with V. But the action is assumed faithful.

Ad 2. It suffices to show that every non-trivial element a of G lies in a non-trivial connected definable abelian subgroup. We proceed by induction on the rank of G.

If $d(a)$ is connected then we are done so suppose $d(a) = C \times d^\circ(a)$ with C finite and cyclic and write $a = a_C a'$ with $a_C \in C$, $a' \in d^\circ(a)$.

By Fact 2.12 a_C is contained in a Π-torus T_a. Thus $a_C \in C_G^\circ(a_C)$. As $d^\circ(a) \leq C_G^\circ(a_C)$ we have $a \in C_G^\circ(a_C)$. If a_C is not central in G we can conclude by induction.

If a_C is central in G then it lies in every maximal Π-torus of G. Hence a_C lies in every Borel subgroup. but $d^\circ(a)$ lies in some Borel subgroup, so a lies in some Borel subgroup.

This proves (2).

Ad 3. With T a maximal Π-torus and B a Borel subgroup containing T, we have $N(B) \cap C(T) = N(B) \cap C^\circ(T) = B$ and hence $N(B)/B$ embeds into $N(T)//T$.

But in view of Fact 2.22 this group is trivial. This proves (3).

Ad 4. As $\sigma^\circ(G) = Z^\circ(G)$, the inverse image in G of $G/\sigma^\circ(G)$ is $Z^\circ(G)E(G)$. Also $E(G)$ centralizes $\sigma(G)/\sigma^\circ(G)$, so $E(G)$ centralizes $\sigma(G)$. □

2.4. **Co-prime actions.**

Fact 2.31 ([ABC99, Prop. 2.43], [ABC08, Prop. I.9.12]). *Let $G = H \rtimes T$ be a group of finite Morley rank, $Q \lhd H$, and π a set of primes, such that Q, H, and T are definable and*

(1) Q and T are solvable;

(2) T is a π-group of bounded exponent;

(3) Q is a T-invariant $\pi^\perp$-subgroup.

Then

$$C_{H/Q}(T) = C_H(T)Q/Q.$$

Fact 2.32 ([ABC08, I.10.4]). *Let G be a group of finite Morley rank without involutions, and α a definable involutory automorphism of G. Then*

$$G = C_G(\alpha) \times G^-$$

(i.e., the multiplication map from right to left is bijective) where G^- is the subset inverted by α.

Lemma 2.33. *Let H be a connected D-group of finite Morley rank and odd type satisfying* NTA_2. *Let T be a Sylow 2-subgroup of H. Then*

$$H = UF(H) \cdot C_H^\circ(T).$$

Proof. $C_H(T) = C_H(A)$ for some finite 2-subgroup $A \le T$.

Apply Fact 2.8 to H and Fact 2.31. to $H \rtimes A$. Taking $\bar{H} = H/UF(H)$, we have

$$\bar{H} = C_{\bar{H}}(A) = \overline{C_H(A)} = \overline{C_H(T)},$$

and the lemma follows. □

Fact 2.34 ([ABCC03], [Bur09, Lemma 3.5])**.** *Let G be a connected solvable $p^\perp$-group of finite Morley rank, and let P be a finite p-group of definable automorphisms of G. Then $C_G(P)$ is connected.*

If in addition G is a nilpotent $U_{0,r}$-group then $C_G(P)$ is a $(0,r)$-unipotent group.

The following is a variation on [BN94, Prop. 13.4] (a simple bad group has no definable involutive automorphism).

Fact 2.35. *Let G be a connected group of finite Morley rank of degenerate type, and let α be an involutive automorphism of G. Then $\bigcup C_G(\alpha)^G$ is disjoint from $G^- \setminus \{1\}$, where $G^- = \{x \in G \mid x^\alpha = x^{-1}\}$.*

Proof. By Theorem 1.1, G has no involutions. By Fact 2.32 we have $G = C_G(\alpha)G^-$.

Suppose $g \in G$ and $x \in G^- \cap C_G(\alpha)^g$. We may take $g \in G^-$. So

$$x^{g^{-1}} = (x^{g^{-1}})^\alpha = (x^\alpha)^{(g^{-1})^\alpha} = (x^{-1})^g;$$
$$x^{g^{-2}} = x^{-1}.$$

Thus $g^4 \in C(x)$, but $g \in d(\langle g^4 \rangle)$, so also $g \in C(x)$. Thus $x^{-1} = x$, $x = 1$. □

Fact 2.36 ([ABC08, Prop. II 11.7])**.** *Let H be a connected solvable $\pi^\perp$-group of finite Morley rank acting faithfully and definably on a nilpotent π-group V of bounded exponent. Then H is a good torus.*

2.5. Generic subsets.

Fact 2.37 ([BBC07, Lemma 4.1]). *Let G be a group of finite Morley rank, H a definable subgroup of G, and X a definable subset of G. Suppose that*

$$\mathrm{rk}(X \setminus \bigcup_{g \notin H} X^g) \geq \mathrm{rk}(H)$$

Then $\mathrm{rk}(\bigcup X^G) = \mathrm{rk}(G)$.

Fact 2.38 ([BC09, Thm. 1 (1,2)]). *Let G be a connected group of finite Morley rank, p a prime, and let a be a generic element of G. Then*

(1) the element a commutes with a unique maximal p-torus T_a of G, and

(2) the definable hull $d(\langle a \rangle)$ contains T_a.

2.6. The Uniqueness Case.

Fact 2.39 ([BC08, Lemma 6.2]). *Let G be a simple L^*-group of finite Morley rank and odd type, with $\mathrm{pr}_2(G) \geq 2$. Suppose that G has a strongly embedded subgroup M.*

Then M is a D-group, and hence its Sylow 2-subgroups of G are connected.

The second point is found in Fact 2.3.

Corollary 2.40. *Let G be a simple L^*-group of finite Morley rank and odd type with*

$$\mathrm{pr}_2(G) = 2; \qquad\qquad m_2(G) \geq 3.$$

Then G has no proper definable strongly embedded subgroup.

This gives a useful generation result.

Definition 2.41. Let G be a group of finite Morley rank, and V a subgroup (in practice, an elementary abelian 2-subgroup with $m_2(V) \geq 2$). We set

$$\Gamma_V = \langle C^\circ(U) \,:\, U \leq V,\, [V:U] = 2 \rangle \,.$$

Fact 2.42 ([BC08, Thm. 4.3])**.** *Let G be a connected simple L^*-group of finite Morley rank and odd type with*

$$m_2(G) \geq 3.$$

Suppose that $\Gamma_V(G) < G$ for some elementary abelian 2-subgroup V of rank 2. Then the normalizer

$$N(\Gamma_V(G))$$

is a strongly embedded subgroup.

Corollary 2.43. *Let G be a simple L^*-group of finite Morley rank and odd type with*

$$\mathrm{pr}_2(G) = 2; \qquad\qquad m_2(G) \geq 3.$$

Then for any elementary abelian 2-subgroup V of rank 2 we have

$$\Gamma_V = G.$$

The following result is similar, but it has a more elementary character, as it concerns L-groups rather than L^*-groups.

Fact 2.44 ([BC08, Theorem 2.1])**.** *Let G be a connected L-group of finite Morley rank and odd type. Let V be an elementary abelian 2-group of rank 2 acting definably on G.*

Then $\Gamma_V = G$.

2.7. **Algebraic groups.** A key tool in our program is the fact that a group of finite Morley rank acting faithfully as a group of automorphisms of a quasi-simple algebraic group must itself be algebraic.

Given a quasi-simple algebraic group G, a maximal torus T of G, and a Borel subgroup B of G which contains T, we define the group Γ of *graph automorphisms* associated to T and B to be the group of algebraic automorphisms of G which normalize both T and B.

Fact 2.45 ([BN94, Theorem 8.4])**.** *Let $G \rtimes H$ be a group of finite Morley rank where G and H are definable, G an infinite quasi-simple algebraic group over an algebraically closed field, and $C_H(G)$ is trivial. Then, viewing H as a subgroup of* $\mathrm{Aut}(G)$*, we have $H \leq \mathrm{Inn}(G)\Gamma$ where* $\mathrm{Inn}(G)$ *is the group of inner automorphisms of G and Γ is the group of graph automorphisms of G, relative to a fixed choice of Borel subgroup B and maximal torus T contained in B.*

An algebraic group is said to be *reductive* if it has no unipotent radical. Such a group is a central product of semisimple algebraic groups and algebraic tori. The centralizer of an involution in a reductive algebraic group over a field of characteristic $\neq 2$ is itself reductive.

Fact 2.46 ([Ste68, Theorem 8.1])**.** *Let G be a quasisimple algebraic group over an algebraically closed field. Let ϕ be an algebraic automorphism of G whose order is finite and relatively prime to the characteristic of the field. Then $C_G^\circ(\phi)$ is nontrivial and reductive.*

Proof. We shall replace G by its universal central extension, so our original group is now $\bar{G} = G/Q$ for some $Q \leq Z(G)$. We also replace ϕ by an automorphism of our new G, which exists say by Fact 2.45. There is a homomorphism $C_G(\phi \bmod Q) \to Q$ given by $x \mapsto [\phi, x]$. As Q is finite, ϕ centralizes $C_G^\circ(\phi \bmod Q)$. So it suffices to prove the result for our G which is its own universal central extension.

Since ϕ is algebraic and has finite order, the group $G \rtimes \langle\phi\rangle$ is an algebraic group which contains ϕ as an inner automorphism. Since the order of ϕ is finite and relatively prime to the characteristic, ϕ is a semisimple automorphism of G. So the result follows from Theorem 8.1 of [Ste68]. $\square$

Lemma 2.47. *Let G be a connected group of finite Morley rank and $t \in G$. Suppose that $K \lhd C(t)$ is a definable normal quasisimple algebraic subgroup of $C(t)$ and $t \in K$.*

Then $[C_G(t) : C_G^\circ(t)] \leq |\Gamma|$.

Proof. We argue that $C_G(t)/C_G^\circ(t) \hookrightarrow \Gamma$. Otherwise, some $x \in C_G(t) \setminus C_G^\circ(t)$ centralizes K by Fact 2.45.

Now x has finite order modulo $C_G^\circ(t)$, so by Fact 2.9 we may assume that x has finite order. As $t \in Z(K)$ by hypothesis, t lies inside a maximal torus T of K. As x centralizes K, by Fact 2.13 we find

$$x \in C(T) = C^\circ(T) \leq C^\circ(t). \qquad \square$$

Table 1 below contains necessary information about conjugacy classes of involutory algebraic automorphisms, as well as their centralizers, in Lie rank one or two quasi-simple groups (see [GLS98, Table 4.3.1 p. 145 and Table 4.3.3 p. 151]). Note that where there are central involutions in the group they are not shown, as the corresponding automorphisms are trivial. Furthermore, in such cases some of the involutory automorphisms are represented by elements of order 4 in the group.

Here the second involutory automorphism of Sp_4 is represented by an element of order four.

Fact 2.48 ([Poi01, Thms. 1,4])**.** *Let k be a field of finite Morley rank (as usual, in any language) and let G be a definable subgroup of $\mathrm{GL}_n(k)$. Then the following hold.*

(1) If the characteristic of k is non-zero and G is simple, then G is definably isomorphic to an algebraic group.

G	$\bar{\Gamma}$	Z	i	$C^\circ_G(i)$
SL_2	1	$\mathbb{Z}/2\mathbb{Z}$	inner	k^*
PSL_2	1	1	inner	k^*
SL_3	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	inner	$\mathrm{SL}_2 * k^*$
			graph	PSL_2
PSL_3	$\mathbb{Z}/2\mathbb{Z}$	1	inner	$\mathrm{SL}_2 *_2 k^*$
			graph	PSL_2
Sp_4	1	$\mathbb{Z}/2\mathbb{Z}$	inner	$\mathrm{SL}_2 \times \mathrm{SL}_2$
			inner	$\mathrm{SL}_2 *_2 k^*$
PSp_4	1	1	inner	$\mathrm{SL}_2 *_2 \mathrm{SL}_2$
			inner	$\mathrm{PSL}_2 \times k^*$
G_2	1	1	inner	$\mathrm{SL}_2 *_2 \mathrm{SL}_2$

TABLE 1. Data on Chevalley Groups

(2) *If $n = 2$, then either G is solvable by finite, or G contains $\mathrm{SL}_2(k)$.*

The following is a useful consequence of the second point.

Corollary 2.49. *Let k be a field of finite Morley rank, and T_1, T_2 two nontrivial connected definable subgroups of $\mathrm{PSL}_2(k)$ not contained in a single Borel subgroup. Then $\mathrm{PSL}_2(k)$ is generated by $T_1 \cup T_2$.*

Proof. The subgroup generated by T_1, T_2 is connected and definable. Hence the preimage in $\mathrm{SL}_2(k)$ is either $\mathrm{SL}_2(k)$ or solvable and connected. In the latter case its Zariski closure is a solvable group which is connected in the algebraic sense. □

Lemma 2.50. *Let G be an algebraic group of finite Morley rank, T a torus, and U a definable subgroup of a unipotent group which is normalized by an infinite subgroup of T acting faithfully on U. Then U is Zariski closed in G.*

Proof. We may suppose that U is connected, and proceeding inductively, we may suppose that U is abelian and T-minimal. We may suppose further that the Zariski closure of U is 1-dimensional, and we come down to the case where U is a subgroup of the additive group of the base field k which is invariant under an infinite subgroup T of the multiplicative group. Then the stabilizer of U under multiplication in k is an infinite subring of k, hence is all of k. $\square$

2.8. **Covering groups.**

Definition 2.51. Let L be a group of finite Morley rank. A *covering group* L^* of L is any group of finite Morley rank such that

(1) $L^*/\sigma^\circ(L^*) \simeq L$.

(2) Any proper definable normal subgroup of L^* is solvable.

Lemma 2.52. *Let H be a group of finite Morley rank which is a covering group of a quasi-simple algebraic group L Then $\sigma^\circ(H) = F^\circ(H)$.*

Proof. Let $\bar{H} = H/F^\circ(H)$. By Fact 2.4 $\sigma^\circ(\bar{H})$ is central in $\bar{H}$ and thus $\bar{H}$ has a normal subgroup with quotient L. By the minimality of H, we find $\sigma^\circ(\bar{H}) = 1$ and $\sigma^\circ(H) = F^\circ(H)$. $\square$

Lemma 2.53. *Let H be a group of finite Morley rank of odd type which is a covering group of a quasi-simple algebraic group L with base field k. Then $\sigma^\circ(H) = OF(H)$. Furthermore we have the following.*

(1) If k has characteristic $p > 0$ then $OF(H)$ is a unipotent p-group.

(2) If k has characteristic 0 and $\mathrm{rk}(k) = r$ then $OF(H)$ is a homogeneous $(0, r)$-unipotent subgroup.

Proof. We know $\sigma^\circ(H) = F^\circ(H)$. Let B be the preimage in H of a Borel subgroup $\mathbb{B}$ of $H/F^\circ(H)$. Then $U = B'$ is a nilpotent group covering the unipotent radical $\mathbb{U}$ of $\mathbb{B}$. Thus $U \leq F(B)$ and $UF^\circ(H)$ is nilpotent.

Depending on the characteristic of k, $\mathbb{U}$ is either p-unipotent or $(0, r)$-unipotent. If the corresponding subgroup

$$U_p(F^\circ(H)) \text{ or } U_{(0,r)}(F^\circ(H))$$

is proper in $F^\circ(H)$, then after factoring out a maximal proper definable normal subgroup Q of $F^\circ(H)$ containing that subgroup, the action of $\mathbb{U}$ induced by U on the quotient $F^\circ(H)/Q$ is trivial. This applies to all unipotent subgroups of L and hence $[H, F^\circ(H)] \leq Q$. Therefore Q is normal in H and $F^\circ(H)/Q$ is normal in the quotient. But then by the minimality of H we find $F^\circ(H) = Q$, a contradiction. So $F^\circ(H)$ must in fact reduce to $U_p(F^\circ(H))$ or $U_{(0,r)}(F^\circ(H))$ respectively.

If the characteristic is $p > 0$ (odd) the proof is complete.

If the charactertistic is zero we have $F^\circ(H) = U_{(0,r)}(F^\circ(H))$. By Fact 2.27 the group $[H, F^\circ(H)]$ is a homogeneous $U_{(0,r)}$-group. Taking the quotient by this subgroup and applying minimality shows that $F^\circ(H) = [H, F^\circ(H)]$ is a homogeneous $U_{(0,r)}$-group. In particular $F^\circ(H) = O(H)$ and we conclude. □

Lemma 2.54. *Let H be a group of finite Morley rank and odd type, L a quasi-simple algebraic group, and L_1 a definable subgroup of H which is a covering group of L with $OF(L_1) \leq O(H)$. With $\bar{H} = H^\circ/O(H)$, suppose that $\bar{H} = E_{alg}(\bar{H}) * \bar{K}$ with $\bar{K}$ a D-group.*

Then the normal closure L_2 of L_1 in H is a covering group of $\bar{L}_2$.

Proof. In $\bar{H}/Z(E_{alg}(\bar{H}))$ the central product decomposition becomes a direct product and the projection of $\bar{L}_1$ to the image of $\bar{K}$ is a D-group.

Hence the kernel is not solvable and by minimality it is L_1. It follows that $\bar{L}_1 \leq E_{alg}(\bar{H})$ and the same appies to $\bar{L}_2$, which is a product of components of $E_{alg}(\bar{H})$. The normal closure of $\bar{L}_1$ in $\bar{L}_2$ is again a product of components of $E_{alg}(\bar{H})$ and hence is normal in $\bar{H}$. Thus this normal closure in $\bar{L}_2$. In other words, the normal closure of L_1 in L_2 covers $\bar{L}_2$.

Thus $O(H) \cap L_2 = \sigma^\circ(L_2)$ and $L_2/\sigma^\circ(L_2) \simeq \bar{L}_2$.

Now suppose N is a proper definable normal subgroup of L_2. Then $N \cap L_1$ is a proper definable normal subgroup of L_1, hence lies in $OF(L_1) \leq O(H)$. But N is normal in L_2. If L_2^* is the normal closure of L_1 in L_2, then $[N, L_2^*] \leq O(H)$.

But L_2^* covers $\bar{L}_2$, and it follows that $N \leq O(H)$. Thus N is solvable and L_2 is a covering group of $\bar{L}_2$. $\square$

3. Special topics

We will next bring in some more specialized topics which will come into play as we encounter particular pathological configurations which need to be eliminated. Then in §3.4 we will review the tools developed in [BC22a] for finding algebraic components in centralizers of involutions, with reference to the particular case of Prüfer rank 2. This is where the main line of our analysis begins.

3.1. **Thompson $A \times B$.** We give a version of Thompson's $A \times B$ lemma suitable for our present purposes—it will be applied repeatedly in §§5,6. Related results see found in [Suz86, Chap. 4, §1] and [ABC08, §I.12].

Lemma 3.1. *Let G be a group of finite Morley rank, π a set of primes, and HK a subgroup with H, K definable. Assume the following.*

(1) K normalizes H.

(2) No non-trivial definable quotient of K is definably isomorphic to a definable section of H.

Then

$$[K,[K,H]] = [K,H]$$

Proof. We may take $G = H \rtimes K$.

By [ABC08, Cor. I.3.29] the groups $[K,H]$ and $[K,[K,H]]$ are definable.

Let $N = [K,H]K$.

Claim 1. *N is the smallest normal subgroup of G for which G/N is definably isomorphic to a definable section of H.*

On one hand, $G/N \simeq H/[K,H]$. On the other hand, if $N^* \triangleleft G$ and G/N^* is isomorphic to a quotient of H, then $K \leq N^*$ and hence $[K,H] \leq N^*$.

Thus the claim holds. In particular N is definably characteristic in G.

Similarly, $N_1 = [H, K, K]K$ is the smallest definable normal subgroup of N such that N/N_1 is definably isomorphic to a definable section of H. Hence N_1 is definably characteristic in N, and thus normal in G.

As N_1 contains K, it follows that $N_1 = N$, and this yields the claim. □

Lemma 3.2 (Thompson $A \times B$ Lemma)**.** *Let G be a group of finite Morley rank. Let $U \leq Q$ be definable subgroups of G, and T any subgroup of G. Suppose the following.*

(1) Q is nilpotent.

(2) $C_Q(U) \leq U$.

(3) T normalizes Q and centralizes U.

(4) No non-trivial definable quotient of $d(T)$ is definably isomorphic to a definable section of Q.

(5) Either $d(T)$ is connected, or U and Q are connected.

Then T centralizes Q.

Proof. Let K be the definable hull $d(T)$. Then $C_Q(T) = C_Q(K)$. We suppose toward a contradiction that

$$Q > C_Q(K).$$

Claim 1. *Without loss of generality, U is normal in Q.*

Suppose first that K is connected. Set $U_1 = C_Q(K)$ and $Q_1 = N_Q(U_1)$. Then our hypotheses on U and Q apply to U_1 and Q_1, including $Q_1 > U_1$, this last by the normalizer condition in Q. So we may replace U, Q by U_1, Q_1, and add to our hypotheses the condition $U \triangleleft Q$.

Now suppose that U and Q are connected. Set $U_1 = C_Q^\circ(K)$ and $Q_1 = N_Q^\circ(U)$. Then our hypotheses on U and Q—including connectedness—again apply to U_1 and Q_1.

So in either case we may suppose $U \triangleleft Q$.

Now we have

$$[[K,U],Q] = [1,Q] = 1;$$
$$[[U,Q],K] \leq [U,K] = 1.$$

By the *three subgroups lemma* we get

$$[[Q,K],U] = 1.$$

As $C_Q(U) \leq U$ this gives $[Q,K] \leq U$ and $[[Q,K],K] = 1$.

By Lemma 3.1 this gives $[Q,K] = 1$ as required. □

Corollary 3.3. *Let G be a group of finite Morley rank, and U, Q, K definable nilpotent subgroups satisfying the following conditions.*

(1) $U \leq Q$.

(2) $C_Q(U) \leq U$.

(3) K normalizes Q and centralizes U.

(4) K is connected.

(5) Q is a homogeneous $U_{0,r}$-group.

(6) $U_{0,r}(K)$ centralizes Q.

Then K centralizes Q.

Proof. In view of Fact 2.25, it suffices to prove the result with K replaced by its subgroups of the following forms.

(1) $U_p(K)$ with p a prime;

(2) $U_{(0,s)}(K)/U_{(0,r)}(U_{(0,s)}(K))$ for $s \neq r$;

(3) the maximal decent torus of K.

These are connected subgroups of K, and none has a definable quotient definably isomorphic to a definable section of Q. So Lemma 3.2 applies. □

3.2. **Linearization.** A general result on linearization of actions of algebraic groups in the finite Morley rank category has been attained only recently.

Fact 3.4 ([Bor20, Theorem 3])**.** *Let K be an algebraically closed field of characteristic $p > 0$ and G the group of points over K of a simple algebraic group defined over K. Assume that G acts definably and irreducibly on an elementary abelian p-group V of finite Morley rank. Then V can be given the structure of a finite dimensional K-vector space V_K in a manner compatible with the action of G, and G becomes a Zariski closed subgroup of* $\mathrm{GL}(V_K)$*.*

.

This result was preceded by very special cases dealing with actions of SL_2 in low rank situations, which so far have been adequate for the applications to identification theorems [CD12]. That state of affairs actually continues to be the case here. But with the stronger result on hand one may hope to put it go good use as well—if not for classification results, then in the study of permutation groups of finite Morley rank.

On the other hand what one may call the "characteristic zero" version is classical, given originally in the superstable context, but we specialize in various respects, as follows.

Fact 3.5 ([LW93, Theorem 4]). *Suppose that there is an infinite definable set S of automorphisms of the abelian, torsion free group A, such that A is S-minimal, and the structure (A, S) together with the action has finite Morley rank. Then there is a subgroup $A_1 \leq A$ and a field K such that $A_1 \simeq K_+$ definably. Furthermore, S embeds into a matrix ring over K.*

Implicit in this statement is the structure of a K-vector space on A.

3.3. **Around the Z^* theorem.** The Z^* theorem of finite group theory has no real analog in the finite Morley rank context, as far as is known, but the following is related, and useful.

Fact 3.6 ([BBC07, Thm. 6]). *Let G be a connected group of finite Morley rank, S a Sylow 2-subgroup in G, and $i \in S$ an involution. Then either*

(1) i is conjugate in G to another involution in S; or

(2) $C_G(i)$ is connected.

Below we will put considerable effort into eliminating specific configurations that would immediately violate the Z^*-theorem.

3.4. **Algebraic components in centralizers of involutions.** For what follows, we will be making use of the following hypotheses.

Hypothesis 3.7. Let G be a group of finite Morley rank.
We suppose the following.

(1) G is a connected simple L^*-group of odd type satisfying the condition NTA_2.

(2) $m_2(G) \geq 3$.

(3) G has no proper non-trivial definable strongly embedded subgroup.

Soon we will focus on the case of Prüfer rank 2 (Hypothesis 4.1 below) but for the present section this is the right setting.

We are interested now in the existence of algebraic quasi-simple components in centralizers of involutions. We first state two general results from [BC22a], and then fill in the relevant definitions.

The main thing to retain from what follows is that the notation $\mathcal{E}_A$ refers to a certain set of particularly well=behaved algebraic quasisimple components of centralizers $C_G(i)$ as i varies over the involutions of an elementary abelian 2-subgroup A. See Definition 3.15. The notation m_2° refers to a variant of 2-rank which is introduced in Definition 3.12.

Fact 3.8 ([BC22a, Lemma 2.15])**.** *Let G be a group of finite Morley rank and odd type. Suppose $m_2(G) \geq 3$. Then $m_2^\circ(G) \geq 3$. More precisely, if U is an elementary abelian 2-group of rank 2 which is contained in a 2-torus, then there is an elementary abelian 2-subgroup A of 2-rank 3 containing U, and an involution $i \in I(U)$ which is co-toral with every involution of A.*

Fact 3.9 ([BC22a, Lemma 4.17])**.** *Let G be a group of finite Morley rank satisfying Hypothesis 3.7. Let $A \leq G$ be an elementary abelian 2-subgroup of 2-rank 3 such that the co-torality graph on $I(A)$ is connected, and let $U \leq A$ be a subgroup of 2-rank 2 which is contained in a 2-torus.*

Then there is an involution $i \in U$ such that $C(i)$ contains a quasi-simple component belonging to $\mathcal{E}_A$. Furthermore, for any quasi-simple algebraic component L belonging to $\mathcal{E}_A$, there is an involution $i \in U$ such that L is contained in a product of quasi-simple components belonging to $\mathcal{E}_U$.

There are two points to clarify here: the *co-torality* condition, and the definition of $\mathcal{E}_A$.

Definition 3.10. Let G be a group of finite Morley rank and $i, j \in G$ two involutions. We say that i and j are *co-toral* if there is a 2-torus containing both.

For $A \leq G$ a subgroup, the *co-torality graph* on A has as its vertices the involutions in A, and as its edges the co-toral pairs of involutions in A.

Fact 3.11 (cf. [BC22a, §2.1])**.** *Let G be a group of finite Morley rank and odd type, and $i, j \in I(G)$. Then the following are equivalent.*

(1) i, j are co-toral.

(2) $j \in C_G^\circ(i)$.

Indeed, if $j \in C_G^\circ(i)$ then by [BC09, Thm. 3] one may take maximal 2-tori T_i, T_j containing i, j respectively, and they are conjugate in $C_G^\circ(i)$ (e.g., by conjugacy of Sylow 2-subgroups).

For applications of signalizer functor theory the following modified 2-rank is important.

Definition 3.12. The *co-toral* 2*-rank* $m_2^\circ(G)$ is defined as the maximal 2-rank of an elementary abelian 2-subgroup A such that the graph on $I(A)$ whose edges are the co-toral pairs of involutions in A is connected.

Now we turn to the definition of $\mathcal{E}_A$. This makes use of the unipotence theory of §2.3, and in particular the parameter $\bar{\mathrm{r}}_0(H)$ briefly recalled there; this is the largest value of r for which H contains a $(0, r)$-unipotent subgroup (or zero).

Definition 3.13. Let H be a group of finite Morley rank.

A *component* of H is a quasi-simple subnormal subgroup ([BN94, p. 118 (2)]).

$E_{alg}(H)$ denotes the product of the algebraic components of H°.

Definition 3.14. Let G be a group of finite Morley rank, i an involution, and A a subgroup of G.

We set

$$\begin{aligned} r_{0,i} &= \bar{r}_0(O^\sigma(C_G(i))); \\ r_{f,i} &= \max(\bar{r}_0(k^\times) : k \text{ the base field of a component} \\ &\qquad\qquad \text{of } E_{alg}(C_G(i)/O^\sigma(C_G(i)))). \end{aligned}$$

and

$$r_{0,A} = \max(r_{0,i} \,:\, i \in I(A)); \qquad r_{f,A} = \max(r_{f,i} \,:\, i \in I(A)).$$

Here the subscript "f" stands for "field."

Definition 3.15 (The family $\mathcal{E}_A$)**.** Let G be a group of finite Morley rank, H a definable subgroup, and $\rho \geq 0$.

$\Delta_\rho(H)$ denotes the definable subgroup of H generated by all p-unipotent subgroups with p prime, together with all $(0,r)$-unipotent subgroups for $r > \rho$ (briefly: "all sufficiently unipotent subgroups").

For A an abelian 2-subgroup of G, we let $\mathcal{E}_A$ denote the family of all quasi-simple algebraic components of any of the subgroups $\Delta_r(C_G(i))$ as i varies over $I(A)$ and $r \geq r_{f,A}$.

This definition is clarified by the following.

Fact 3.16 ([BC22a, Lemma 4.9])**.** *Let L be a quasi-simple algebraic group of finite Morley rank (in any language), with base field k. Let $r \geq 1$.*

Then the following are equivalent.

(1) $\Delta_r(L) > 1$.

(2) $\Delta_r(L) = L$.

(3) $\mathrm{rk}(k) > r$, *or the characteristic of k is non-zero.*

A closely related point of independent interest is the following.

Fact 3.17 ([BC22a, Cor. 2.26], [Poi87, Cor. 3.3])**.** *Let k be a field of finite Morley rank and characteristic* 0*, and $k_+, k^\times$ the additive and multiplicative groups. Then*

$$\bar{r}_0(k_+) = \mathrm{rk}(k) > \bar{r}_0(k^\times)$$

and k_+ is a $U_{0,\mathrm{rk}(k)}$-group.

The connection with components comes through the following.

Fact 3.18 ([BC22a, Lemma 4.13])**.** *Let G be a group satisfying Hypothesis 3.7. Let i be an involution of G and $\rho = r_{f,G}$.*

Then

$$\Delta_\rho(C_G(i)) = \Delta_\rho(E_{alg}(C_G(i)))$$

is the product of algebraic components of $C_G(i)$ whose base field k either has non-zero characteristic or satisfies $\mathrm{rk}(k) > \rho$*.*

Note here that there must be some base field for k for which $\bar{\mathrm{r}}_0(k^\times) = r_{f,G}$ and hence $\mathrm{rk}(k) > r_{f,G}$, and an involution i so that k occurs as the base field of some algebraic component of the group $C_G(i)/OFC_G(i)$. In this case the previous fact gives an algebraic component of $C_G(i)$ with the same base field.

Definition 3.19. We may also define $\mathcal{E}_G$ similarly to $\mathcal{E}_A$ (considering all involutions in G and their centralizers). We write $\mathcal{E}$ for $\mathcal{E}_G$.

One of the points of the various definitions made is to ensure the following.

Fact 3.20 ([BC22a, Lemma 4.15])**.** *Let G be a group of finite Morley rank satisfying Hypothesis 3.7. Let $A \le G$ be an elementary abelian 2-subgroup of 2-rank 3 such that the co-torality graph on $I(A)$ is connected, and let i be an involution of A. Let L be a definable quasi-simple algebraic subgroup of $C_G(i)$ over a base field which is either of characteristic p or has a multiplicative group of reduced rank $r_{f,A}$.*

Then L is a subgroup of $E_{\mathcal{E}}(C_G^\circ(i))$.

We record a more technical point which underlies the theory.

Definition 3.21. Let G be a group of finite Morley rank, i an involution, and A a subgroup of G.

For $i \in A$, and ρ either a prime or a symbol $(0, r)$, we set

$$\theta_\rho(i) = U_\rho(O^\sigma(C_G(i))).$$

Fact 3.22 ([BC22a, Lemma 4.4])**.** *Let G be an L^*-group of finite Morley rank of odd type, satisfying* NTA_2.

Let ρ be either a prime or a symbol $(0, r)$ satisfying the conditions

$$r > r_{f,i}; \qquad\qquad r \geq r_{0,i}.$$

Let i, j, k be three commuting involutions in G satisfying the following conditions.

(1) i and j are co-toral in G (i.e., there is 2-torus T containing i and j).

(2) $\theta_\rho(k) \cap C_G(j) \leq \theta_\rho(j)$.

Then

$$\theta_\rho(k) \cap C_G(i) \leq \theta_\rho(i).$$

An important consequence of the signalizer functor theory is the following structural result.

Fact 3.23 ([BC22a, Lemma 4.6])**.** *Let G be a group of finite Morley rank satisfying Hypothesis 3.7. Let A be an elementary abelian 2-subgroup of 2-rank 3.*

Suppose that the graph on $I(A)$ with edges (i, j) for co-toral pairs of involutions is a connected graph.

Then for $i \in I(A)$ an involution, we have

$$U_p(O^\sigma(C_G(i))) = 1 \textit{ for all primes } p;$$
$$U_{(0,r)}(O^\sigma(C_G(i))) = 1 \textit{ for all } r > r_{f,A}.$$

Note that when $U_p(O^\sigma(C_G(i))) = 1$ for all primes p, then $UF(C_G(i))$ is torsion free.

We also give the general result on signalizer functors, as it has further applications.

Fact 3.24 ([BC22a, Lemma 3.19])**.** *Let G be a connected simple L^*-group of finite Morley rank and odd type. Suppose that for some rank 3 elementary abelian 2-subgroup A of G, there is a nontrivial connected nilpotent A-signalizer functor θ satisfying the naturality condition*

$$(\star) \qquad \theta(i)^g = \theta(i^g) \text{ when } g \in G \text{ and } i, i^g \in I(A).$$

Then G has a proper definable strongly embedded subgroup.

In particular, this is excluded in the case of Prüfer 2-rank 2 and 2-rank greater than 2.

4. Component analysis: Preliminaries

We now come to our actual subject matter, the analysis of components of centralizers of involutions, and notably the components in $\mathcal{E}$ for groups satisfying the following conditions.

Hypothesis 4.1. Let G be a group of finite Morley rank.

We assume the following.

(1) G is a simple L^*-group of odd type satisfying the condition NTA_2.

(2) G has Prüfer 2-rank 2 and 2-rank at least 3.

Note that Hypothesis 4.1 implies Hypothesis 3.7, in view of Corollary 2.40.

Our eventual goal is Theorem 1.1.

4.1. **The point of departure.**

First we show the following, as our point of departure.

Lemma 4.2. *Let G be a group satisfying Hypothesis 4.1. Then $\mathcal{E}$ is non-empty, and every component in $\mathcal{E}$ is of Prüfer rank* 1.

Lemma 4.3. *Let G be a group satisfying Hypothesis 4.1. Then the following hold.*

(1) The associated family $\mathcal{E}$ is non-empty.

(2) For any involution i in G we have

$$U_p(O^\sigma(C_G(i))) = 1 \text{ for all primes } p;$$
$$U_{(0,r)}(O^\sigma(C_G(i))) = 1 \text{ for all } r > r_{f,G}.$$

In particular, $UF(C_G(i))$ is torsion-free.

Proof.

Ad 1. By Facts 3.8 and 3.9, there are elementary abelian 2-subgroups A of rank 3 on which the co-torality graph is connected. Then Fact 3.20 applies.

Ad 2.

By Fact 3.8 any involution i belongs to an elementary abelian 2-subgroup A of rank 3 on which the co-torality graph is connected. Then Fact 3.23 gives (2). □

At this point the hypothesis NTA_2 has been invoked via its connection with the background material.

Lemma 4.4. *Let G be a group satisfying Hypothesis 4.1. Then any component in $\mathcal{E}$ has Prüfer rank 1 (type* $(\mathrm{P})\mathrm{SL}_2$*).*

Proof. Suppose the contrary. Let i be an involution and $L \in \mathcal{E}$ be a component of $C^\circ(i)$ of Prüfer rank 2. Then $i \in L$ is a central involution and so

$$L = E_{alg}(C^\circ(i)) \simeq \mathrm{Sp}_4(k),$$

with k algebraically closed (cf. Table 1). Let T be a maximal 2-torus of $C^\circ(i)$, and S a Sylow 2-subgroup of G containing T.

Claim 1. *All involutions of G are conjugate.*

Let j be an involution of G. We will show that j is conjugate to i.

By Fact 2.12 we may suppose $j \in T$. We may also suppose $j \neq i$. Then

$$C_L(j) \simeq \mathrm{SL}_2(k) \times \mathrm{SL}_2(k)$$

(Table 1).

By the definition of $\mathcal{E}$ and Fact 3.20 we have

$$C_L(j) \leq E_{alg}(C^\circ(j)).$$

Hence we have the following two possibilities.

(a) $E_{alg}(C^\circ(j)) > C_L(j)$; then $E_{alg}(C^\circ(j)) \simeq \mathrm{Sp}_4(k)$.

(b) $E_{alg}(C^\circ(j)) = C_L(j)$; then $i \in Z(C^\circ(j))$.

The same applies to $j' = ij$.

By Corollary 2.43, G is generated by the groups $C^\circ(t)$ for $t \in \Omega_1(T)$. Hence i cannot centralize both $C^\circ(j)$ and $C^\circ(j')$. As j, j' are conjugate in $C^\circ(i)$ it follows that i centralizes neither, and thus

$$E_{alg}(C^\circ(j')) \simeq \mathrm{Sp}_4 .$$

Then i, j are conjugate in $C^\circ(j')$. The claim follows.

Claim 2. *A Sylow 2-subgroup of L is a Sylow 2-subgroup of G.*

As all involutions are conjugate, $C(i)$ contains a Sylow 2-subgroup S of G. As $L \simeq \mathrm{Sp}_4(k)$ has no graph automorphisms, the centralizer $C(i)$ is connected by Lemma 2.47.

But then $C(i)/E_{alg}(C^\circ(i))$ is connected and of degenerate type, hence contains no involution. This proves the claim.

Now $C_L(j) \lhd C_G(i,j)$ and so $C_L(j) = \mathrm{SL}_2(k) \times \mathrm{SL}_2(k) = E_{alg}(C(i,j))$, with i lying in neither factor. But (i,j) is conjugate to (j,i) and so j lies in neither factor, and as $i \neq j$ this is impossible. □

Proof of Lemma 4.2. Lemmas 4.3 and 4.4. □

4.2. **Isolated components.**

Definition 4.5. Let G be a group satisfying Hypothesis 4.1.

For any definable subgroup H we write $E_{\mathcal{E}}(H)$ for the product of the $\mathcal{E}$-components of $E(H)$.

An involution $i \in I(G)$ is an $\mathcal{E}$*-involution* if $E_{\mathcal{E}}(C_G(i)) > 1$.

Definition 4.6. Let G be a simple L^*-group of finite Morley rank and odd type satisfying Hypothesis 4.1 and let L be an $\mathcal{E}$-component of the centralizer of an involution. We say L is *isolated* if it is a component of the centralizer of any involution which commutes with L.

Remark 4.7. Any $\mathcal{E}$-component of type SL_2 is isolated.

An $\mathcal{E}$-component L of type PSL_2 is isolated if there is no involution i with $E_{\mathcal{E}}(C_G(i)) \simeq \hat{L} *_2 \hat{L}$ with $\hat{L} \simeq \mathrm{SL}_2$ over the same base field as L.

In particular, isolated components must exist.

Lemma 4.8. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Suppose that L is an isolated component of G.*

Then no 4-group in G centralizes L.

Proof. Suppose toward a contradiction that the 4-group V centralizes L.

Since L is isolated, L is a component of $C^\circ(i)$ for $i \in I(V)$, by Fact 3.20. Hence Γ_V normalizes L, contradicting Corollary 2.43. $\square$

Definition 4.9. Let G be a group satisfying Hypothesis 4.1. If L is an $\mathcal{E}$-component for G, and i is an involution centralizing L, set $K_{L,i} = C^\circ_{C_G(i)}(L)$.

We generally will write K_L for this, when the involution i is fixed.

Remark 4.10. When L is of type SL_2 with central involution i, then i is the only involution centralizing L and $K_{L,i} = C^\circ_G(L)$.

Lemma 4.11. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Suppose that L is an isolated $\mathcal{E}$-component for G, i is an involution of G centralizing L, and $K_L = K_{L,i}$.*

Then we have the following.

(1) $C^\circ_G(i) = LK_L$ with both factors of Prüfer rank 1, and with $L \cap K_L \leq \langle i \rangle$.

(2) More precisely, we have the following.

(a) If L is of type PSL_2 then $C^\circ_G(i) = L \times K_L$.

(b) If L is of type SL_2 then $C^\circ_G(i) = L *_2 K_L$ with intersection $\langle i \rangle$.

(3) i is the unique involution of K_L, and the Sylow 2-subgroup of K_L is either connected or as in SL_2.

(4) $N_G(K_L) = N_{C_G(i)}(L)$.

Proof.
Ad 1. As $C_G(i)$ permutes its components, $C_G^\circ(i)$ normalizes them.

As $C_G^\circ(i)$ acts on L by inner automorphisms we find

$$C_G^\circ(i) = LK_L.$$

As $C_G^\circ(i)$ has Prüfer rank 2 and L has Prüfer rank 1, K_L has Prüfer rank 1.

The involution i is central in K_L. By Lemma 4.8, $C_{C_G(i)}(L)$ contains no 4-group, so i is its only involution. As $L \cap K_L \leq Z(L)$ we have $L \cap K_L \leq \langle i \rangle$.

Ad 2. Considering the possibilities according to the type of L gives $(2a, 2b)$.

Ad 3. We saw above that i is the unique involution of K_L.

Let S be a Sylow 2-subgroup of K_L and $T = S^\circ$, 2-torus of Prüfer rank 1. As S has a unique involution, any element of $S \setminus T$ must invert the elements of order 4 in T. It follows that $S = T$ or $S = T\langle w \rangle$ where w inverts T and $w^2 = i$.

Ad 4. The claim is that $N_G(K_L) = N_{C_G(i)}(L)$.

Certainly $N_{C_G(i)}(L) \leq N(K_L)$.

Conversely, if T is a maximal 2-torus of K_L, then

$$N_G(K_L) \leq K_L N_G(T) \leq C_G(i).$$

If $C_G(i)$ normalizes L there is nothing more to prove, and if not then $E_{alg}(K_L)$ is a conjugate of L, so $N_G(K_L)$ fixes that component of $C_G(i)$. and hence also fixes the component L. □

Lemma 4.12. *Let G be a group satisfying Hypothesis 4.1 and suppose that L an isolated $\mathcal{E}$-component of $C_G(i)$. Let i be an involution centralizing L.*

If A is an elementary abelian 2-group of rank 3 contained in $N_{C_G(i)}(L)$, then

$$A \cap C_G^\circ(A) \leq \langle i \rangle .$$

If L is of type PSL_2 then this intersection is $\langle i \rangle$, while if L is of type SL_2 the intersection is trivial.

Proof. A acts on L as a group of inner automorphisms. As no 4-group centralizes L, A must contain i and induce a 4-group acting faithfully on L.

Claim 1. $C_G^\circ(A) \leq K_L$

We have

$$C_G^\circ(A) = C_{C_G(i)}^\circ(A) = C_{LK_L}^\circ(A).$$

Working modulo $L \cap K_L = Z(L)$ we have a direct product, and one finds

$$C_G^\circ(A) \leq C_{L \bmod Z(L)}^\circ(A) \cdot C_{K_L \bmod Z(L)}^\circ(A).$$

But $C_{L \bmod Z(L)}^\circ(A) = 1$, so $C_G^\circ(A) \leq K_L$, as claimed.

Hence $A \cap C_G^\circ(A) \leq A \cap K_L \leq \langle i \rangle$.

If L is of type PSL_2 then $A = \langle V, i \rangle$ with $V \leq L$ and $C_{K_L}(A) = K_L$, so in this case the intersection is $\langle i \rangle$.

Conversely, if $i \in C_G^\circ(A)$, then i lies in some 2-torus T of K_L centralizing A. By Fact 2.13 we have $A \leq C_G^\circ(T) \leq C_G^\circ(i)$. But then TA is contained in a Sylow 2-subgroup of $C_G^\circ(i)$. If L is of type SL_2 then there is no such subgroup A. So in this case L is of type PSL_2. $\square$

Lemma 4.13. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and let i be an involution of G for which $E_{\mathcal{E}}(C_G(i))$ has more than one component.*

Then $E_{\mathcal{E}}(C_G(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$, possibly over different base fields.

Proof. Set $L = E_{\mathcal{E}}(C_G(i))$. Then L is a central product of two components $L_1 * L_2$, each of Prüfer rank 1.

Claim 1. *$E_{\mathcal{E}}(C_G(i))$ has a component of type SL_2.*

Let T be a maximal 2-torus of L. Then $i \in C(T)$ and hence $i \in T \leq L$. So $Z(L)$ is nontrivial and the claim follows.

We may suppose that L_1 is of type SL_2. Then L_1 is isolated and by Lemma 4.8 no 4-group centralizes it. So L_2 is also of type SL_2 and the product cannot be direct.

The result follows. □

4.3. **Isolated components of type PSL_2.**

Lemma 4.14. *Let G be a group satisfying Hypothesis 4.1. Suppose that L is an isolated $\mathcal{E}$-component of type PSL_2 and let i be an involution centralizing L.*

Then the following hold.

(1) *The involutions of $C_G(i)$ lie in $L\langle i\rangle$, hence in $C^\circ_G(i)$.*

(2) *The involution i is co-toral with any involution of $C_G(i)$.*

(3) *The only $\mathcal{E}$-involution in $C_G(i)$ is i.*

(4) *A Sylow 2-subgroup of $C_G(i)$ is a Sylow 2-subgroup of G.*

(5) *$C_G(i)$ is connected.*

(6) A Sylow 2-subgroup of G has the form

$$T_1 \langle w \rangle \times S_2$$

with T_1 a 2-torus, w an involution inverting T_1, and S_2 either a 2-torus or as in SL_2.

In particular, G has 2-rank 3.

(7) $E_{\mathcal{E}}(C_G(i)) = L$.

Proof.

Ad 1. Consider an involution j in $C_G(i)$. As j acts on L by an inner automorphism there is an element t in L, either an involution or the identity, for which jt is an involution centralizing L.

Then $\langle jt, i \rangle$ centralizes L. By Lemma 4.8 no 4-group centralizes L, so $jt \in \langle i \rangle$. Point (1) follows.

Ad 2. This follows from (1) by Fact 3.11.

Ad 3. Let j be an $\mathcal{E}$-involution in $C_G(i)$ (hence in $C_G^\circ(i)$). In view of the structure of $C_G^\circ(i)$ there is an elementary abelian 2-group A of rank 3 containing i and j.

Suppose first that $E_{\mathcal{E}}(C_G(j))$ has an isolated component. Then Lemma 4.12 applies to A and to both i and j, giving $\langle i \rangle = \langle j \rangle$ and $i = j$.

In particular, if $E_{\mathcal{E}}(j)$ has a component of type SL_2 we have a contradiction. Therefore all $\mathcal{E}$-components are of type PSL_2, and are isolated. So the argument applies generally.

Ad 4. Let S be a Sylow 2-subgroup of $C_G(i)$. As the only $\mathcal{E}$-involution in S is i, we find

$$N_G(S) \leq C_G(i)$$

and hence S is a Sylow 2-subgroup of G.

Ad 5. Since i is not conjugate to any other involution in $C_G(i)$, by Fact 3.6 $C_G(i)$ must be connected.

Ad 6. We work with a Sylow 2-subgroup of $C_G(i) = L \times K_L$. Here K_L is connected of Prüfer rank 1 and has a unique involution i.

If K_L is a D-group then its Sylow 2-subgroup is connected and the structure is as stated with S_2 a 2-torus.

Otherwise, with $\bar{K}_L = K_L/OF(K_L)$, we have $\bar{K}_L = E_{alg}(\bar{K}_L)\bar{K}_L^*$ with $E_{alg}(\bar{K}_L)$ of type SL_2 and $\bar{K}_L^*$ of degenerate type. We get the stated structure with S_2 as in SL_2.

Ad 7. By Lemma 4.13, $E_{\mathcal{E}}(C_G(i)) = L$. □

Lemma 4.15. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and suppose that $L \in \mathcal{E}$ is of type* PSL_2 *and isolated.*

Then there are exactly three conjugacy classes of involutions in G. These may be characterized by the following properties of the involution $t \in I(G)$.

(1) *$E_{\mathcal{E}}(C_G(t))$ is of type* PSL_2*; then $C_G(t)$ is connected.*

(2) *t commutes with an $\mathcal{E}$-involution t' with $t \in E_{\mathcal{E}}(C_G(t'))$; then $E_{\mathcal{E}}(C_G(t)) = 1$ and $C_G(t)$ is disconnected.*

(3) *t commutes with an $\mathcal{E}$-involution $t' \neq t$ with $t \notin E_{\mathcal{E}}(C_G(t'))$; then $E_{\mathcal{E}}(C_G(t)) = 1$ and $C_G(t)$ is disconnected.*

Furthermore:

($*$) *A pair of distinct commuting involutions s, t in G are co-toral if and only if there is an $\mathcal{E}$-involution in the 4-group $\langle s, t\rangle$.*

Proof. We fix an $\mathcal{E}$-involution i, and we apply the information given in Lemma 4.14.

Claim 1. *There are exactly three conjugacy classes of involution in G, represented by i, the involutions of L, and the involutions of Li other than i.*

Let T be a maximal 2-torus of $C_G(i)$. Every involution is conjugate to an involution of T (Fact 2.12). We know that i is the only $\mathcal{E}$-involution in T. In particular $N_G(T) \leq C_G(i)$.

If the other two involutions in T are conjugate, then as fusion in T is controlled by $N_G(T)$ (Fact 2.14), there is a 2-element of $N_G(T)$ which swaps them. Hence this conjugacy takes place in S; but S centralizes the involutions of T. This proves the claim.

Now according to Lemma 4.14 the $\mathcal{E}$-involutions t are conjugate to i and have $C_G(t)$ connected, so point (1) is taken care of, and in addition $E_{\mathcal{E}}(C_G(t)) = 1$ in the other two cases. And in addition we have most of (2,3), apart from the question of connectedness of $C_G(t)$, to which we will return.

Claim 2. *If s, t are distinct commuting involutions, then they are co-toral if and only if the group $\langle s, t\rangle$ contains an $\mathcal{E}$-involution.*

If $\langle s, t\rangle$ contains an $\mathcal{E}$-involution then we may suppose that involution is i and then look again at $C_G(i)$ to conclude.

Conversely, if s, t are co-toral then as the Prufer rank is 2 the group $\langle s, t\rangle$ contains a conjugate of i.

The claim follows.

Now looking again at representatives of conjugacy classes of involutions in $C_G(i)$, each involution t which is not an $\mathcal{E}$-involution commutes with an involution which is not co-toral with it. Thus by Fact 3.11, $C_G(t)$ is not connected.

So now all points (1–3), as well as point $(*)$, have been verified. $\square$

4.4. **Components of type** SL_2.

Definition 4.16. If G is a group of finite Morley rank satisfying Hypothesis 4.1, then an involution $t \in I(G)$ is called an SL_2-*involution* iff $C_G(t)$ has a component in $\mathcal{E}$ of type SL_2.

Of course, we expect to have SL_2-involutions. Here we work toward Lemma 4.21 below, which lays out the main configurations that arise in that case, provisionally; one of them will be eliminated later.

Lemma 4.17. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Suppose that i is an SL_2-involution and L is a component of $C_G(i)$ of type SL_2.*

Then

(1) $C_G^\circ(i) = N_G(L)$.

(2) If $C_G(i)$ is disconnected then $E_{alg}(C_G(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ with the factors conjugate in $C_G(i)$.

Proof.

Ad 1. The connected group $C_G^\circ(i)$ must normalize its components. On the other hand, $N_G(L) = L \cdot C_G(L)$ and if T is a 2-torus of L then $C_G(L) \le C_G(T) \le C_G^\circ(i)$ (Fact 2.13).

Ad 2. If $C_G(i)$ is disconnected then L has a conjugate $K \neq L$ in $C_G(i)$ and it follows that $E_{alg}(C_G(i)) = L *_2 K = E_{\mathcal{E}}(C_G(i))$. The result follows. □

Lemma 4.18. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Suppose that i is an SL_2-involution.*

Then either

(1) $E_{alg}(C_G(i)) = E_{\mathcal{E}}(C_G(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ with components conjugate in $C_G(i)$, or

(2) $C_G(i)$ is connected. Letting $K_L = C_G^\circ(L)$ we have

(a) $C_G(i) = L *_2 K_L$ *with* K_L *of Prüfer rank* 1 *and unique involution* i*.*

(b) $K_L = C_G(L)$*.*

(c) $E_{alg}(K_L/OF(K_L))$ *is of type* SL_2 *(possibly with a different base field from* L*, and not necessarily in* $\mathcal{E}$*).*

(d) *A Sylow* 2*-subgroup of* $C_G(i)$ *is a Sylow* 2*-subgroup of* G*, and is isomorphic to the Sylow* 2*-subgroup of* $\mathrm{SL}_2 *_2 \mathrm{SL}_2$*.*

In particular, G *has* 2*-rank* 3*.*

Proof. If $C_G(i)$ is disconnected then point (1) simply rephrases Lemma 4.17.

So we will suppose the contrary.

$$C_G(i) \text{ is connected.}$$

We then aim at point (2), clauses (a–d).

Clause (a) is covered by Lemma 4.11, bearing in mind Remark 4.10. .As we suppose $C_G(i)$ is connected, (b) then follows.

Let S be a Sylow 2-subgroup of $C_G(i)$.

Claim 1. *S is a Sylow* 2*-subgroup of G.*

Any involution of $C_G(i)$ other than i can be written as ab with $a \in L$, $b \in K_L$ of order 4. It follows that i is the only involution in $Z(S)$. Therefore i is central in $N(S)$ and the claim follows.

Now we consider $\bar{K}_L = K_L/OF(K_L) = E_{alg}(\bar{K}_L)\bar{K}_D$ with $K_D \leq K_L$ a D-group (Fact 2.7).

Claim 2. $E_{alg}(\bar{K}_L) \neq 1$*.*

Supposing the contrary, $K_L = K_D$ is a D-group and its Sylow 2-subgroup is connected. Then inspection of involutions in S shows that the 2-rank is 2, a contradiction. The claim follows.

Now as $E_{alg}(\bar{K}_L)$ has Prüfer 2-rank one and no involution other than i, it is of type SL_2. This gives us point (c).

It then follows that S has the isomorphism type of a Sylow 2-subgroup of $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ (but with no assumption of any connection between the base fields of L and of $E_{alg}(\bar{K}_L)$).

The result is proved. $\square$

Lemma 4.19. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1, and having an SL_2-involution.*

Then G has at most two conjugacy classes of involutions.

Furthermore, all SL_2-involutions of G are conjugate, and all non-SL_2-involutions of G (if any) are conjugate.

Proof. Let i be an SL_2-involution of G, S a Sylow 2-subgroup of $C_G(i)$, and $T = S^\circ$. As usual the conjugacy classes of involutions in G have representatives in T. By Lemma 4.18 if j is another involution of T then j, ij are conjugate in S.

So there are at most two conjugacy classes of involutions in G. If there are any non-SL_2-involutions in G then everything is clear.

On the other hand, if all involutions of G are SL_2-involutions, then it follows similarly that ij and i are conjugate in $C_G(j)$, and thus there is just one conjugacy class of involutions. $\square$

Lemma 4.20. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1, and having an SL_2-involution and two conjugacy classes of involutions. Suppose that the centralizer of an SL_2-involution is connected.*

Then the centralizer of a non-SL_2-involution is a D-group.

Proof. Let i, t be commuting involutions with i an SL_2-involution and t a non-SL_2-involution, and T a maximal 2-torus containing i, t. We have case (2) of Lemma 4.18. We claim that $E_{alg}(C_G(t)/OFC_G(t)) = 1$.

Claim 1. *There is no component $\bar{L}$ of $C_G(t)/OF(C_G(t))$ of type* PSL_2*.*

Assuming the contrary, there is an involution inverting a Prüfer rank 1 subgroup of T and centralizing a Prüfer rank 1 subgroup. This contradicts the structure of the Sylow 2-subgroup of G. The claim follows.

Now if $E_{alg}(C_G(t)/OFC_G(t))$ has two components then its Sylow 2-subgroup must be a Sylow 2-subgroup of G with t central, and hence t is conjugate to i, for a contradiction.

So if $E_{alg}(C_G(t)/OFC_G(t)) > 1$ we are left with the following possibility.

$E_{alg}(C_G(t)/OFC_G(t))$ is a single component, of type SL_2.

Let S be a Sylow 2-subgroup of $C_G(t)$ containing T. Then S centralizes the involutions of T and in particular $S \leq C_G(i)$. So $S = T \cdot \langle w \rangle$ with w an element of order 4 inverting T. But $C_G(t)$ contains an element of order 4 with a different nontrivial action on T, and we have a contradiction. □

We now combine Lemmas 4.20 and 4.18 into a somewhat more explicit list of possibilities.

Lemma 4.21. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and let i be an* SL_2*-involution of G.*

Then one of the following applies.

(1) $E_{alg}(C_G(i)) = E_{\mathcal{E}}(C_G(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ with components conjugate in $C_G(i)$, or

(2) $C_G(i)$ is connected, contains a Sylow 2-subgroup of G, and has the form $L *_2 K_L$ with L of type SL_2 and K_L of Prüfer rank 1 and unique involution i; $E_{alg}(K_L/OF(K_L))$ is of type SL_2. Furthermore we have one of the following.

(2a) There are two conjugacy classes of involution. For t not an SL_2*-involution, $C_G(t)$ is a D-group.*

(2b) There is one conjugacy class of involutions, and they satisfy

$$E_{alg}(C_G(i)/OFC_G(i)) = \mathrm{SL}_2 * \mathrm{SL}_2,$$

(possibly with differing base fields).

This gives us a PSp_4-configuration, a G_2-configuration, and a pathological configuration to consider.

5. Component analysis: The case of no PSL_2-component

We pursue the component analysis in the case in which there are SL_2-involutions and there is no $\mathcal{E}$-component of type PSL_2. Our aim is to show that there is just one conjugacy class of involutions in this case. So we will devote this section to eliminating the configuration that arises if this fails.

Thus we consider the following pathological configuration.

Hypothesis 5.1. G is a group satisfying Hypothesis 4.1.

In addition, all components in $\mathcal{E}$ are of type SL_2, and there are two conjugacy classes of involutions.

Notation 5.2. In the context of Hypothesis 4.1 we will also set $\rho = r_{f,G}$.

5.1. Generalities.

Recapitulating, we start with the following.

Lemma 5.3. *Let G be a group satisfying hypothesis 5.1.*
Then

(1) *For i an SL_2-involution, $E_{alg}(C_G(i)/OF(C_G(i))) \simeq \mathrm{SL}_2 *_2 \mathrm{SL}_2$*

(2) *For t a non-SL_2-involution, $C_G^\circ(t)$ is a D-group, and in particular its Sylow 2-subgroup is a 2-torus.*

Proof. Lemma 4.21. $\square$

Lemma 5.4. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1. Let i be an SL_2-involution, $L = E_{alg}(C_G(i))$.*
Then

(1) *Case (2) of Lemma 4.18 applies. In particular, $C_G(i)$ is connected and contains a Sylow 2-subgroup of G, isomorphic to the Sylow 2-subgroup of $SL_2 *_2 \mathrm{SL}_2$.*

(2) *A pair of distinct commuting involutions s, t in G are co-toral if and only if there is an SL_2-involution in $\langle s, t\rangle$.*

Proof.
Ad 1. Suppose toward a contradiction that $E_{\mathcal{E}}(C_G(i))$ has two conjugate components L_1, L_2. Then $C_G(i)$ contains involutions not in $C_G^\circ(i)$; these are the involutions swapping the two components. If j is such an involution, then $C_G(j)$ contains a group of type PSL_2, so in view of Lemma 5.3 j must be an SL_2-involution. In particular j and ij are both SL_2-involutions, hence are conjugate. On the other hand, for $j \in C_G^\circ(i)$, j and ji are also conjugate.

Applying what was just proved to j rather than i, it follows that any involution t in $C_G(j)$ is conjugate to tj.

With $\mathbb{T}_1$ an algebraic torus of L_1, j inverts the group $\mathbb{T}^* = \{(t(t^{-1})^j) : t \in \mathbb{T}_1\}$. Let w be the involution of $\mathbb{T}^*$ and w' a square root of w in $\mathbb{T}_1$. Then

$$j^{w'} = jw.$$

On the other hand w and wj are conjugate in $C_G(j)$ and so w, j are conjugate. This gives a contradiction.
Ad 2.

Let s, t be commuting involutions in G, and $V = \langle s, t \rangle$. If s, t belong to a 2-torus T then $V = \Omega_1(T)$ meets every conjugacy class of involutions, hence contains an SL_2-involution.

Conversely, if $V \leq L$ contains an SL_2-involution then we may suppose $i \in V$. Since $C_G(i)$ is connected the claim follows. $\square$

Remark 5.5. Let G be a group satisfying Hypothesis 5.1. Then for i an SL_2-involution of G and $E = E_{alg}(C_G(i))$, we have

$$N(E) = C_G(i).$$

Indeed, $N(E) \leq N(Z(E)) = C_G(i)$ and the reverse inclusion is clear.

In what follows, one must bear in mind particularly the conclusion that $C_G(i)$ is connected. In particular, the involutions of $C_G(i)$ other than i are conjugate.

Lemma 5.6. *Let G be a group satisfying Hypothesis 5.1. Let i be an* SL_2*-involution and L an associated $\mathcal{E}$-component. If H is a definable proper subgroup of G of 2-rank at least 2 containing L, then $H \le C_G(i)$.*

Proof.

Claim 1. $\Delta_\rho(OF(H)) \le C_G(i)$.

$\Delta_\rho(OF(H))$ is the product of unipotent subgroups $U_p(F(H))$ and $U_{0,r}$-subgroups $U_{0,r}(F(H))$ with $r > \rho$. So let U be one of the subgroups $U_p(F(H))$ or $U_{0,r}(F(H))$ with $r > \rho$.

Take a 4-group V in H containing i. The involutions of V other than i are not SL_2-involutions.

Fix an involution $t \in V$ other than i, and consider the group $U_t = C_U(t) \le \Delta_\rho(C_G(t))$. As t is not an SL_2-involution this must be trivial. So t inverts U. The same applies to ti. Thus i centralizes U.

Claim 2. *L centralizes $OF(H)$.*

We have $\Delta_\rho(OF(H)) \le \Delta_\rho(C_G(i)) \le E_{alg}(C_G(i))$. It follows that $\Delta_\rho(OF(H))$ normalizes L, so $[L, \Delta_\rho(OF(H))] \le L \cap OF(H) = 1$.

On the other hand L centralizes $U_{0,r}F(H)$ for $r \le \rho$, so L centralizes $OF(H)$, as claimed.

Now let $\hat{L}$ be the normal closure of L in H. Then $\hat{L}$ centralizes $OF(H)$. The image of $\hat{L}$ in $H/OF(H)$ is contained in $E_{alg}(H/OF(H))$, and normal. Hence $\hat{L}/Z(\hat{L})$ is quasi-simple and $\hat{L} \le E_{alg}(H)$.

If $E_{alg}(H)$ has Prüfer 2-rank 1, or more than one component, then $L = \hat{L}$ is a component of H and the result follows.

If $E_{alg}(H)$ is quasi-simple with Prüfer 2-rank 2 then as it has two conjugacy classes of involution but only one type of $\mathcal{E}$-component and we reach a contradiction. $\square$

5.2. $UFC_G(t)$ **and** K_L**.** We consider $UFC_G(t)$ for involutions t which are not SL_2-involutions.

Lemma 5.7. *Let G be a group satisfying Hypothesis 5.1. Then the following hold.*

(1) For any involution t which is not an SL_2-involution, we have

$$\bar{\mathrm{r}}_0(UF(C_G(t))) > 0.$$

(2) $\rho > 0$.

Proof.
Ad 1. If we suppose (1) fails then for any involution t which is not an SL_2-involution we have $\bar{\mathrm{r}}_0(UF(C_G(t))) = 0$, $UF(C_G(t))$ is a good torus, and $UF(C_G(t))$ is central in $C_G(t)$.

Take a maximal 2-torus T and an SL_2-involution $i \in T$. Then for $t \neq i$ an involution in T, we find $T \leq Z(C^\circ_G(t))$ by Fact 2.6. Thus $C^\circ_G(t) \leq C_G(i)$. So with $V = \Omega_1(T)$ we have $\Gamma_V \leq C_G(i) < G$, and a strongly embedded subgroup, for a contradiction.

Thus $\bar{\mathrm{r}}_0(UF(C_G(t))) > 0$ for such involutions t.
Ad 2. In particular,

$$\rho \geq \bar{\mathrm{r}}_0(OF(C_G(t))) > 0. \qquad \square$$

Lemma 5.8. *Let G be a group satisfying Hypothesis 5.1.*

Let i be an SL_2-involution of G and t an involution of $C_G(i)$ other than i. Let T be a Sylow 2-subgroup of $C^\circ_G(t)$ and let L be a component of $E_{alg}(C_G(i))$.

Then

(1) $C_G(T) \leq N(L)$.

(2) $C^\circ_G(t) = UF(C_G(t)) \cdot C_G(T)$.

(3) If $UF(C_G(t)) \le N(L)$ *then* $C_G^\circ(t) \le C_G(i)$.

Proof.
Ad 1. $C_G(T) \le C_G(i) = N(L)$.
Ad 2. This holds by Lemma 2.33.
Ad 3. This follows from (1) and (2). □

Lemma 5.9. *Let G be a group satisfying Hypothesis 5.1. Let i be an* SL_2*-involution of G and t an involution of $C_G(i)$ other than i.*

Then $UF(C_G(t))$ is not contained in $C_G(i)$.

Proof. Assuming the contrary, by Lemma 5.8 we find

$$C_G^\circ(t) \le C_G(i).$$

Let V be a 4-group in $C_G(i)$ not containing i. As the involutions of V are conjugate to t in $C_G(i)$, it follows that $\Gamma_V \le N(L)$, contradicting Corollary 2.43. □

Lemma 5.10. *Let G be a group satisfying Hypothesis 5.1. Let i be an* SL_2*-involution, L a component of $E_{\mathcal{E}}(C_G(i))$, and $K_L = C_G^\circ(L)$. Let $t \in C_G(i)$ be a non-*SL_2*-involution.*

Then for all r we have

$$U_{0,r}OF(C_G(t)) \cap K_L = 1.$$

Proof. Suppose toward a contradiction that

$$X = U_{0,r}OF(C_G(t)) \cap K_L > 1.$$

As this intersection is also the centralizer in $U_{0,r}F(C_G(t))$ of a finite set of involutions in L, the group X it is a $U_{0,r}$-group (Fact 2.34).

Claim 1. *For $s \neq r$,*

$$U_{0,s}(F(C_G(t))) \leq N(L).$$

The group $C_G(X)$ contains L and has 2-rank at least 2, so by Lemma 5.6 we have $C_G(X) \leq N(L)$. Since $U_{0,s}(F(C_G(t)))$ commutes with X for $s \neq r$, the claim follows.

Let $\mathbb{T}_t = C_L(t)$ and let T be a maximal 2-torus of $\mathbb{T}_t$.

Claim 2. *The 2-torus T centralizes $U_{0,r}(FC^\circ_G(t))$.*

We apply Lemma 3.2 to $Q = U_{0,r}(F(C_G(t)))$, T, and $U = Q \cap N(L)$. Here T normalizes Q, and centralizes U. Furthermore $C_Q(U) \leq U$ since

$$C_G(U) \leq C_G(X) \leq N(L).$$

So the lemma applies, and T centralizes Q.

Thus $U_{0,r}(FC_G(t)) \leq C_G(T) \leq C_G(i) = N(L)$. So $UF(C_G(t)) \leq N(L)$ and we contradict Lemma 5.9.

This contradiction completes the proof. □

Lemma 5.11. *Let G be a group satisfying Hypothesis 5.1. Let i be an* SL_2*-involution, L a component of $E_{\mathcal{E}}(C_G(i))$, and $K_L = C^\circ_G(L)$. Let $t \in C_G(i)$ be a non-*SL_2*-involution.*

$$F(C_G(t)) \cap K_L = 1.$$

Proof. If $X = F(C_G(t)) \cap K_L > 1$ then as $C(X)$ contains L and has 2-rank at least 2, we have $C(X) \leq C_G(i)$. On the other hand by Lemma 5.10, $U_{0,r}(X) = 1$ for all r and X° is a good torus. Here X may be finite, but in any case contains a nontrivial torsion element a, which commutes with $UF(C_G(t))$. So by considering $C(a)$ we find $UF(C_G(t)) \leq C_G(i)$ and again reach a contradiction. □

5.3. $FC_G(t)$ and $C_G(i)$.

Lemma 5.12. *Let G be a group satisfying Hypothesis 5.1. Let i be an* SL_2*-involution, $t \in C_G(i)$ a non-*SL_2*-involution, and T a 2-torus containing $\langle i, t\rangle$.*

Then $C_G(i,t) = C(T)$.

Proof. Let $H = C_G(\langle i, t\rangle)$. As t is represented as a product of two elements of order 4 with square i in $E_{alg}(C_G(i)/OFC_G(i))$, one from each component of $E_{\mathcal{E}}(C_G(i))$, we find that H acts on $E_{alg}(C_G(i)/OFC_G(i))$ like a subgroup of $C(T)$, and thus we have $[T, H] \leq OFC_G(i) \leq K_L$.

But by Fact 2.8, also $[T, H] \leq UF(C_G(t))$ and so by Lemma 5.10, $[T, H] = 1$. □

Lemma 5.13. *Let G be a group satisfying Hypothesis 5.1. Let i be an* SL_2*-involution, L an $\mathcal{E}$-component of $C_G(i)$, and $t \in C_G(i)$ a non-*SL_2*-involution. Let T be a 2-torus of $C_G(i)$ containing $\langle i, t\rangle$.*

Let

$$Q = F(C_G(t)) \cap C_G(i).$$

Then $Q \leq C_L(T)$.

Proof. Let T_1 be the 2-torus $T \cap L$ and let $w \in L$ invert the algebraic torus $\mathbb{T}_t = C_L(t)$ of L containing T_1.

The group Q is w-invariant. As w inverts $Q \cap L$, and $C_Q(w) \leq K_L$, the group Q decomposes as $(Q \cap L)(Q \cap K_L)$. But $Q \cap K_L = 1$ by Lemma 5.10, so $Q \leq L$.

In view of Lemma 5.12, the result follows. □

Lemma 5.14. *Let G be a group satisfying Hypothesis 5.1. Let i be an* SL_2*-involution, $t \in C_G(i)$ a non-*SL_2*-involution, and $r \geq 1$.*

If $U_{0,r}(FC_G(i)) > 1$ then $U_{0,r}(FC_G(t)) \leq C_G(i)$.

Proof. Let

$$\begin{aligned} U_i &= U_{0,r}(FC_G(i)) \cap C_G(t); \\ U_t &= U_{0,r}(FC_G(t)) \cap C_G(i); \\ U &= U_i \cdot U_t. \end{aligned}$$

Claim 1. $U_i > 1$.

There is a 4-group V in $C_G(i)$ whose involutions are not SL_2-involutions, and some involution $v \in V$ centralizes a nontrivial subgroup of $U_{0,r}(UFC_G(i))$. By conjugacy, the same applies to t.

The claim follows by Fact 2.34.

Now $U_{0,r}(FC_G(t)) \cdot U$ is a solvable $U_{0,r}$-group, hence nilpotent.

Let $Q = N_{U_{0,r}(FC_G(t)) \cdot U}(U)$. Note that $Q \leq C_G(t)$.

By Lemma 5.12, T centralizes U.

By Lemma 5.6, $C_G(U) \leq C_G(U_i) \leq C_G(i)$. So $C_Q(U) \leq C_Q(i) = U$. Now by Lemma 3.2, T centralizes Q.

Accordingly $Q = U$ and hence $U_{0,r}(FC_G(t)) \leq U \leq C_G(i)$. The result follows. □

Lemma 5.15. *Let G be a group satisfying Hypothesis 5.1. For i, t commuting involutions with i an SL_2-involution and t a non-SL_2-involution, and any parameter r for which $U_{0,r}(OFC_G(t)) > 1$, we have*

$$C_{U_{0,r}(OFC_G(t))}(i) > 1.$$

Proof. We suppose on the contrary that $C_{U_{0,r}(OFC_G(t))}(i) = 1$. By Lemma 5.14 we find $U_{0,r}(FC_G(i)) = 1$.

Furthermore i inverts $U_{0,r}(OFC_G(t))$, and this applies to all involutions t of $C_G(i)$ other than i.

Claim 1. *The function $\theta_r(s) = U_{0,r}(OFC_G(s))$ defines a signalizer functor.*

We have to check the balance condition $\theta_r(t_1) \cap C_G(t_2) \le OF(C_G(t_2))$ for commuting involutions t_1, t_2. This is trivial if the intersection in question is trivial, which covers the case in which either one of the involutions t_1, t_2 is i.

So we may suppose that t_1, t_2 are non-SL_2-involutions commuting with i, in which case our assumptions imply that i inverts $OF(C_G(t_1))$. But then $\theta_r(t_1) \cap C_G(t_2) \le [i, C_G(t_2)] \le OF(C_G(t_2))$ and the balance condition holds.

This proves the claim.

By Fact 3.24 this signalizer functor must be trivial, contradicting our assumptions. This contradiction completes the proof. □

5.4. L_2.

Lemma 5.16. *Let G be a group satisfying Hypothesis 5.1. For i an SL_2-involution of G, $E_{\mathcal{E}}(C_G(i))$ consists of a single component L.*

Proof. If there are two $\mathcal{E}$-components then Lemma 5.13 implies that for t a non-SL_2-involution commuting with i we have

$$UFC_G(t) \cap C_G(i) = 1.$$

This contradicts Lemma 5.15 □

In particular, going forward, L may be defined as $E_{\mathcal{E}}(C_G(i))$, unambiguously.

Definition 5.17. Let G be a group satisfying Hypothesis 5.1 and let i be an SL_2-involution of G.

Then we set $L = E_{\mathcal{E}}(C_G(i))$, $K_L = C_G^\circ(L)$, and

$$\bar{L}_2 = E_{alg}(K_L/OF(K_L)).$$

The group $\bar{L}_2$ is of type SL_2 by Lemma 5.3.

Let L_2 be a minimal normal definable subgroup of $C_G(i)$ covering $\bar{L}_2$. In particular L_2 is a perfect group and $L_2/OF(L_2) \simeq \bar{L}_2$.

Notation 5.18. Let k_2 be the base field for $\bar{L}_2$ and set $r_2^\times = \bar{\mathrm{r}}_0(k_2^\times)$, $r_2^+ = \mathrm{rk}(k_{2+})$.

According to Lemma 5.16, L_2 is not an $\mathcal{E}$-component of $C_G(i)$. Thus k_2 has characteristic 0 and r_2^+ can also be defined as $\bar{\mathrm{r}}_0(k_{2+})$. Furthermore $r_2^+ \leq r_{f,G} = \bar{\mathrm{r}}_0(k^x)$ with k the base field for L. It is possible, but not certain, that $OF(L_2) > 1$. (In fact we might be able to ensure this just by widening the definition of $\mathcal{E}$ a bit we do not take that approach, which would require revisiting earlier material.)

Lemma 5.19. *Let G be a group satisfying Hypothesis 5.1. Let i be an* SL_2*-involution. Suppose also that $OF(L_2) > 1$.*

Then any definable proper subgroup H of G containing L_2 is contained in $C_G(i)$.

Proof. Let $\hat{L}_2$ be the normal closure in H of L_2.

Claim 1. $\hat{L}_2 = L_2 OF(\hat{L}_2)$.

In $\bar{H} = H/OF(H)$, $\bar{L}_2$ is contained in $E_{alg}(\bar{H})$. If $E_{alg}(\bar{H})$ has Prüfer rank 2 and a single component then as it has two conjugacy classes of involutions and none of type PSL_2 we have a contradiction. If $E_{alg}(\bar{H})$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ then $\bar{L}_2$ is one of the components and hence $\hat{L}_2$ is not the normal closure, for a contradiction.

So $\hat{L}_2$ has Prüfer 2-rank 1 and $\bar{L}_2$ covers $\hat{L}_2/OF(L_2)$. That is, $L_2 OF(\hat{L}_2) = \hat{L}_2$ as claimed.

Furthermore $\bar{L}_2 \simeq L_2/OF(L_2)$, so $OF(L_2) \leq OF(\hat{L}_2) \leq OF(H)$.

Let $U = OF(H) \cap C_G(i)$. Then $C_G(U) \leq C_G(OF(L_2)) \leq C_G(i)$ by Lemma 5.6. Hence $C_{OF(H)}(U) \leq U$. By Lemma 3.2 we find $OF(H) \leq C_G(i)$.

Hence $\hat{L}_2 \leq C_G(i) \leq N(L_2)$ and $\hat{L}_2 = L_2$. Thus $H \leq N(L_2) = C_G(i)$ by Lemma 5.6. □

Lemma 5.20. *Let G be a group satisfying Hypothesis 5.1. If i is an* SL_2*-involution then we have the following.*

(1) $OF(L_2) = 1$.

(2) $E_{alg}(C_G(i)) = LL_2$.

Proof. It suffices to show the first point. Suppose the contrary, and consider $Q = UFC_G(t) \cap C_G(i)$.

By Lemma 5.15, $Q > 1$. By Lemma 5.13 $Q \leq C_L(T)$. As we suppose $OF(L_2) > 1$, by Lemma 5.19 we have $N_{UFC_G(t)}(Q) \leq C_G(i)$, and Q is self-normalizing in $UFC_G(t)$. Hence i centralizes $UFC_G(t)$, giving a contradiction. □

Lemma 5.21. *Let G be a group satisfying Hypothesis 5.1. Let i be an* SL_2*-involution.*

Then any definable proper subgroup H of G containing L_2 is contained in $C_G(i)$.

Proof. By Lemma 5.19 we may suppose

$$OF(L_2) = 1.$$

Let L_2^* be the normal closure of L_2 in H°, and $\bar{H} = H/OF(H)$.

Then $\bar{L}_2 \leq \bar{L}_2^* \leq E_{alg}(\bar{H})$.

Claim 1. $\bar{L}_2 = \bar{L}_2^*$.

Otherwise, $\bar{L}_2^*$ has Prüfer 2-rank 2. As it has two conjugacy classes of involutions and no component of type PSL_2, it must be of the type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$, in which case $\bar{L}_2$ is one of the factors and is normal in $\bar{H}^\circ$. Hence $L_2O(H)$ is normal in H° and $\bar{L}_2^* = \bar{L}_2$ after all (which is a contradiction in this case).

The claim follows. That is,

$$L_2^* = L_2 \cdot OF(L_2^*).$$

If $L_2^* \leq C_G(i)$ then L_2 is normal in L_2^* and hence $OF(L_2^*)$ centralizes L_2. But as L_2^* is the normal closure of L_2 in H°, $OF(L_2^*)$ is then central in L_2^*. But then $L_2^* = L_2 \times OF(L_2^*)$ $OF(L_2^*) = 1$, $L_2^* = L_2$, and $L_2 \lhd H^\circ$. In this case $H^\circ \le N(L_2) \le C_G(i)$ by Lemma 5.6. So i is the unique SL_2-involution of H°. As H normalizes H°, we have $H \le C_G(i)$ in this case.

So we will suppose

$$L_2^* \text{ is not contained in } C_G(i).$$

It follows that $OF(L_2^*)$ is not contained in $C_G(i)$. This will lead to a contradiction.

Let $A = F(L_2^*) \cap C_G(i)$. Note that $O(A) = A^\circ$ and $A = A^\circ \langle i \rangle$.

Claim 2. $A = C_{OF(L_2^*)}(L_2)$.

A normalizes L_2, so $[A, L_2] \le L_2 \cap A = 1$. Thus A centralizes L_2.
Conversely $C_G(L_2) \le N(L_2) = C_G(i)$ by Lemma 5.6.
The claim follows.

Claim 3. $O(A) \ge O(Z(L_2^*)) > 1$

Let $\mathbb{B}$ be a Borel subgroup of L_2. Then $OF(L_2^*)\mathbb{B}$ is connected and solvable, and its commutator subgroup lies in its Fitting subgroup. Hence the unipotent radical $\mathbb{U}$ of $\mathbb{B}$ is in the Fitting subgroup and $OF(L_2^*)\mathbb{U}$ is nilpotent. Hence $A_0 = C^\circ_{ZOF(L_2^*)}(\mathbb{U})$ is nontrivial. On the other hand $C(\mathbb{U}) \le N(\mathbb{U}) \le C_G(i)$ by Lemma 5.6. So $A_0 \le O(A)$. In particular A_0 centralizes L_2 as well as $OF(L_2^*)$, so $A_0 \le Z(L_2^*)$.

The claim follows.

We now consider

$$Q = N^\circ_{OF(L_2^*)}(A^\circ) > A^\circ.$$

By Fact 2.32 we have

$$Q = A^\circ \times Q^-.$$

Thus i inverts Q/A°, and Q/A° is abelian.

Let Q_0 be the centralizer in Q of $\mathbb{U}$ modulo A°. Then $Q_0 > A^\circ$. For $u \in \mathbb{U}$, $x \in Q_0$ we have

$$[u, x] = [u, x]^i = [u, x^i] = [u, x^{-1}]; \qquad [u, x^2] = 1; \qquad [u, x] = 1.$$

So $x \in C(\mathbb{U}) \leq C_G(i)$ and $Q_0 = A^\circ$, for a contradiction. □

5.5. **Two conjugacy classes implies PSL_2-components.** Now we may conclude.

Proposition 5.22. *Let G be a group satisfying Hypothesis 4.1, with an SL_2-involution, and with two conjugacy classes of involutions.*

Then some $\mathcal{E}$-component of an involution is of type PSL_2.

Proof. With $Q = OFC_G(t) \cap C_G(i)$, by Lemma 5.15 we have $Q > 1$, and by Lemma 5.13 we have $Q \leq C_L(T)$. By Lemma 5.21 we have $N_{OFC_G(t)}(Q) \leq C_G(i)$, and Q is self-normalizing in $OFC_G(t)$. Hence i centralizes $OFC_G(t)$, giving a contradiction. □

Lemma 5.23. *Let G be a group satisfying Hypothesis 4.1, with an SL_2-involution i, and with two conjugacy classes of involutions.*

Then

(1) $C_G(i)$ is disconnected.

(2) $E_{\mathcal{E}}(C_G(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ with the two components conjugate by an involution of type SL_2.

(3) The Sylow 2-subgroup is as in PSp_4.

Proof.

Ad 1.

Let S be a Sylow 2-subgroup of $C^\circ_G(i)$, $T = S^\circ$, and $\hat{S}$ a Sylow 2-subgroup of G containing S.

Claim 1. *There is an involution j in $\hat{S} \backslash S$.*

Let t be a PSL_2-involution and T_t a maximal 2-torus in $C_G(t)$. Then there is an involution w in $C_G(t)$ normalizing T_t and centralizing a Prüfer 2-rank 1 subgroup of T_t. Accordingly in a Sylow 2-subgroup $\hat{S}$ of G containing S, there is an involution j acting similarly on T. By inspection, j is not in $C^\circ_G(i)$.

This proves the claim.

Now i is the unique SL_2-involution in T, so j centralizes i. Therefore $C_G(i)$ is disconnected.

Ad 2,3. Let $E = E_{\mathcal{E}}(C_G(i))$.

Claim 2. *The normalizer of the components of E in $\hat{S}$ is S.*

Let $s \in \hat{S}$ normalize the components of E. Then there is some $s' \in S$ so that ss' centralizes E. But $C_S(T) = T$, so this forces $s \in S$. This proves the claim.

In particular, j switches the components of E and $\hat{S} = S \langle j \rangle$. This determines the structure of $\hat{S}$ and, in particular, this must match the structure of a Sylow 2-subgroup of PSp_4. $\square$

6. Component analysis: The case of no SL_2-components

We need to show that when $\mathcal{E}$-components of type PSL_2 occur, then we have two conjugacy classes of involutions, and that the two classes correspond to $\mathcal{E}$-components of type PSL_2 and SL_2, respectively. The essential point may also be phrased as follows.

Proposition 6.1. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1.*

Then some component in $\mathcal{E}$ is of type SL_2*.*

Accordingly we work toward a contradiction under the following hypothesis, to be expanded by some notational conventions in Hypothesis 6.4 below.

Hypothesis 6.2. G is a group of finite Morley rank satisfying Hypothesis 4.1. In addition, there is no component in $\mathcal{E}$ of type SL_2.

This will take an extensive analysis.

6.1. **Preliminaries.**

We are assuming that all $\mathcal{E}$-components are of type PSL_2, and, in particular, the terms "$E_{\mathcal{E}}$-involution" and "PSL_2-involution" are synonyms.

Furthermore, under this assumption the $\mathcal{E}$ components are isolated, so Lemmas 4.14 and 4.15 apply. For convenience we repeat those results here, in a different order.

(1) There are three conjugacy classes of involutions. These have the following form.

  (a) PSL_2-involutions;

  (b) component involutions: involutions in an $\mathcal{E}$-component;

  (c) products of a PSL_2-involution i with an involution in $E_{\mathcal{E}}(C_G(i))$.

(2) For i a PSL_2-involution and $L = E_{\mathcal{E}}(C_G(i))$, we have the following.

(a) L is a single $\mathcal{E}$-component, of type PSL_2.

(b) $C_G(i)$ is connected and contains a Sylow 2-subgroup of G, of the form

$$S_1 \times S_L$$

where $S_1 = T_t \langle w \rangle$ is a Sylow 2-subgroup of L, T_t a 2-torus of Prüfer 2-rank 1, w an involution inverting T_t, and $S_L = C_S(L)$ is either a Prüfer 2-rank 1 2-torus or as in SL_2. In particular, G has 2-rank 3.

(c) i is the only PSL_2-involution in $C_G(i)$.

(d) The involutions of $C_G(i)$ are in $L \langle i \rangle$.

(3) Commuting involutions t, t' of G are co-toral iff $\langle t, t' \rangle$ contains a PSL_2-involution.

As the analysis leading to the identification of PSp_4 might naturally begin with the assumption that the 2-rank is at least 4, none of what follows would be needed in that context.

We go over some closely related and familiar ground not explicitly contained in the above.

Lemma 6.3. *Let G be a group of finite Morley rank satisfying Hypothesis 6.2. Let i be an $\mathcal{E}$-involution, $L = E_{\mathcal{E}}(C_G(i))$, $K_L = C_{C_G(i)}(L)$, and $\bar{K}_L = K_L/OF(K_L)$.*

Then

$$\bar{K}_L = E_{alg}(\bar{K}_L) * \bar{K}_D$$

with K_D a D-group and $E_{alg}(\bar{K}_L)$ either trivial or of type SL_2.

Correspondingly, in the representation of a Sylow 2-subgroup S of $C_G(i)$ as $S_1 \times S_L$, the factor S_L is

(1) a Prüfer 2*-group if* $E_{alg}(\bar{K}_L) = 1$*;*

(2) the Sylow subgroup of SL_2 *otherwise.*

Proof. $C_G(i)$ is connected and $C_G(i) = L \times K_L$.

The main point here is that K_L is an L-group. So by Fact 2.6 we have

$$\bar{K}_L = E_{alg}(\bar{K}_L) * \bar{K}_D$$

with $\bar{K}_D/Z^\circ(\bar{K}_L)$ of degenerate type. In particular K_D is a D-group.

We know that the Sylow 2-subgroup of K_L contains a unique involution, so if $E_{alg}(\bar{K}_L)$ is non-trivial then it is of type SL_2, and contains S_L.

On the other hand if $E_{alg}(\bar{K}_L)$ is trivial then $\bar{K}_L$ is a connected D-group and its Sylow 2-subgroup is connected (Fact 2.3).

This covers all claims made. □

We now update our hypothesis and notation to include the structural information just given.

Hypothesis 6.4. G is a group satisfying Hypothesis 6.2.

i is an $\mathcal{E}$-involution in G. $L = E_{\mathcal{E}}(C_G(i))$. $K_L = C_{C_G(i)}(L)$.

$\tilde{E}_{alg}(K_L)$ is the preimage in K_L of $E_{alg}(K_L/OF(K_L))$. K_D is the preimage in K_L of $\bar{K}_D$.

Our focus now is on centralizers of other involutions, and on $N_G(L)$.

Lemma 6.5. *Let the group G be as in Hypothesis 6.4, with the associated notational conventions. Let t be an involution of $C_G(i)$ other than i.*

Set $H = C_G^\circ(t)$ and $\bar{H} = H/OH$. Then the following hold.

1. If T_t is a maximal 2*-torus of $C_L^\circ(t)$ and S_L is a Sylow* 2*-subgroup of K_L, then $T_t \times S_L$ is a Sylow* 2*-subgroup of H.*

2. $\bar{H}$ *decomposes as*

$$\bar{H} = E_{alg}(\bar{H}) * \bar{H}_D$$

with $\bar{H}_D$ *a D-group, and the image of* $\tilde{E}_{alg}(K_L)$ *in* $\bar{H}$ *is* $E_{alg}(\bar{H})$*.*

3. $H \leq UF(H) \cdot C^\circ_{N(L)}(t) = UF(H)(\mathbb{T}_t \times \hat{K}^\circ_L)$*.*

4. If $UF(H) \leq N(L)$*, then* $H \leq N(L)$*.*

Proof.

Ad 1. L contains t or ti, so $C^\circ_L(t)$ is an algebraic torus of L and T_t is its 2-torsion subgroup. T_t is inverted by an involution w of L.

$T_t \langle w \rangle \times S_L$ is a Sylow 2-subgroup of $C_G(t)$. As w, t are not co-toral, w is not in $C^\circ_G(t)$. So $T_t \times S_L$ is a Sylow 2-subgroup of H.

Ad 2. By Fact 2.6, $\bar{H}$ has the structure

$$E_{alg}(\bar{H}) * \bar{H}_D$$

with $\bar{H}_D$ a D-group.

The group $\tilde{E}_{alg}(K_L)$ is contained in H. Its image in $\bar{H}$ is isomorphic to $E_{alg}(\bar{K}_L)$. if $E_{alg}(\bar{K}_L)$ is non-trivial then in view of the structure of $T_t \times S_L$, it covers $E_{alg}(\bar{H})$. If $E_{alg}(\bar{K}_L)$ is trivial then $T_t \times S_L$ is connected and $E_{alg}(\bar{H})$ is trivial. So the second point follows in all cases.

Ad 3. As K_L normalizes (and centralizes) L, the same applies to its subgroup $\tilde{E}_{alg}(K_L)$. So it suffices to check that

$$H_D \leq UF(H)(H \cap N(L)).$$

Applying Lemma 2.33 to the group SH_D with $S = T_t \times S^\circ_L$ a maximal 2-torus in $C_G(i)$, we find

$$H_D \leq UF(H)C_H(S).$$

But $i \in S$ and so $C_H(S) \leq C_G(i) \leq N(L)$. The claim follows.

Ad 4. Apply (3). □

Lemma 6.6. *Let G be as in Hypothesis 6.4, with the associated notational conventions. Let t be an involution of L.*

Then $UFC_G(t)$ is not contained in $N(L)$.

Proof. Assuming the contrary, by Lemma 6.5 we have

$$C_G^\circ(t) \leq N(L).$$

Let $V \leq L$ be a 4-group. As the involutions of V are conjugate to t in L, it follows that $\Gamma_V \leq N(L)$, contradicting Corollary 2.43. □

The starting point for our analysis is Lemma 4.3 (part (2)), which we repeat for convenience. Namely, for any involution i in G we have

$$U_p(O^\sigma(C_G(i))) = 1 \text{ for all primes } p;$$
$$U_{(0,r)}(O^\sigma(C_G(i))) = 1 \text{ for all } r > r_{f,G}.$$

In particular, $UF(C_G(i))$ is torsion-free.

Lemma 6.7. *Let G be as in Hypothesis 6.4, with the associated notational conventions.*

Let k be the base field of a component in $\mathcal{E}$. Then k has characteristic 0.

Proof. Suppose on the contrary that k has positive characteristic. Then the algebraic tori in L are good tori, and hence have no non-trivial torsion free subgroups. Thus $r_f = 0$. It then follows from Lemma 4.3 that $O^\sigma(C_G(i))$ is a good torus and $UF(C_G(i)) = 1$.

This contradicts Lemma 6.6. □

Notation 6.8. We will write $E_{deg}(H)$ for the product of the quasi-simple components of $E(H)$ of degenerate type.

Lemma 6.9. *Let G be as in Hypothesis 6.4, with the associated notational conventions. Let H be a definable subgroup of G containing L.*

Then L is a component of $E_{alg}(H)$, and is the only component of PSL_2 *type. In particular,*

$$H \leq N(L).$$

Proof. We make use of the notation introduced in Definition 3.15.

Claim 1.

$$[L, \Delta_{r_f}(F(H))] = 1.$$

$\Delta_{r_f}(F(H))$ is the central product of the subgroups $U_{0,r}(F(H))$ with $r > r_f$. Let $U = U_{0,r}(F(H))$ be one such.

By Fact 2.34 the group $C_U(t)$ is also a $(0, r)$-group for any involution $t \in I(L)$. L contains a 4-group V and U is generated by $C_U(t)$ for t an involution of V (Fact 2.44), so it suffices to check that

$$[L, C_U(t)] = 1.$$

Let T be a maximal 2-torus of $C_G(t)$ containing i. Then by Fact 2.27, $[T, U]$ is contained in $U_{0,r}(FC_G(t))$, which is trivial by Lemma 4.3. Thus T centralizes $C_U(t)$. So $C_U(t) \leq C_G(i)$ and $C_U(t)$ acts on L, acting like a subgroup of $C_L^\circ(t)$. But then as $r > r_f$ this action is trivial, and $[L, C_U(t)] = 1$.

The claim follows.

Claim 2. *L centralizes $F^\circ(H)$ and $E_{deg}(H)$.*

By Fact 2.33 the 2-tori of L centralize $E_{deg}(H)$, so the second point is clear.

The root subgroups of L are copies of the additive group of the base field k_+. By Fact 3.17, k_+ is a $(0, r)$-group with $r = \mathrm{rk}(k) > r_f$.

A maximal divisible torsion subgroup of $F(H)$ is central in H and the root subgroups of L act trivially on $U_p(F(H))$ for any prime p, so it suffices to show that L centralizes $U_{0,r}(F(H))$ for all r. The case $r > r_f$ was dealt with in the previous claim and for $r \le r_f < \mathrm{rk}(k_+)$ it follows from the unipotence theory.

This proves the claim.

Claim 3. *$L \le E_{alg}(H)$.*

Set

$$H_1 = C_H^\circ(F^\circ(H)E_{deg}(H)).$$

Then $L \le H_1$ and

$$F^{*\circ}(H_1) = E_{alg}(H)$$

In particular $C_{H_1}(E_{alg}(H)) \le E_{alg}(H)$.

But H_1 induces inner automorphisms on $E_{alg}(H)$ and hence

$$H_1 \le E_{alg}(H)C_{H_1}(E_{alg}(H)) = E_{alg}(H).$$

So $L \le E_{alg}(H)$, as claimed.

Claim 4. *L is the unique component of $E_{alg}(H)$ of PSL_2 type.*

In view of the structure of the Sylow 2-subgroup of G, H can only have components of type PSL_2 and SL_2, and at most one of each. The claim follows.

From the last claim we infer that $H \le N(L)$. □

6.2. **Case 1. Involution centralizers and $C_G(L)$.** We focus now on the case where we have an involution $t \in C_G(i)$ with $OFC_G(t)$ not contained in $N(L)$, but $OFC_G(t)$ meets $C_G(L)$. So for the present we work with the following hypotheses and notation.

Hypothesis 6.10. G is a simple L^*-group of finite Morley rank and odd type of Prüfer 2-rank 2 and 2-rank at least 3, satisfying NTA_2. There is no component in $\mathcal{E}$ of type SL_2.

Fix i an $\mathcal{E}$-involution and set $L = E_{\mathcal{E}}(C_G(i))$, $\hat{K}_L = C_G(L)$, and $r_{\hat{K}_L} = \bar{\mathrm{r}}_0(\hat{K}_L)$.

Take an involution $t \in I(C_G(i))$, $t \neq i$. Let $\mathbb{T}_t$ be the algebraic torus $C_L^\circ(t)$.

Suppose

$$OFC_G(t) \cap \hat{K}_L > 1;$$
$$OFC_G(t) \not\leq N(L).$$

Note that by Lemma 6.5 the condition $OFC_G(t) \not\leq N(L)$ can be sharpened to

$$UFC_G(t) \not\leq N(L).$$

We will make use of this form of the hypothesis withour further mention.

Lemma 6.11. *Let G, i, and t satisfy Hypothesis 6.10. Then $t \in L$.*

Proof. Suppose toward a contradiction that $t \in Li$. Let

$$Q = OF(C_G(t)) \cap N(L).$$

Claim 1. $Q = OF(C_G(t)) \cap C_G(i)$.

Since $C_G(i) \leq N(L)$, one inclusion is clear. In the reverse direction, as Q is i-invariant and contains no involutions we have

$$Q = (Q \cap C_G(i)) \times Q^-$$

where Q^- is the set of elements inverted by i. However an element q of $N(L)$ inverted by i for which $d(q)$ contains no involutions must lie in $[i, N(L)] = [i, \hat{K}_L] \le \hat{K}_L$ and hence commute with the element ti of L. Hence $Q^- \subseteq C_G(t, ti) \le C_G(i)$ and $Q \le C_G(i)$.

This proves the claim.

Now let $Q_1 = N_{OF(C_G(t))}(Q)$. Then $C_{Q_1}(i) = Q$ and $Q_1 = Q \times Q_1^-$, with Q_1^- the subset inverted by i.

Claim 2. *Q_1^- centralizes Q.*

We take $q \in Q$, $x \in Q_1^-$, and we have

$$[q, x] = [q, x]^i = [q, x^{-1}]; \qquad\qquad [q, x^2] = 1$$

and thus x centralizes q, proving the claim.

To conclude, we have $Q_1^- \subseteq C_G(Q) \le N(L)$ since Q_t meets $\hat{K}_L$. Thus $Q_1 = Q$ and $Q = OF(C_G(t))$. That is, $OF(C_G(t)) \le N(L)$, for a contradiction.

Thus $t \in L$. □

Lemma 6.12. *Let G, i, L, $\hat{K}_L$, t, and $\mathbb{T}_t$ satisfy Hypothesis 6.10. Let T_t be the maximal 2-torus of $\mathbb{T}_t$. Then the following hold.*

(1) $U_{0,r_{\hat{K}_L}}(FC_G(t))$ is not contained in $N(L)$.

(2) $U_{0,s}(FC_G(t))$ is contained in L for $s \ne r_{\hat{K}_L}$.

(3) $OFC_G(t) \cap \hat{K}_L$ is a homogeneous $U_{0,r_{\hat{K}_L}}$-group.

Furthermore,

$$T_t \le Z(C_G^\circ(t))$$

Proof. Let $Q = OF(C_G(t)) \cap \hat{K}_L$.

Claim 1. *Q is a homogeneous $U_{0,r}$-group for some r, and $U_{0,s}(FC_G(t)) \leq N(L)$ for $s \neq r$.*

By Lemma 6.9, $C(Q) \leq N(L)$, and the same applies to any nontrivial subgroup of Q.

If Q_t meets $ZOF(C_G(t))$ then $OF(C_G(t)) \leq N(L)$ and we have a contradiction. So Q is a product of its subgroups $U_{0,r}(Q)$ for certain values of r. If more than one such subgroup is nontrivial then $OF(C_G(t))$ is generated by the centralizers of nontrivial subgroups of Q and we arrive at the same contradiction. Thus Q is a homogeneous $U_{0,r}$-subgroup for some value of r, and the claim follows.

We continue to work with the fixed parameter r.

Claim 2. *$U_{0,r}(FC_G(t))$ is not contained in $N(L)$.*

Since $OFC_G(t)$ is the central product of the subgroups $U_{0,s}(FC_G(t))$ with s varying, this follows from the previous claim and our hypotheses.

Claim 3. *For $s \neq r$,*

$$U_{0,s}(FC_G(t) \cap C_G(i)) \leq L$$

Let $\mathbb{T}_t$ be the algebraic torus of L containing t and w an involution of L inverting $\mathbb{T}_t$.

The group $Y = U_{0,s}(FC_G(t))$ is a w-invariant subgroup of $L \times \hat{K}_L$, hence is of the form

$$(Y \cap \mathbb{T}_t) \times (Y \cap \hat{K}_L).$$

As Y meets $\hat{K}_L = 1$, we find $Y \leq \mathbb{T}_t \leq L$.

Claim 4. $r = r_{\hat{K}_L}$.

By definition, $r \le r_{\hat{K}_L}$. If $r < r_{\hat{K}_L}$ then let A be an abelian $U_{0,r_{\hat{K}_L}}$-subgroup of $\hat{K}_L$. By Lemma 6.11 we have $t \in L$ and thus A normalizes $U_{0,r}(FC_G(t))$. As $r_{\hat{K}_L} > r$, A must centralize $U_{0,r}(FC_G(t))$.

Hence $U_{0,r}(FC_G(t)) \le C_G(A) \le N(L)$, a contradiction to Claim 2.

We have proved points (1–3). Now we consider the 2-torus T_t.

Claim 5. *T_t centralizes $OF(C_G(t))$.*

Certainly T_t centralizes $ZC_G^\circ(t)$ so it suffices to consider the action of T_t on subgroups of the form $U_{0,r}(OF(C_G(t)))$.

It follows from Claim 1 that T_t centralizes $U_{0,s}(FC_G(t))$ for $s \ne r_{\hat{K}_L}$ by looking at the action of that group on L.

For $U_{0,r_{\hat{K}_L}}(FC_G(t))$ we apply Lemma 3.2 to the groups $Q = OF(C_G(t))$, T_t, and $U = Q_t \cap N(L)$. Here T_t normalizes Q, and centralizes U. Furthermore $C_{Q_t}(U) \le U$ since $C_G(U \cap \hat{K}_L) \le N(L)$. So the lemma applies, and T_t centralizes Q.

Claim 6. *T_t centralizes $C_G^\circ(t)$.*

Let $H = C_G^\circ(t)$. By Lemma 6.5, part (2), $H \le (H \cap \hat{K}_L)H_D$ with H_D a D-group. As T_t centralizes $\hat{K}_L$, it suffices to check that T_t centralizes H_D. Then Lemma 2.33 and the previous claim complete the proof of the claim.

Now all points of our lemma have been proved. $\square$

6.3. **Case 1, continued.** $U_{0,r_{\hat{K}_L}}(OF(C_G(t)))$**.** We work graduatlly toward the following.

Proposition 6.13. *Let G be as in Hypothesis 6.4, with the associated notational conventions. Let t be an involution of L, and $\hat{K}_L = C_G(L)$.*

Then $OFC_G(t) \cap \hat{K}_L = 1$.

We first make a detailed analysis of $U_{0,r_{\hat{K}_L}}(OF(C_G(t)))$. We extend the notation of Hypothesis 6.10 as follows.

Notation 6.14. Let G, i, L, $\hat{K}_L$, t, $\mathbb{T}_t$ satisfy Hypothesis 6.10.

We set $Q_t = U_{0,r_{\hat{K}_L}}(FC_G(t))$ and

$$A_t = Q_t \cap L; \quad B_t = C_{Q_t}(L); \quad U_t = N_{Q_t}(L); \quad Q_1 = N_{Q_t}(U_t).$$

By assumption B_t is nontrivial.

Lemma 6.15. *Let G, i, L, $\hat{K}_L$, t, and $\mathbb{T}_t$ satisfy Hypothesis 6.10. Then the torsion in Q_t lies in A_t.*

Proof. The torsion subgroup X in Q_t is Π-torus, hence central in $C_G^\circ(t)$. Since Q_t meets $\hat{K}_L$ it follows that X normalizes L. Furthermore X is w-invariant, where $w \in L$ inverts $\mathbb{T}_t$. So X decomposes as $(X \cap L) \times (X \cap \hat{K}_L)$ and the second factor is trivial. The lemma follows. $\square$

Lemma 6.16. *Let G, i, L, $\hat{K}_L$, t, and $\mathbb{T}_t$ satisfy Hypothesis 6.10. Let $w \in I(L)$ invert $\mathbb{T}_t$.*

Then

(1) $A_t \leq \mathbb{T}_t$.

(2) $U_t = A_t \times B_t$ *is abelian.*

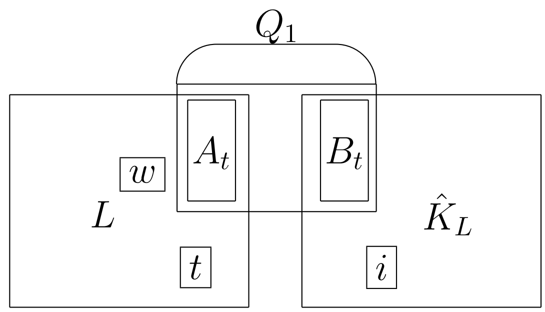
(3) i *and* w *invert* Q_1/U_t.

(4) $C_{Q_t}(Q_1) = Z(Q_1) \leq A_t$.

(5) $Z_2(Q_1) \cap U_t$ *decomposes as*

$$Z(Q_1) \times (Z_2(Q_1) \cap B_t),$$

with both factors non-trivial.



Proof.

Claim 1. *$U_t = A_t \times B_t$ and $A_t \leq \mathbb{T}_t$.*

The group U_t is w-invariant. It acts on L like a subset of $\mathbb{T}_t$ with B_t the kernel of the action. Here B_t is centralized by w and $\mathbb{T}_t$ is inverted by w. The claim follows.

Claim 2. *$Q_1 > U_t$.*

This holds since Q is not contained in $N(L)$.

Claim 3. *The involution i inverts Q_1/U_t. In particular, Q_1/U_t is abelian.*

$C_{Q_t}(i) \subseteq Q_t \cap N(L) = U_t$. The claim now follows by Fact 2.31.

Claim 4. *$C_{Q_t}(Q_1) = Z(Q_1) \leq A_t$.*

We have

$$C_{Q_t}(Q_1) \leq C_{Q_t}(B_t) \leq Q_t \cap N(L) = U_t \leq Q_1,$$

so $C_{Q_t}(Q_1) = Z(Q_1)$. As $Z(Q_1)$ is w-invariant and contained in $U_t = A_t \times B_t$ we get

$$C_{Q_t}(Q_1) = C_{A_t}(Q_1) \times C_{B_t}(Q_1).$$

Now as Q_1 is not contained in $N(L)$, we find $C_{B_t}(Q_1) = 1$. The claim follows.

Claim 5. *U_t is abelian.*

A_t is abelian.

We have $U_t' = B_t'$. So B_t' is normal in Q_1. If B_t' is non-trivial then it meets $Z(Q_1)$, a contradiction. So B_t is abelian and the claim follows.

Now consider the subset Q_1^- of Q_1 consisting of elements which are inverted by i.

Claim 6. $Q_1 = C_{Q_1}(i) \times Q_1^-$

This is Fact 2.32.

Claim 7. *$Z_2(Q_1) \cap A_t = Z(Q_1)$.*

Let $b \in Z_2(Q_1) \cap A_t$. Since $C_G(i) \leq N(L)$ we have $C_{Q_t}(i) \leq U_t$ and thus b centralizes $C_{Q_1}(i)$.

We show now that b centralizes Q_1^-.

Let $q \in Q_1$ with $q^i = q^{-1}$. Then $[b, q] \in Z(Q_1) \leq A_t$, so

$$[b, q] = [b, q]^i = [b, q^{-1}] = [b, q]^{-1}.$$

Hence $[b, q] = 1$ as required. The claim follows.

Claim 8. *$Z_2(Q_1) \cap U_t = Z(Q_1) \times (Z_2(Q_1) \cap B_t)$, with both factors non-trivial.*

By w-invariance and the previous claim the factorization holds, and as U_t is normal in Q_1 and properly contains $Z(Q_1)$, we have $Z_2(Q_1) \cap U_t > Z(Q_1)$.

Claim 9. *w inverts Q_1/U_t.*

Let T_w be the 2-torus of L containing w. By Lemma 6.12 T_w centralizes $C_G^\circ(w)$. If w does not invert Q_1/U_t then w centralizes some element q of $Q_1 \setminus U_t$. Then $d(q)$ is a torsion-free group, so $d(q) \leq C_G^\circ(w)$. Hence T_w centralizes q.

But the 2-torus T_t containing t also centralizes $d(q)$ and thus L centralizes q. So $q \in N(L) \cap Q_t = U_t$, a contradiction.

The claim follows.

With this, the proof of the lemma is complete. □

Lemma 6.17. *Let G, i, L, $\hat{K}_L$, t, and $\mathbb{T}_t$ satisfy Hypothesis 6.10. Let $w \in I(L)$ invert $\mathbb{T}_t$. Then*

$$Q_1 = U_t \cdot (Q_1 \cap OF(C_G(wi))).$$

In particular, $Q_1 \cap OF(C_G(wi))$ is not contained in $N(L)$.

Proof.

Claim 1. $Q_1 = U_t \cdot C_{Q_1}(wi)$.

By Lemma 6.16, wi centralizes Q_1/U_t. By Fact 2.31, $C_{Q_1}(wi)$ covers Q_1/U_t. This proves the claim.

As w inverts Q_1/U_t, according to Fact 2.32, the group $H = C_{Q_1}(wi)$ decomposes as the product of two sets

$$C_H(i) \times H^-,$$

where H^- is the subset inverted by w (or, equivalently, by i).

Claim 2. $H^- \subseteq OF(C_G(wi))$.

By Lemma 6.5, part (3) applied to $C_G^\circ(wi)$, we have

$$[w, H^-] \leq OF(C_G(wi))[w, C_{N(L)}(wi)].$$

Now

$$C_{N(L)}(wi) = C_L(wi) \times C_{\hat{K}_L}(wi) \leq C_L(w) \times \hat{K}_L.$$

Thus

$$[w, H^-] \leq OF(C_G(wi)).$$

Since w inverts H^- this gives the claim.

Hence

$$C_{Q_1}(wi) = C_H(i) \times H^- \leq (Q_1 \cap N(L)) \cdot OF(C_G(wi))$$
$$= U_t \cdot OF(C_G(wi)),$$

and the result follows from Claim 1. □

Lemma 6.18. *Let G, i, L, $\hat{K}_L$, t, and $\mathbb{T}_t$ satisfy Hypothesis 6.10.*

Then $OF(C_G(ti)) \cap \hat{K}_L = 1$.

Proof. By Lemma 6.11 if we assume the contrary then $OF(C_G(ti)) \leq N(L)$. However this contradicts Lemma 6.17. □

Lemma 6.19. *Let G, i, L, $\hat{K}_L$, t, and $\mathbb{T}_t$ satisfy Hypothesis 6.10. Let $w \in I(L)$ invert $\mathbb{T}_t$.*

Then

$$Z(OF(C_G(wi))) \cap N(L) = 1.$$

Proof. By Lemma 6.18, $OF(C_G(wi))$ meets $\hat{K}_L$ trivially.

As $Z(OF(C_G(wi)))$ is w-invariant, it suffices therefore to consider the group $Z_L = Z(OF(C_G(wi))) \cap L$, which is contained in the algebraic torus of L containing w.

As $Q_1 \cap OF(C_G(wi))$ commutes with Z_L and (by Lemma 6.12) with the 2-torus containing t, if Z_L is non-trivial then $Q_1 \cap OF(C_G(wi))$ centralizes L, contradicting Lemma 6.17. □

6.4. **Case 1, continued.** $U_t \triangleleft Q_t$.

We continue the study of Q_1 and eventually show that $Q_1 = Q_t$.

Lemma 6.20. *Let G, i, L, t satisfy Hypothesis 6.10. Let w be an involution inverting the algebraic torus of L containing t.*

Let $Q_1^+ = C_{Q_1}(wi)$. Then

(1) $Q_1 = U_t \rtimes Q_1^+$.

(2) $Q_1^+ = Q_1 \cap OF(C_G(wi))$.

(3) Q_1^+ *is abelian and inverted by* i.

(4) i *inverts* B_t.

(5) $C_{Q_t}(i) = A_t$.

(6) $Q_1' = (Q_1' \cap A_t) \times (Q_1' \cap B_t) = [Q_1^+, U_t]$.

Proof. Let $B_t^+ = C_{B_t}(i)$.

Claim 1. $Q_1^+ = B_t^+ \times (Q_1 \cap OF(C_G(wi)))$.

Recall that $Q_1 = U_t \cdot (Q_1 \cap OF(C_G(wi)))$, so

$$Q_1^+ = (Q_1^+ \cap U_t)(Q_1 \cap OF(C_G(wi))).$$

We have $Q_1^+ \cap U_t = C_{A_t \times B_t}(wi) = B_t^+$ and thus

$$Q_1^+ = B_t^+ \cdot (Q_1 \cap OF(C_G(wi))).$$

As $OF(C_G(wi)) \cap \hat{K}_L = 1$ and the factors are normal in Q_1^+ we have

$$Q_1^+ = B_t^+ \times (Q_1 \cap OF(C_G(wi))).$$

This proves the claim.

Claim 2. *i inverts B_t, and $Q_1^+ = Q_1 \cap OF(C_G(wi))$.*

If $B_t^+ > 1$ then $Q_1^+ \leq C_G(B_t^+) \leq N(L)$ and hence $Q_1 \leq N(L)$, a contradiction. So $B_t^+ = 1$. Therefore i inverts B_t (point (4)) and we have $Q_1^+ = Q_1 \cap OF(C_G(wi))$, which is point (2).

Claim 3. *$C_{Q_t}(i) = A_t$. (Point (5).)*

Since $C_{Q_t}(i) \le U_t = A_t \times B_t$ and i inverts B_t this is immediate.

Claim 4. *Q_1^+ is abelian, inverted by i and w. (Point (3).)*

As $Q_1^+ \cap \hat{K}_L = 1$ and $Q_1^+ \cap L = 1$, we find $Q_1^+ \cap N(L) = 1$. So i inverts Q_1^+. Hence Q_1^+ is abelian and w also inverts Q_1^+.

Claim 5. *$Q_1 = U_t \rtimes Q_1^+$. (Point (1).)*

We have $Q_1 = U_t Q_1^+$ with $U_t \lhd Q_1$. Furthermore $Q_1^+ \cap U_t = B_t^+ = 1$. The claim follows.

At this point we have covered the first five points in the statement of the lemma.

Claim 6. *$Q_1' = (Q_1' \cap A_t) \times (Q_1' \cap B_t) = [Q_1^+, U_t] = [Q_1^+, A_t][Q_1^+, B_t]$.*

As $Q_1' \le U_t$ the first equation holds by w-invariance.

For the second, take $x_1 = q_1 u_1$, $x_2 = q_2 u_2$ with $u_1, u_2 \in U_t$ and $q_1, q_2 \in Q_1^+$, and compute the commutator bearing in mind that U_t and Q_1^+ are abelian and $Q_1' \le U_t$.

$$\begin{aligned}
[x_1, x_2] &= [q_1 u_1, q_2 u_2] = [q_1, q_2 u_2]^{u_1} [u_1, q_2 u_2] = [q_1, q_2 u_2][u_1, q_2 u_2];\\
[q_1, q_2 u_2] &= [q_1, u_2][q_1, q_2]^{u_2} = [q_1, u_2];\\
[u_1, q_2 u_2] &= [u_1, u_2][u_1, q_2]^{u_2} = [u_1, q_2];\\
[q_1 u_1, q_2 u_2] &= [q_1, u_2][u_1, q_2] = [q_1, u_2][q_2, u_1]^{-1};\\
Q_1' &= [Q_1^+, U_t].
\end{aligned}$$

Finally since $U_t = A_t \times B_t$ we find $[Q_1^+, U_t] = [Q_1^+, A_t][Q_1^+, B_t]$ This proves the claim, and completes the proof of the lemma. □

Now we move on from Q_1 to Q_t.

Lemma 6.21. *Let G, i, L, t satisfy Hypothesis 6.10. Let w be an involution inverting the algebraic torus of L containing t.*

Let $Q_t^+ = C_{Q_t}(wi)$ and $Q_t^- = \{q \in Q_t : q^{wi} = q^{-1}\}$. Then

(1) $U_t = Q_t^-$.

(2) $Q_t = U_t \cdot Q_t^+$ *and* $U_t \cap Q_t^+ = 1$.

(3) Q_t^+ *is an abelian group inverted by* i *and* w.

(4) If P *is a* (wi)*-invariant subgroup of* Q_t*, then*

$$P = (P \cap U_t) \cdot (P \cap Q_t^+).$$

Proof. By Lemmas 6.16 and 6.20 $U_t = A_t \times B_t \leq Q_t^-$.

Claim 1. $Q_t = Q_t^- \times Q_t^+$ *as a product of sets.*

This is Fact 2.32.

Claim 2. $Q_t^- = U_t$.

Bear in mind that the torsion in Q_t lies in A_t (Lemma 6.15).

Let $q \in Q_t^-$. Decompose Q_t with respect to the action of i; then $q = q_+ q_-$ with q_+ centralized by i and q_- inverted by i.

Now $q_+ \in C_{Q_t}(i) = A_t$, so w also inverts q_+. Hence

$$\begin{aligned} q^{-1} &= q^{iw} = q_+^{-1}(q_-^w)^{-1}; \\ q_+ q_- &= q_-^w q_+; \\ (q_-)^{wq_+} &= q_-. \end{aligned}$$

If $q_- \in U_t$ then $q \in U_t$. Supposing the contrary, then, q_- has infinite order and $q_- \in d(q_-)^\circ$.

Now wq_+ is an involution of L and $q_- \in C_G^\circ(wq_+)$. By Lemma 6.12 the 2-torus of L containing wq_+ commutes with q_-. The 2-torus containing t also centralizes q_-. These 2-tori are distinct, so q_- must centralize L. Accordingly $q_- \in B_t$ and $q \in U_t$.

This proves the claim.

At this point, we have

$$Q_t = U_t \cdot Q_t^+; \quad U_t \cap Q_t^+ = 1.$$

Claim 3. *Q_t^+ is an abelian group inverted by i and w.*

Since $Q_t^+ \cap U_t = 1$, we have $C_{Q_t^+}(i) = 1$, and thus i inverts Q_t^+. So w does as well, and the group Q_t^+ is abelian.

Claim 4. *If P is a (wi)-invariant subgroup of Q_t, then*

$$P = (P \cap U_t) \cdot (P \cap Q_t^+).$$

We have the decomposition as a product of sets with respect to the action of wi:

$$P = P^- P^+,$$

so the claim follows from the corresponding claim for Q_t. □

Lemma 6.22. *Let G, i, L, t satisfy Hypothesis 6.10.*

Then the following hold.

(1) $Z_2(Q_t) \le Q_1$.

(2) $Z_2(Q_t) \cap A_t = Z(Q_t)$.

Proof. By Lemma 6.16 we have $A_t \le \mathbb{T}_t$ and $C_{Q_t}(Q_1) = Z(Q_1) \le A_t$. We continue to work with the decomposition

$$Q_t = Q_t^- \cdot Q_t^+ = U_t \cdot Q_t^+.$$

Ad 1. We have $Z(Q_t) \le C_{Q_t}(Q_1) \le A_t$ and hence $Z_2(Q_t)$ normalizes U_t. Thus $Z_2(Q_t) \le Q_1$, as claimed.

Ad 2. We know $Z(Q_t) \le A_t$. We must show that $Z_2(Q_t) \cap A_t \le Z(Q_t)$.

Since A_t commutes with U_t and $U_t = Q_t^-$, it suffices to show that $Z_2(Q_t) \cap A_t$ also centralizes Q_t^+.

Let $a \in Z_2(Q_t) \cap A_t$ and $q \in Q_t^+$. Then $[a, q] \in Z(Q_t) \le A_t$, and i inverts Q_t^+, so

$$[a, q] = [a, q]^i = [a, q^{-1}] = [a, q]^{-1}.$$

So $[a, q] = 1$. This proves the claim. □

Lemma 6.23. *Let G, i, L, t satisfy Hypothesis 6.10. Suppose that $Q_1 < Q_t$.*

Then the following hold.

(1) $Z(Q_t) = Z_2(Q_t) \cap U_t$.

(2) $Q_1' \leq A_t = Z(Q_1)$.

Proof.

Claim 1. $Z_2(Q_t) \cap U_t = Z(Q_t)$.

Suppose $Z_2(Q_t) \cap U_t > Z(Q_t)$. Then $Z_2(Q_t)$ meets B_t. But Q_t' centralizes $Z_2(Q_t)$, so $Q_t' \leq C_{Q_t}(Z_2(Q_t) \cap B_t) = U_t$. But then U_t is normal in Q_t and $Q_t = Q_1$, a contradiction.

Claim 2. $Q_1' \leq A_t$

We know $Q_1' = (Q_1' \cap A_t) \times (Q_1' \cap B_t)$.

If Q_1' is not contained in A_t, then Q_1' meets B_t. As $Z_2(Q_t)$ centralizes Q_1', it follows that $Z_2(Q_t) \leq U_t$. But then $Z_2(Q_t) = Z(Q_t)$ and Q_t is abelian, a contradiction.

Claim 3. $A_t = Z(Q_1)$.

We know $Z(Q_1) \leq A_t$ (Lemma 6.16) and we must prove the reverse inclusion.

Since $Q_1' \leq A_t$ we have

$$[Q_1^+, A_t] \leq A_t.$$

Then for $q \in Q_1^+$, $a \in A_t$ we find

$$[q, a] = [q, a]^i = [q^{-1}, a].$$

This then gives

$$[q^2, a] = 1; \qquad [a, d(q^2)] = 1; \qquad [a, q] = 1.$$

Hence Q_1^+ centralizes A_t. Since $Q_1 = U_t Q_1^+$ with U_t abelian it follows that $A_t \le Z(Q_1)$. The claim follows.

This completes the proof. □

Lemma 6.24. *Let G, i, L, t satisfy Hypothesis 6.10. Then $U_t \lhd Q_t$.*

Proof. We suppose on the contrary that

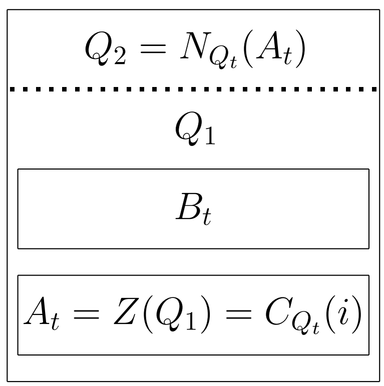
$$Q_1 < Q_t.$$

Let $Q_2 = N_{Q_t}(A_t)$.

Claim 1. *$Q_1 \le Q_2$ and Q_2/A_t is abelian, inverted by i.*

As $A_t = Z(Q_1)$ we have $Q_1 \le Q_2$.

Now $C_{Q_2}(i) = A_t$, so i inverts Q_2/A_t. Thus Q_2/A_t is abelian.



Claim 2. $A_t = Z(Q_2)$.

We have $Z(Q_2) \le Z(Q_1) = A_t$. We need to show $A_t \le Z(Q_2)$.

We have $Q_2 = A_t Q_2^-$ with Q_2^- the subset inverted by i, so it suffices to show that $[A_t, Q_2^-] = 1$.

This goes as before. For $a \in A_t$, $q \in Q_2^-$ we have

$$[a,q] = [a,q]^i = [a,q^i] = [a,q^{-1}],$$
$$a = a^{q^2}; \quad [a,q] = 1.$$

Claim 3. $Q_2 = Q_t$.

As A_t is characteristic in Q_2, A_t is normal in $N_{Q_t}(Q_2)$. It follows that Q_2 is self-normalizing in Q_t, so $Q_2 = Q_t$.

Now we arrive at a contradiction. Since $Q_t = Q_2$ we find $Q_t' \leq A_t$ and hence Q_t normalizes U_t, so

$$Q = Q_1,$$

in spite of our assumption to the contrary.

This concludes the proof. □

Corollary 6.25. *Let G, i, L, t satisfy Hypothesis 6.10.*

Then

$$C_{Q_t}(wi) = Q_t \cap OF(C_G(wi)).$$

Proof. Apply point (2) of Lemma 6.20. □

6.5. **Proof of Proposition 6.13.** Going forward, the main items of notation are the following.

$$\begin{aligned} Q_t = U_{0,r_{\hat{K}_L}}(FC_G(t)); \qquad A_t &= Q_t \cap L = C_{Q_t}(i); \\ Q_t^+ &= C_{Q_t}(wi) = Q_t \cap OF(C_G(wi)); \\ Q_{tw}^+ &= C_{Q_{tw}}(wi) = Q_{tw} \cap OF(C_G(wi)), \end{aligned}$$

with $w \in L$ inverting $\mathbb{T}_t$.

To this we add the following.

Notation 6.26. With G, i, L, t satisfying Hypothesis 6.10, fix $t \in L$, and let T_t be the 2-torus of L containing t. Let w be an involution of L inverting T_t, and define the following.

$$\hat{A}_w = U_{0,r_{\hat{K}_L}}(C_L(w)); \qquad R = U_{0,r_{\hat{K}_L}}(FC_G(wi))\hat{A}_w.$$

Note that $A_w \leq \hat{A}_w$.

Lemma 6.27. *Suppose G is a simple L^*-group of finite Morley rank and odd type satisfying Hypothesis 6.10.*

Then R is a nilpotent group with

$$R = \left\langle Q_t^+, Q_{tw}^+, \hat{A}_w \right\rangle$$

and

$$C_R(t) = Q_t^+;\ C_R(tw) = Q_{tw}^+;\ C_R(w) = \hat{A}_w.$$

Each of these three groups is non-trivial.

Proof. We set

$$R_0 = U_{0,r_{K_L}}(FC_G(wi)).$$

Recall that the torsion of R_0 lies in A_w.

We have $\hat{A}_w \leq C_G(wi)$ so $\hat{A}_w$ normalizes R_0 and R is a connected solvable group. R is in fact nilpotent by Fact 2.26.

Claim 1. $\left\langle Q_t^+, Q_{tw}^+, \hat{A}_w \right\rangle \leq R$

As Q_t^+ and Q_{tw}^+ are $U_{0,r_{\hat{K}_L}}$-groups, Corollary 6.25 covers Q_t^+ and Q_{tw}^+. The group $\hat{A}_w$ is included by definition.

Claim 2. $C_R(w) = \hat{A}_w$

It suffices to check that $C_{R_0}(w) \leq \hat{A}_w$. As $C_{R_0}(w) \leq C_G(i) \leq N(L)$, we find

$$C_{R_0}(w) \leq \mathbb{T}_w \times \hat{K}_L.$$

As R_0 is t-invariant and $R_0 \cap \hat{K}_L = 1$ (Lemma 6.18), we find

$$C_{R_0}(w) = R_0 \cap \mathbb{T}_w.$$

On the other hand by Fact 2.34, $C_{R_0}(w)$ is itself a $U_{0,r_{\hat{K}_L}}$-group, and thus this intersection lies in $\hat{A}_w$.

The claim follows.

Claim 3. $C_R(t) = Q_t^+$.

The group $C_R(t)$ reduces to $C_{R_0}(t)$, so this is a $U_{0,r_{\hat{K}_L}}$-subgroup of $FC_G(t)$. That is,

$$C_R(t) \leq U_{0,r_{\hat{K}_L}}(FC_G(t)) = Q_t.$$

As $C_{Q_t}(wi) = Q_t^+$, the claim follows.

Correspondingly, $C_R(tw) = Q_{tw}^+$.

Now by Fact 2.44 the group R is generated by its subgroups

$$C_R^\circ(t), C_R^\circ(tw), C_R^\circ(w).$$

Only the final point (non-triviality) remains to be checked.

By our assumptions Q_t^+ is non-trivial, and Q_{tw}^+ is a conjugate. Furthermore $\hat{A}_w$ contains A_w and we know $Z(Q_w) \leq A_w$. So the result follows. □

Now we come to the proof of Proposition 6.13.

Proof of Proposition 6.13. Now G satisfies Hypothesis 6.4, and t denotes an involution of L. The claim is that $OFC_G(t) \cap \hat{K}_L = 1$.

Suppose on the contrary that $OFC_G(t) \cap \hat{K}_L > 1$. Then by Lemma 6.6, Hypothesis 6.10 applies to t. We take $w \in L$ inverting the torus $\mathbb{T}_t$ of L containing t.

Claim 1. *There is a non-trivial connected subgroup X of Q_t^+ commuting with a non-trivial connected subgroup Y of $\hat{A}_w$.*

Consider the group R as in Lemma 6.27.

$$R = \left\langle Q_t^+, Q_{tw}^+, \hat{A}_w \right\rangle,$$

with all three subgroups non-trivial.

The connected component of the center of R is generated similarly by

$$C^\circ_{Z(R)}(t), C^\circ_{Z(R)}(tw), C^\circ_{Z(R)}(w),$$

where now some of these subgroups may be trivial.

However, as t and tw are interchangeable for our present purpose, we may suppose that at least one of the groups $C^\circ_{Z(R)}(t)$ or $C^\circ_{Z(R)}(w)$ is non-trivial. Any central subgroup of R commutes with Q_t^+ and with $\hat{A}_w$. Therefore this gives us either a non-trivial connected subgroup of $\hat{A}_w$ commuting with Q_t^+, or a non-trivial connected subgroup of Q_t^+ commuting with $\hat{A}_w$.

In either case, the claim follows.

But Q_t commutes with the 2-torsion subgroup T_t of $\mathbb{T}_t$ (Lemma 6.12), and thus X commutes with T_t and Y, forcing $X \leq C(L)$, and thus $X \leq Q_t^+ \cap N(L) = 1$, a contradiction. □

6.6. **Case 2. Preliminaries.**

Hypothesis 6.28 (Hypothesis and notation)**.**

Hypotheses

G is a simple L^* group of finite Morley rank and odd type satisfying NTA$_2$. We suppose that there is no component in $\mathcal{E}$ of type SL$_2$.

Notation

Let i be an $\mathcal{E}$-involution.
$L = E_{\mathcal{E}}(C_G(i))$, $\hat{K}_L = C_G(L)$, $K_L = C_G(L, i)$.
σ is maximal so that $U_{0,\sigma}(FC_G(t)) \not\leq N(L)$ for $t \in I(L)$.
For t, t' commuting involutions in $C_G(i)$ set

$$Q_t = U_{0,\sigma}(FC_G(t)); \qquad A_t = Q_t \cap L;$$
$$Q_t^+(t') = C_{Q_t}(t').$$

Remark 6.29. In the context of Hypothesis 6.28, if t is an involution of L, we have the following.

(1) $OFC_G(t)$ has maximal reduced rank at most r_f and $U_p(FC_G(t)) = 1$ for p prime. (Lemma 3.23).

(2) $UFC_G(t)$ is not contained in $N(L)$ (Lemma 6.6).

By (1), $OFC_G(t)$ decomposes as a product of groups $U_{0,r}(FC_G(t))$ and a central subgroup $d(T)$ with T a Π-torus. By (1,2), σ is well-defined and is at most r_f.

By Proposition 6.13 we have $Q_t \cap \hat{K}_L = 1$. Furthermore, as Q_t is w-invariant contains no involutions we find

$$\begin{aligned} Q_t \cap N(L) &= A_t \times (Q_t \cap \hat{K}_L) \\ &= A_t. \end{aligned}$$

Also $A_t = C_{Q_t}(i)$ is a connected $U_{0,\sigma}$-subgroup of the algebraic torus $\mathbb{T}_t$ of L containing t.

Lemma 6.30. *Let G, i satisfy Hypothesis 6.28.*

Then for s an involution in $C_G(i)$ we have

$$[i, C_G^\circ(s)] \le OF(C_G(s)).$$

Proof. This follows from Lemma 6.5 since $E_{alg}(C_G(s))$ is either trivial or of type SL_2 and $C_{C_G^\circ(s)}(i)$ covers the centralizer of i in the quotient modulo $OFC_G(s)$. $\square$

Lemma 6.31. *Let G, i satisfy Hypothesis 6.28. Let t, w be distinct commuting involutions in L, and suppose*

$$C_G^\circ(wi) \le N(L).$$

Then wi inverts Q_t and Q_t is abelian.

Proof. We have

$$C^\circ_{Q_t}(wi) \leq Q_t \cap N(L) = A_t$$

and $C_{A_t}(wi) = 1$, so $C^\circ_{Q_t}(wi) = 1$ and wi inverts Q_t.

Hence Q_t is abelian. □

Lemma 6.32. *Let G, i satisfy Hypothesis 6.28.*

Then for any involution $s \in C_G(i)$ with $s \neq i$ we have

$$UF(C_G(s)) \not\leq N(L).$$

Proof. Supposing the contrary, we have $s = it$ with t an involution of L. With $w \in L$ inverting $\mathbb{T}_t$, by conjugacy we have $UF(C_G(wi)) \leq N(L)$. By Lemma 6.5 we find $C^\circ_G(wi) \leq N(L)$ and then by Lemma 6.31 wi inverts Q_t and Q_t is abelian.

Therefore the decomposition of Q_t with respect to the action of i can be written as

$$Q_t = Q_t^+(i) \times Q_t^+(w) = A_t \times Q_t^+(w).$$

Here i inverts $Q_t^+(w)$ and in particular $[i, Q_t^+(w)] = Q_t^+(w)$. By Lemma 6.30 we have

$$Q_t^+(w) \leq OF(C_G(w)).$$

But $Q_t^+(w)$ is a $U_{0,\sigma}$-group and so $Q_t^+(w) \leq Q_w$.

If A_t is non-trivial then A_w is non-trivial and both centralize $Q_t^+(w)$, so $Q_t^+(w)$ centralizes L and then $Q_t^+(w) = 1$, a contradiction. So A_t, A_w are trivial and $Q_t = Q_t^+(w) \leq Q_w$. Hence $Q_t = Q_w$ is normalized by the algebraic tori of L containing t and w respectively, and L normalizes Q_t.

But L cannot centralize Q_t, so L acts faithfully on Q_t. In particular the root subgroups of L, which are copies of the additive group of the base field k, act faithfully on Q_t. This forces the characteristic of k to be zero and the rank of the additive group (which is also its reduced rank) to be at most σ. But σ is at most the reduced rank of the multiplicative group of k, so this is a contradiction.

The result follows. $\square$

Remark 6.33. Our hypotheses on $t \in I(L)$ now apply to all involutions of $C_G(i)$ other than i. In particular we have

$$Q_s \cap N(L) = A_s.$$

In the case $s = ti$ with $t \in L$, A_s is a subgroup of the algebraic torus of L containing t.

Lemma 6.34. *Take the hypotheses and notation as in Hypothesis 6.28.*

Let s be any involution in $C_G(i)$ other than i, and let V be a 4-group in $C_G(i)$ commuting with s and not containing s.

Then

(1) For $v, v' \in I(V)$ distinct, $Q_s^+(v)$ is an abelian $U_{0,\sigma}$-group inverted by v'.
In particular, $Q_s^+(v) \cap Q_s^+(v') = 1$.

(2) $Q_s = \langle Q_s^+(v) \, : \, v \in I(V) \rangle$.

(3) If $v \in I(V) \setminus \langle s, i \rangle$ then $Q_s^+(v) = Q_s \cap Q_v$.

(4) $U_{0,\sigma}(\hat{K}_L) = 1$.

(5) $Q_i = 1$.

Proof. By Proposition 6.13 we have

$$OF(C_G(s)) \cap \hat{K}_L = 1$$

for all such s.

Ad 1. v' inverts $Q_s^+(v)$.

By Fact 2.34, the groups $Q_s^+(v)$ are $U_{0,\sigma}$-groups.

Suppose first that $i \in \langle s, v\rangle$. Then $Q_s^+(v) \le C_{Q_s}(i) \le Q_s \cap N(L) = A_s$. But A_s is contained in an algebraic torus of L inverted by the elements of $\langle V, s\rangle \setminus \langle s, i\rangle$, hence by v'.

Now suppose $i \notin \langle v, s\rangle$. Then $Q_s^+(v) \cap N(L) = Q_s^+(v) \cap L \le C_L(s, v)$, and $C_L(s, v)$ is finite. Since $C_G(i) \le N(L)$, this implies that the group $Q_s^+(v)$ is inverted by i, and is abelian. Furthermore the elements of $I(V) \setminus \{v\}$ belong to $\langle v, s\rangle\, i$ and hence invert $Q_s^+(v)$.

Ad 2. $Q_s = \langle Q_s^+(v) \,:\, v \in I(V)\rangle$.

This point is simply an application of Fact 2.44, but it should be kept in mind.

Ad 3. If $v \in I(V) \setminus \langle s, i\rangle$, then $Q_s^+(v) = Q_s \cap Q_v$.

Evidently $Q_s \cap Q_v \le Q_s^+(v)$.

Conversely, as $Q_s^+(v)$ is inverted by i, it follows from Lemma 6.30 that $Q_s^+(v) \le F(C_G(v))$ and thus $Q_s^+(v) \le Q_s \cap Q_v$.

This proves the claim.

Ad 4. $U_{0,\sigma}(\hat{K}_L) = 1$.

Claim 1. $U_{0,\sigma}(\hat{K}_L) \le OF(\hat{K}_L)$.

Suppose X is a non-trivial abelian $U_{0,\sigma}$-subgroup of $\hat{K}_L$.

Let $t \in I(L)$ and let $w \in L$ invert the algebraic torus of L containing t. Then X normalizes $Q_t^+(w)$ and as both X and $Q_t^+(w)$ are $U_{0,\sigma}$-groups, the group $Q_t^+(w)X$ is nilpotent.

Let Z be the center of $Q_t^+(w)X$. Since Z commutes with X we have $Z \le N(L)$. By w-invariance

$$Z = (Z \cap L) \times (Z \cap \hat{K}_L).$$

But $Z \cap L \le Q_t^+(w) \cap L = 1$. Thus $Z \le \hat{K}_L$. Now since Z commutes with $Q_t^+(w)$ we find $Q_t^+(w) \le N(L)$, and then $Q_t^+(w) = 1$. Thus

$$Q_t = \left\langle Q_t^+(i), Q_t^+(wi) \right\rangle .$$

If $C_X(i) > 1$ then similarly we get $Q_t^+(wi) = 1$ and $Q_t = Q_t^+(i) \leq N(L)$, for a contradiction. So i inverts X. In particular $X \leq OF(\hat{K}_L)$. This proves the claim.

Claim 2. *$U_{0,s}(FC_G(t))$ commutes with $U_{0,\sigma}(\hat{K}_L)$ for $s \neq \sigma$.*

For $s < \sigma$, as $U_{0,\sigma}(\hat{K}_L)$ normalizes $FC_G(t)$, $U_{0,\sigma}(\hat{K}_L)$ centralizes $U_{0,s}(FC_G(t))$.

For $s > \sigma$ we have $U_{0,s}(FC_G(t)) \leq N(L)$, so $U_{0,s}(FC_G(t))$ normalizes $U_{0,\sigma}(\hat{K}_L)$ and hence centralizes it.

This proves the claim.

The group $\hat{K}_L$ normalizes $Q_t = U_{0,\sigma}(FC_G(t))$. Let R be the group $Q_t U_{0,\sigma}(\hat{K}_L)$.

Then R is normalized by $\hat{K}_L$ and by $C_L^\circ(t)$, hence by $C_G^\circ(t) \cap N(L)$.

Claim 3. *R is normalized by $OFC_G(t)$.*

Certainly R is normalized by $U_{0,s}(FC_G(t))$ for $s \neq \sigma$, and R contains $Q_t = U_{0,\sigma}(FC_G(t))$, so the claim follows.

So R is normalized by $OFC_G(t) \cdot (C_G^\circ(t) \cap N(L))$. By Lemma 6.5 it follows that R is normalized by $C_G^\circ(t)$, and thus $R = Q_t$.

But then Q_t meets $\hat{K}_L$, a contradiction. Point (4) follows.

Ad 5. $Q_i = 1$.

As $Q_i \leq U_{0,\sigma}(N(L))$ and $U_{0,\sigma}(L)$ is either 1 or L, we have $Q_i \leq U_{0,\sigma}(\hat{K}_L) = 1$ by the previous claim. □

Now we return to consideration of

$$A_s = Q_s \cap L.$$

Lemma 6.35. *Take the hypotheses and notation as in Hypothesis 6.28. Let $s \in I(C_G(i))$, $s \neq i$ and set $\mathbb{T}_s = C_L^\circ(s)$.*

Then $U_{0,\sigma}(\mathbb{T}_s) = A_s$. In particular, $A_s = A_{si}$.

Proof. Let $A_s^* = U_{0,\sigma}(\mathbb{T}_s)$. We argue as in the analysis of $U_{0,\sigma}(\hat{K}_L)$ above.

A_s^* centralizes $\hat{K}_L$. As A_s^* lies in $C_G(s)$ it centralizes $U_{0,r}(F(C_G(s)))$ for $r < \sigma$. For $r > \sigma$ we have $U_{0,r}(F(C_G(s))) \le C^\circ_{N(L)}(s) = \mathbb{T}_s \times \hat{K}_L$ so also in this case A_s^* centralizes $U_{0,r}(FC_G(t))$.

Now the group $Q_s A_s^*$ is normalized by $OF(C_G(s))$ and by $\hat{K}_L$, so by $C^\circ_G(s)$. As it is a nilpotent $U_{0,\sigma}$-group, $Q_s A_s^* \le Q_s$ and so $A_s^* \le Q_s \cap L = A_s$. The result follows. $\square$

Summary

Taking $t, w \in I(L)$, $s \in I(C_G(i))$ other than (possibly) i.

—Lemma 6.5, p. 73. Set $H = C_G^\circ(t)$ and $\bar{H} = H/OH$.
1. $T_t \times S_L$ is a Sylow 2-subgroup of H.
2. $\bar{H}$ decomposes as $\bar{H} = \overline{E_{alg}(K_L)} * \bar{H}_D$ with $\bar{H}_D$ a D-group.
3. $H \le UF(H) \cdot C_{N(L)}^\circ(t)$.
4. If $UF(H) \le N(L)$, then $H \le N(L)$.

—Hypothesis 6.28, p. 96. G simple L^*-group, odd type, with NTA_2. No $\mathcal{E}$-component of type SL_2.

—Remark 6.29, p. 97. For $t \in I(L)$ we have

(1) $\bar{\mathrm{r}}_0(OFC_G(t)) \le r_f$; $U_p(FC_G(t)) = 1$. (Lemma 3.23).

(2) $OFC_G(t)$ is not contained in $N(L)$ (Lemma 6.6).

—Lemma 6.30, p. 97. $[i, C_G^\circ(s)] \le OF(C_G(s))$.

—Lemma 6.32, p. 98. $UF(C_G(s)) \not\le N(L)$.

—Remark 6.33, p. 99. $Q_s \cap N(L) = A_s$.

—Lemma 6.34, p. 99. $s \notin V$, $v, v' \in V$ distinct.

(1) $Q_s^+(v)$ is an abelian $U_{0,\sigma}$-group inverted by v'.
In particular, $Q_s^+(v) \cap Q_s^+(v') = 1$.

(2) $Q_s = \langle Q_s^+(v) \,:\, v \in I(V) \rangle$.

(3) If $v \in I(V) \setminus \langle s, i \rangle$ then $Q_s^+(v) = Q_s \cap Q_v$.

(4) $U_{0,\sigma}(\hat{K}_L) = 1$.

(5) $Q_i = 1$.

—Lemma 6.35, p. 101. $U_{0,\sigma}(\mathbb{T}_s) = A_s$.

6.7. **Normalizers of subgroups of Q_t.** *Revised 10/2022. Review.*
Next we apply signalizer functor theory.

Lemma 6.36. *Take the hypotheses and notation as in Hypothesis 6.28. Let $s \in I(C_G(i))$, $s \neq i$. and set $\mathbb{T}_s = C_L^\circ(s)$.*
Then the following hold.

(1) $Q_s^+(i) = A_s = U_{0,\sigma}(\mathbb{T}_s)$.

(2) $A_s > 1$.

Proof.
Ad 1. $Q_s^+(i)$ is a $U_{0,\sigma}$-group (Fact 2.34).
We have

$$Q_s^+(i) \leq Q_s \cap N(L) = Q_s \cap L = A_s.$$

So (1) follows by Lemma 6.35.
Ad 2. Suppose toward a contradiction that

$$A_s = 1.$$

Then i inverts Q_s.
Also by Lemma 6.35 we have $A_{si} = A_s = 1$, and as $Q_i = 1$ our hypothesis becomes

$$A_s = 1 \text{ for all involutions } s \in C_G(i).$$

Now take an elementary abelian 2-group A of 2-rank 3 containing the involution i and define θ on $I(A)$ by

$$\theta(s) = Q_s.$$

Claim 1. *The function θ defines an invariant nilpotent signalizer functor on A.*

We need to check the balance condition on A:

$$Q_s^+(s') = Q_s \cap C_G(s') \leq Q_{s'}.$$

As i inverts Q_s for all $s \in A$, $(Q_s \cap C_G(s')) = [i, Q_s \cap C_G(s')]$ lies in $OF(C_G(s'))$. Furthermore $Q_s^+(s')$ is again a $U_{0,\sigma}$-group, so the balance condition follows and the claim holds.

It follows that the signalizer functor θ is trivial.

$$Q_s = 1 \text{ for } s \text{ an involution in } C_G(i).$$

But this contradicts Lemma 6.32. $\square$

Lemma 6.37. *Take the hypotheses and notation as in Hypothesis 6.28. Then we have the following.*

(1) $\hat{K}_L$ has abelian Borel subgroups.

(2) $\sigma^\circ(\hat{K}_L) = Z^\circ(\hat{K}_L)$.

(3) $OF(\hat{K}_L) = 1$.

(4) $\hat{K}_L^\circ$ is a D-group with $\hat{K}_L^\circ / Z^\circ(\hat{K}_L)$ of degenerate type.

(5) $\hat{K}_L^\circ$ has a connected Sylow 2-subgroup of Prüfer rank 1.

(6) $\hat{K}_L = K_L$ is connected.

Proof.

Ad 1. Let $H \leq \hat{K}_L$ be definable, connected, and solvable. Let $Q = F(Q_t H)$. By Lemma 6.34 $U_{0,\sigma}(Q) = 1$. Hence Q commutes with Q_t. If $Q > 1$ this forces $Q_t \leq N(L)$, a contradiction.

Thus $F(Q_tH) = Q_t$ and $H' \le Q_t \cap H = 1$. Hence H is abelian.

Ad 2. This is given by Lemma 2.29.

Ad 3. As $U_{0,\sigma}(\hat{K}_L) = 1$ and $OF(\hat{K}_L) \le OF(C_G(t))$, the group $OF(\hat{K}_L)$ commutes with Q_t. Hence it is trivial.

Ad 4. By Fact 2.7 we have

$$\hat{K}_L = E_{alg}(\hat{K}_L) * K_D$$

with $K_D/Z^\circ(K_D)$ of degenerate type.

We show that $E_{alg}(\hat{K}_L) = 1$. Let $\mathbb{B}$ be a Borel subgroup of $Ealg(\hat{K}_L)$ and $\mathbb{U}$ its unipotent radical. Then $\mathbb{U} \le \mathbb{B}' \le F(Q_t\mathbb{B})$. As $U_{0,\sigma}(\hat{K}_L) = 1$ the group $\mathbb{U}$ centralizes Q_t, and hence must be trivial. So $E_{alg}(\hat{K}_L) = 1$.

Thus (4) follows.

Ad 5. This is immediate from (4) .

Ad 6. $\hat{K}_L$ has the unique involution i so $N(\hat{K}_L) \le C_G(i)$. Thus $\hat{K}_L = K_L \le C_G(i)$.

But $C_G(i)$ is connected, and $C_G(i) = L \times K_L$. So K_L is connected. $\square$

Remark 6.38. It follows that a Sylow 2-subgroup of G has the form $S_L \times T_i$ with S_L a Sylow 2-subgroup of L and T a 2-torus containing i.

Lemma 6.39. *Take the hypotheses and notation as in Hypothesis 6.28. Then for s an involution which is not a* PSL_2*-involution, $C_G(s)$ is a D-group.*

Proof. We may suppose $s = t$ or ti and w inverts $\mathbb{T}_t$. Then s, w are not cotoral so $w \notin C_G^\circ(s)$. It follows that the Sylow 2-subgroups of $C_G^\circ(s)$ are connected. Thus with $H = C_G^\circ(s)$ and $\bar{H} = H/OF(H)$, we have $E_{alg}(\bar{H}) = 1$ and the result follows. $\square$

Lemma 6.40. *Take the hypotheses and notation as in Hypothesis 6.28. Then $Z^\circ(Q_t)$ is inverted by i.*

Proof. Supposing the contrary we have

$$Z^\circ(Q_t) \cap A_t > 1.$$

As t is conjugate to w in L we find

$$Z^\circ(Q_w) \cap A_w > 1.$$

Claim 1. $Q_{t,w} = 1$.

The group $Q_{t,w}$ is centralized by nontrivial connected subgroups of A_t and A_w, and hence by L, which must act trivially, forcing $Q_{t,w} = 1$.

Thus w inverts Q_t and Q_t is abelian.

Let $B_{wi} = N_{Q_{wi}(A_w)}$. Writing $B_{wi}(s)$ for $C_{B_{wi}}(s)$ we have

$$B_{wi} = \langle B_{wi}(t), B_{wi}(tw), B_{wi}(w) \rangle .$$

Here $B_{wi}(w) = B_{wi}(i) = A_w$. Accordingly $B_{wi}(t)$ or $B_{wi}(tw)$ is nontrivial, so by conjugacy both are nontrivial.

Now $B_{wi}(t) \cap A_w$ is trivial and hence i inverts $B_{wi}(t)$. Hence $B_{wi}(t) \le Q_t$. Therefore $B_{wi}(t)$ is normalized by A_t.

Claim 2. $B_{wi}(t)$ *centralizes* A_w

For $b \in B_{wi}(t)$ and $a \in A_w$ we have $[b, a] \in A_w$ and thus

$$[b, a] = [b, a]^w = [b^{-1}, a]$$

and as usual $[b, a] = 1$.

This proves the claim.

As $B_{wi}(t)$ centralizes A_w and A_t, it centralies L, giving a contradiction. □

Notation 6.41. Let G satisfy Hypothesis 6.28.

We let T_i denote the 2-torus of $\hat{K}_L$.

For t, w commuting involutions, we set

$$\begin{aligned} Q_{t,w} &= Q_t \cap Q_w; \\ Q_t(w) &= C_{Q_t}(w). \end{aligned}$$

In the next lemma we will probably take $t, w \in L$ but it may be useful to have it a little more generally.

Lemma 6.42. *Take the hypotheses and notation as in Hypothesis 6.28. Let t, w be distinct commuting involutions in $C_G(i)$, distinct from i, with w inverting A_t and conjugate to t. Then $Q_{t,w} = Q_t(w) > 1$.*

Proof. $Q_t(w)$ is inverted by i hence lies in $Q_{t,w}$. If $Q_t(w)$ is trivial then w inverts Q_t and Q_t is abelian. But this contradicts Lemma 6.40. □

This last point is one to bear in mind as it will tend to be applied without explicit mention.

Lemma 6.43. *Take the hypotheses and notation as in Hypothesis 6.28. Let t, w be distinct commuting involutions in $C_G(i)$, distinct from i, with w inverting A_t and conjugate to t.*

Then $N^\circ_{A_t}(Q_{t,w}) = 1$.

Proof. Assuming the contrary, then as (t, v) is conjugate to (v, t) we find

$$N^\circ_{A_t}(Q_{t,v}), N^\circ_{A_v}(Q_{t,v}) > 1.$$

Hence L normalizes $Q_{t,v}$ as well. On the other hand the reduced rank of the additive group of the base field is bigger than σ and hence L centralizes $Q_{t,v}$, giving $Q_{t,v} = 1$, and a contradiction. □

Lemma 6.44. *Take the hypotheses and notation as in Hypothesis 6.28. Let B be a non-trivial subgroup $\langle T_i, t, w\rangle$-invariant subgroup of Q_t inverted by i, with $N^\circ_{A_t}(B) > 1$. Then the following hold.*

(1) $N_G(B) \cap L \leq \mathbb{T}_t \langle w \rangle$.

(2) $N_G^\circ(B) = OF(N_G(B)) \cdot N^\circ_{C_G(i)}(B)$.

(3) $C_G^\circ(B) = OF(C_G(B)) \cdot N^\circ_{\mathbb{T}_t}(B)$.

(4) $U_{0,\sigma}(N_G(B)) = U_{0,\sigma}F(N_G(B))$.

Proof. Let $H = N_G^\circ(B)$ and $\bar{H} = H/OF(H)$. By Fact 2.7 we have

$$H = \tilde{E}_{alg}(H) \cdot H_D$$

with $\tilde{E}_{alg}(H)$ the inverse image of $E_{alg}(\bar{H})$

Ad 1. We have $N^\circ_{A_t}(B) > 1$, so if (1) fails then L normalizes B. As $\mathrm{rk}(k) > \sigma$ it then follows that L centralizes B, and $B = 1$.

Ad 2.

Claim 1. *$\bar{H}$ is a D-group.*

Suppose on the contrary that $L_1 = E_{alg}(\bar{H}) > 1$. Then in view of the structure of a Sylow 2-subgroup it is of type PSL_2. The 2-torus T_i acts on L_1 and centralizes at least one algebraic torus $\mathbb{T}_1$ of L_1. If T_1 is the preimage in H then $C^\circ_{T_1}(i)$ covers $\mathbb{T}_1$. The involutions of L_1 are conjugate and L_1 contains a 4-group, so the involutions of L_1 are conjugate to involutions in L. In particular an involution t_1 of $C_{T_1}(i)$ lies in L. Then $C_{L_2}(t_1 i)$ covers $\bar{L}_1$ and we contradict Lemma 6.39.

This proves the claim.

It follows that $[T_i, H] \leq UF(H)$ (Fact 2.8). Hence $C^\circ_H(i)$ covers $\bar{H}$. This gives (2).

Ad 3. This follows freom $(1, 2)$.

Ad 4.

We have

$$U_{0,\sigma}(H) \leq OF(H)U_{0,\sigma}(C_H(i)) \leq OF(H)U_{0,\sigma}(N_{\mathbb{T}_t}(B)).$$

Let $U = U_{0,\sigma}(H)$.

Claim 2. *$U_{0,r}F(U)$ centralizes $U \cap \mathbb{T}_t$ for $r \neq \sigma$.*

For $r < \sigma$, U centralizes $U_{0,r}F(U)$.
Suppose now that $r > \sigma$ and let $U_r = U_{0,r}F(U)$. Then

$$U_r = \left\langle C^{\circ}_{U_r}(t), C^{\circ}_{U_r}(w), C^{\circ}_{U_r}(tw) \right\rangle .$$

Here the centralizers of w and tw meet $\mathbb{T}_t$ trivially. Hence they are inverted by i. Thus these groups lie in Q_w and Q_{tw} respectively. But then as $r > \sigma$ they must lie in A_w or A_{tw}. But U cannot meet A_w or A_{tw} nontrivially, so we find $C_{U_r}(w), C_{U_r}(tw) = 1$ and $U_r \leq C(t)$ with U_r inverted by w. But then w inverts $UN_{\mathbb{T}_t}(B)$ and hence U_r centralizes $N_{\mathbb{T}_t}(B)$.

It follows that $U_{0,\sigma}(FU)U_{0,\sigma}N_{\mathbb{T}_t}(B)$ is normal in U. But FU covers the quotient of U by this subgroup and hence $U_{0\sigma}(FU)$ covers the quotient (since $U = U_{0,\sigma}(U)$. Thus $U = U_{0,\sigma}(FU)U_{0,\sigma}N_{\mathbb{T}_t}(B)$. This is then nilpotent, and hence contained in $F(N(B))$.

This proves (4). □

Lemma 6.45. *Take the hypotheses and notation as in Hypothesis 6.28. Let $t, w \in L$ be involutions with w inverting $\mathbb{T}_t$. Let $B_{t,w} = U_{0,\sigma}N(Q_{t,w})$.*

Then $B_{t,w}$ is abelian and inverted by i.

Proof. Let $B_{t,w}(i)$ be $C_{B_{t,w}}(i)$. By Lemma 6.44 $B_{t,w}(i) \leq \mathbb{T}_t$. As this is a $U_{0,\sigma}$-group and $N^{\circ}_{A_t}(Q_{t,w}) = 1$ we find that i inverts $B_{t,w}$. Hence $B_{t,w}$ is abelian. □

Lemma 6.46. *Take the hypotheses and notation as in Hypothesis 6.28. Let $t, w \in L$ be involutions with w inverting $\mathbb{T}_t$. Let $B_{t,w} = U_{0,\sigma}N(Q_{t,w})$. and let $B_{t,w}(t) = B_{t,w} \cap Q_t$.*

Let $A^*_t = U_{0,\sigma}N_{A_t}(B_{t,w}(t))$. Then $A^*_t > 1$.

Proof. Let $H = U_{0,\sigma}N_{Q_t}(B_{t,w}(t))$. If $C_H(i)$ is trivial then H is abelian and hence contained in $B_{t,w}$, for a contradiction.

So $C_H(i)$ is a nontrivial $U_{0,\sigma}$-subgroup contained in A^*. □

Lemma 6.47. *Take the hypotheses and notation as in Hypothesis 6.28. Let $t, w \in L$ be involutions with w inverting $\mathbb{T}_t$.*

Then $Q_{t,w} = Q_{t,tw} = Q_{w,tw}$.

Proof. $Q_{t,w} = Q_t(w) = Q_t(tw) = Q_{t,tw}$, and similarly for $Q_{w,tw}$. □

Accordingly we might also use the notation $Q_{\langle t,w\rangle}$ and, similarly, $B_{\langle t,w\rangle}$, but we prefer the lighter, less symmetric notation.

Lemma 6.48. *Take the hypotheses and notation as in Hypothesis 6.28. Let $t, w \in L$ be involutions with w inverting $\mathbb{T}_t$, and $B_{t,w} = U_{0,\sigma}N_G(Q_{t,w})$. Then $U_{0,\sigma}N(B_{t,w}) = B_{t,w}$.*

Proof. Let $H = U_{0,\sigma}N(B_{t,w})$. $C_H(i)$ is generated by $C_H(i,t)$, $C_H(i,w)$, and $C_H(i,tw)$. and these groups are conjugate.

If $C_H(i,t)$ is trivial then $C_H(t)$ is abelian and contained in

$$U_{0,\sigma}C_G(Q_{t,w}) = B_{t,w}.$$

The same then applies to $C_H(i,w)$ and $C_H(i,tw)$, and $H = B_{t,w}$.

Otherwise, $A_t^* = U_{0,\sigma}N_{\mathbb{T}_t}(B_{t,w})$ is nontrivial, and A_w^* similarly. So L normalizes $U_{0,\sigma}N_{\mathbb{T}_t}(B_{t,w})$. So L normalizes $B_{t,w}$. But the rank of the base field of F is greater than σ and then L centralizes $B_{t,w}$ and hence also $Q_{t,w}$, for a contradiction. □

We can apparenty set the next point aside and head directly for the final argument.

Lemma 6.49. *Take the hypotheses and notation as in Hypothesis 6.28. Let $t, w \in L$ be involutions with w inverting $\mathbb{T}_t$, and $B_{t,w} = U_{0,\sigma}N_G(Q_{t,w})$.*

Let $H(t,w)$ *be* $U_{0,\sigma}N_G(B_{t,w}(t))$ *and* $H_0(t,w)$ *the preimage in* $H(t,w)$ *of* $U_{0,\sigma}Z(H(t,w)/B_{t,w}(t))$.

Then $H_0(t,w) \le B_{t,w}$.

Proof. $B_{t,w}$ centralizes $B_{t,w}(t)$ and hence lies in $H(t,w)$. Thus

$$[B_{t,w}, H_0(t,w)] \le B_{t,w}(t) \le B_{t,w},$$

and $H_0(t,w)$ normalizes $B_{t,w}$. Thus $H_0(t,w) \le B_{t,w}$. □

6.8. **Existence of components of type** SL_2**.** We can now prove the existence of components of type SL_2 in $\mathcal{E}$.

Proof of Proposition 6.1. Supposing the contrary, we arrive at Hypothesis 6.28 and we consider a 4-group $V = \langle t, w\rangle$ contained in L.

With the notation of Lemma 6.46 we have

$$A_t^* = U_{0,\sigma}(A_t^*) \le N_G(B_{t,w}(t)).$$

On the other hand $U_{0,\sigma}C_G(B_{t,w}(t)) = B_{t,w}$ and thus $A_t^* \le N_G(B_{t,w})$. There is a conjugate subgroup A_w^* of $N_G(B_{t,w})$ and thus L normalizes $B_{t,w}$. Then as usual L centralizes $B_{t,w}$ and we arrive at a contradiction. □

7. The main results

Now we will prove the main result, Theorem 1.1, and add some further details about the structure of centralizers of involutions.

For the main result, this is largely a matter of assembling the prior results, with some additional argument in the manner of [FW69, Won69].

7.1. Theorem 1.1.

Lemma 7.1. *Let G be a connected simple L^* group of finite Morley rank of odd type satisfying the condition* NTA_2*, with Prüfer 2-rank 2 and*

$$m_2(G) \geq 3.$$

Then then there is an SL_2-involution (an involution whose centralizer has an $\mathcal{E}$-component of type SL_2). For any such involution i, one of the following applies.

(1) $E_{alg}(C_G(i)) = E_{\mathcal{E}}(C_G(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ with components conjugate in $C_G(i)$, and $C_G(i)$ disconnected;

or:

(2) $C_G(i)$ is connected, contains a Sylow 2-subgroup of G, and has the form $L *_2 K_L$ with L of type SL_2 and K_L of Prüfer rank 1 and unique involution i; $E_{alg}(K_L/OF(K_L))$ is of type SL_2. Furthermore we have one of the following.

(2a) There are two conjugacy classes of involution. For t not an SL_2-involution, $C_G(t)$ is a D-group.

(2b) There is one conjugacy class of involutions, and they satisfy

$$E_{alg}(C_G(i)/OFC_G(i)) = \mathrm{SL}_2 *_2 \mathrm{SL}_2,$$

(possibly with differing base fields).

Proof. The analysis begins with Lemma 4.2 (also subsumed under later results): $\mathcal{E}$ is non-empty and consists of Prüfer 2-rank 1 groups. According to Lemma 4.13, if $E_{\mathcal{E}}(C_G(i))$ has more than one component for some involution i, then it has the structure

$$\mathrm{SL}_2 *_2 \mathrm{SL}_2,$$

allowing for the possibilty of unrelated base fields. In particular there is a clear distinction between SL_2-involutions and PSL_2-involutions, when both types exist.

By Proposition 6.1, there are SL_2-involutions.

By Lemma 4.19 there are at most two conjugacy classes of involutions.

By Lemma 4.21 the possibilities associated with an SL_2-involution are as described. □

Continuing, we divide the analysis according to the number of conjugacy classes of involution present.

Lemma 7.2. *Let G be a connected simple L^* group of finite Morley rank of odd type satisfying the condition* NTA_2*, with Prüfer 2-rank 2 and*

$$m_2(G) \geq 3.$$

Suppose that G has two conjugacy classes of invoution.

Then the following hold.

(1) *The 2-rank of G is 4.*

(2) *One conjugacy class of involutions satisfies $E_{\mathcal{E}}(C_G(i)) = \mathrm{PSL}_2$, and the other satisfies $E_{\mathcal{E}}(C_G(i)) = \mathrm{SL}_2 *_2 \mathrm{SL}_2$, the two components of $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ are conjugate.*

(3) *The Sylow 2-subgroup is as in PSp_4.*

(4) All components of centralizers of involutions have the same base field.

Note that we have arrived at case (1) from the statement of Theorem 1.1.

Proof. By Proposition 5.22 one class of involutions consists of PSL_2-involutions, the other of SL_2-involutions. (In particular, Case (2a) above is eliminated.)

Then by Lemma 5.23, the SL_2 involutions satisfy $E_{\mathcal{E}}(C_G(i)) \simeq \mathrm{SL}_2 *_2 \mathrm{SL}_2$ with components conjugate by an involution, and the Sylow 2-subgroup is as in PSp_4. In particular the 2-rank is 4.

By Lemma 4.13, if i is a PSL_2-involution then $E_{\mathcal{E}}(C_G(i))$ is a single component, of type PSL_2.

This gives points (1–3). The various components of type SL_2 have the same base field, and it remains to consider components of type PSL_2. We can take an involution swapping two compoments of type $\mathrm{SL}_2(k)$. Then we have PSL_2 over the same base field in the centralizer. □

Lemma 7.3. *Let G be a connected simple L^* group of finite Morley rank of odd type satisfying the condition* NTA_2*, with Prüfer* 2*-rank* 2 *and*

$$m_2(G) \geq 3.$$

Suppose that G has one conjugacy class of invoution.

Then $C_G(i)$ is connected.

Proof. Suppose the contrary. Then by Lemma 4.18, $E_{\mathcal{E}}(C_G(i))$ is of the form $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ with the components conjugate. Now we argue more or less as in [FW69] (with some overlap with [Won69]).

Claim 1. *There is an element $j \in C_G(i)$ with $j^2 \in \langle i \rangle$ which swaps the two components.*

Take j to be any element which swaps the two components L_1, L_2 and let

$$j^2 = a_1 a_2 c$$

with $a_1 \in L_1$, $a_2 \in L_2$, $c \in C_G(L_1 L_2)$,

We make two computations of $(ja^{-1})^2$.

$$\begin{aligned}(ja_1^{-1})^2 &= (j^2 a_1^{-1})^j a_1^{-1} \in L_1 c^j;\\ (ja_1^{-1})^2 &= (a_1^{-1})^{j^{-1}} (j^2 a_1^{-1}) \in L_2 c,\end{aligned}$$

so $(ja_1^{-1})^2 = c^j = c$. and thus replacing j by ja_1^{-1} gives $j^2 \in C_G(L_1 L_2)$.

Let T be a maximal 2-torus of $L_1 L_2$. Then $C_G(T)$ is connected and any 2-element of $C_G(T)$ lies in T. Thus $C_G(L_1 L_2)/\langle i \rangle$ contains no involutions. Hence $jC_G(L_1 L_2)$ contains an element whose square lies in $\langle i \rangle$, as claimed.

Now consider the group

$$S = T \langle w, j \rangle,$$

with w inverting T and commuting with j. This is a Sylow 2-subgroup of $C_G(i, t)$. We will show that $\langle i \rangle$ is characteristic in S.

The connected component of S is T and $T \langle w \rangle$ is the subgroup acting trivially or by inversion on T. We consider the fourth powers of elements of the coset $(T \langle w \rangle) j$.

Choose notation as follows. For elements of T we write $a = a_1 a_2 \in T$ with $a_1 \in L_1$, $a_2 \in L_2$; we also write $a = (a_1, a_2)$ but must allow for possible adjustments by i.

We have the two cosets Tj and Twj to consider. Then we have the following.

$$\begin{aligned}(aj)^2 &= aj^2a^j = j^2(a_1a_2, a_1a_2); & (aj)^4 &= ([a_1a_2]^2, [a_1a_2]^2);\\ (ajw)^2 &= a(jw)^2a^{jw} & &\\ &= j^2(a_1a_2^{-1}, a_1^{-1}a_2); & (ajw)^4 &= ([a_1a_2^{-1}]^2, [a_1^{-1}a_2]^2).\end{aligned}$$

In other words, the fourth powers run over a 2-torus containing t and a 2-torus containing ti. If follows that $\langle i\rangle$ is characteristic in S.

From this it follows that S is self-normalizing in $C_G(t)$ and hence S is a Sylow 2-subgroup of $C_G(t)$. As the structure of a Sylow 2-subgroup of $C_G(i)$ is different, we find that i and t are not conjugate, giving a contradiction.

This proves the lemma. □

Proof of Theorem 1.1. We have a connected simple L^* group G of finite Morley rank of odd type satisfying the condition NTA_2, with Prüfer 2-rank 2 and

$$m_2(G) \geq 3.$$

By Lemma 7.1 there are at most two conjugacy classes of involutions.

If there are two conjugacy classes of involutions then Lemma 7.2 gives point (1) of Theorem 1.1.

Suppose therefore that there is one conjugacy class of involutions. By Lemma 7.1 the involutions are SL_2-involutions and case (2b) of that lemma applies. That is,

$$E_{alg}(C_G(i)/OFC_G(i)) = \mathrm{SL}_2 *_2 \mathrm{SL}_2,$$

possibly with different base fields.

The rest of clause (2) of Theorem 1.1 is then clear.

Thus in either case the relevant clause of Theorem 1.1 is verified. □

7.2. **Continuation: one conjugacy class.** We push the structural analysis of centralizers of involutions a little further in the two configurations presented in Theorem 1.1. We begin with the case of one conjugacy class.

Lemma 7.4. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and having one conjugacy class of involutions. Let i be an involution and let L be an $\mathcal{E}$-component of $C_G(i)$. Let H be a definable proper subgroup of G containing L and having 2-rank at least 2.*

Let $\hat{L}$ be the normal closure of L in H.

Then either $L \leq E_{\mathcal{E}}(H)$ or $\hat{L} = OF(\hat{L}) \cdot L$

Proof. Let k be the base field of L and let $\hat{L}$ be the normal closure of L in H. Then $\hat{L}/OF(\hat{L})$ is of type G_2, SL_3, $\mathrm{SL}_2 *_2 \mathrm{SL}_2$, or SL_2, with base field k.

But if $\hat{L}/OF(\hat{L})$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ then L covers one of the components and some element of H conjugates L to a group L^* covering the second component. Then the conjugating element may be taken to lie in $C_G(i)$ and this gives a contradiction. So the quotient $\hat{L}/OF(\hat{L})$ is of type G_2, SL_3, or SL_2.

$\hat{L}$ is a covering group of $\hat{L}/OF(\hat{L})$. Let $Q = OF(\hat{L})$. If Q is trivial then $L \leq \hat{L} \leq E_{\mathcal{E}}(H)$.

So we assume that

$$Q > 1.$$

By Lemma 2.53, $O(\hat{L})$ is a either a p-unipotent group, if the characteristic of the base field k is $p > 0$, or else a $U_{0,r}$-unipotent group with $r = \mathrm{rk}(k)$, if the characteristic is zero.

Let $Q_i = C_Q(i)$.

Case 1. $Q_i > 1$.

By Fact 3.18 $Q_i \leq E_{alg}(C_G(i))$. Then Q_i normalizes L and so Q_i centralizes L. Hence $E_{alg}(C_G(i))$ has two components L, L_2 of type SL_2 and $Q_i \leq L_2$. Here the base fields of the components have the same characteristic, and the same rank if the characteristic is zero.

If $\hat{L}/Q$ is of type G_2 then $C_{\hat{L}}(i)$ covers $E_{alg}(C_G(i))$ and we have a contradiction as $Q_i \leq E_{alg}(C_G(i))$.

Suppose next that $\hat{L}/Q$ is of type $\mathrm{SL}_3(k)$.

The Sylow 2-subgroup of $\hat{L}$ has the form $T\langle t_i \rangle$ where T is a 2-torus with $\Omega_1(T) = \langle i, j \rangle$ and t_i is an involution inverting a torus T_i of L containing i and swapping 2-tori containing j and ij.

In $C_G(i)$ the torus T is a central prodict of 2-tori T_1, T_2 from the factors L, L_2 and t_i is the product of an element of order 4 in L which inverts T_1 by an element of order 4 in T_2. This element must centralize Q_i, which is contained in a unipotent subgroup of L_2. This is impossible.

Therefore we find $\hat{L}/OF(\hat{L})$ is of type SL_2 and $\hat{L} = Q \cdot L$.

Case 2. $Q_i = 1$.

Suppose that $\hat{L}/Q$ is of type G_2 or SL_3 and $Q_i = 1$. Then the involutions are conjugate and the claim applies to a 4-group in $\hat{L}$, showing that $Q = 1$, a contradiction. Thus $\hat{L}/Q$ is of type SL_2 and $\hat{L} = Q \cdot L$. $\square$

The following is more or less found in the preceding, but we bring it out.

Lemma 7.5. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and having one conjugacy class of involutions. Let i be an involution and let L be an $\mathcal{E}$-component of $C_G(i)$ with base field k. Let H be a definable proper subgroup of G containing L and having 2-rank at least 2.*

Let $\hat{L}$ be the normal closure of L in H and $Q = OF(\hat{L})$. If $\hat{L} = OF(\hat{L}) \cdot L$ and $C_Q(i) > 1$ then $E_{\mathcal{E}}(C_G(i))$ has two components of type SL_2 *whose base fields have the same characteristic, and, in the case of characteristic zero, the same rank.*

Proof. We review the relevant part of the previous argument. We set $Q_i = C_Q(i)$, which is p-unipotent if k haa characteristic p and is $U_{0,r}$-unipotent if k has characteristic zero and rank r.

Then Q_i is a unipotent subgroup of $E_{alg}(C_G(i))$ commuting with L, so if $Q_i > 1$ then it lies in a root subgroup of a second component of $E_{alg}(C_G(i))$. So the characteristics agree, the rank of the second base field is at least the rank of k, and in characteristic zero the ranks agree. $\square$

In the same situation, if i inverts Q which is non-trivial and not the natural module, one will arrive at the same conclusion regarding the structure of $E_{alg}(C_G(i))$.

Lemma 7.6. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and having one conjugacy class of involutions. Let i be an involution and let L be an $\mathcal{E}$-component of $C_G(i)$. Let H be a definable proper subgroup of G with $L \leq E_{alg}(H)$.*

Then either $E_{\mathcal{E}}(H)$ is of type SL_3 *or $H \leq C_G(i)$.*

Proof. We suppose that $E_{\mathcal{E}}(H)$ is not of type SL_3.

If $E_{alg}(H)$ is of type SL_2 or $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ then L is a component of $E_{alg}(H)$ and $H \leq N(L) = C_G(i)$.

So it remains only to eliminate the case in which $E_{alg}(H)$ is of type G_2. With $\hat{L}$ the normal closure of L in H we then have

$$(\star) \qquad \hat{L} = E_{alg}(H) = E_{\mathcal{E}}(H) \simeq \mathrm{G}_2\,.$$

In this case for $i \in L$ a central involution, $C_G(i) = L_i \times K$ with $L_i \simeq \mathrm{SL}_2 *_2 \mathrm{SL}_2$ and K of degenerate type. Then K acts on $C_G(t)$ for t

an involution of L_i and centralizes $C_G(i,t)$, which contains a maximal algebraic torus of $C_G(t)$. So on the one hand K acts on $E_{\mathcal{E}}(C_G(t))$ as a subgroup of this torus and on the other hand K centralizes an element w which inverts it. As K contains no involutions, K centralizes $E_{\mathcal{E}}(C_G(t))$. If follows that K centralizes $E_{\mathcal{E}}(H)$ and $E_{\mathcal{E}}(H)$ is normalized by $C_G(i)$, hence by Γ_V for $V \leq L_i$ a 4-group. So $\Gamma_V \leq N(E_{\mathcal{E}}(H))$, a contradiction.

This eliminates case $(\star)$ and completes the proof. $\square$

Lemma 7.7. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and having one conjugacy class of involutions.*

Let i be an involution and suppose that $E_{alg}(C_G(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$. Then $C_G(i) = E_{alg}(C_G(i))$.

Proof. We know that $C_G(i)$ is connected. As the involutions are conjugate our assumption on $C_G(i)$ applies to all involutions.

Let $E_i = E_{\mathcal{E}}(C_G(i))$ and factor $C_G(i)$ as $E_i \times K_i$ with K_i of degenerate type. For t another involution of $C_G(i)$, consider $U = C_{K_i}(t)$.

Decompose $C_G(t)$ similarly as $E_t \times K_t$.

U acts on E_t as a subgroup of degenerate type commuting with an elementary abelian 2-group $\langle i, t, w \rangle$ of rank 3. Hence the action is trivial and $U \leq K_t$. So U commutes with E_i and E_t.

As there is a four-group in E_i with involutions conjugate over i, if $K_i > 1$ then we may suppose that $U > 1$. We let $H = C_G(U)$. As H contains E_i and E_t the only possible structure for $E_{alg}(H)$ is G_2, and this must then be $E_{\mathcal{E}}(H)$.

But this was eliminated in Lemma 7.6. $\square$

We put the last few lemmas together.

Lemma 7.8. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and having one conjugacy class of involutions. Let i be an involution and let L be an $\mathcal{E}$-component of $C_G(i)$.*

Let H be a definable proper subgroup of G containing L and having 2-rank at least 2. Let $\hat{L}$ be the normal closure of L in H and $Q = OF(\hat{L})$.

Then one of the following applies.

(1) *$Q = 1$ and $\hat{L} \leq E_{alg}(H)$.*
   (a) *$H \leq C_G(i)$, or*
   (b) *$E_{alg}(H)$ is of type SL_3.*

   or

(2) *$Q > 1$ and $\hat{L} = Q \cdot L$.*
   (a) *i inverts Q, or*
   (b) *$C_G(i) = E_{alg}(C_G(i)) = E_{\mathcal{E}}(C_G(i))$ has two components of type SL_2. Their base fields have the same characteristic and, in the case of characertistic zero, the same rank.*

Lemma 7.9. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and having one conjugacy class of involutions.*

Let i be an involution. Suppose that $C_G(i)$ has a component L of type SL_2 with base field k of characteristic $p > 0$. Then $E_{alg}(C_G(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ with both components of characteristic p.

Proof. Let $K_L = C_{C_G(i)}(L)$, let $L_2 = E_{alg}(K_L/OF(K_L))$. and let k_2 be the base field of L_2. If k_2 has characteristic p then $E_{\mathcal{E}}(C_G(i))$ has the desired structure. So suppose the characteristic of k_2 is not p.

Then a maximal p-torus P of $C_G(i)$ has Prüfer rank 1, lies in L_2, and commutes with a maximal 2-torus T of $C_G(i)$. Taking t an involution other than i in T, since t and i are conjugate it follows that P lies in a conjugate $L_{2,t}$ of L_2 contained in $C_G(t)$. Then $H = C_G(P)$ contains L and a component of $C_G(t)$ conugate to L.

Referring to Lemma 7.8, we do not have $H \leq C_G(i)$ or $\hat{L} = OF(\hat{L}) \cdot L$, so this leaves the possibility that $E_{alg}(H)$ is of type SL_3, generated by compoents of E_i and E_t. P then centralizes an involution of $E_{alg}(H)$ which lies in $C_G(i)$ and inverts an algebraic torus of L. But viewed in $C_G(i)$ this involution cannot centralize P.

So this gives a contradiction, and the result follows. □

Lemma 7.10. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and having one conjugacy class of involutions.*

Let i be an involution. Suppose that $C_G(i)$ has a component L of type SL_2 with base field k of characteristic zero. Then one of the following holds.

(1) $E_{alg}(C_G(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ with both base fields k, k_2 of characteristic zero and with $\bar{\mathrm{r}}_0(k^\times) = \bar{\mathrm{r}}_0(k_2^\times)$. In particular

$$E_{alg}(C_G(i)) = E_{\mathcal{E}}(C_G(i)).$$

(2) $E_{alg}(C_G(i))$ is of type SL_2 and $\bar{\mathrm{r}}_0(k^\times) = \mathrm{rk}(k_2)$.

Proof. The group $E_{alg}(C_G(i)/OF(C_G(i)))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$. We call the base fields k, k_2. Let $\rho = \bar{\mathrm{r}}_0(k)$.

If k_2 has caracteristic p then $E_{\mathcal{E}}(C_G(i))$ has a component of characteristic p and Lemma 7.9 applies to give a contradiction. So both k and k_2 have characteristic zero.

Case 1. $\mathrm{rk}(k_2) > \rho$.

Suppose first that $\mathrm{rk}(k_2) > \rho$. Then $\Delta_\rho(C_G(i)) \leq E_{\mathcal{E}}(C_G(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$. In particular the role of the two components is symmetrical, so we may suppose $\bar{\mathrm{r}}_0(k_2) \leq \bar{\mathrm{r}}_0(k)$.

If $\bar{\mathrm{r}}_0(k_2) = \rho$ we are done, so suppose $\bar{\mathrm{r}}_0(k_2) < \rho$. We let T be a maximal 2-torus of $C_G(i)$ and $\mathbb{T} = C^\circ_{E_{alg}(C_G(i))}(T)$. Then $U_{0,\rho}(\mathbb{T}) \leq L$. Similarly $U_{0,\rho}(\mathbb{T})$ is contained in a conjugate of L in $C_G(j)$ for each involution of T.

Hence $H = C_G(U_{0,\rho}(\mathbb{T}))$ contains components of $E_{alg}(C_G(t))$ for each such involution t. Then by Lemma 7.8 $E_{alg}(H)$ is of type SL_3. Then as before, an involution which inverts $\mathbb{T}$ should centralize $U_{0,r}(\mathbb{T})$ and we have a contradiction.

So this case is eliminated.

Case 2. $\mathrm{rk}(k_2) < \rho$.

Then we may argue similarly in terms of T, $\mathbb{T}$, and $U_{0,r}(\mathbb{T})$, though in place of the components of $E_{alg}(C_G(i))$ we must work with the inverse images of the components of $E_{alg}(C_G(i)/OFC_G(i))$.

Case 3. $\mathrm{rk}(k_2) = \rho$.

Writing L_2 for a normal subgroup of $C_{C_G(i)}(L)$ covering $E_{alg}(C_{C_G(i)}(L))$, and minimal such, if $OF(L_2) = 1$ then we may proceed as above and arrive at a contradiction. Otherwise, we have $|Ealg(C_G(i)) = L$ and the second alternative applies. □

Lemma 7.11. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and having one conjugacy class of involutions.*

Let i be an involution. Suppose that $C_G(i)$ has a component L of type SL_2 *with base field k of characteristic zero. Then $E_{alg}(C_G(i))$ is of type* $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ *with both base fields k, k_2 of characteristic zero and with $\bar{\mathrm{r}}_0(k^\times) = \bar{\mathrm{r}}_0(k_2^\times)$. In particular*

$$E_{alg}(C_G(i)) = E_{\mathcal{E}}(C_G(i)).$$

Proof. Let $\rho = \bar{\mathrm{r}}_0(k^\times)$ and let k, k_2 be the base fields of the components of $E_{alg}(C_G(i))/OFC_G(i))$. Let L_2 be a normal subgroup of $C_G(i)$ covering $E_{alg}(C_G(L))$. We have to eliminate the case of Lemma 7.10 for which

$$\mathrm{rk}(k_2) = \rho$$

and $OF(L_2) > 1$ (a homogeneous $U_{0,\rho}$-group).

We use the signalizer functor theory. Define $\theta(t) = U_{0,\rho}(OFC_G(t))$. We need to check the balance condition

$$\theta(i) \cap C_G(t) \leq OFC_G(t)$$

for t an involution in $C_G(i)$.

In $C_G(t)$ we have a conjugate L_t of L. The involution i acts on L_t like an element i_1 of order 4 and centralizes an algebraic torus $\mathbb{T}_i$. The subgroup $Q = \theta(i) \cap C_G(t)$ acts on L like a $U_{0,\rho}$-group centralizing i_1, hence as a subgroup of $\mathbb{T}$. If w is an involution in $C_G(t)$ inverting $\mathbb{T}_i$ then Q is w-invariant and is hence a product $Q_1 \times Q_2$ with $Q_1 \leq \mathbb{T}_i$ and Q_2 centralizing L_t.

Claim 1. $Q_1 = 1$.

The centralizer H_1 of Q_1 contains both L and a conjugate L_2^* of L_2 contained in L_t. If $Q_1 > 1$ then H_1 must fall under one of the cases listed under (1) in Lemma 7.8: $H_1 \leq C_G(i)$ or $E_{alg}(H_1)$ is of type SL_3.

As the notmral closure of a Sylow 2-subgroup of H_1 contains L_2^*, we cannot have $E_{alg}(H_1)$ of type SL_3. So this forces $L_2^* \leq C_G(i)$. But L_2* centralizes j so this is impossible.

The claim is proved.

Claim 2. $Q = 1$.

At this point, $Q = Q_2$ centralizes L_j. Suppose $Q > 1$. The centralizer H of Q contains L and L_j, so again we find either $H \leq C_G(i)$ or $E_{alg}(H)$ is of type SL_3. But L_j is not contained in $C_G(i)$.

So $E_{alg}(H)$ must be of type SL_3. In particular Q centralizes the normal closure of a Sylow 2-subgroup of H. Thus Q centralizes L_2^*. But then $L_2^* \leq E_{alg}(H)$ and again we have a contradiction.

Thus the claim is proved, and with this, the balance condition is proved. But then θ gives a non-trivial nilpotent signalizer functor and we have a contradiction. $\square$

Lemma 7.12. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1 and having one conjugacy class of involutions.*

Let i be an involution. Then $C_G(i)$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ where the base fields k_1, k_2 of the two components have the same characteristic, and in addition $\bar{\mathrm{r}}_0(k_1^\times) = \bar{\mathrm{r}}_0(k_2^\times)$. In parciular $C_G(i) = E_{\mathcal{E}}(C_G(i))$.

Proof. Putting together Lemmas 7.9 and 7.11, the structure of $E_{alg}(C_G(i))$ is as stated, and furthermore $E_{\mathcal{E}}(C_G(i)) = E_{alg}(C_G(i))$.

Then Lemma 7.7 applies. □

One should perhaps prove $\mathrm{rk}(k_1) = \mathrm{rk}(k_2)$ as well, at least in characteristic zero, but it does not seem necessary. Having all components be $\mathcal{E}$-components seems like the main point (along with the fact that two components occur in $E_{alg}(C_G(i))$).

7.3. **Continuation: two conjugacy classes.** We now turn to the case of two conjugacy classes, beginning with the configuration as described by Theorem 1.1.

Hypothesis 7.13.

(1) G is a simple L^*-group of odd type, satisfying NTA_2.

(2) There are two conjugacy classes of involution, as follows.

  (a) There is an involution i with $E_{\mathcal{E}}(C_G(i)) \simeq \mathrm{SL}_2(k) *_2 \mathrm{SL}_2(k)$, and with the components conjugate by an involution.

  (b) There is an involution t with $E_{\mathcal{E}}(C_G(t)) \simeq \mathrm{PSL}_2(k)$.

(3) The Sylow 2-subgroup is as in PSp_4. In particular the 2-rank is 4.

Lemma 7.14. *Let G be a group of finite Morley rank satisfying Hypothesis 7.13 and leet i be an SL_2-involution of G. Then the following hold.*

(1) i is the unique SL_2*-involution of* $C_G^\circ(i)$*.*

(2) The PSL_2*-involutions are precisely the involutions lying in a copy of* PSL_2 *in G.*

(3) The PSL_2*-involutions are precisely the involutions lying in a component of the centralizer of a* PSL_2*-involution.*

(4) For t a PSL_2*-involution, and $L_t = E_{alg}(C_G(t))$, the involutions of $C_G^\circ(t)$ are those of $L\langle t\rangle$. Those in $L_t \cup \{t\}$ are* PSL_2*-involutions and the rest are* SL_2*-involutions.*

Proof.

Ad 1. The involutions in a given torus consist of one SL_2-involution and two PSL_2-involutions. All the involutions in $C_G^\circ(i)$ are co-toral with i. So apart from i they are PSL_2-involutions and point (1) follows.

Ad 2. There is an involution in $C_G^\circ(i)$ which lies in a copy of PSL_2, and all PSL_2-involutions are conjugate.

On the other hand if i were to lie in a copy L of PSL_2 then there would be an involution w in the centralizer of i inverting the torus of L containing i, and conjugate to i in L. Such an involution would be an SL_2-involution and so cannot lie in $C_G^\circ(i)$. It must then swap the two components of $\mathbb{C}_G^\circ(i)$. But then the tori of $C_G^\circ(i)$ inverted by w do not contain i.

Thus point (2) follows.

Ad 3. This is more or less immediate from point (2). There are involutions which lie in components of centralizers of PSL_2-involutions. Such involutions must be PSL_2-involutions, and all PSL_2-involutions are conjugate.

Ad 4. This adds a bit more to the above.

Let s be an involution of $C_G^\circ(t)$. Then s, t are co-toral. Let T be a 2-torus containing s, t. Then T is a maximal 2-torus of $C_G^\circ(t)$ and hence meets L_t in a nontrivial 2-torus T_1. Accordingly $s \in T_1\langle t\rangle \leq L_t\langle t\rangle$. As we have seen the involutions of $L \cup \{t\}$ are PSL_2-involutions. On the other hand there is an SL_2-involution cotoral with t which must then lie in $L \cdot t$, and all of the latter, apart from t, are connugate under the action of L_t. This completes the argument.

$\square$

Now we argue in the vein of Lemma 7.4

Lemma 7.15. *Let G be a group of finite Morley rank satisfying Hypothesis 7.13. Let i be an involution and let L be an $\mathcal{E}$-component of $C_G(i)$. Let H be a definable proper subgroup of G containing L and having 2-rank at least 2.*

Let $\hat{L}$ be the normal closure of L in H.

Then either $L \leq E_{\mathcal{E}}(H)$ is of type PSp_4 or $\hat{L} = OF(\hat{L}) \cdot L$

Proof. Let k be the base field of L.

Then $\hat{L}/OF(\hat{L})$ is of type PSp_4, G_2, SL_3, $\mathrm{SL}_2 *_2 \mathrm{SL}_2$, or SL_2, with base field k.

In the cases of G_2 or SL_3 all toral involutions would be SL_2-involutions, and hence all involuitons would be SL_2-involutions, a contradiction.

If $\hat{L}/OF(\hat{L})$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ then L covers one of the components and some element of H conjugates L to a group L^* covering the second component. Then the conjugating element may be taken to lie in $C_G(i)$ and this gives a contradiction.

So the quotient $\hat{L}/OF(\hat{L})$ is of type PSp_4 or SL_2. In the latter case $\hat{L} = OF(\hat{L}) \cdot L$.

So suppose that $\hat{L}/OF(\hat{L})$ is of type PSp_4 and let $Q = OF(\hat{L})$. This is a p-unipotent or a $U_{0,r}$-unipotent group with $r = \mathrm{rk}(k)$. For $i \in \hat{L}$ an SL_2-involution, $C_{\hat{L}}(i)$ covers $E_{alg}(C_G(i))$ and $C_Q(i) \leq E_{alg}(C_G(i))$, which forces $Q_i = 1$. There is a 4-subgroup V in $\hat{L}$ whose involutions are SL_2-involutions . So $Q = 1$ and $\hat{L} \leq E_{alg}(H)$ is of type PSp_4. $\square$

Lemma 7.16. *Let G be a group of finite Morley rank satisfying Hypothesis 7.13 and let i be an* SL_2*-involution. Then $C_G^\circ(i)$ is of type* $\mathrm{SL}_2 *_2 \mathrm{SL}_2$.

Proof. We write $C_G^\circ(i) = EK$ with $E = E_{alg}(C_G(i))$ and K the centralizer of E. Let $H = C_G(K)$.

For $t \neq i$ an involution in E, K acts on the component $L_t = E_{\mathcal{E}}(C_G(t))$ of type PSL_2 and centralizes a 4-group in L_t, so the action is trivial and $L_t \leq H$.

If $K > 1$ then in view of Lemma 7.15 $E_{alg}(H)$ must be of type PSp_4.

Now let t_1, t_2 be commuting involutions of $E_{alg}(H)$, $E_\ell = E_{alg}(C_G(t_\ell))$, and $K_\ell = C_{C_G(t_\ell)}(E_\ell)$. Then K_1 centralizes t_2, hence acts on E_2, and centralizes a 4-subgroup. Hence K_1 acts trivially on E_2, and $K_1 \leq K_2$. Thus $K_1 = K_2$. Taking a 4-group V in $E_{alg}(H)$, the associated group K_V is normalized by Γ_V, so $\Gamma_V < G$, and we have a contradiction.

This shows that $K = 1$. □

Lemma 7.17. *Let G be a group of finite Morley rank satisfying Hypothesis 7.13. Suppose that t is a* PSL_2*-involution of G commuting with an* SL_2*-involution i, and let $E_i = E_{alg}(C_G(i))$, $L_t = E_{alg}(C_G(t))$.*

Then $C_G^\circ(L_t) = C_{E_i}(L_t)$ is a 1-dimensional algebraic torus of E_i.

Proof. Fix a maximal 2-torus T of E_i. Take j an involution of $N(T)$ swapping the factors of E_i. We may suppose that t is the involution of T in the diagonal subgroup with respect to j. Let $K = C_G^\circ(L_t)$.

We may suppose that $j \in L_t$, replacing j by jt if needed. Then j is a Weyl group element with respect to $T \cap L_t = T^-$ and $T^+ \leq K$. Furthermore j commutes with K.

Then $T = T^+ \times T^-$ where $T^+ = C_T^\circ(j)$ and T^- is inverted by j. Also j acts on L_t, so T^- is a maximal 2-torus of L_t and either j or jt is a Weyl group element in L_t.

As $i \in C^\circ_G(t)$ is an SL_2-involution, ti lies in L_t. Thus j and ti both centralize K.

As j is a PSL_2-involution, $L_j = E_{alg}(C_G(j))$ is the diagonal copy of PSL_2 with respect to the action of j on E_i. Here K acts on L_j and $t \in L_j$.

Furthermore the algebraic torus $\mathbb{T}_t$ of L_j containing t lies in $C^\circ_G(t)$ and commutes with both T and the Weyl group element j of L_t, so $\mathbb{T}_t \leq K$ and K acts on L_j like $\mathbb{T}_t$.

Let $K_0 = C^\circ_K(L_j)$. Then

$$K = \mathbb{T}_t K_0$$

where K_0 centralizes L_t and L_j. Since $i \in L_t \langle t \rangle$, it follows that K_0 centralizes i.

So K_0 acts on E_i like a subgroup of the algebraic torus containing T. But K_0 commutes with the Weyl group element of L_j which inverts T, and K_0 is connected, so K_0 is trivial.

Thus $K = \mathbb{T}_t \leq L_j \leq E_i$. □

7.4. **Theorem 1.2.** We may now conclude. We have two configurations, one involving two conjugacy classes of involutions and 2-rank 4, the other involving one class of involuations and 2-rank 3, with fairly detailed information concerning the structure of centralizers of involutions.

Proof of Theorem 1.2. The difference between the prior Theorem 1.1 and Theorem 1.2 lies in a more precise description both of $E_{alg}(C_G(i))$ and of the full centralizer $C_G(i)$ in each of the two cases arising.

In the case of one conjugacy class of involutions the additional information is found in Lemma 7.12.

In the case of two conjugacy classes of involutions it is found mainly in Lemma **??**. For the further statements about the SL_2-involutions and the PSL_2-involutions one refers to Lemma 7.14.

□

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This is part [1] of a series of four papers whose earliest versions are ca. 2008 by the first author, treating groups of 2-rank at least 3 (up to a point) in a context broader than, but similar to, the K^* setting.

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