HOMOGENEOUS ORDERED GRAPHS, METRICALLY HOMOGENEOUS GRAPHS, AND BEYOND

VOL. I

ORDERED GRAPHS AND DISTANCED GRAPHS

GREGORY CHERLIN
Dept. of Mathematics
Hill Center, Busch Campus
Rutgers University
Piscataway, NJ 08854
cherlin.math@gmail.com

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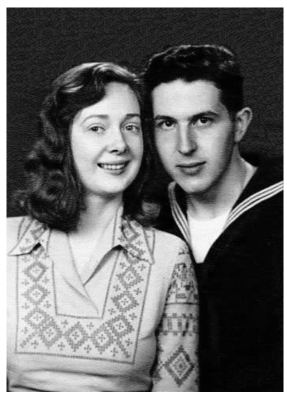
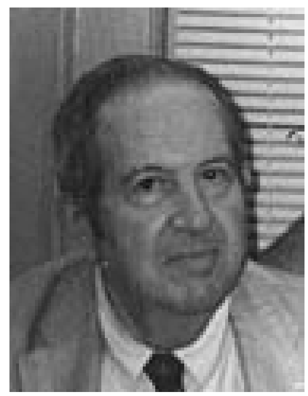
In memory of George Cherlin, *1924-1992.*
Enthralled by the beauty of mathematics, and its power for good or evil.

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ABSTRACT

VOLUME I

Part I: A complete classification of homogeneous ordered graphs is given: up to a change of language each is either a generically ordered homogeneous graph or tournament, or a generic linear extension of a homogeneous partial order.

Part II: A catalog of the currently known metrically homogeneous graphs is given, with proofs of existence and some evidence for the completeness of the catalog. This includes a reduction of the problem to what may be considered the generic case, and some tools for the analysis of the generic case.

Some related developments are discussed in an appendix.

VOLUME II

Here the impact of the results of Parts I and II and of related work in Amato, Cherlin, and Macpherson [2021] on the classification of homogeneous structures for a language with two anti-symmetric 2-types or with 3 symmetric 2-types is worked out in detail.

An appendix to Volume II discusses some further advances in related areas, and a wide variety of open problems.

An extensive bibliography of related literature and a quick survey of that literature, organized by topic, will be made available online (Cherlin [2021]).

The method used in Part I of Volume I is due to Alistair Lachlan. The method used in Part II of Volume I and throughout Volume II is a direct application of Fraïssé's theory of amalgamation classes.

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PREFACE

The notion of homogeneity that concerns us here was first noticed by Urysohn in a metric context, a few days before his tragic death in a swimming accident in 1924. As transcribed by his companion Alexandrov into German for the benefit of Hausdorff, and into French for the benefit of the mathematical public (Hušek [2008], Urysohn [1925]), Urysohn's remark goes roughly as follows.

> U is *homogeneous* in this sense: the finite and congruent sets A, B (lying in U) being arbitrarily chosen, there is an isometric map of U onto itself transforming A into B.[1]

One can trace the notion back further, since it amounts to saying that the Euclidean viewpoint and Felix Klein's coincide on finite configurations. Indeed, this is more or less the point of view taken by Freudenthal in a survey of work on two-point homogeneity (Freudenthal [1956]), which places the issue firmly in the context of ideas of Riemann, Helmholtz, and Lie on the foundations of geometry.

In an algebraic or combinatorial context we speak of *isomorphism* rather than congruence or isometry; and to be precise, we require that *any* given isomorphism extend to an automorphism.

Erdős and Rényi remarked in 1963, with what I take to be some bemusement, that infinite structures with rich automorphism groups appear as natural limits (in a probabilistic sense) of rigid finite structures.

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[1]"U est *homogène* en ce sens que, les ensembles finis et congruents A et B (situés dans U) étant quelconques, il existe une représentation isométrique de U sur lui-même transformant A en B."
" … eine recht starke Homogenitätsbedingung … letzterer besteht darin, daß man den ganzen Raum (isometrisch) so auf sich selbst abbilden kann, daß dabei eine beliebige endliche Menge M in eine ebenfalls beliebige, der Menge M kongruente Menge M_1 übergeführt wird."

> *Thus there is a striking contrast between finite and infinite graphs: while „almost all" finite graphs are asymmetric, „almost all" infinite graphs are symmetric.* Erdős and Rényi [1963]

Fraïssé's theory of *amalgamation classes* from the 1950s would suggest a more extreme example of the same phenomenon: the rational order is the *Fraïssé limit* of the finite orderings. Fraïssé pointed out the role of the amalgamation property as the counterpart at the finite level of the rich automorphism group in the infinite limit, giving one possible answer to the question implicit in the remark of Erdős and Rényi.

The automorphism group of a homogeneous structure carries a natural topology, and with this topology the automorphism group of the ordered set $\mathbb{Q}$ has a remarkable fixed-point property : any continuous action on a compact set has a fixed point. Pestov pointed out in 2002 that this property is equivalent to Ramsey's theorem.

Meanwhile, Nešetřil had observed that under mild conditions, the generalized Ramsey property for a class of finite structures implies that the homogeneous Fraïssé limit exists (Nešetřil [1989], cf. Nešetřil [2005]). And in 2005 Kechris, Pestov, and Todorcevic closed this circle of ideas by showing that for homogeneous structures, the fixed point property for the automorphism group is equivalent to structural Ramsey theory for the finite substructures. Presumably Erdős and Rényi would have found all this very illuminating.

Since then, things have been generally lively, and a good deal of young talent has taken up the subject from a bewildering multiplicity of points of view. We will revisit a little of that in the Appendices to Volumes I and II.

In the 1970's Henson observed that Fraïssé's method gave more examples than one would necessarily want to have[2], notably an uncountable family of homogeneous directed graphs. In the 1980's Lachlan and Woodrow developed techniques based on Fraïssé's theory, sometimes using the classical Ramsey theorem as well, to *classify* homogeneous structures for restricted languages, showing that all homogeneous graphs and tournaments were known.

In the 1990's I decided to put Henson's examples together with the Lachlan/Woodrow technique in a cage match, uncountably many

[2]"This book tells me more about penguins than I wanted to know."

against classification, and the latter won:[3] the homogeneous directed graphs can be classified, and in fact most of them are the ones Henson originally constructed—that is, the full list contains countably many additional structures (Cherlin [1998]). In the appendix to that work I took a very tentative look at the classification problem for homogenous structures in slightly more complicated languages, having either two anti-symmetric 2-types, or three symmetric 2-types, and later I classified the imprimitive examples of the latter kind.

VOLUME I

In recent years, I have taken up two more classification problems for homogeneous structures that struck me as of particular interest, which are the subject of the first volume of this monograph.

The first of these problems is the classification of the *homogeneous ordered graphs,* which was suggested to me in 2012 by Lionel Nguyen Van Thé as a natural problem from a Ramsey theoretic point of view, and potentially the source of interesting new examples. The second problem is the classification of *metrically homogeneous graphs* (equipping a connected graph with its path metric,[4] and requiring metric homogeneity). This problem was noted in passing in Moss [1992] and more explicitly by Cameron, in terms that I find memorable:

> The theory of infinite distance-transitive graphs is open. Not even the countable metrically homogeneous graphs have been determined. —Cameron [1998]

The complete solution to our first problem, the classification of the homogeneous ordered graphs, will be found in Part I.

The second problem is not yet completely solved. A catalog and a corresponding conjecture as to the classification of the metrically homogeneous graphs will be found in Part II, together with a reduction of the problem to what I call "generic type," and some general structural results which apply in the case of generic type.

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[3]This was unexpected. Chapter IV of Cherlin [1998] provides a missing link between the case of homogeneous graphs and the case of homogeneous digraphs which might perhaps have altered my expectations if it had been known beforehand. That chapter is also the main source of the strategy for Part I of this monographs.

[4]Such a structure is called a *distanced graph,* in L.Moss' terminology. Thus a metrically homogeneous graph is a homogeneous distanced graph.

In the conjectured classification, the class of 3-*constrained* metrically homogeneous graphs plays a leading role; these are the metrically homogeneous graphs whose finite induced metric substructures can be characterized by forbidden *metric triangles.* I give a completely explicit classification in the 3-constrained case, and somewhat more. The main classification conjecture amounts to a reduction of the classification to the 3-constrained case, in a sense to be discussed below.

Other results in Part II concern the treatment of the *bipartite* case and the case of *infinite diameter:* if all metrically homogeneous non-bipartite graphs of finite diameter are in our catalog, then in fact all metrically homogeneous graphs are known.

Not given in this monograph, but very relevant to it, is the verification of the conjectured classification in the case of diameter 3, which was carried out jointly with Amato and Macpherson in Amato, Cherlin, and Macpherson [2021]. That work uses a certain amount of material from the present monograph, but only for the sake of convenience—the general theory supplies some initial reductions which would not be difficult in diameter 3, and that theory also predicts—and explains—the classification found.

Later we realized that one can take the diameter 3 treatment as an indication of an inductive approach to a full proof of the classification conjecture (and then the full content of Chapters 13, 14 of the present work also becomes relevant, beyond the 3-constrained case). This new approach is being actively explored and is discussed further in the Appendix to this volume, in §18B.1.

That appendix discusses four directions in which there has been substantial recent progress, and which are connected fairly directly to the material of this volume. The reader interested in a broader view of open problems in the area of homogeneity and related parts of model theory will no doubt want to explore the appendix to Volume II as well.

The first two topics dealt with in the appendix to Volume I concern classification problems: namely, the classification of the homogeneous "multi-orders," also called *finite-dimensional permutation structures*, by Braunfeld and Simon, and the ongoing classification project for metrically homogeneous graphs and its relationship to the strategy developed in Amato, Cherlin, and Macpherson [2021], the latter already mentioned.

The other two topics discussed there involve the closer study of the automorphism groups of metrically homogeneous graphs of generic type. One of the goals of a classification project is to uncover interesting, and possibly exotic, examples which are suitable for further study. Certainly the known metrically homogeneous graphs of generic type fall under that heading.

There is now a very rich general theory relating the study of the automorphism groups of homogeneous relational structures to finite combinatorics, following on a breakthrough in the seminal paper by Kechris, Pestov, and Todorcevic [2005]. There is another very interesting line coming from Tent and Ziegler [2013]. Typically these theories reduce the study of automorphism groups of homogeneous structures, viewed as abstract groups, as topological groups, or as permutation groups, to combinatorial problem concerning the associated classes of finite structures.

As far as the theory of automorphism groups of metrically homogeneous graphs is concerned, we confine ourselves in the appendix to Volume I to the combinatorial side of the problem. That is, we discuss the relevant combinatorial properties of finite substructures of metrically homogeneous graphs. These properties relate to *completion procedures* for partial metric spaces embedding in metrically homogeneous graphs.

The desired completion procedures lead to exotic notions of *shortest path completion* in generalized metric spaces which promise to reshape the whole theory of metrically homogeneous graphs conceptually. Among the sources for this material are Aranda et al. [2017], Konečný [2019a], [2019b]; the first two articles mentioned do not use the language of generalized metric spaces, while the third exploits that point of view enthusiastically, but gives less detail for the case of metrically homogeneous graphs. The published version of Aranda et al. [2017] is Aranda et al. [2021]; this is rather compressed and does not give the most general form of the results.

Finally, we revisit a question of Cameron and Tarzi on splitting the group of "twisted automorphisms" over the group of ordinary automorphisms, taking up the question in the context of metrically homogeneous graphs. The result found in that case provides an interesting, though anecdotal, counterpart to their prior results. What form such results might take in a more general setting, and how broad such a setting should be, remains unclear. But this is an area which invites further investigation.

The first two chapters of Volume I present an overview of the results obtained and the methods used in Parts I and II. That is, the first chapter presents the *results* of Parts I and II in detail, and the second chapter discusses the *methods* used in both parts. Thus readers who have a definite interest in just one of the two topics treated should read these two chapters selectively; depending on their needs or interests, they may then possibly be spared reading the rest of the monograph—but will certainly want to look at the appendix, and very likely at the appendix to Volume II as well.

VOLUME II

At the end of my earlier monograph on homogeneous directed graphs (Cherlin [1998]) the logical next phase of that project was briefly considered: the classification project for homogenous structures in languages with two pairs of anti-symmetric 2-types, or with three symmetric 2-types. We will refer to structures of the first kind as 2-*multi-tournaments* and to structures of the second kind as 3-multi-graphs. With a little computer assistance (specifically, a home computer of 1990s vintage), the 3-constrained 2-multi-tournaments and 3-multi-graphs were found, and were listed in tabular form in Cherlin [1998], with some trivial cases omitted.

The work presented in Part I of the present volume falls wholly within the first (anti-symmetric) setting, while the work in Amato, Cherlin, and Macpherson [2021] falls wholly within the second (symmetric) setting. So it is natural to ask how far this work advances us toward a solution of either of those more general classification problems. The answer is not immediate.

In fact, this question is the subject of Volume II. To be clear, we do not seek a solution to these classification problems in the near term, but rather a road map and an understanding of where the current results actually leave us with respect to the broader problems.

Our experience in Part II of this volume strengthens our sense that the study of 3-constrained cases is an important part of the classification process in its cataloguing phase. In a systematic approach to the classification problem for all homogeneous structure in a fixed, small, binary relational language, one expects to proceed according to the following scheme, which has some theoretical justification (mostly conjectural).

1. Identify the 3-constrained structures.
2. Show that with few exceptions the triangle constraints in a homogeneous structure agree with those in some 3-constrained structure.
3. Classify the homogeneous structures whose triangle constraints do not define a free amalgamation class.
4. Classify the homogeneous structures whose triangle constraints do define a free amalgamation class.

For the last two points, one expects to encounter Henson constraints (suitably understood) as well as some truly exceptional or even sporadic cases.

See §18.B in Volume II for a fuller discussion of this.

For our purposes point (1) is part of the pre-history, and the present volume (and related work) bears on instances of (3). It seems that something has been skipped!

The missing point (2) turns out to be challenging. The whole of Volume II attempts to address it. That is, we attempt to find all patterns of forbidden triangles in homogeneous 2-multi-tournaments or 3-multi-graphs, with the known classification in the 3-constrained case providing the target for the analysis. At the end, certain recalcitrant cases remain open, which we believe can be eliminated, ideally with some further computer assistance of a more substantial (interactive) type.

Chapter 18 (the first chapter of Volume II) surveys the results obtained on this problem in considerable detail. This chapter serves as an extended introduction to the whole volume.

In chapter 19 we give a satisfactory treatment of point (2) in the case of homogeneous 3-multi-graphs. We know the possible patterns of triangle constraints, and we find in fact that an unknown homogeneous 3-multi-graph must be infinite and primitive and have triangle constraints compatible with free amalgamation. In other words, we arrive at the start of what should be the generic case, with all of the obvious special cases treated. It is very convenient for our purposes that the imprimitive case has been analyzed separately in earlier work.

Homogeneous 2-multi-tournaments are more recalcitrant. We do not have a prior classification in the imprimitive case, so we next address that point, in Chapter 20. The analysis there is similar to what was done previously in the case of imprimitive homogeneous digraphs.

Next we give the classification of the 3-constrained homogeneous 2-multi-tournaments in detail, without relying on computer computations. Indeed, if we wish to work out the general patterns of constraints on triangles which are compatible with homogeneity, then we need to have such a treatment as a point of departure. So this is the subject of Chapter 21.

Finally in Chapter 22 we arrive at the problem of the determination of the possible triangle constraints for homogeneous 2-multi-tournaments. This is incompletely resolved. Four cases which do not correspond to 3-constrained examples remain elusive. We believe these cases can be excluded with substantial computer assistance, or possibly by a very elaborate line of argument, which we investigated just far enough to see that it has some promise. One would probably do best to combine the approaches: the individual steps of such an analysis are of a type which lends itself to proof by computer, with a lucid proof resulting as output, via a tree search which is very tedious by hand, and has no a priori bound on depth, though in practice the depth seems very small, while the width is exponentially large. These cases are eliminated in the 3-constrained case using amalgamation diagrams of order 6 (factors of order 5). They are the only cases in which the class defined by the constraints allows amalgamation of all diagrams of order at most 5, but is not an amalgamation class.

The numerical point here is that $6 = 4+2$ with 4 being the number of 2-types. In the case of the known metrically homogeneous graphs of generic type, of any diameter, amalgamation up to order 5 implies amalgamation. But in general one expects to need amalgamation up to order $r+2$, where r is the number of 2-types.

An appendix to Volume II considers, very broadly, some open problems in the theory of homogeneous structures, with references to other similar surveys. This may be viewed as a continuation of the appendix to Volume I. This appendix began its life as a short note intended for graduate students, but has evolved since.

We will elaborate further on the contents of Volume II, and on the classification of binary homogeneous structures generally, in a separate introduction to Volume II.

Each volume has its own biliography and index (the latter with few cross references across volumes).

General Remarks

As combinatorialists occasionally remind me, their concern is not so much with the classification of homogeneous structures of a particular kind, but rather with the identification of novel examples.

From that point of view, Part I is a failure (or, if one prefers, it is only due diligence)—there is nothing new to be found in that direction. Part II on the other hand is very successful from that point of view. It contains a rich catalog of new examples and this catalog has itself been studied and to a degree explained by a considerable body of combinatorial work already alluded to.

As we observed, the main feature of this catalog—as such—is the classification of the *3-constrained* metrically homogeneous graphs of generic type, and the associated *variants with Henson constraints.* In practice the Henson variations do not much complicate matters, and the main combinatorial issues arise already in the study of the 3-constrained structures.

Apparently the assumption of 3-constraint is not in itself useful combinatorially, and what is useful is the reinterpretation of the classification of the 3-constrained metrically homogeneous graphs in terms of a theory of generalized metric spaces with values in a partially ordered semigroup, as a step toward the characterization of the *partial* (i.e., *weak*) substructures of a given 3-constrained metrically homogeneous graph. For the moment, at least, this conceptual description still depends on an explicit classification by direct methods.

So the lesson I would take from this, currently, is that one should focus more on understanding the 3-constrained case in general (possibly under a hypothesis of strong amalgamation and perhaps also primitivity).

One tantalizing feature of recent developments is that the proofs of the amalgamation property under suitable (and quite arcane) conditions given in Chapter 12 have since been reinterpreted as a shortest path completion in a generalized metric space with values in a partially ordered semigroup. We say more about this point of view in the appendix to this volume.

With reference to the arcane numerical conditions (called "admissibility") which turn out to be equivalent to the amalgamation property for 3-constrained classes, an ex post facto justification of a sort can be found in Hubička, Konečný, and Nešetřil [2020a], which may provide a useful heuristic for the classification of 3-constrained homogenous binary structures in other contexts. (From another point of view, these conditions are the result of applying quantifier elimination to a formula in Presburger arithmetic; this accounts for their general form but does not elucidate their content.) While clarifying, this interpretation in terms of generalized metric spaces does not immediately provide a precise explanation of all of the conditions found, though many of them are required for the construction of the semigroups $D^{\delta}_{M,C}$ described in the appendix, and all of them come in eventually in the treatment of the completion procedure (inevitably).

My motivation for the work in Parts I and II is touched on again in the acknowledgments below, and is expanded on in Chapter 1. While I had not expected to return to the broader classification problems considered in Part III after taking up these problems, it now seems very natural to do so—-at least, now that the work in Amato, Cherlin, and Macpherson [2021] is complete.

Fans of the Ramsey theoretic approach[5] to the classification of homogeneous structures will be pleased to see it carrying the burden of the argument in Part I. I had thought it might come in also to the general classification project for metrically homogeneous graphs, but it seems not (see the appendix to this volume for current thinking on this point).

Part II introduces a new family of metrically homogeneous graphs. This family was described in Cherlin [2011], but not actually proved to exist there. Here at last the existence proof is given (Chapter 12), along with some useful general theory.

The rest of Part II makes a start on the problem of showing that the catalog of metrically homogeneous graphs found is complete, that is,

[5]It deserves more fans—or practitioners; the method is very powerful. Though it may lend itself more easily to machine-assisted work.

that the catalog is exhaustive. We carry out reductions of the classification problem to what we call generic type, as well as to the non-bipartite and finite diameter cases (under suitable inductive hypotheses). We also develop, and apply, some general methods of *local analysis,* by which we mean the study of the substructures induced on the locus of a 1-type relative to a fixed parameter (basepoint). The immediate prospects for completion of the classification project are discussed in the appendix.

To close, I add a few words about the development of the material presented here.

I began working seriously on the classification of metrically homogeneous graphs in 2006, starting on the side of what I now call *exceptional local type.* The classification for exceptional local type was given in Cherlin [2011] along with the catalog for generic type, including a statement of the classification in the 3-constrained case and a description of the amalgamation procedure. For reasons of space (not to mention a deadline) I did not include the proof that the amalgamation procedure works for the classes in question, nor that these classes exhausted the 3-constrained ones of generic type.

The question addressed in Part I came up in Summer 2012, and it occurred to me soon afterward that the method of Chapter IV of Cherlin [1998] was very likely applicable. The details were worked out in 2013.

At the time, it seemed reasonable to put these two projects together into one monograph, with (as I thought) two Parts, a project initiated in 2015. A third part (now a second volume) made its appearance in 2016. The monograph was submitted in 2017, but from time to time I revisited the third part. That part became both more intelligible but also longer as the results were completed and the proofs somewhat expanded, partly in response to remarks by referees.

Ultimately a projected chapter on homogeneous 2-multi-tournaments became three chapters. In parallel, the treatment of the diameter three case of metrically homogeneous graphs jointly with Amato and Macpherson also expanded, and then led to further developments discussed in the appendix. At present there is good reason to think that we are on the right track for a full proof of the metrically homogeneous graphs, building on the methods used in diameter 3.

The state of knowledge has shifted considerably over this period. This is most visible in the discussion of open problems, whose perspective is mainly that of 2016, with modest revisions in 2020–21, mainly with respect to our new approach to the proof of the classification of the metrically homogeneous graphs.

The division into volumes, each with its own appendix, is a late decision. There may well be some remarks in the text more reminiscent of 2016 than of 2021, but in any case these would concern matters that remain to be settled.

Princeton, August 2021

ACKNOWLEDGMENTS

I am grateful to Lionel Nguyen Van Thé for raising the question treated in Part I of Volume I in summer 2012. This did not strike me immediately as a reasonable question, but my inability to articulate a concrete objection soon forced me to take it seriously. I am also grateful to Miodag Sokić for drawing my attention to the article by Dolinka and Mašulović Dolinka and Mašulović [2012] in Fall 2013.

With regard to the material of Part II of Volume I, I first encountered the problem of classifying countable metrically homogeneous graphs in Moss [1992], then found the discussion in Cameron [1998] very stimulating, and found evidence in Kechris, Pestov, and Todorcevic [2005] that the problem was natural from several other points of view.

Much of my fascination with the classification of homogeneous structures can be traced back to Alistair Lachlan. I have long enjoyed his understated enthusiasm, his gift for turning examples into theories, and his fearlessness.

In the course of preparing this text, remarks by Dugald Macpherson on the content and Jan Hubička on the literature were very helpful.

Lately Braunfeld, Coulson, Evans, Hubička, Konečný, Nešetřil, and Simon have been among those who have complicated my life by telling me interesting things outside the scope of the monograph that nonetheless deserved mention here and there. A good deal of that found its way into the appendix to Volume I. That list could be longer, but one has to stop somewhere.

I greatly appreciate the careful work done by a number of anonymous referees, as should the reader. (This applies with particular force to Volume II.)

For that matter, I also appreciate the work done by an anonymous referee on Amato, Cherlin, and Macpherson [2021] and the impetus provided to us jointly to reflect further on the path forward with regard to the classification metrically homogeneous graphs.

I thank Stewart Cherlin for editing the family photographs used as a frontispiece to be suitable for reproduction in black and white.

And I take this opportunity to acknowledge, for once, my lifelong debt to Donald Knuth for the years he took off to make TEX work.

Amélie and Grégoire provided regular and varied distractions; le petit Nicolas came in at the end with music and dance. S. L. Huang was a source of additional entertainment. Christiane and Rufus provided their famous hospitality, and Chantal kept it all together.

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CHAPTER 1

RESULTS

1A. Introduction

We deal with structures in a relational language (often, but not always, a finite language). The problem that concerns us is the classification of the countable homogeneous structures in some natural cases. The two cases discussed here are the following.

- ⋆ Countable homogeneous ordered graphs.
- ⋆ Countable metrically homogeneous graphs.

Everything will be countable, and it is tempting to drop out that word throughout, but we will try to resist the impulse.

A structure in a relational language is called *homogeneous* (or *ultra-homogeneous,* for emphasis) if every isomorphism between finite substructures is induced by an automorphism. An ordered graph is a graph with a linear ordering; there are no additional constraints.

A *metrically homogeneous* graph is a connected graph with the property that the associated metric space is a homogeneous metric space. The associated metric space has the vertices as points, with distance the minimal path length connecting two vertices.

Metrically homogeneous graphs of diameter 2 are just connected homogeneous graphs. The main examples were given in Henson [1971] using Fraïssé's method, and Lachlan and Woodrow showed that they were indeed the main examples (Lachlan and Woodrow [1980]), by completing the classification of the countable homogeneous graphs, building on a prior classification of the finite ones by Sheehan and Gardiner (Sheehan [1974], Gardiner [1976]). The Lachlan/Woodrow classification plays a fundamental role in our analysis (see §1D).

In Part I we will give an explicit, and surprisingly simple, classification of the countable homogeneous ordered graphs. The main ingredient of the proof was found (hidden) in Cherlin [1998, Chapter IV]. In addition, one of the three cases that must be treated was covered in its entirety by Dolinka and Mašulović [2012].

In Part II we will give an explicit, and not particularly simple (but still simple enough), *conjecture* concerning the classification of the countable metrically homogeneous graphs. This conjecture has already been discussed in Cherlin [2011], but we got ahead of ourselves there, stating a good deal more than we proved, for reasons of space and balance—our earlier discussion was intended to be a broad one, though it evolved in a more technical direction. Here we give a full account of the conjecture, which to begin with requires that we prove that the family of new examples described in Cherlin [2011] actually exists, after which we present a number of results which provide some support for the conjecture that our explicitly given list of examples is complete, or very nearly so.

In this introductory chapter we will state our main results on both of these classification problems in detail, and in the next chapter we will indicate the main lines of argument used. Much of the work involved in problems of this kind lies in finding a suitable inductive framework for the proof, and then working out the scaffolding of supporting lemmas on which the argument ultimately depends. This definitely requires a top-down approach, so in the next chapter we will begin at the top.

Before describing the results to be obtained, we review the Fraïssé theory, on which everything done here depends—both the existence of many of the structures in question, as well as their classification.

1B. Fraïssé limits and amalgamation classes

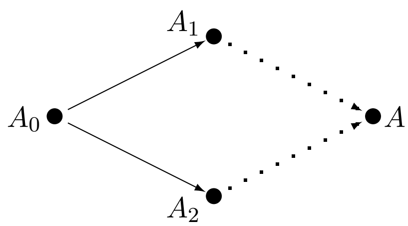
Fraïssé theory plays a fundamental role in classifications of homogeneous structures. We recommend Macpherson [2011] for a survey in the modern spirit. We now give a synopsis of that theory.

Fraïssé observed that a countable homogeneous structure is uniquely determined by the isomorphism types of its finite substructures, and gave an explicit recipe for deriving the structure as a kind of limit of its finite substructures, now called the Fraïssé limit. This limit structure occasionally has a probabilistic interpretation (or more than occasionally, if one forces matters as in Petrov and Vershik [2010], Ackerman, Freer, and Patel [2016], Ackerman, Freer, Nešetřil, and Patel [2016]).

The main property required for the construction is the *amalgamation property*: there must be an amalgamation procedure which extends any diagram

$$f_1, f_2 : A_0 \hookrightarrow A_1, A_2$$

involving embeddings of structures in the given class to an amalgam $\hat{A}$, also in the given class, and embeddings $g_1, g_2 : A_1, A_2 \hookrightarrow \hat{A}$, making a commutative diagram.



The fact that the amalgamation property holds for the class of finite structures embedding into a given homogeneous structure is an elementary but useful fact. In both Parts we will use many explicit amalgamation arguments.

In the context of structures of combinatorial type (e.g., purely relational structures), Fraïssé correlates countable homogeneous structures with *amalgamation classes* of *finite* structures. These classes are characterized by the following properties.

- Closure under isomorphism and substructure;
- Joint embedding: any two embed in a third;
- Amalgamation over an arbitrary base;
- Only countably many isomorphism types are represented.

The last condition is superfluous when the relational language is finite.

The structure associated by Fraïssé's construction to an amalgamation class is called its *Fraïssé limit.* We note some examples of Fraïssé limits, some of which have natural probabilistic constructions.

— Finite linear orders: limit, the rational order;
— Finite graphs or tournaments: limit, the random graph (also called the Rado graph) or tournament;

— Finite ordered graphs or ordered tournaments: limit, the randomly ordered random graph or tournament.
— Finite graphs with no n-clique: limit, the generic K_n-free graph (the Henson graph H_n);
— Finite ordered graphs with no ordered n-clique: limit, the generic $\vec{K}_n$-free graph.
— Pairs of linear orders on a single finite set: limit, the generic permutation;
— All partial orders: limit, the generic partial order;
— All finite structures consisting of a partial order and a linear extension of it: limit, the generic linear extension of a generic partial order;
— All integer valued finite metric spaces: limit, the homogeneous universal connected graph with respect to embeddings preserving the path metric (Urysohn graph)
— All integer valued finite metric spaces containing no triangles of *odd* perimeter: limit, the homogeneous universal connected *bipartite* graph with respect to embeddings preserving the path metric.

A technique introduced by Lachlan and Woodrow makes use of a special kind of induction over amalgamation classes to prove classification theorems. This technique was subsequently modified by Lachlan in a way that brings the Ramsey theorem to bear. This method builds on the Fraïssé theory, and provides a systematic approach to the proof of classification theorems for homogeneous structures, notably those whose associated amalgamation class has some form of free amalgamation. Lachlan's version of this method plays an essential role in Part I.

In Cherlin [1988] we gave a streamlined account of Lachlan's original application of his method to the classification of homogeneous tournaments. This provides a detailed introduction to the method in a context less encumbered by exceptional cases.

For the construction of homogeneous ordered structures, the notion of *strong amalgamation* is useful. Strong amalgamation is a sharper version of the amalgamation property, in which we require that every amalgamation problem $A \to A_1, A_2$ have a completion (or "amalgam") in which the images of A_1 and A_2 are disjoint modulo the base. In other words, at the level of the underlying sets, the amalgamation process should be free amalgamation.

Since the amalgamation procedure at the level of the underlying sets is canonical, this condition allows a number of strong amalgamation classes to be combined into one. In particular, the class of finite linear orders is a strong amalgamation class, and hence any other countable homogeneous structure whose associated amalgamation class has strong amalgamation can be equipped with a *generic linear order,* with the result being unique up to isomorphism.

For similar reasons, any countable homogeneous graph whose amalgamation class has strong amalgamation can be *generically oriented* to give a canonical homogeneous directed graph which when symmetrized gives back the original graph.

Strong amalgamation for an amalgamation class is equivalent to a condition on the Fraïssé limit known as triviality of algebraic closure. We will not elaborate here; see Cameron [1990, (2.15), p. 37]. In fact, we will avoid this terminology, and say in this case that the structure has strong amalgamation—an abuse of language, and a confusion of categories.

In another direction, strong amalgamation also permits the probabilistic representations mentioned above (Petrov and Vershik [2010], Ackerman, Freer, and Patel [2016], Ackerman, Freer, Nešetřil, and Patel [2016]).

Finite homogeneous structures with more than one element cannot have strong amalgamation; and since a finite linear order is rigid, a finite homogeneous structure cannot be expanded to a homogeneous structure in the language with an additional linear order unless it was rigid to begin with.

Being homogeneous, infinite Fraïssé limits of finite structures of combinatorial type tend to have very rich automorphism groups (with some exceptions when the language is infinite). We have noted the example $(\mathbb{Q}, <)$, the Fraïssé limit of finite linear orders, a limit of rigid structures with a rich automorphism group, as well as the example of the random graph, remarked on by Erdős and Rényi (see the Preface).

Some of the graphs we consider in Part II have infinite diameter, and are considered as metric spaces in the path metric. This takes us out of the comfortable setting of finite relational languages, $\aleph_0$-categorical structures, and oligomorphic permutation groups.

Combinatorially, a metric space is an edge labeled complete graph with some constraints. Model theoretically, a metric space is a relational structure with one relation for each possible distance—and with every pair carrying a unique label. As we do not allow unlabeled edges, this class of structures is not axiomatizable.

Fortunately, the Fraïssé theory does not rely on axiomatizability; but one must pay attention to the requirement that there are only countably many finite structures. In the context of metric spaces, it suffices to restrict the values of the metric to a countable set— this is why Urysohn built his universal complete separable metric space as the completion of a rationally valued metric space. As the metric spaces considered here are integer valued, no difficulties arise from this quarter.

1C. The classification of countable homogeneous ordered graphs

Homogeneous ordered graphs are homogeneous ordered tournaments, but ordered homogeneous graphs are not ordered homogeneous tournaments.

One of the motivations for studying homogeneous structures of combinatorial type is that they tend to be associated with classes for which a structural Ramsey theorem holds, or in dynamical terms, structures whose automorphism groups are extremely amenable.

More precisely, these structures tend to have metrizable universal minimal flows, which can typically be realized as the space of expansions of the structure by predicates required for the structural Ramsey theorem to hold. Thus the universal minimal flow for the random graph is the space of its linear orders, and the class of finite ordered graphs has the Ramsey property. The existence of a definable order is necessary for the Ramsey property, and frequently a Ramsey theorem is obtained by adjoining a suitable ordering—but not always, and it is unclear what one can say in general about the required expansion.

In the case of homogeneous graphs or homogeneous directed graphs, the natural expansions to Ramsey classes have been found by a systematic study of all cases. The question naturally arises, and was put by Nguyen Van Thé in conversation in the summer of 2012, whether a systematic classification of homogeneous ordered graphs would uncover any "sporadic" cases not familiar to Ramsey theorists. As it turns out—perhaps surprisingly, given past experience—all homogeneous ordered graphs have been noticed already (though not always explicitly given as ordered graphs). The task of Part I is to prove this.

Our present task is merely to state the result to be proved, in a form that suggests the general structure of the proof.

For our purposes, the categories of ordered graphs and ordered tournaments are interchangeable, as we will now explain.

If a structure Γ carries two binary relations R_1 and R_2, then the structure (Γ, R_1, R_2) is equivalent to the structure (Γ, R_1', R_2) where R_1' is the symmetric difference$R_1 \Delta R_2$. If the second relation R_2 gives Γ the structure of a tournament, then (Γ, R_1') will be a graph if and only if (Γ, R_1) is a tournament. This trivial change of language preserves the automorphism group and takes homogeneous structures to homogeneous structures.

In particular, a linear order on Γ defines a tournament, so with R_2 given by $<$, this transformation gives an equivalence between the classification of homogeneous ordered graphs and the classification of homogeneous ordered tournaments.

Similarly, the replacement of a tournament by its reversal, or a graph by its complement, preserves homogeneity. In translating between ordered tournaments and ordered graphs we will actually combine these two transformations, taking a tournament to the graph complement of the symmetric difference with the order (or, in the other direction, reversing the tournament relation obtained). In more explicit terms, this means that the translation between ordered graphs and tournaments used here is the following.

DEFINITION 1.1.

(a) If $(\Gamma, \to, <)$ is an ordered tournament, then the associated ordered graph $(\Gamma, -, <)$ has edge relation defined by

$$x - y \iff \to \text{ and } < \text{ agree on } x, y$$

(b) If $(\Gamma, -, <)$ is an ordered graph, then the associated ordered tournament $(\Gamma, \to, <)$ has arc relation defined by

$$x \to y \iff x - y \text{ and } x < y, \text{ or } x > y \text{ and } x \not- y$$

.

For example, the cyclic tournament C_3 of order 3 has two ordered forms, $\vec{C}_3^+$ and $\vec{C}_3^-$, while as ordered graphs, these are a path and its complement. Compare Figure 1 below, which shows the translation between a pair of ordered oriented cycles, and the corresponding pair consisting of an ordered path on three vertices and its graph complement. When taken as *constraints* (forbidden structures), either as ordered graphs or as ordered tournaments, they will play an important role in the case division to be presented below. At early stages of the analysis it will be useful to view them as ordered tournaments, and later it will be more convenient to view them as ordered graphs.

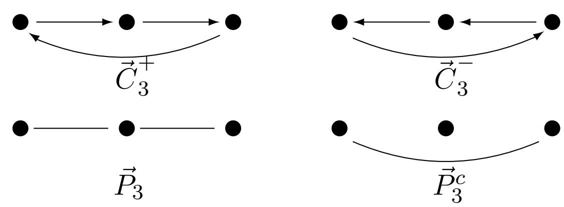


Figure 1. Ordered Tournaments and Ordered Graphs

So for our purposes, homogeneous ordered tournaments may be viewed as homogeneous ordered graphs, and conversely. But in general, ordered homogeneous tournaments are not ordered homogeneous graphs! We next address this point.

A natural way to produce examples of homogeneous ordered graphs is to begin with a homogeneous graph or a homogeneous tournament with strong amalgamation, and then generically add a linear ordering; in terms of Fraïssé theory, we replace a given strong amalgamation class $\mathcal{A}$ by all ordered forms of the structures in $\mathcal{A}$.

The homogeneous ordered graphs arising from homogeneous graphs or homogeneous tournaments in this way are not the same, though there is a little overlap: generically ordering a random tournament or a random graph gives the same structure, up to a change of language.

In addition to these two sources of examples, there is a third and less obvious source of homogeneous ordered graphs. If Γ is a homogeneous partial order, then we may take a generic linear extension of the partial order—again, under the hypothesis that the corresponding amalgamation class has strong amalgamation. The resulting structure will be homogeneous, and may be viewed as an ordered graph. The graph structure is obtained by symmetrizing the partial order, in other words the edge relation is comparability in the ordering; conversely, the partial order is the intersection of the linear order with this edge relation.

We may also pass to the complementary graph. For examples of the first two kinds, this corresponds to taking the complement of the original graph, or the reversal of the original tournament; so this gives nothing new. But if we take complements of examples of the third kind, we get a fourth kind.

The classification theorem states that we have now described all homogeneous ordered graphs.

Theorem 1.2 (Classification of Homogeneous Ordered Graphs). *The countable homogeneous ordered graphs are the following, up to a change of language.*

(a) $(\Gamma, \prec, <)$ *with* $(\Gamma, \prec)$ *a countable homogeneous partial order with strong amalgamation and* $<$ *a generic linear extension of* $\prec$*;*
(b) $(\Gamma, \to, <)$ *with* $(\Gamma, \to)$ *a countable homogeneous tournament with strong amalgamation and* $<$ *a generic linear ordering of it;*
(c) $(\Gamma, -, <)$ *with* $(\Gamma, -)$ *a countable homogeneous graph with strong amalgamation and* $<$ *a generic linear ordering of it.*

We can easily make this statement more explicit, since there are known classification results for countable homogeneous partial orders, graphs, and tournaments. Indeed, we *must* make the statement more explicit in order to prove it, since our method of proof involves an exhaustive treatment of all possibilities.

We then arrive at the catalog shown below as Table 1.1, which is organized according to the natural order of proof, in terms of the complexity of the minimal forbidden structures for the structure.

The *Type* label refers to the four types of structures involved: *EPO* and *EPO*^{c} stand for linear extensions of homogeneous partial orders, and their complementary graphs (once these structures are coded as graphs); *LT* and *LG* stand for generic linear extensions of homogeneous tournaments or graphs, with the ambiguous case noted. Subscripts on *EPO* refer to the particular partial order involved—details are given in the text, in §3B. The notations for the minimal constraints in the *Forbidden* column are also described in that section. From the ordered graph theoretic point of view, the symbol $A \perp B$ represents a disjoint sum, with A preceding B.

I	*Graphs Omitting $\vec{I}_2$ or $\vec{K}_2$*		
Label	*Structure*	*Forbidden*	*Type*
I.1	$\lvert\Gamma\rvert = 1$	$\vec{K}_2$, $\vec{I}_2$	Triv
I.2	$(\mathbb{Q}, <) = \vec{K}_\infty$	$\vec{I}_2$	EPO_0, LT, LG
I.2^c	$(\mathbb{Q}, >) = \vec{I}_\infty$	$\vec{K}_2$	EPO_0, LT, LG
II	*Graphs containing $\vec{I}_2$ and $\vec{K}_2$, but not both $\vec{C}_3^+$ and $\vec{C}_3^-$*		
Label	*Structure*	*Forbidden*	*Type*
II.1	$\mathbb{Q}[\mathbb{Q}^{\mathrm{op}}] = \vec{K}_\infty[\vec{I}_\infty]$	$\vec{C}_3^+$,$\vec{I}_1 \perp \vec{K}_2$, and $\vec{K}_2 \perp \vec{I}_1$, $\vec{C}_3^-$	$\mathrm{EPO}_\perp$
II.2	Generic permutation	$\vec{C}_3^+$, $\vec{C}_3^-$	LT
II.3$_n$	$\vec{I}_n * \vec{K}_\infty$ dense, with each class dense ($n \cdot \mathbb{Q}$, shuffled); $n \geq 2$	$\vec{C}_3^+$, $[\vec{I}_1, \vec{I}_2]$, $[\vec{I}_2, \vec{I}_1]$ and $\vec{I}_{n+1}$ (if $n < \infty$) and $\vec{K}_{n+1}$ (if $n < \infty$)	$\mathrm{EPO}_\to$
II.4	$\vec{\mathcal{P}}$=Generic linear extension of generic p.o.	$\vec{C}_3^+$	EPO_g
II.1^c	$\mathbb{Q}^{\mathrm{op}}[\mathbb{Q}] = \vec{I}_\infty[\vec{K}_\infty]$	$\vec{C}_3^+$,$[\vec{I}_1, \vec{I}_2]$, $[\vec{I}_2, \vec{I}_1]$, $\vec{C}_3^-$	$\mathrm{EPO}_\perp^c$
II.3$_n^c$	$\vec{K}_n * \vec{I}_\infty$ dense, with each class dense	$\vec{C}_3^-$, $\vec{I}_1 \perp \vec{K}_2$, $\vec{K}_2 \perp \vec{I}_1$	$\mathrm{EPO}_\to^c$
II.4^c	Reversal (complement) of II.4	$\vec{C}_3^-$	EPO_g^c
III	*Graphs containing both $\vec{C}_3^+$ and $\vec{C}_3^-$*		
Label	*Structure*	*Forbidden*	*Type*
IIIA	$\vec{\mathbb{S}}(2)$ = Generically ordered $S(2)$	$[I_1, C_3]$ and $[C_3, I_1]$ (all ordered forms)	LT
IIIB$_n$	$\vec{H}_n$ = Generically ordered Henson graph ($n < \infty$)	$\vec{K}_{n+1}$	LG
IIIB$_n^c$	$\vec{H}_n^c$	$\vec{I}_{n+1}$	LG
IIIC	$\vec{\Gamma}_\infty$ = Generically ordered random graph	none	LT, LG

TABLE 1.1. HOMOGENEOUS ORDERED GRAPHS

We remark that our notational conventions thoroughly mix notations for ordered tournaments ($\vec{C}_3^{\pm}$) with notation for ordered graphs ($\vec{I}_1 \perp \vec{K}_2$, $[\vec{I}_1, \vec{I}_2]$, etc.), In particular, even when we adopt the point of view of ordered tournaments, the illustrations are more legible when presented as ordered graphs, because fewer edges are required (the order, left-to-right, is implicit).

One remarkable byproduct of our analysis is that the classification of countable homogeneous tournaments is closely related to the classification of countable homogeneous graphs. The main point in the classification of homogeneous tournaments is the fact that a countable homogeneous tournament which is not a local order is the random tournament. If we restrict our attention to homogeneous structures corresponding to strong amalgamation classes, this may be proved as follows: add a linear order generically and then see that we fall under one of the cases $(IIIB, C)$ in Group (III). By inspection, one may show that the only one of these with a homogeneous tournament underlying it is the generically ordered random graph, and that the relevant tournament is the random tournament.

We will describe the proof of this classification theorem in more detail in §2A. But everything in groups (I, II) is either covered by Dolinka and Mašulović [2012], or is reducible to it by passing to the complement. The odd-looking entry at $(II.2)$, the generic permutation, is most naturally thought of as the generic linear extension of the tournament $(\mathbb{Q}, <)$, and is classified as type LT. But it happens to be a linear extension of a non-homogeneous partial order (namely, the intersection of the two orders) and as such is picked up by Dolinka and Mašulović [2012].

So the main points are how to organize matters so as to take advantage of the case covered by Dolinka and Mašulović [2012]—via complementation this gives two cases—and how to deal with and to distinguish the two kinds of examples that both fall under Group (III). Indeed, the four cases in the catalog correspond to distinct portions of the analysis, with no overlap between them.

We will return to the discussion of the methods used to classify the homogeneous ordered graphs in §2A. In the remainder of this chapter we will describe the results to be obtained in Part II on the classification of countable metrically homogeneous graphs. This requires a lengthier presentation, spanning four sections.

1D. Countable metrically homogeneous graphs of known type: a catalog and a conjecture

The main goals of Part II are to present a conjecture on the classification of the countable *metrically homogeneous graphs* and some supporting evidence. In this section we discuss the conjecture, and in the following sections, the supporting evidence. Then in the final section of this chapter we will describe a body of results of a more general character on which our more concrete results depend, which fall under the heading of *local analysis.*

The main conjecture may be stated as follows, using some notation which requires elucidation.

Conjecture 1 (Metric Homogeneity Classification Conjecture). The countable metrically homogeneous graphs are the following.

1. In diameter $\delta \leq 2$: the connected homogeneous graphs, classified by Lachlan and Woodrow Lachlan and Woodrow [1980]; Fact 1.4.
2. In diameter $\delta \geq 3$:
   (a) The finite ones, classified by Cameron Cameron [1980]; Fact 1.7.
   (b) Macpherson's regular tree-like graphs $T_{m,n}$ with $m, n \leq \infty$, $m, n \geq 2$; §1D.3.
   (c) The Fraïssé limits of amalgamation classes of the form
   $$\mathcal{A}_3 \cap \mathcal{A}_H$$
   with $\mathcal{A}_3$ 3-constrained and $\mathcal{A}_H$ of Henson type or antipodal Henson type; §§2B, 1D.5.

This formulation is compact, but also unintelligible without further explanation. We will make the conjecture completely explicit on a line-by-line basis. It will turn out that the meaning of clauses (1) and ($2c$) is essential throughout Part II, while the meaning of clauses ($2a, b$) is entirely irrelevant.

The Lachlan/Woodrow classification remains important throughout, because we can use it to formulate a simple notion of metrically homogeneous graph of *generic type* (Definition 1.17), and then prove a classification theorem for all metrically homogeneous graphs which are not of generic type—these are the graphs of types ($2a, b$) and some of the graphs of type (1). The non-generic classification was completed in Cherlin [2011], and the upshot is that our classification conjecture becomes the following (as we will explain in more detail below).

Conjecture 2. (Metric Homogeneity Classification Conjecture for Generic Type). The countable metrically homogeneous graphs of generic type are the Fraïssé limits of amalgamation classes of the form

$$\mathcal{A}_3 \cap \mathcal{A}_H$$

with $\mathcal{A}_3$ 3-constrained, and with $\mathcal{A}_H$ of Henson type or antipodal Henson type.

So once we state precisely what we mean by the terms *generic type* and *3-constrained* (Definition 1.10), and the two forms of *Henson type,* (given by Henson constraints in the sense of Definition 1.14) we will know precisely what the subject matter of Part II is.

The rest of the present section is devoted to elucidation of the following points.

— The classification of homogeneous graphs.
— The classification of finite metrically homogeneous graphs.
— Macpherson's regular tree-like graphs.
— 3-constrained amalgamation classes.
— Henson constraints and antipodal Henson constraints.
— The notion of generic type.

Once these points are dealt with, we will clarify how, exactly, Conjecture 1 reduces to Conjecture 2.

But before going into this, we should clarify one further point of language. A metrically homogeneous graph can be viewed either as a graph or as a metric space. On a technical level we usually need to view these structures as metric spaces (since that is what is required to formulate and to apply the homogeneity condition). But since these structures can also be viewed as graphs, we use graph theoretic terminology freely as well, with an eye out for the occasional conflicts of language that might result.

In particular, given a metrically homogeneous graph—as a graph—we immediately view it as a metric space in the path metric. The converse direction is also important: the edges of the graph are defined metrically by the condition

$$d(x, y) = 1.$$

In particular, in the construction of metrically homogeneous graphs alluded to above, the Fraïssé theory is applied to classes of finite metric spaces and yields a homogeneous integer valued metric space. But we are of course claiming more: namely, that the underlying graph with edge relation "$d(x, y) = 1$" has the given metric as its path metric, and is therefore a metrically homogeneous graph. For this, the following is critical, but elementary. It is given in greater generality, in a form applying to connected distance transitive graphs, in Cameron [1998, Prop. 5.1].

Fact 1.3. Let Γ be a homogeneous integer valued metric space of diameter δ (possibly infinite), and let Γ^g be the graph on the points of Γ with edges given by the relation "$d(x, y) = 1$." Then the following are equivalent.

1. The graph Γ^g is metrically homogeneous.
2. For any finite $d \leq \delta$ there is a geodesic path $(a_0, \ldots, a_d)$ of length d (i.e., a path in Γ^g with $d(a_0, a_d) = d$ in Γ).
3. The metric on Γ is the path metric associated to the graph Γ^g.

What matters for us here is the equivalence of the first two points; for us, the third point is more properly part of the proof.

1D.1. Homogeneous graphs and the graph Γ_1. The finite homogeneous graphs were classified in Sheehan [1974], Gardiner [1976], and the infinite ones in Lachlan and Woodrow [1980]. We refer to the full classification as the Lachlan/Woodrow classification.

Fact 1.4. A countable homogeneous graph falls into one of the following categories.

(a) Degenerate: complete or edgeless.
(b) Imprimitive: a disjoint union of at least two complete graphs of fixed size, or the complement of such a graph, which is complete multipartite.
(c) Finite, primitive, and nondegenerate: a pentagon or a certain self-complementary graph $K_3 \square K_3$ of order 9 (called the *cartesian product* in graph theoretic terminology), the latter representable also as the 2-dimensional vector space $\mathbb{F}_3^2$ equipped with an anisotropic quadratic form Q and edge relation $Q(x-y) = 1$.[6] Another description (in graph theoretic terminology) of $K_3 \square K_3$

[6] If $\mathbb{F}_3$ is replaced by any finite field, this remains homogeneous for a finite binary relational language—but not in the language of graphs

which is of some interest is as the line graph of the complete bipartite graph $K_{3,3}$; here it is interesting that the line graph of $K_{n,n}$ is not homogeneous in any binary relational language for $n > 3$.

(d) Infinite, primitive, and nondegenerate: a Henson graph H_n (the generic graph omitting an n-clique) with $n \geq 3$, or its complement; or the random graph, which is self-complementary.

A metrically homogeneous graph of diameter at most 2 is a homogeneous graph (and the converse holds for connected graphs); so the problem remains of classifying the countable metrically homogeneous graphs of diameter $\delta \geq 3$. The Lachlan/Woodrow result remains highly relevant also in this case, because of the following.

Definition 1.5. Let Γ be a countable metrically homogeneous graph, v_* a basepoint, and let $\Gamma_1 = \Gamma_1(v_*)$ be the metric space induced on the set of neighbors of v_* in Γ.

This is well-defined, up to isomorphism, independent of the basepoint.

If Γ is a countable metrically homogeneous graph, then Γ_1 is a homogeneous metric space with distances $1, 2$, and may also be viewed as a homogeneous graph with edges given by $d(x, y) = 1$. This is worth recording in the following form.

Lemma 1.6. *If Γ is a countable metrically homogeneous graph, then Γ_1 has of one of the following forms.*

(a) Finite or imprimitive;

(b) Infinite, complete or edgeless;

(c) A Henson graph H_n or its complement H_n^c, with $3 \leq n < \infty$

(d) The random graph G_∞.

This gives us an important invariant to consider as a point of departure for any analysis, namely, *the isomorphism type of* Γ_1, and as we go through the list of metrically homogeneous graphs of known type, we should consider what values this invariant takes on.

1D.2. Finite metrically homogeneous graphs. The finite metrically homogeneous graphs were classified in Cameron [1980].

Fact 1.7. The finite metrically homogeneous graphs of diameter at least 3 are of the following two forms.

1. An n-cycle with $n \geq 6$ ($\delta = \lfloor n \rfloor$).

2. Diameter 3: An antipodal double cover of one of the graphs
   (a) C_5,
   (b) $K_3 \square K_3$, or
   (c) I_n (an independent set of order $n \geq 2$).

The antipodal double cover of a graph G is defined as follows. Take two copies of G, say G and G', and for $v \in G$ let v' denote the corresponding element of G'. Add two vertices $*, *'$ and form a graph $\Gamma(G)$ on the vertex set

$$G \cup G' \cup \{*, *'\}$$

with the following edge relation.

— The induced graphs on G and G' are the given copies of G.
— The neighbors of $*$ are the vertices of G; for $*'$, the neighbors are the vertices of G'.
— For $g \in G$, $h' \in G'$, we have g adjacent to h' iff (g, h) is a non-edge with $g \neq h$.

These graphs are called *antipodal* because each vertex v is paired with a unique vertex v' at distance 3. In addition, they satisfy the following "distance-reversing" condition.

$$d(u, v') = 3 - d(u, v)$$

Notice that the antipodal double cover of I_n is the bipartite complement of a perfect matching between two sets of order $n + 1$.

For our purposes, antipodality is defined in general as follows.

DEFINITION 1.8. A graph Γ of finite diameter $\delta \geq 3$ will be called *antipodal* if for each vertex v there is a unique vertex v' satisfying

$$d(v, v') = \delta$$

This definition would also be meaningful in diameter 2 but not very helpful there. The related notion of antipodality which is adopted in the theory of finite distance transitive graphs is broader than the one we use here.

As we have mentioned, we will be paying attention to the structure of Γ_1 as we proceed. In the present case this reads as follows.

Structure of Γ_1: finite, and in case (2), isomorphic to G.

1D.3. Tree-like graphs $T_{m,n}$.

DEFINITION 1.9. For $2 \leq m, n \leq \infty$, the graph $T_{m,n}$ is a regular tree-like graph in which the blocks are cliques of order n and every vertex is a cut vertex, lying in precisely m blocks.

Alternatively, consider the bi-regular tree $T(m, n)$ in which vertices have degree m or n, alternately, and consider the vertices of degree m[7] with edge relation

$$d(x, y) = 2$$

This is a concrete realization of $T_{m,n}$. The vertices of degree n then represent the blocks.

Using the fact that the tree $T(m, n)$ is homogeneous when equipped with a labeled partition (A, B) into vertices of degree m and n respectively, one can show that A is homogeneous in the induced metric, and after rescaling by a factor of $1/2$, A becomes $T_{m,n}$ in the path metric. Thus $T_{m,n}$ is metrically homogeneous.

These graphs have infinite diameter. We will argue later that our classification conjecture can be reduced to the case of finite diameter.

With $\Gamma = T_{m,n}$, we find that $\Gamma_1 = m \cdot K_{n-1}$, where $\infty - 1 = \infty$. Thus Γ_1 tends to be imprimitive, except when $n = 2$ and $T_{m,n}$ is a regular tree, in which case Γ_1 is an independent set of order m.

If Γ_1 is an infinite independent set, then it provides an excessively weak invariant, as it does not distinguish the tree $T_{\infty,2}$ from more typical metrically homogeneous graphs encountered as Fraïssé limits elsewhere in the catalog. Thus in the definition of *generic type* below (Definition 1.17), we will need to look beyond the structure of Γ_1.

1D.4. 3-constrained amalgamation classes. A *constraint* (or *forbidden subspace*) for a metric space Γ is a finite metric space A which does not embed isometrically into Γ. In particular, a *minimal constraint* is a constraint A for which every proper subspace of A *does* embed into Γ.

According to the Fraïssé theory, a countable homogeneous metric space is determined up to isomorphism by the isomorphism types of its minimal constraints; and in practice, this is a useful way to describe such spaces.

[7] If $m = n$, one should say more properly, of the first kind.

At the level of amalgamation classes $\mathcal{A}$, the conditions are that $A \notin \mathcal{A}$, and $B \in \mathcal{A}$ for B any proper induced subspace of A.

DEFINITION 1.10. A metrically homogeneous graph is *k-constrained* if all of its minimal constraints have order at most k; similarly for amalgamation classes.

The 3-constrained amalgamation classes play a major role in the classification conjecture. We say that they are determined by *forbidden triangles.* This terminology neglects the point that a bound on the diameter involves constraints of order 2. But we generally treat the diameter as fixed in advance.

Every triple of points in a metrically homogeneous graph is a triangle in the *metric* sense. The *type* of a triangle (a, b, c) is the triple of distances (i, j, k) occurring between its vertices (taken in any order). A triangle in the *graph theoretic* sense is a metric triangle of type $(1, 1, 1)$. The term "triangle" will customarily refer to metric triangles here.

As a metrically homogeneous graph Γ is connected, it follows that every geodesic triangle (i.e., metric triangle with the longest edge length the sum of the other edge lengths) whose diameter is at most the diameter δ of Γ embeds into Γ. Therefore the only triangle constraint relevant to the structure of Γ_1 is the triangle of type $(1, 1, 1)$. In particular, for a 3-constrained metrically homogeneous graph, either Γ_1 is the random graph, or it is an independent set.

While our formulation of the classification conjecture in terms of the notion of 3-constraint is efficient, it leaves aside the question as to what metrically homogeneous 3-constrained graphs actually exist. We will settle this point. This allows us to present a completely explicit version of the classification conjecture, which however requires considerably more space to lay out.

We will have a good deal more to say about this in the present chapter, and throughout much of Part II.

1D.5. Henson constraints and antipodal variants. The Henson graphs H_n are the generic K_n-free graphs; and their complements are the generic I_n-free graphs, with I_n an independent set.

We will generalize this concept and arrive at a notion of Henson metrically homogeneous graph. We discuss the motivation first.

The Henson graphs are associated with notions of free amalgamation. Free amalgamation of graphs is usually taken to mean free join with no added edges. In metric terms this means that pairs of points not in one of the two factors of the amalgamation are given distance 2. As the numbers 1,2 play no distinguished role here, one can equally well consider the complementary notion of free amalgamation, where such pairs are given distance 1.

In the first case, if we forbid an n-clique in the factors, it will not appear in the free amalgam. In the second case, if we forbid an independent set of size n in the factors, it will not appear in the free (or "anti-free") amalgam.

Accordingly we will call an n-clique, or an independent set of size n, a Henson constraint (relative to one or the other notion of free amalgamation of graphs); and a Henson graph is the generic graph associated with a free amalgamation class and an associated Henson constraint.

Here we work with metric variants of these graph theoretic notions. A metric clique K_n is a set of points with mutual distance 1, while a metric independent set I_n is a set of points at mutual distance 2. But when we pass to diameter $\delta \geq 3$, we replace the notion of independent set by the notion of *anticlique* or *δ-clique,* namely a set of points mutually at distance δ.

More generally, when we have $\delta \geq 3$, the situation becomes more symmetrical and we will also consider finite metric spaces in which both distances 1 and δ may occur, but no others, as Henson constraints. Namely, once we take $\delta \geq 3$, most 3-constrained amalgamation classes have an amalgamation procedure which avoids creating new pairs at either of the extreme distances 1 or δ, and so in such case we may forbid an arbitrary set of finite $(1, \delta)$-spaces. In the simplest case, for $\delta = 3$, we may amalgamate by giving all pairs not lying in one of the factors the distance 2, and we again have a notion of free amalgamation. On the other hand, if $\delta \geq 4$, then the triangle inequality forces more care to be taken, but one may still avoid the extreme values of the metric. Further variations on the notion of Henson constraint occur in the antipodal case, of a slightly more technical nature, but a very similar character.

Now we give the precise definitions.

Definition 1.11. Suppose $\delta \geq 3$.

A $(1, \delta)$-*space* is a metric space in which all distances are 1 or δ; thus the relation $d(x, y) \leq 1$ is an equivalence relation, and the classes lie at mutual distance δ.

A $(1,\delta)$-space will also be called a *Henson constraint* (or more precisely, an *ordinary* Henson constraint).

Fact 1.12. For any $\delta \geq 3$ and any set $\mathcal{S}$ of $(1,\delta)$-spaces, the collection $\mathcal{A}_{\mathcal{S}}$ of finite $\mathcal{S}$-free metric spaces of diameter at most δ is an amalgamation class.

It is enough to check that any amalgamation problem in the category of finite metric spaces of diameter δ has a solution in which no additional points are given distance 1 or δ. This is easily checked, and is discussed in more detail in Cherlin [2011].

Thus, we do not encounter any difficulties in understanding the range of possibilities afforded by the use of Henson constraints. When we combine this construction with a set of triangle constraints, the issue must be reexamined, but again it turns out to be straightforward.

There is another way to impose Henson constraints which applies only when the graph in question is antipodal.

DEFINITION 1.13. Let $\delta \geq 4$.

1. For A a finite metric space of diameter at most δ, an *antipodal companion* of A is a metric space A^* obtained by replacing a subset B of A by the set $B' = \{b' \mid b \in B\}$ in the following way:

— $A^* = (A \setminus B) \cup B'$
— The metric on $A \setminus B$ is the metric in A.
— B' is isometric with B under $b' \mapsto b$.
— $d(a,b') = \delta - d(a,b)$ for $a \in A \setminus B$ and $b \in B$.

2. The class

$$\mathcal{A}^{\delta}_{a,n}$$

is defined as the set of finite metric spaces A of diameter δ which satisfy the *antipodal law*

$$d(a,b) = \delta \Longrightarrow d(b,x) = \delta - d(a,x) \text{ for all } x$$

and the following *antipodal Henson constraint* of order n:

$$A \text{ contains no copy of an antipodal companion of } K_n.$$

This constraint forbids certain $(1,\delta-1)$-spaces.

Here the subscript a stands for the antipodal law. In terms of numerical parameters, one can show that for metrically homogeneous graphs the antipodal law is equivalent to the conditions $C = 2\delta + 1$, $C' = 2\delta + 2$. The point of the special notation used here is that the notion of Henson constraint differs in this case from the usual notion.

For reference, we record here the two definitions of Henson constraint.

DEFINITION 1.14. For $\delta \geq 3$, a δ-Henson constraint of ordinary type is a finite $(1, \delta)$-space.

For $\delta \geq 4$, a δ-Henson constraint of antipodal type is an antipodal companion of a finite 1-clique.

These are referred to also as ordinary δ-Henson constraints and antipodal δ-Henson constraints. When the diameter δ is fixed, reference to δ may be omitted.

The antipodal companions of K_n have the form K_{n_1,n_2}, consisting of two cliques of orders n_1, n_2 with $n_1 + n_2 = n$, at distance $\delta - 1$. In particular they are $(1, \delta - 1)$-spaces rather than $(1, \delta)$-spaces.

It was shown in Cherlin [2011, Theorem 14] that the class $\mathcal{A}^\delta_{a,n}$ is an amalgamation class for $\delta \geq 4$.

DEFINITION 1.15. The Fraïssé limit of the class $\mathcal{A}^\delta_{a,n}$ is denoted

$$\Gamma^\delta_{a,n}$$

Again, $\Gamma^\delta_{a,n}$ is defined as a homogeneous metric space, but may be viewed as a metrically homogeneous graph.

The only influence of Henson constraints or antipodal Henson constraints on the invariant Γ_1 arises when a clique K_{n+1} occurs among the minimal constraints for Γ. Then Γ_1 is the corresponding Henson graph H_n.

This is a point well worth recording.

Remark 1.16. Let Γ be a countable metrically homogeneous graph which when viewed as a metric space is the Fraïssé limit of a class of the form

$$\mathcal{A}_3 \cap \mathcal{A}_H$$

with $\mathcal{A}_3$ 3-constrained and $\mathcal{A}_H$ governed by Henson constraints or antipodal Henson constraints. Then Γ_1 is either an independent set, a Henson graph, or the random graph.

Note the symmetry-breaking: the complement H_n^c of a Henson graph H_n does not occur here.

1D.6. Generic type. We have now encountered all of the known types of metrically homogeneous graphs, though in the case of 3-constrained classes our description is still formal and vague at this point. We have also taken note of the associated invariants of the form Γ_1 arising.

A review of these graphs shows that the various special cases which arise have Γ_1 imprimitive, finite, complete, or an infinite independent set, the last when Γ is a regular tree of infinite degree, while the Fraïssé limits of classes

$$\mathcal{A}_3 \cap \mathcal{A}_H$$

have Γ_1 a Henson graph, a random graph, or an infinite independent set.

We have settled on the following as a clean (and useful) way of separating the special cases from the rest.

DEFINITION 1.17. Let Γ be a countable metrically homogeneous graph of diameter $\delta \geq 2$.

1. The *local type* of Γ is the isomorphism type of Γ_1.
2. Γ is of *exceptional local type* if Γ_1 is finite, imprimitive, or complete.
3. Γ is of *generic type* if Γ_1 is primitive, and for any vertex v at distance 2 from the basepoint, the set of neighbors of v in Γ_1 contains an infinite independent set. In particular, Γ_1 contains an infinite independent set.

This definition is justified by the following result, which builds on a body of prior work. We rely on the Lachlan/Woodrow classification to read off the situation in diameter 2.

Fact 1.18 (Cherlin [2011, Theorem 10, Lemma 8.6]). Suppose that Γ is a countable metrically homogeneous graph (in particular, Γ is connected). Then one of the following applies.

(a) Γ is finite.
(b) Γ is complete multipartite with at least two classes (possibly complete).[8]
(c) Γ is the complement H_n^c of a Henson graph with $3 \leq n < \infty$.
(d) Γ is one of the tree-like graphs $T_{m,n}$.
(e) Γ is of generic type.

[8]The edgeless case is excluded as we consider only connected graphs.

In particular, if Γ is infinite and of finite diameter $\delta \geq 3$, then Γ is of generic type.

This statement includes the claim that Γ_1 cannot be the complement of a Henson graph when $\delta \geq 3$, which is a point that is established separately along the way.

Corollary 1.18.1. Conjecture 1 is equivalent to Conjecture 2.

Thus we are concerned here with understanding the case of generic type.

As we shall see in Chapter 15, the definition of generic type is precisely what we need to begin the higher level "local analysis" of metrically homogeneous graphs, which involves the study of the metric spaces Γ_i induced on the set of vertices at fixed distance i from the basepoint, for all $i \leq \delta$—mainly in the case in which Γ_i contains at least one edge of Γ.

1E. Part II: 3-Constraint

We take up trigonometry

In Part II we will take the first steps in the direction of a proof of the conjectured classification of countable metrically homogeneous graphs, or so we hope. Rather late in the development of this monograph, that hope became substantially more concrete, a point we leave for the preface and Appendix 18B.1, both of which were put in final form substantially later than the bulk of the text.

Another way to view the results given here would be as due diligence: we examine some of the points where examples missed in the construction of the catalog might be suspected to lurk, and we find nothing unexpected. One of the steps in this "due diligence" is the classification in the 3-constrained case, and in fact we did find some unexpected items there, but early enough to include them in our first published catalog. Thus on the first pass our "due diligence" led to an improved version of the conjecture, and since then nothing unexpected has turned up.

The main points to be established in Part II are the following.

(I) The precise description of the 3-constrained metrically homogeneous graphs (Chapters 12, 13, and 14).

(II) The classification of the bipartite metrically homogeneous graphs and the metrically homogeneous graphs of infinite diameter, under a suitable inductive hypothesis (Chapters 16 and 17).

(III) Some fundamental results of *local analysis* (Chapter 15).

The first step toward the classification of 3-constrained metrically homogeneous graphs is the explicit definition of a family of such graphs as Fraïssé limits of certain amalgamation classes depending on five numerical parameters $(\delta, K_1, K_2, C_0, C_1)$, and denoted accordingly $\Gamma^\delta_{K_1,K_2,C_0,C_1}$ (for the graphs) or $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ (for the amalgamation classes). This definition will be given shortly. The first point is the following.

THEOREM 1.19. *Let $\mathcal{A}$ be a 3-constrained amalgamation class of integer metric spaces, corresponding to some countable metrically homogeneous graph of generic type. Then $\mathcal{A} = \mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ for some values of the numerical parameters*

$$\delta, K_1, K_2, C_0, C_1,$$

with C_0 even and C_1 odd.

The precise description of the 3-constrained metrically homogeneous graphs is fairly complicated, and the theorem does not say that these parameters can be chosen randomly; this formulation leaves entirely open the question as to which of the classes $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ are amalgamation classes, and only in such cases can we speak of the Fraïssé limit $\Gamma^\delta_{K_1,K_2,C_0,C_1}$. Given the possibility of mixing triangle constraints with Henson constraints, to put this in a more satisfactory form we must solve both of the following problems.

Problem.

(I) Determine the 3-constrained amalgamation classes;

(II) Determine the pairs $\mathcal{A}_3$, $\mathcal{A}_H$ with $\mathcal{A}_3$ a 3-constrained amalgamation class and $\mathcal{A}_H$ a Henson or antipodal Henson amalgamation class, so that

$$\mathcal{A}_3 \cap \mathcal{A}_H$$

is an amalgamation class.

Fortunately, while the solution to the first part is complicated, the solution to the second part of the problem follows quickly from the solution to the first part. In fact, this is the point of Henson constraints—they allow an existing amalgamation procedure to be applied without further adjustment.

Now we pass to the main technical point, the explicit definition of the classes

$$\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$$

in terms of the given numerical parameters. (Afterward, we will provide a variety of concrete examples.)

This 5-parameter family began life as a 2-parameter family in Cameron [1998], with one parameter the diameter, and the other parameter suggested by one of the results from Komjáth, Mekler, and Pach [1988]. Taking into account another parameter suggested by reading farther in Komjáth, Mekler, and Pach [1988], I thought this 2-parameter family should really be a 3-parameter family, and my initial formulation of the classification conjecture (unpublished) took that form. That formulation turned out to be wrong, and after further mitosis I arrived at a 5-parameter family which covers all possibilities. Since one parameter, the diameter δ, is just along for the ride, the phylogenetic branching is not so much $2 \mapsto 3 \mapsto 5$ as $1 \mapsto 2 \mapsto 4$.

The 3-parameter version of the family was denoted

$$\Gamma^{\delta}_{K,C}$$

while the 5-parameter version of the family is denoted

$$\Gamma^{\delta}_{K_1,K_2,C_0,C_1}.$$

As we know, the parameter δ controls the diameter. The 4 remaining parameters control the set of forbidden triangles

$$\mathcal{T}(\delta, K_1, K_2, C_0, C_1)$$

(among all metric triangles of diameter at most δ, with integral sides). The corresponding class of structures

$$\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$$

consists of all finite integral metric spaces of diameter at most δ in which the forbidden triangles (those in $\mathcal{T}(\delta, K_1, K_2, C_0, C_1)$) do not embed. This may or may not be an amalgamation class. When it is an amalgamation class, then the Fraïssé limit may be called

$$\Gamma^{\delta}_{K_1,K_2,C_0,C_1}.$$

There are four points to deal with.

1. What is the set $\mathcal{T}(\delta, K_1, K_2, C_0, C_1)$ corresponding to a given choice of parameters?
2. What are the relevant values of the numerical parameters?
3. What are the numerical conditions on these parameters ensuring that $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ has the amalgamation property, and thus that $\Gamma^\delta_{K_1,K_2,C_0,C_1}$ exists (at least, as a metric space)?
4. Is this metric space, given as a Fraïssé limit, in fact the metric space associated to its underlying graph, with edges defined as usual by "$d(x, y) = 1$"?

The fourth point is dealt with by invoking Fact 1.3 and observing that all metric triangles embedding into a geodesic path in the sense of that fact are themselves geodesic triangles, and are not among the forbidden triangles in the set $\mathcal{T}(\delta, K_1, K_2, C_0, C_1)$. In fact it will be seen that no triangle of even diameter at most 2δ is forbidden.

So we now turn to the answers to the first three questions, and in particular we place ourselves in the context of metric spaces (and mainly finite metric spaces).

DEFINITION 1.20. Let $(\delta, K_1, K_2, C_0, C_1)$ be given. Then

$$\mathcal{T}(\delta, K_1, K_2, C_0, C_1)$$

is the set of triangles whose edge lengths (i, j, k) satisfy one of the following conditions, writing $p = i + j + k$ for the perimeter of the triangle.

$$\begin{array}{ll} p < 2K_1 \text{ and } p \text{ is odd} & p > 2K_2 + 2\min(i, j, k) \text{ and } p \text{ is odd} \\ p \geq C_0 \text{ and } p \text{ is even} & p \geq C_1 \text{ and } p \text{ is odd} \end{array}$$

In other words, a finite integral metric space A of diameter at most δ will be in the class $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ if and only if all of its triangles have types (i, j, k) satisfying the following conditions, where $p = i + j + k$.

$$\begin{array}{ll} p \geq 2K_1 + 1 \text{ if } p \text{ is odd;} & p < 2K_2 + 2\min(i, j, k) \text{ if } p \text{ is odd;} \\ p < C_0 \text{ if } p \text{ is even;} & p < C_1 \text{ if } p \text{ is odd.} \end{array}$$

Note that with p odd, the alternative $p = 2K_2 + 2\min(i, j, k)$ is excluded.

This definition only makes good sense for certain parameter sequences

$$(\delta, K_1, K_2, C_0, C_1),$$

called *acceptable sequences.*

This definition always struck me as a little puzzling (notably the way the parameter K_2 involves not just the perimeters but all the side lengths). This point has been somewhat elucidated by combinatorial investigations since then, offering at least two reasonable interpretations of these conditions (Hubička, Konečný, and Nešetřil [2020b], Hubička, Kompatscher, and Konečný [2018], Hubička, Konečný, and Nešetřil [2020a]). As we will see, the parameter K_1 controls the length of the shortest cycle of odd length. The other parameters can also be interpreted naturally in terms of forbidden cycles, and some of the inequalities become more transparent. But all of this—apart from the meaning of K_1–lies outside the scope of our discussion here.

DEFINITION 1.21. A sequence of parameters $(\delta, K_1, K_2, C_0, C_1)$ is *acceptable* if it satisfies the following conditions.

- $3 \leq \delta \leq \infty$;
- $1 \leq K_1 \leq K_2 \leq \delta$, or $K_1 = \infty$ and $K_2 = 0$;
- $2\delta < C_0, C_1$ and $C_0, C_1 \leq 3\delta + 2$; here C_0 is even, and C_1 is odd;
- If $K_1 = \infty$ (the bipartite case) then $C_1 = 2\delta + 1$.

We could allow $\delta = 2$ as well, but this would bring more exceptional cases into the theory.

The conditions on K_1, K_2 are transparent, except for the possibility $K_1 = \infty$ and $K_2 = 0$, which we use to specify the bipartite case (no triangles of odd perimeter). Another way to code this case would be with $K_1 = \delta + 1$ and $K_2 = 0$.

The upper bounds on C_0, C_1 refer to the fact that the largest possible perimeter is 3δ, and that we need to have values of C_0 or C_1 available which correspond to the absence of the corresponding restriction on the perimeter. The lower bound reflects the fact (discussed later) that geodesic triangles can have any even perimeter up to 2δ, while the possibilities for triangles of small odd perimeter are controlled in another way by K_1 and K_2.

With these notions in hand, we can address the main issue: among the acceptable sequences, which ones correspond to amalgmation classes?

THEOREM 1.22 (Admissibility). *Let* $(\delta, K_1, K_2, C_0, C_1)$ *be an acceptable sequence of parameters (in particular,* $\delta \geq 3$*). Then the associated class* $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ *is an amalgamation class if and only if one of the following three groups of conditions is satisfied, where we write* C *for* $\min(C_0, C_1)$ *and* C' *for* $\max(C_0, C_1)$.

(I) $K_1 = \infty$ *(the bipartite case; so* $K_2 = 0$ *and* $C_1 = 2\delta + 1$*).*

(II) $K_1 < \infty$, $C \leq 2\delta + K_1$ *and*

— $C = 2K_1 + 2K_2 + 1$;

— $K_1 + K_2 \geq \delta$; *and*

— $K_1+2K_2 \leq 2\delta-1$. *Furthermore, one of the following applies.*

(IIA) $C' = C + 1$ *or*

(IIB) $C' > C + 1$, $K_1 = K_2$, *and* $3K_2 = 2\delta - 1$.

(III) $K_1 < \infty$, $C > 2\delta + K_1$ *and*

— $K_1 + 2K_2 \geq 2\delta - 1$ *and* $3K_2 \geq 2\delta$;

— *if* $K_1 + 2K_2 = 2\delta - 1$ *then* $C \geq 2\delta + K_1 + 2$;

— *if* $C' > C + 1$ *then* $C \geq 2\delta + K_2$.

A sequence of acceptable parameters $(\delta, K_1, K_2, C_0, C_1)$ will be called *admissible* iff it satisfies one of the three sets of conditions enumerated in Theorem 1.22.

Now it may be useful to see some examples of admissible parameters. We begin with a table of examples for the case $\delta = 3$, from Amato, Cherlin, and Macpherson [2021, Table 2], given as Table 1.2. This table includes a specification of Henson constraints in the final column, which may be set aside for the present.

In Table 1.2 we see in type (I) some bipartite graphs with a bound on the perimeters of triangles; that bound may be vacuous. In Type (II) we see very little, just the generic antipodal graph. In particular we miss type (IIB) entirely. Indeed, this type requires

$$\begin{aligned} \delta &\equiv 2 \pmod 3 \\ K_1 = K_2 &= (2\delta - 1)/3 \\ C &= 4K_1 + 1 = 2\delta + K_1 \end{aligned}$$

And for $\delta = 3$ one sees that the upper and lower bounds associated with type (IIA) force $K_1 = 1$, $K_2 = 2$.

Similarly, in type (III) we see no examples with $C' > C + 1$, as this would require both $C \leq 8$ and $C \geq 6 + K_2$, leading quickly to $K_2 = 2$, $K_1 = 1$, and then $C \geq 2\delta + K_1 + 2$ for a contradiction.

Type	Case	K_1	K_2	C	C'	$\mathcal{S}$
Bipartite	(I)	∞	0	7	8 or 10	empty
Antipodal	(II)	1	2	7	$C+1$	empty
Primitive	(III)	1	2	9 or 10	$C+1$	cliques and anticliques
"	(III)	2	2	9 or 10	$C+1$	anticliques
"	(III)	1	3	8, 9, or 10	$C+1$	If $C = 8$ then $\mathcal{S}$ is empty.
"	(III)	2	3	9 or 10	$C+1$	anything not involving K_3
"	(III)	3	3	10	$C+1$	empty

Table 1.2. Admissible parameters for $\delta = 3$

To illustrate the cases with $C' > C + 1$, in both types (II) and (III), we add the minimal examples in larger diameter.

Type	δ	K_1	K_2	C	C'
(IIB)	5	3	3	13	16
(III)	4	1 or 2	3	11	14

We make one further remark on the complexity of the explicit definition of admissibility, from the point of view of logic: logicians may note that this definition consists of a quantifier-free set of conditions in the language of Presburger arithmetic (the theory of $\mathbb{Z}$ as an ordered group, enriched with the function "$x \bmod n$" for each n, separately). We explore this line of thought a little farther, as follows.

Definition 1.23. Let $\mathcal{A}$ be a class of finite structures closed under substructure and isomorphism, and $k \geq 1$.

Then $\mathcal{A}$ is a *k-amalgamation class* iff every amalgamation problem $A_0 \to A_1, A_2$ in $\mathcal{A}$ with $|A_1 \cup A_2| \leq k$ has a completion in $\mathcal{A}$.

Remark 1.24. Suppose that $\mathcal{F}$ is a uniformly definable family of sets of finite metric spaces in the language of Presburger arithmetic; that is, for some finite k, m there is a formula $\mu(\mathbf{d}, \mathbf{n})$ in the language of Presburger arithmetic, with the variables $\mathbf{d} = (d_{i,j} \mid 1 \leq i, j \leq k)$ and $\mathbf{n} = (n_i \mid 1 \leq i \leq m)$, such that

$$\mathcal{F} = \{\mathcal{F}(\mathbf{n}) \mid n_1, \ldots, n_m \geq 0\}$$

where $\mathcal{F}(\mathbf{n})$ is the set of metric spaces A of order at most k such that in some enumeration, the distances $d_{i,j}$ between the i-th and j-th points of A satisfy the condition $\mu(\mathbf{d}, \mathbf{n})$. (E.g., $\mathbf{n} = (\delta, K_1, K_2, C_0, C_1)$ and $\mathcal{F} = \mathcal{T}$.) Correspondingly, let $\mathcal{A}(n_1, \dots, n_m)$ be the class of finite metric spaces defined by forbidding all substructures lying in the constraint set $\mathcal{F}(n_1, \dots, n_m)$.

Then for fixed k, the set of parameter sequences $\mathbf{n}$ for which

$$\mathcal{A}(\mathbf{n}) \text{ is a } k\text{-amalgamation class.}$$

is definable in Presburger arithmetic by some formula $\phi_k(\mathbf{n})$ (namely, write out what this means, quantifying over the distances occurring in the relevant amalgamation diagrams of order at most k).

In particular, the property that $\mathcal{A}(n_1, \dots, n_m)$ is an amalgamation class will be definable in Presburger arithmetic iff for all classes $\mathcal{A}(n_1, \dots, n_m)$ the amalgamation property is equivalent to k-amalgamation.

In our 3-constrained case with $\mathcal{F} = \mathcal{T}$ the analysis gives the bound on k explicitly.

THEOREM 1.25. *For any choice of parameters* $(\delta, K_1, K_2, C_0, C_1)$ *the following conditions are equivalent.*

1. $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ *is an amalgamation class.*
2. $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ *has the* 5*-amalgamation property.*

We deal with the characterization of 3-constrained metrically homogeneous graphs in Chapters 12, 13, and 14, and at the same time we take care of the related issue of characterizing the amalgamation classes of the form

$$\mathcal{A}_3 \cap \mathcal{A}_H$$

with $\mathcal{A}_3$ 3-constrained, and $\mathcal{A}_H$ one of the two possible variations on the Henson construction.

1F. Part II: The bipartite case, and infinite diameter

It remains to discuss the content of Chapters 15–17. We first present the two concrete results of Chapters 16 and 17, and later return to the more foundational material of Chapter 15.

In Chapter 17 we reduce the classification of metrically homogeneous graphs of infinite diameter to the case of finite diameter, in the following sense.

Theorem 1.26. *Let* Γ *be a countable metrically homogeneous graph of infinite diameter and suppose that every countable metrically homogeneous graph of finite diameter is of* known type *(according to the catalog given in §1D). Then* Γ *is of known type.*

Thus, to show that the conjectured classification is in fact correct, it would suffice to prove it for graphs of finite diameter. This opens up the possibility of proceeding inductively throughout the analysis.

Of course, some additional examples may turn up in small finite diameter without drastically altering the character of the conjecture. In that case, the proof of Theorem 1.26 would need to be reexamined, and adapted to a larger context.

The main result of Chapter 16 is a similar reduction theorem for the case of metrically homogeneous bipartite graphs. We would like a similar result for imprimitive metrically homogeneous graphs in general. In the case of distance transitive graphs, the point of departure for the treatment of imprimitive graphs is called *Smith's Theorem.*

Fact 1.27 (cf. Alfuraidan and Hall [2006, Theorem 2.2]). Let Γ be a connected distance transitive graph of diameter $\delta \geq 3$, with vertex degrees at least 3, and let E be a nontrivial congruence of Γ.

1. E is either the relation E_2 defined by "$d(x, y)$ is even," or the relation E_δ defined by "$d(x, y)$ is a multiple of δ" (i.e., 0 or δ, with δ finite).
2. If $E = E_2$, then Γ is bipartite

In the context of distance transitive finite graphs, the second alternative above is called the *antipodal* case. We have defined this term in a more restrictive way, but in the metrically homogeneous context the distinction is moot, in view of the following.

Fact 1.28 (Cherlin [2011, Theorem 11]). Let Γ be a countable metrically homogeneous graph, of diameter $\delta \geq 3$. Suppose that the relation

$$d(x, y) \in \{0, \delta\}$$

defines an equivalence relation on Γ. Then for each vertex $u \in \Gamma$, there is a unique vertex $u' \in \Gamma$ at distance δ from u, and we have the "antipodal law"

$$d(u,v) = \delta - d(u',v) \text{ for } u,v \in \Gamma$$

In particular, the map $u \mapsto u'$ is a central involution in $\mathrm{Aut}(\Gamma)$.

The distinction between primitive and imprimitive cases generally represents a major dividing line in the classification of infinite homogeneous structures of combinatorial type, so one of the issues that one might like to deal with early on is the verification of the classification conjecture in the imprimitive case (under a suitable inductive hypothesis).[9]

In the context of finite distance transitive graphs, there are good inductive approaches to both the bipartite and the antipodal cases. Unfortunately, the reduction used in the antipodal case leaves the category of metrically homogeneous graphs. But the reduction used in the bipartite case is available. This goes as follows

Definition 1.29. Let Γ be a countable metrically homogeneous bipartite graph. Then $B\Gamma$ is the graph induced on either part of Γ by the edge relation

$$d(x,y) = 2$$

The resulting path metric is the induced metric on that part of Γ, rescaled by a factor of $1/2$. Clearly the isomorphism type of $B\Gamma$ is independent of which part is used. As the induced metric structure is again homogeneous, and coincides with the path metric of $B\Gamma$ up to a scale factor, the graph $B\Gamma$ is metrically homogeneous and of diameter at most $\delta/2$.

Theorem 1.30 (Bipartite Classification). *Let Γ be a countable metrically homogeneous bipartite graph and suppose that $B\Gamma$ is of known type (according to the catalog given in §1D). Then Γ is of known type.*

Note that this is a reduction from diameter δ to diameter $\lfloor \delta/2 \rfloor$, if δ is finite, and the result is satisfactory as stated.

On the other hand, if Γ is a bipartite graph of infinite diameter, then we avoid difficulties by invoking Theorem 1.26 in a careful way.

[9] Our current thinking divides the problem up a little differently. See Appendix 18B.1.

Namely, in the proof of Theorem 1.26, the proof first treats the non-bipartite case. Then for Γ bipartite of infinite diameter, rather than tackling the problem directly, first Theorem 1.26 is applied to $B\Gamma$, and then Theorem 1.30 may be applied to Γ to conclude. Thus the two theorems combine to give a satisfactory reduction of all bipartite cases to cases of smaller diameter.

One important ingredient in the proof of Theorem 1.30 is the following, which we quote in a compressed form from Cherlin [2011].

Fact 1.31 (Cherlin [2011, Theorem 13]). Let Γ be a connected, bipartite, and metrically homogeneous graph of generic type, of vertex degree at least 3. Then either $(B\Gamma)_1$ is isomorphic to the random graph, or $B\Gamma$ and Γ are in the catalog, with Γ of diameter at most 5.

We do not have a similar treatment of the antipodal case.

Problem 1. Complete the treatment of the imprimitive case under suitable inductive hypotheses by similar results for antipodal graphs.[10]

There are some good prospects for a direct inductive argument in this case, except in one case: δ even with $K_1 = \delta/2$. To see where the problem lies, we need to discuss *local analysis,* the subject of Chapter 15.

1G. Part II: Toward a proof of the classification conjecture

Before taking up the reductions of the bipartite and infinite diameter cases to smaller diameter in Chapters 16 and 17, we aim in Chapter 15 to provide some broadly useful results applicable in the generic type case. These results concern *local analysis.*

Local analysis refers to the study of the metric space Γ_i induced on the set of vertices of Γ at distance i from a fixed basepoint. This is a homogeneous metric space, and when there are points at distance 1 in Γ_i, then Γ_i inherits a graph structure from Γ.

Until quite recently there was no clear strategy for proving the completeness of the catalog described above. But the catalog itself,

[10]Our current thinking does not make a sharp separation between this special case and the general classification project; see §18B.1.

together with the reduction to the case of generic type and the reduction of infinite diameter to finite diameter, provides the broad framework for an approach to the problem.

Problem 2. Under an appropriate inductive hypothesis, show the following for finite diameter $\delta \geq 3$.

(a) A countable metrically homogeneous graph of generic type and diameter δ has as its set of forbidden triangles one of the collections

$$\mathcal{T}(\delta, K_1, K_2, C_0, C_1)$$

with admissible parameters;

(b) If the forbidden triangles of a countable metrically homogeneous graph of generic type and diameter δ are those in

$$\mathcal{T}(\delta, K_1, K_2, C_0, C_1)$$

then the other minimal constraints are Henson constraints, or (in the antipodal case) antipodal Henson constraints.

This two-step approach has been completed in the case of diameter 3, in Amato, Cherlin, and Macpherson [2021], with the second step the more difficult one by far, in that case.

Local analysis is our point of departure for the treatment of these problems. The classification conjecture implies that Γ_i should carry very useful information about Γ when Γ_i contains an edge, and the goal of local analysis is to verify some of the relevant properties of Γ_i.

For $i < \delta/2$, if Γ_i contains an edge then it is of smaller diameter and turns out to be a metrically homogeneous graph in its own right, so one may assume inductively that Γ_i is of known type. This relies on the following point.

THEOREM 1.32. *Let Γ be a countable metrically homogeneous graph of generic type and of diameter δ, and suppose $i \leq \delta$. Suppose that Γ_i contains an edge. Then Γ_i is a countable metrically homogeneous graph (and, in particular, is connected).*

Furthermore, Γ_i is primitive and of generic type, apart from the following two cases.

1. *$i = \delta$;*
   $K_1 = 1$; $\{C_0, C_1\} = \{2\delta + 2, 2\delta + 3\}$;
   Γ_δ is an infinite complete graph (hence not of generic type).
2. *$\delta = 2i$;*
   Γ is antipodal (hence Γ_i is imprimitive, namely antipodal).

The case in which Γ_δ is complete is not inherently exceptional, but as complete graphs do not fall under our definition of generic type, this case must be listed separately.

For this result to be useful, we need to be able to identify cases in which Γ_i contains an edge. The classification conjecture implies a good deal more than we can prove at present along these lines. But the following is a step in that direction.

Proposition 1.33. Let Γ be a countable metrically homogeneous graph of diameter $\delta \geq 3$. Suppose

$$K_1 \leq 2.$$

Then for $2 \leq i \leq \delta - 1$, Γ_i contains an edge, unless $i = \delta - 1$, $K_1 = 2$, and Γ is antipodal.

This completes our extended presentation of the contents of the present monograph. We had intended to end with a more detailed discussion of the prospects for a general proof (which originally ran into some difficulties already in diameter 4) but in view of recent developments suggesting that the natural line of attack flows out of Amato, Cherlin, and Macpherson [2021], we defer this discussion to the updated Appendix 18 found at the end of this Volume.

We will not summarize the results of Volume II here, leaving that to the first chapter of Volume II, but we have touched on this in the preface.

CHAPTER 2

METHODS

This chapter is devoted to an overview of the methods used to prove the results described in the previous chapter. The details as presented in Parts I and II are lengthy.

In the previous chapter we described the classification of the homogeneous ordered graphs, and then, at considerably greater length, what we know about the classification of the metrically homogeneous graphs. We proceed similarly in this chapter. Thus §2A concerns the methods used in the classification of the homogeneous ordered graphs, and the methods used to analyze metrically homogenous graphs are discussed in the remaining three sections of the chapter. The material in the first section is independent of the remainder. Similarly, the treatments in Parts I and II are self-contained and independent.

2A. Homogeneous ordered graphs vs. ordered homogeneous graphs

The classification of the homogeneous ordered graphs begins with the case completely treated in Dolinka and Mašulović [2012] and ends with an application of Lachlan's Ramsey theoretic argument along lines sketched in a different context in Cherlin [1998, Chapter IV]. Along the way we vary the choice of language used, according to the particular identification we are aiming at: if we are aiming to show we have an expansion of a homogeneous tournament, we use the language of ordered tournaments, and if we are aiming to show that we have an expansion of a homogeneous graph, we switch to the language of ordered graphs; we then have to reread some of our earlier lemmas in the new language.

To begin with, we consider homogeneous ordered graphs which are expansions of a partial order by a linear order extending it, a case treated fully in Dolinka and Mašulović [2012]. We need only settle where exactly this classification fits in to the analysis in general, when viewed as an ordered graph.

In such cases, the initial partial order will be the intersection of the edge relation in the graph with the order. Since we require this relation to be transitive, this means that there should be no path of length 2 occurring as an induced graph, with the midpoint of the path in the middle of the order.

The graph complement of a homogeneous ordered graph is also a homogeneous graph, so we get a second family of examples by taking the graph complements of the ordered graphs associated to linear extensions of partial orders. So this corresponds to forbidding the graph complement of a path of length 2, with an ordering putting the isolated vertex in the middle.

Thus, in view of Dolinka and Mašulović [2012], we begin our analysis with the assumption that these two specific ordered graphs of order 3 do occur in our homogeneous structure. Under this hypothesis, we will see that we have either a generically ordered homogeneous tournament or a generically ordered homogeneous graph (both fall under Group (III) in Table 1.1).

We work first in the language of tournaments.

There are three infinite homogeneous tournaments: the rational order $\mathbb{Q}$, the generic local order $\mathbb{S}$, and the random tournament $\mathbb{T} = T^\infty$. Two of these cases may be set aside: a generically ordered copy of $(\mathbb{Q}, <)$ (the "generic permutation"') already occurs as a linear extension of a partial order (namely, the intersection of the two orders), while the random tournament will reappear later on in the guise of the (ordered) random graph, up to a change of a language. So at this point we aim at a characterization of a generically ordered copy of $\mathbb{S}$ (Chapter 4). Once we dispose of this case, we will find it convenient to return to the language of ordered graphs.

The generic local order $\mathbb{S}$ is characterized, within the class of homogeneous tournaments which are not linearly ordered, by the absence of a certain tournament of order 4, and its dual; so the generically ordered copy of $\mathbb{S}$ should be characterized by the omission of all (or, equivalently—we may hope—one) of the ordered versions of these two constraints.

We will have a lengthy series of lemmas at that point showing that the absence of one particular form of these ordered tournaments implies the absence of all of them. As there are some symmetries, each constraint gives rise to 8 ordered forms rather than 24, for a total of 16 configurations (or 8 complementary pairs).

It should not much matter which of these 16 configurations we select as our starting point, but for the sake of the proof we do need to select one, and treat its presence or absence as the principal case division at this stage.

The conclusion takes the following form. The notations used here for specific ordered tournaments are arcane, but not of great importance just now.[11]

Lemma 2.1 (4.14, §4D). *Let* Γ *be a homogeneous ordered tournament which contains* $\vec{I}_3$, $\vec{C}_3^{\,+}$, *and* $\vec{C}_3^{\,-}$, *but does not contain* $(\vec{C}_3^{\,+} \to 1)$. *Let* A *be an ordered tournament of order at most* 4. *Then* A *embeds into* Γ *if and only if the underlying tournament of* A *is a local order.*

It then remains to be checked that under the indicated constraints, we arrive at $\mathbb{S}$ with a generic ordering. All of this is done by direct amalgamation arguments, and it is clear throughout what is needed. Once we reach this point, we can argue much as in the case of homogeneous permutations (where $\mathbb{S}$ is replaced by $\mathbb{Q}$), as treated in Cameron [2002/03)].

Once past this stage of the classification, we assume that one specific ordered tournament of order 4 which is not a local order embeds into our homogeneous ordered tournament Γ; and we switch to the language of ordered graphs.

2A.1. Recognizing the underlying homogeneous graphs. Now we aim to show that the underlying graph is one of those found by Lachlan and Woodrow in their classification of homogeneous graphs. That is, we must now reprove their classification theorem with an order in place. As their argument made use of the symmetry of the edge relation, this presents some difficulties, in principle.

But as it happens, I gave another approach to their result in Cherlin [1998, Chap. IV], intended to illustrate the methods used in the classification of homogeneous directed graphs; the approach taken is a symmetrized form of an analysis originally carried out in an asymmetric setting (with its roots in the classification of homogeneous

[11] Just *bleep* right over them—Linus, 1964

tournaments), and so it turns out to be well suited for our present purpose. Thus we proceed by de-symmetrizing the symmetrized version of an older argument in a way that gives a new result.

It seems clear a priori that any method which can be used to classify the homogeneous directed graphs ought to be applicable to the symmetric case as well—for one thing, if a homogeneous graph has the strong amalgamation property, then it has a "generically oriented" variant which is a homogeneous directed graph. Chapter IV of Cherlin [1998] (the last chapter written) worked out the symmetrized version of the argument given in subsequent chapters of that monograph. At the time my sense of the argument was expressed as follows.

> "The proof given here is more complex than the one given [by Lachlan and Woodrow], but it generalizes ... "

While the generalization I had in mind was to the context of directed graphs, there is a more direct generalization, to the context of ordered graphs—once we have eliminated the cases leading to extensions of partial orders or a generically ordered local order.

The details as presented in Part I of this volume take up considerably more space than the corresponding treatment of homogeneous graphs, particularly in the preparatory phase. One has first to build a library of small configurations known to occur before one can really begin.

But after some lengthy preliminaries, we run closely parallel to the line of Chapter IV of Cherlin [1998]. One has to make various refinements of the main definitions to incorporate the effect of the order. We then transfer to our setting an elaborate inductive framework which requires the simultaneous proofs of nine propositions, of which the last four are proved with some elegance on formal grounds (these are the points where one sees that the basic definitions have been made in a useful way). Each of the first five propositions requires additionally work, much of it brute force. The last of these five propositions brings us back, finally, to the setting where Lachlan's Ramsey argument is the only tool available—after some further reductions.

To quote scripture: "All of this has happened before, and all of this will happen again." Lachlan's methods, as applied in the last stage of our analysis, involve three distinct technical ideas that play well together, to which one must add a fourth which does not appear in the context of tournaments. These ideas may be described as follows.

1. Auxiliary amalgamation classes $\mathcal{A}^r$.
2. The Ramsey argument, reducing to 1-point extensions of "stacks."
3. A shift to partitioned structures (called ordered 2-graphs).
4. A "1-2 punch" in the space of types that occurs just at the end, to set up the final amalgamation argument.

These four points were the main points in Cherlin [1998], both in the context of the classification of the homogeneous directed graphs, as well as the symmetrized version given in Chapter IV. They remain the main points of our Part I as well. So we will describe them in detail here.

While these points underly the inductive strategy outlined in Chapter 5 below, some are more in evidence than others in the presentation there: namely, the first and third points are built into the inductive framework explicitly, while the second and fourth points will appear as a punch line in Chapter 10.

2A.2. The inductive framework. We begin our discussion with the inductive framework shown in Table 2.1 below. We make use of the following notation: $\mathbb{H} = (H_1, H_2)$ is a homogeneous *2-partitioned ordered graph*, i.e., an ordered graph with a partition into two labeled pieces H_1 and H_2, with $\mathbb{H}$ homogeneous in that language.

There are a number of additional notations and some unfamiliar terminology which will require elucidation. The 2-graph $\mathbb{H}$ is supposed to be *ample,* which means that certain finite configurations of the form $(\emptyset, A_2)$ or (a_1, A_2) embed in $\mathbb{H}$. In the latter case, we suppose $a_1 < A_2$ and we call such a configuration (or its isomorphism type over A_2) an *initial* 1-*type* over A_2.

We pay a price for moving to the category of ordered 2-graphs—first we pay a price to get into that category, and then our statements become more ambitious when interpreted in that category. To see why that price is worth paying, we examine the transition from Propositions IV_n and V_n (the last of our concrete propositions) to Proposition VI_n (the first of our more general assertions).

Before entering into this, we discuss some additional points of notation and terminology occurring in these propositions. We have the inductive parameter n and correspondingly a family of finite ordered graphs $\mathcal{A}(n)$. The effect of the parameter n is to include an ordered n-clique in the family $\mathcal{A}(n)$. There is also a kind of "direct sum" or "stacking" operator (an ordered disjoint union) at work in both of Propositions V_n and VI_n. Namely, $\vec{K}_n \perp \vec{K}_n$ is the ordered disjoint

(I)	If $a \in \Gamma$ then the ordered 2-graph $\mathbb{H}_a = (a^{\perp -}, a^{\perp +})$ is ample.
(II_n)	If all elements of $\mathcal{A}(n)$ embed in Γ, and $B = baK$ satisfies – $K \cong \vec{K}_n$ – $b < a < K$ – $a \perp bK$ – B does not contain $\vec{K}_{n+1}$ then B embeds in Γ.
(III)	If $p = (x,\, A)$ is an $\mathbb{H}$-constrained initial 1-type with $A \in \mathcal{A}(2)$, then p is realized in $\mathbb{H}$.
(IV_n)	If $A \in \mathcal{A}(n)$ and $p = (x,\, A)$ is an initial 1-type over A which is realized in $\mathbb{H}$ with $x \in H_1$, $A \subseteq H_2$, then the ordered 2-graphs $(A^p, A^{\perp -})$ and $(A^p, A^{\perp +})$ are ample.
(V_n)	If $p = (x, \vec{K}_n \perp \vec{K}_n)$ is an $\mathbb{H}$-constrained initial 1-type, then p is realized in $\mathbb{H}$.
(VI_n)	If $p = (x,\, A)$ is an $\mathbb{H}$-constrained initial 1-type with $A \in \perp \mathcal{A}(n)$, then p is realized in $\mathbb{H}$.
(VII_n)	Suppose that Γ contains every configuration in $\mathcal{A}(n)$. If $B = A \cup \{b\}$ does not contain $\vec{K}_{n+1}$, and $b < A$, with $A \in \perp \mathcal{A}(n)$, then Γ contains B.
(VIII_n)	Suppose Γ contains every configuration in $\mathcal{A}(n)$. If A does not contain $\vec{K}_{n+1}$ then A embeds into Γ.
(IX_n)	If $p = (x,\, A)$ is an $\mathbb{H}$-constrained initial 1-type and A does not contain $\vec{K}_{n+1}$, then p embeds into $\mathbb{H}$.

TABLE 2.1. PROPOSITIONS (I)–(IX)

union of two ordered n-cliques, with the first one preceding the second in the ordering, while $\perp \mathcal{A}(n)$ is the family consisting of all finite ordered sums of configurations in $\mathcal{A}(n)$.

The main case in Proposition VI_n arises when A is an ordered sum of a finite number of n-cliques. Proposition IV_n states that we can do induction on the length of the sum. Proposition V_n says that when the length of the sum is 2, the statement in Proposition VI_n is correct. Proposition VI_n is then immediate by induction on the length of the sum, as long as we work in the category of ordered 2-graphs. This argument will not work in the category of ordered graphs. This is the reason—the only reason—we change categories.

Part of the (modest) price we pay for this change in viewpoint is the need to prove Proposition I, which gives us the transition from homogeneous ordered graphs to ample homogeneous ordered 2-graphs. Here $a^{\perp-}$ and $a^{\perp+}$ are two specific a-definable sets.

$$a^{\perp-} = \{x \in \Gamma \mid a \perp x, x < a\}$$
$$a^{\perp+} = \{x \in \Gamma \mid a \perp x, a < x\}$$

The notation "$a \perp x$" means that (a, x) is not an edge.

Proposition II is of a similar nature, and can be interpreted as a statement about the associated ordered 2-graph $\mathbb{H}_a$.

At this point, we have accounted for technical point (3) above (the shift to partitioned structures), and indicated that Propositions I–V are the concrete results that must be proved before an argument of any generality gets underway. After the transitional argument from Proposition V to Proposition VI, the main ideas emerge in the proofs of the remaining propositions. Here Proposition VII is just a variant of Proposition VI in the category of ordered graphs, and the last two propositions are two forms of our main theorem (limited to cases covered by the parameter n), again covering both categories. Here we deal first with the category of homogeneous ordered graphs, which is all we care about, ultimately, but we continue on in the final proposition to deal with ordered 2-graphs again: to keep the induction going we must stay in that category to the end.

The transition from Proposition VI_n to Proposition VII_n contains the two main technical points, namely $(1, 2)$ above. The first of these originates with the Lachlan/Woodrow argument, the second with Lachlan's treatment of countable homogeneous tournaments. We carry over the latter treatment to our setting with no substantial alteration.

The connection between VI_n and VII_n is not evident, so we go into this here.

First, if $\mathcal{A}$ is an amalgamation class of ordered graphs, and r is a 2-type (an isomorphism type of ordered graphs of order 2), we will say that an ordered graph L is r-*Ramsey* if it can be labeled in such a way that any pair of elements has the specified 2-type. This means that L is either an ordered clique or an ordered independent set (and we arrange its elements in increasing or decreasing order; but we may suppose that they are arranged in increasing order). Admittedly there is something strange about ordering an already ordered set, but this notion also makes sense in contexts in which no ordering is provided in advance.

We let $\mathcal{A}^r$ be the subset of $\mathcal{A}$ defined by the following condition.

> A is in $\mathcal{A}^r$ iff every extension of A by a finite r-Ramsey ordered graph L with $L < A$ belongs to $\mathcal{A}$.

Points to observe:

— On formal grounds, $\mathcal{A}^r$ is again an amalgamation class, and is contained in $\mathcal{A}$.

— If we are minimally optimistic—that is, if we believe the theorem we are trying to prove—then $\mathcal{A}^r$ should be $\mathcal{A}$ for some choice of r.

This means that it is reasonable to try to prove that any assumptions we have made about $\mathcal{A}$ will apply to $\mathcal{A}^r$, for some choice of r. The relevant assumptions are that certain special configurations belong to $\mathcal{A}$, such as n-cliques for some definite value of n.

At this point, we are in the vicinity of the Lachlan/Woodrow argument, except that they manufactured a different class $\mathcal{A}^*$ to play the role of $\mathcal{A}^r$ in the case of symmetric graphs, and the Ramsey theoretic construction we are now describing was introduced later, by Lachlan, along with the following point, which makes it work.

We stack up multiple copies of n-cliques, and some other special ordered graphs, and study extensions of such stacks by single vertices.

LEMMA 2.2. *Let $\mathcal{A}$ be an amalgamation class of ordered graphs. Suppose a finite set of finite ordered graphs $\mathcal{A}'$ has the following property.*

> *For any finite ordered sum $A \in \perp \mathcal{A}'$, and any further extension $B = A \cup \{b\}$ of A by one additional vertex, B belongs to $\mathcal{A}$.*

Then $\mathcal{A}' \subseteq \mathcal{A}^r$ for some 2-type r.

For the proof, one takes a long sum $\hat{A} = A_1 \cup \cdots \cup A_n$ and a large number of 1-point extensions $a_1, \ldots, a_N$, and one amalgamates all structures $\hat{A}_i = \hat{A} \cup \{a_i\}$ over the base $\hat{A}$. If N is large enough. then Ramsey's theorem gives a large Ramsey ordered graph of some type r (with the ordering on it coinciding either with the one we began with or its reversal). Since we know this, we can set up the 1-point extensions in advance so that whatever the amalgam results, the structure obtained includes a copy of $A \cup L$, for any (specific) desired extension of that type.

This argument is not in itself sufficiently uniform to prove the lemma as stated, with a fixed choice of r, but as there are only finitely many possibilities for r, one of them will work uniformly, regardless of the size of L.

This is the short version. We will give a more detailed exposition in the text. But this is the main technical point in Lachlan's classification of the countable homogeneous tournaments and it remains the main technical point in Cherlin [1998], ultimately shaping the form of the inductive argument, which aims at giving this argument a place to operate.

Further difficulties arise in the proof of Proposition V, the last of the propositions whose proofs rely on direct amalgamation arguments, rather than artfully constructed definitions. And this is where the last of our four points comes into play. Recall the statement of Proposition V.

> Let $\mathbb{H}$ be a countable ample homogeneous ordered 2-graph and $p = (x, \vec{K}_n \perp \vec{K}_n)$ an $\mathbb{H}$-constrained initial 1-type. Then p is realized in $\mathbb{H}$.

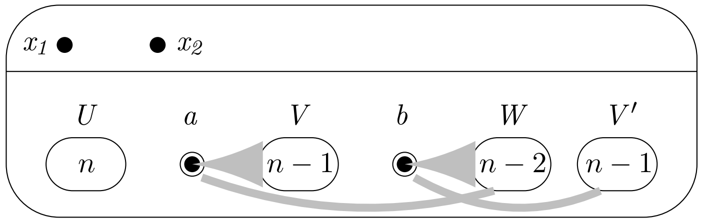
A similar statement is given as Lemma 8.7, for the case $n = 2$ (and more than two copies of $\vec{K}_2$, but the argument reduces quickly to the case of two copies). The main ingredients of the argument for general n occur already in the proof of Lemma 8.7.

We now discuss the version of the argument needed for the proof of Proposition V.

Let us write $\vec{K}^1, \vec{K}^2$ for the two ordered n-cliques, and P_i for $(x, \vec{K}^i)$. (So $p = P_1 \perp P_2$, in a certain sense.) The hypothesis of $\mathbb{H}$-*constraint* means that P_1 and P_2 both embed into $\mathbb{H}$. We must realize the type p in $\mathbb{H}$, relying in the critical cases on explicit amalgamation arguments.

The comparatively simple asymmetrical case in which $P_1 \neq P_2$ can be handled by fairly direct methods and some extension of the inductive framework to take into account the number of initial 1-types occurring in $\mathbb{H}$ over $\vec{K}_n$. So we come down quickly to the symmetric case $p = P + P$, that is $P_1 = P_2 = P$.

The treatment of this symmetrical case is delicate, and the fourth of our key technical points mentioned at the outset emerges only at this



$$x_1/U, aV, bV' = Q, P, P$$
$$x_2/U, abW = P, P$$

stage of the analysis. Our discussion of this occupies the remainder of the present subsection.

Another way to see what is at stake here would be to examine the special case of these arguments which occurs in the lengthy preparation for the proof of Proposition III in §8C, alluded to above.

In order to see how we proceed at this point, it is necessary to begin with the final amalgamation argument which produces a realization of the type p, and then to back up several steps to see what conditions are required to ensure that suitable factors of this final diagram embed into $\mathbb{H}$.

The concluding amalgamation argument that we have in mind looks as follows (Lemma 10.8). Certain features of the factors are determined as the diagram is constructed. Others are either given in advance or are constructed in preliminary lemmas.

This represents an amalgamation problem in the category of ordered 2-graphs, with the components H_1 and H_2 arranged as horizontal layers.

The base of the amalgamation consists of everything except the two circled vertices a and b, and the amalgam will determine the structure of the pair (a, b).

In this diagram, we have the following specifications.

— The type of x_1 over aV or bV', and the type of x_2 over U or abW, is the given type P.
— The type of x_1 over U is an as yet unknown type Q, which we will need to construct carefully, once we have worked out the properties required of it.
— The type of x_1 over W is not specified and will be worked out in our concluding argument.

— The type of x_2 over aV or bV' will also be worked out in the concluding argument, but the type of x_2 over a or b is already determined.

In the amalgamation diagram, once it is completed, as $a < V < b$, we must have $a < b$. So the question to be settled by amalgamation is whether or not (a, b) is to be an edge.

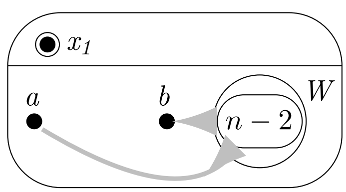
The factors have been constructed to ensure that in either case, a copy of the type $p = P + P$ appears. If (a, b) is not an edge, then (x_1, aV, bV') will be of type p, while if (a, b) is an edge, then $(x_2, U \perp abW)$ will be of type p.

So we see how we intend to complete the argument. Now it is necessary to work out how the two (compatible) factors of this diagram are to be constructed so as to ensure that each of them embeds into $\mathbb{H}$, and what properties the type Q must have to make this possible.

The fact that we have two elements x_1, x_2 to deal with in the first component would appear to pose a separate problem, but the Lachlan Ramsey argument will give us a choice r of 2-type for (x_1, x_2) which in some cases allows us to reduce to the consideration of x_1, x_2 separately. We do not dwell on this point now, as it is one that will have been dealt with well before we reach this configuration, along lines which are typical of Lachlan's approach.

Now assuming for the moment that we know both the type of (x_1, x_2), and what the more mysterious type Q should be, we would proceed as follows to construct the factors of the diagram shown above.

Stage 1: Determination of the type of (x_1, W)*:*
Just amalgamate.



(I)

There is not much to this, as the necessary factors of this amalgamation diagram may be obtained by induction.

Stage 2: Determination of the type P^* of (x_2, aV):

This is a major stage in the argument. For later parts of the argument to succeed, we must choose P^* in accordance with the following.

Claim. Let r be the type of x_1, x_2. Then there is an initial 1-type P^* over aV extending $P \restriction a$ so that any configuration of the form $(R, UaVWV')$ with the following properties embeds into $\mathbb{H}$.

(i) R is r-Ramsey;
(ii) $x_1/UaVW$, x_2/UaW as specified above;
(iii) (x_2, aV_1) realizes P^*;
(iv) (x_2, V') any type realized in $\mathbb{H}$.

We take $R = \{x_1, x_2\}$ when the claim is applied.

One proves such claims by trying all possibilities for P^* and recording a counterexample for each choice of P^*, then putting an ordered sum of all the counterexamples into a single diagram, and amalgamating once more to determine a value of P^* consistent with each of the putative counterexamples.

As one works through this part of the construction, one encounters a condition on the type Q. That is, the type Q must be chosen at the outset so as to make this part of the argument work. The precise conditions on the type Q are given by Lemma 10.7 (summarized loosely below, mainly in terms of a diagram extracted from the treatment of this stage in the proof of Lemma 10.8).

Stage 3: Determination of (x_2, V'):

This is much like Stage 1, but the factors are considerably more complicated (Figure 2).

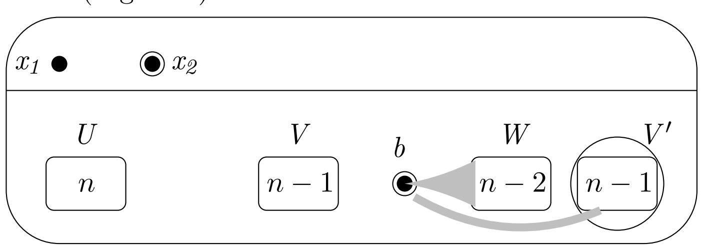


FIGURE 2. $(x_1/U, V, bW, bV') = Q, P, (I), P$
$(x_2/U, V, bW) = P, P^* \restriction V, P$
(III)

To show that both factors embed into $\mathbb{H}$, we again make use of the properties of Q and the type of x_1, x_2.

At the end of this stage, we have proved the existence of a particular form of the second factor in our amalgam, and in the process we have determined the first factor up to isomorphism. Now the only question is whether this first factor embeds into $\mathbb{H}$. For this, we may quote the defining conditions on P^*.

Now we have not been very forthcoming about the necessary properties of Q, as one needs to inspect the relevant diagrams more closely to see them in detail. But in a general way, the goal is to force certain configurations into $\mathbb{H}$ whatever their particular form may be. We quote the two relevant lemmas, but omit the detailed verbal specifications which amount to descriptions of the diagrams that arise in the course of the argument just outlined; these specifications are represented by the accompanying pictures. We stress that we cannot know the precise form of these pictures in advance, only their general properties, as the details are determined via a series of amalgamations in the final argument.

Recall that a 1-type p over a set A is *initial* if its realizations precede A in the order. In the context of ordered 2-graphs, a 1-type is a *cross* type if its parameter set lies on one side, and its realization on the other; our convention is that $\mathbb{H} = (H_1, H_2)$, $A \subseteq H_2$, and $x \in A_1$.

Lemma (10.6). Let $n \geq 3$, and assume Proposition IX_{n-1}. Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. Let p be an initial cross type.

Then there is an initial cross type q with the following property.

> *Assume [something appropriate, suggested by Figure 3 below, which shows only H_2; here Q_1, Q_2, Q_3 are the types over yK, A, B.]*
>
> *Then*
>
> $$Q_1 \perp Q_2 \perp Q_3 \textit{ is realized in } \mathbb{H}$$

Lemma (10.7). Let $\mathbb{H}$ be an ample homogeneous 2-graph and P a 1-type over $\vec{K}_n$ realized in $\mathbb{H}$. Let r be a Ramsey 2-type for $\mathbb{H}^P$ over $\mathcal{A}(n-1)$, and q any initial cross type. Then there is a 1-type Q over $\vec{K}_n$ whose restriction to $a = \min \vec{K}_n$ is q, with the following property.

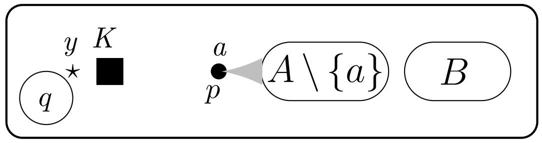
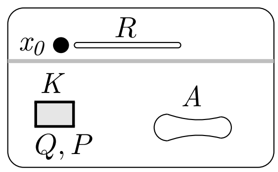


FIGURE 3

($\star$) *For any[thing like the picture below ..., where both sides are shown, as top and bottom,]*
$\mathbb{H}$ *contains the configuration* $(R,\, K \perp A)$ *where [etc. ...]*



When applied, Lemma 10.6 supplies a useful type q of the form (x, y), and then Lemma 10.7 provides a useful extension of q to a type Q of the form $(x, \vec{K}_n)$ with $y = \min \vec{K}_n$.

2B. 3-Constrained metrically homogeneous graphs

In Part II, our analysis requires less substantial machinery than was required in Part I.

The treatment of the case of diameter 2 by Lachlan and Woodrow required a clever induction on amalgamation classes (either the original Lachlan/Woodrow argument, or as in the approach via Lachlan's Ramsey argument sketched above).

On the other hand, in Amato, Cherlin, and Macpherson [2021] we show that the case of diameter 3 succumbs to a series of direct amalgamation arguments with no special technical apparatus, because the triangle inequality comes into play to limit the ways in which amalgamation problems can be completed. Our treatment of the bipartite case in Part II uses similar arguments, in a more straightforward manner, as one can "simplify" configurations by reducing the size of one side (choosing arbitrarily which is the favored side). The problem in general is to arrive at a notion of simplification which does not involve reducing the total number of vertices involved. Very possibly something similar will work in general: cf. Appendix 18B.1.

Our first objective in Part II is the classification of the 3-constrained countable metrically homogeneous graphs of generic type, that is, the ones which are determined solely by constraints on triangles. Actually, the hypothesis of 3-constraint is unnecessarily strong, and we will revisit that below. But we keep that hypothesis in view for the present.

The natural approach to the problem is the following.

1. Let $\mathcal{T}$ be the set of triangles which do not occur in the class $\mathcal{A}$. Calculate parameters $\delta, K_1, K_2, C_0, C_1$ from $\mathcal{T}$.
2. Prove that $\mathcal{T}(\delta, K_1, K_2, C_0, C_1) = T$; so $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2;C_0,C_1}$ by 3-constraint.
3. Work out *necessary* conditions on the parameters $\delta, K_1, K_2, C_0, C_1$ for $\mathcal{A}^{\delta}_{K_1,K_2;C_0,C_1}$ to be an amalgamation class.
4. Show that the conditions obtained are *sufficient* to make $\mathcal{A}^{\delta}_{K_1,K_2;C_0,C_1}$ an amalgamation class.

We extract the numerical parameters as follows.

DEFINITION 2.3. Let $\mathcal{A}$ be a class of finite integral metric spaces. We define the parameters $\delta, K_1, K_2, C_0, C_1$ associated with $\mathcal{A}$ as follows.

— δ is the largest distance occurring, or ∞;
— K_1, K_2 are respectively smallest, and largest, so that triangles of types $(K_1, K_1, 1)$ and $(K_2, K_2, 1)$ occur in $\mathcal{A}$ (where the type of a triangle is the triple of edge lengths involved);
— C_0, C_1 are the least even and odd numbers greater than 2δ such that $\mathcal{A}$ contains is no triangle of perimeter C_0 or C_1, respectively.

When $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$, this definition gives the correct values for the parameters. We will need to prove that the definition is useful also when $\mathcal{A}$ is associated with a metrically homogeneous graph of unknown type. In particular, when $\mathcal{A}$ is associated with a 3-constrained amalgamation class, with forbidden triangle set $\mathcal{T}$, we wish to prove first that $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$, or equivalently that $\mathcal{T} = \mathcal{T}(\delta, K_1, K_2, C_0, C_1)$; for this, we make use of a number of explicit amalgamation arguments. These amalgamation arguments are very direct, and none of the factors occurring in the amalgamation diagrams involved has order greater than 4.

Once we have shown that $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$, we must also derive the various inequalities given in the statement of Theorem 1.22. For example, the first of these is the following.

Lemma (14.3). Let $\mathcal{A}$ be a 3-constrained amalgamation class of finite metric spaces corresponding to a metrically homogeneous graph of diameter δ with associated parameters K_1, K_2. Then one of the following holds.

1. $K_1 + K_2 \geq \delta + 1$;
2. $K_1 = 1$, $K_2 = \delta - 1$;
3. $K_1 > 1$, $K_1 + K_2 = \delta$, and $C_0 = 2\delta + 2$.

Actually, the statement of results of the type of Lemma 14.3 will be given under a weaker hypothesis than 3-constraint, which we call 4-*triviality*. We come back to this point below.

For the proof of Lemma 14.3 one makes another explicit amalgamation argument, under the assumption that none of the three alternatives listed applies. Again, the factors occurring in the amalgamation argument have order at most 4.

The notion of admissibility involves many such numerical constraints. Thus in arguing from the amalgamation property to admissibility we must give a wide variety of explicit amalgamation arguments.

Arguing in the reverse direction, from the numerical constraints to the amalgamation property, also requires the treatment of a large number of cases, but according to a different scheme. Namely, we begin by specifying a complicated but explicit amalgamation strategy, involving a computation of several possible values for the distance that remains to be determined in the amalgam (in the case of a 2-point amalgamation problem over the base), and a comparison of these values with K_1 or K_2.

The explicit amalgamation strategy is given in Table 2.2, p. 54. We write $C = \min(C_0, C_1)$ and $C' = \max(C_0, C_1)$. In most cases, our amalgamation strategy makes use of one of the canonical values $r^+, r^-, \tilde{r}$ described in the caption.

It has also been observed since that one can make this procedure more canonical by first choosing a "default" value M (typically $M = \max(\lceil \delta/2 \rceil, K_1)$ is one suitable choice) and then taking the required distance to be the closest value to M compatible with the lower bound r^- and the upper bounds $r^+, \tilde{r}$. This is not the procedure we follow, but the two approaches naturally have a good deal in common—any valid amalgamation procedures are going to agree to some extent. This point of view leads to interesting developments outside the scope of this volume, but discussed in the Appendix.

We prove on a case by case basis that in each case indicated, following the amalgamation procedure introduces no forbidden substructures. Having done that, we will also want to consider the effect of Henson constraints. With few exceptions, the value chosen by the amalgamation strategy is neither 1 nor δ; this means that the stated procedure tends to be compatible, as it stands, with the imposition of additional Henson constraints. There are some exceptional cases which require further analysis to avoid extreme values.

This description of our analysis follows the approach taken to identify the 3-constrained amalgamation classes in the first place: first, determine the necessary numerical conditions on the parameters, then show that the conditions found are sufficient.

But in our presentation, we will take matters in the reverse order. We have already written out appropriate conditions on the parameters; we will show first that they suffice for amalgamation (Chapter 12), then that they are necessary (Chapters 13, 14).

As we have mentioned, in the portion of the analysis given in Chapters 13 and 14 we actually work with a notion broader than 3-constraint, which we call 4-triviality. So now we will describe the broader setting for these results.

Definition. Let $\mathcal{A}$ be an amalgamation class of finite metric spaces corresponding to some metrically homogeneous graph Γ of diameter δ. We say that $\mathcal{A}$, or Γ, is 4-*trivial* if $\mathcal{A}$ contains every metric space M on 4 vertices satisfying the following two conditions.

(a) M contains no forbidden triangle for Γ; and
(b) M is not an ordinary or antipodal δ-Henson constraint.

Since we expect all metrically homogeneous graphs of generic type to be 4-trivial, this is a more satisfactory assumption than 3-constraint.

Thus the *first goal* in a full proof of the classification conjecture for metrically homogeneous graphs of generic type would be to remove 4-triviality entirely from this part of the argument, or rather, to prove the relevant instances of 4-triviality. This first stage would then show that the triangle constraints and Henson constraints occurring in an arbitrary metrically homogeneous graph of generic type are those

(I) If $K_1 = \infty$:
Use the value r^- in this case.; or any value between r^- and $\min(r^+, \tilde{r}, \delta)$ of the correct parity except possibly δ.

(II) If $K_1 < \infty$ and $C \leq 2\delta + K_1$:
- (a) If $C' = C + 1$ then:
  - (i) If $r^+ \leq K_2$ let $d(a_1, a_2) = \min(r^+, \tilde{r})$. Otherwise:
  - (ii) If $r^- \geq K_1$ let $d(a_1, a_2) = r^-$. And otherwise:
  - (iii) Let $d(a_1, a_2) = K_2$.
- (b) If $C' > C + 1$ then:
  - (i) If $r^+ < K_2$ let $d(a_1, a_2) = r^+$. Otherwise:
  - (ii) If $r^- > K_2$ let $d(a_1, a_2) = r^-$. Otherwise:
  - (iii) Take $d(a_1, a_2) = K_2 - \epsilon$ with $\epsilon = 0$ or 1 defined by

$$\epsilon = \begin{cases} 1 & \text{if there is } x \in A_0 \text{ with } d(a_1, x) = d(a_2, x) = \delta, \\ 0 & \text{otherwise} \end{cases}$$

(III) If $K_1 < \infty$ and $C > 2\delta + K_1$:
- (a) If $r^- > K_1$, let $d(a_1, a_2) = r^-$.
- (b) Otherwise:
  - (i) If $C' = C + 1$:
    - (A) If $r^+ \leq K_1$ let $d(a_1, a_2) = \min(r^+, \tilde{r})$. Otherwise:
    - (B) Let $d(a_1, a_2) = K_1 + \epsilon$; $\epsilon = 0$ or 1 is defined by

$$\epsilon = \begin{cases} 1 & \text{if there is } x \in A_0 \text{ with } d(a_1, x) = d(a_2, x) = \delta, \\ & \text{and } K_1 + 2K_2 = 2\delta - 1; \\ 0 & \text{otherwise} \end{cases}$$

  - (ii) If $C' > C + 1$:
    If $r^+ < K_2$ let $d(a_1, a_2) = r^+$.
    Otherwise, let $d(a_1, a_2) = \min(K_2, C - 2\delta - 1)$.

Here r^+, r^- are the maximal and minimal values for the distance consistent with the triangle inequality, and $\tilde{r}$ is a variant of r^+ taking into account the perimeter bounds. When none of these values is suitable, values close to K_1 or K_2 suffice.

TABLE 2.2. Amalgamation Strategy

which occur in a (unique) metrically homogeneous graph of known type, and also that the triangles realized in the given graph and its twin of known type are the same.

The *second stage* would be the identification of the amalgamation class $\mathcal{A}$, namely $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$, where $\mathcal{S}$ is the class of minimal Henson constraints (or antipodal Henson constraints) for $\mathcal{S}$.

In the treatment of the diameter 3 case in Amato, Cherlin, and Macpherson [2021], the first stage goes very quickly, while the second stage is long and delicate.

At an early stage in the identification of the triangle constraints in a 4-trivial amalgamation class, we derive a property which we call the Interpolation Property (Definition 13.11). This has as a consequence that all triangles of even perimeter at most 2δ are realized, and thus justifies the focus on odd perimeter below that bound (Lemma 13.12).

2C. Reduction theorems for metrically homogeneous graphs

Now we consider the two reduction theorems discussed in the final chapters of this Memoir.

2C.1. The bipartite case. The first of these theorems concerns the bipartite case. If Γ is a homogeneous bipartite graph, equipped with the path metric, then each half of Γ can be viewed as a graph with edge relation given by $d(x, y) = 2$. The effect of this in metric terms is to rescale the metric by a factor of $1/2$, and then each half of Γ becomes a metrically homogeneous graph of diameter $\lfloor \delta/2 \rfloor$, called $B\Gamma$. (This notation comes from the study of finite distance transitive graphs.)

Theorem (Theorem 1.30). Let Γ be a countable bipartite metrically homogeneous graph. Suppose that $B\Gamma$ is one of the graphs in our catalog. Then Γ is also in our catalog. In particular, if Γ is of generic type, then Γ has the form

$$\Gamma^{\delta}_{\infty,0,2C_0,\delta+1,\mathcal{S}}$$

with admissible parameters $(\delta, \infty, 0, C_0, 2\delta + 1, \mathcal{S})$. Here C_0, C_1 are to be omitted if $\delta = \infty$.

This is an interesting trial run at the general classification theorem. In the general case, we can always try to make some use of inductive information (perhaps more usefully once we have $\delta \geq 5$, and Γ_2 may be assumed to be of known type, if it contains an edge). There are specific features of the bipartite case which make the details much easier to handle than the general case, but one sees some elements of a general strategy. And in the cases treated so far, direct amalgamation arguments suffice.

In the bipartite case, we treat the cases of finite or infinite diameter together, since this analysis precedes the reduction of infinite diameter to finite diameter, and plays a role in the latter. An initial reduction (Fact 1.31) takes us down to the case in which

$$(B\Gamma)_1 \text{ is isomorphic to the random graph } G_\infty$$

So we come down to the following.

Proposition (Proposition 16.2). Let Γ be a countable bipartite metrically homogeneous graph for which

$$(B\Gamma)_1 \text{ is the random graph } G_\infty$$

Suppose that $B\Gamma$ is of known type. Then Γ is generic bipartite, subject to a bound on perimeter and some δ-Henson constraints, i.e.

$$\Gamma \cong \Gamma^\delta_{\infty,0,C_0,2\delta+1,\mathcal{S}}$$

for some even C_0 with $2\delta + 2 \leq C_0 \leq 3\delta + 2$. (Here C_0, C_1 are to be omitted if $\delta = \infty$.)

The first step, which is easy, is to work out the parameters of $B\Gamma$ in terms of the parameters of Γ, and to invoke the assumption that $B\Gamma$ is of known type to identify it explicitly.

The rest of the analysis consists of a direct proof of an embedding theorem for Γ—in other words, we determine the associated amalgamation class explicitly. Call a finite metric space A "Γ-constrained" if it embeds into the homogeneous bipartite graph Γ^* with the same parameters. The embedding theorem states that every finite Γ-constrained metric space embeds into Γ, which is a more convenient way of making the claim $\Gamma \cong \Gamma^*$. This must first be proved for triangles (i.e., we need to know first that the forbidden triangles are as the parameter values would suggest), and then in general.

The proof of the embedding theorem proceeds by induction on a particular measure of complexity for the configurations we are trying to embed into Γ, one which pays particular attention to the pairs at distance 1 or δ. Similar measures of complexity are used in the treatment of the corresponding embedding theorem in the diameter 3 case, as well.

2C.2. The case of infinite diameter. In Chapter 17, we take up the reduction of the infinite diameter case to the finite diameter case. That reduction theorem reads as follows.

Theorem (Theorem 1.26). Suppose that every metrically homogeneous graph of finite diameter is of known type. Then every metrically homogeneous graph is of known type.

Since the bipartite case ($K_1 = \infty$) has already been reduced to the opposite case $K_1 = 1$, we may assume here that K_1 is finite. Then by inductive arguments of a general kind, the proof reduces quickly to the following specific point of local analysis.

Lemma (Lemma 17.2). Suppose that every metrically homogeneous graph of finite diameter is of known type. Let Γ be a metrically homogeneous graph of infinite diameter, with $K_1 < \infty$. Let $K = \max(K_1, 2)$. Suppose that Γ_K contains a triangle of type $(K_1, K_1, 1)$. Then Γ is of known type, specifically of the form

$$\Gamma^{\infty}_{K_1,\infty,\infty,\mathcal{S}}$$

with $\mathcal{S}$ either empty or consisting of one clique.

To prove that Γ_K does contain a triangle of type $(K_1, K_1, 1)$, we again make use of explicit amalgamation arguments.

2D. Local analysis of metrically homogeneous graphs

Now we take up the main results of local analysis, which are applied in the proofs of the results sketched above, and are also useful in the case of diameter 3 (though more useful in large diameter). This is the point where the definition of generic type enters directly into the proofs.

We recall the statements of the two main results.

Theorem (Theorem 1.32). Let Γ be a countable metrically homogeneous graph of generic type and of diameter δ, and suppose $i \leq \delta$. Suppose that Γ_i contains an edge. Then Γ_i is a countable metrically homogeneous graph (and, in particular, is connected).

Furthermore, Γ_i is primitive and of generic type apart from the following cases.

(a) If $\delta = 2i$, and Γ is antipodal, then Γ_i is imprimitive (antipodal);
(b) If $i = \delta$, and $C_0, C_1 \leq 2\delta + 3$, then Γ_δ is a complete graph, and either $K_1 = 1$ or Γ is antipodal.

Proposition (1.33). Let Γ be a countable metrically homogeneous graph of diameter δ. Suppose

$$K_1 \leq 2.$$

Then for $2 \leq i \leq \delta - 1$, Γ_i contains an edge, unless $i = \delta - 1$, $K_1 = 2$, and Γ is antipodal.

The proof of Theorem 1.32 takes place in three stages.

1. Assuming Γ_i is connected, show that it is metrically homogeneous (easy) and either of generic type, or falls under one of the exceptional cases mentioned (Lemma 15.1).
2. Show that if Γ_i contains an edge then it is connected (Lemma 15.7).
3. Deal with the issue of primitivity (Lemma 15.10).

We now discuss the second stage of the proof of Theorem 1.32, the proof of connectedness, which makes good use of the hypothesis of generic type and uses arguments typical of our approach to local analysis. We will say no more at this stage about the proof of Proposition 1.33.

In the proof of connectedness, we would like to proceed by induction on i, but for small i the hypothesis concerning the existence of edges is unlikely to hold, so this appears to be challenging. We can deal with this difficulty by first proving an analogous but much weaker statement, which does hold uniformly.

Lemma (Lemma 15.4). Let Γ be a metrically homogeneous graph of generic type and diameter δ. Suppose $i \leq \delta$, and suppose also that if $i = \delta$ then $K_1 > 1$. Then the metric space Γ_i is connected with respect to the edge relation defined by

$$d(x, y) = 2$$

Because we can argue inductively, the proof of this Lemma is very short, but it does rely on the hypothesis of generic type.

Another very useful principle is the following.

Lemma (Lemma 15.5). Let Γ be a metrically homogeneous graph of generic type. Suppose $1 \leq i \leq \delta$. Then for $u \in \Gamma_{i\pm1}$, the set $\Gamma_1(u) \cap \Gamma_i$ is infinite, unless $i = \delta - 1$ and Γ is antipodal.

Apart from one special case, the hypothesis of generic type trivializes the proof of Lemma 15.5, by considering vertices at distance 2 lying in $\Gamma_{i\pm1}$. But the case

$$i = \delta$$

requires direct attention, with a more combinatorial proof.

Assuming these results, the proof that Γ_i is connected when Γ_i contains an edge breaks up into a simple "main line" and various exceptional cases requiring special attention.

In the main line, we suppose that the connected components of Γ_i are not complete; that is, they contain pairs at distance 2. Then one argues easily that for $u \in \Gamma_{i-1}$, the set I_u of neighbors of u in Γ_i is contained wholly within a unique connected component of Γ_i, and that for $u, u' \in \Gamma_{i-1}$ at distance 2, one gets the same connected component. Then connectedness of Γ_{i-1} with respect to the relation $d(x, y) = 2$ gives the uniqueness of the connected component in Γ_i.

So we need only consider the case in which

($\star$) the connected components of Γ_i are complete.

We may divide this case into three subcases.

(a) $i < \delta$, $K_1 = 1$; or
(b) $K_1 > 1$; or
(c) $K_1 = 1$, $i = \delta$.

In the first subcase, where $i < \delta$ and $K_1 = 1$, the contradiction is immediate: it is easy to see that Γ_i contains a copy of Γ_1, so its connected components are not complete.

In the second subcase, with $K_1 > 1$ and the connected components of Γ_i complete, these components must have order 2.

This very special case is analyzed in the proof of Lemma 15.8, arriving in this case at the conclusion that $I_u = I_{u'}$ for $u, u' \in \Gamma_{i-1}$ at distance 2, and hence by connectivity with respect to the relation $d(x, y) = 2$, the set I_u is independent of u, which gives a contradiction.

In the third and last subcase, we have $K_1 = 1$ and $i = \delta$. This is analyzed separately in Lemma 15.9, along the following lines.

First, it is easy to show in this case that $\Gamma_{\delta-1}$ contains an edge, and hence by a case already treated, $\Gamma_{\delta-1}$ is connected.

Now for $u \in \Gamma_{\delta-1}$ we consider, in addition to the set I_u of neighbors of u in Γ_δ, the set $\hat{I}_u$ consisting of all vertices lying in the same connected component of Γ_δ as some vertex in I_u.

If $I_u = \hat{I}_u$ for $u \in \Gamma_{\delta-1}$, one finds easily that Γ is imprimitive; but Γ is not bipartite, and if Γ is antipodal then Γ_δ, consisting of one vertex, is certainly connected.

So for $u \in \Gamma_{\delta-1}$, we may suppose $I_u \neq \hat{I}_u$, and as the connected components of Γ_δ are complete we find that

$$\hat{I}_u = \{v \in \Gamma_\delta \mid d(u,v) \leq 2\}$$

One can then show that for u, u' adjacent in $\Gamma_{\delta-1}$, we have $\hat{I}_u = \hat{I}_{u'}$, and then by connectivity of $\Gamma_{\delta-1}$, the set $\hat{I}_u$ is independent of the choice of u. Thus $\hat{I}_u = \Gamma_\delta$; that is, the only distances occurring between $\Gamma_{\delta-1}$ and Γ_δ are 1 and 2.

This is a very special configuration. Denote by δ' the diameter of Γ_3. Our assumptions lead quickly to $\delta' \leq 3$ and $\delta' = \delta$. As $\delta \geq 3$ we arrive at

$$\delta' = \delta = 3$$

Now a close look at the resulting configuration reveals a contradiction, as follows.

Let v_* be the basepoint of Γ (that is, $\Gamma_3 = \Gamma_3(v_*)$). Let u_1, u_2 be at distance 3 in Γ_3. Then v_*, u_2 are two points in $\Gamma_3(u_1)$ with the property that the connected component of u_2 in $\Gamma_3(u_1)$ is contained in $\Gamma_3(v_*)$.

Cycling (u_1, v_*, u_2) by homogeneity, we conclude that u_1, u_2 have the property that the connected component of u_2 in Γ_3 is contained in $\Gamma_3(u_1)$. But for any neighbor u of u_1 in Γ_2, I_u meets the connected component of u_2 in Γ_3, and this is a contradiction.

Part I. Homogeneous Ordered Graphs

CHAPTER 3

THE CATALOG OF HOMOGENEOUS ORDERED GRAPHS

3A. Preliminaries

We now begin the classification of the countable homogeneous ordered graphs. In Chapters 1 and 2 we have discussed the statement of the classification theorem, our motivation for taking up the problem, and the general methodology of the proof (and its antecedents). In particular we discussed the resulting catalog of homogeneous ordered graphs (Table 1.1), but we have not yet explained all of the notations used there. We add a few general remarks before returning to a discussion of the catalog.

None of the examples in the catalog are exotic.

There is some relatively recent work that can be interpreted as initiating their classification. First, as Cameron pointed out, the model theoretic notion of a permutation is a structure with a pair of linear orders (the corresponding 2-dimensional pictures are one popular representation of permutations in the area of permutation patterns, as they give a good sense of the pattern—the isomorphism type of the permutation). Cameron took up the very natural problem of the classification of homogenous permutations in Cameron [2002/03)]. Since we can trade in one of the two orders for a symmetric relation (by taking its symmetric difference with the other), we can think of these also as homogeneous ordered graphs.

A revisionist reading of Cameron's classification would be the following: homogeneous permutations arise naturally from two sources, namely—

- Generic linear extensions of homogeneous partial orders;
- Generic linear orderings of homogeneous tournaments.

In this particular context, the homogeneous partial orders are just disjoint copies of $\mathbb{Q}$, or dually, a linearly ordered family of independent sets, while the only relevant tournament is the rational order—so this is not the first interpretation of these results that would come to mind.

Any pair of linear orders can be viewed as a structure equipped with one partial order and a linear extension of it; the partial order is their intersection and the order can be either one of the given ones. This motivated Dolinka and Mašulović to classify all the homogeneous structures of that type: a partial order with a linear order extending it (Dolinka and Mašulović [2012]). This includes both the case of a homogeneous partial order with strong amalgamation which is expanded by a generic linear extension, as well as the generic permutation.

This class of homogeneous structures can also be interpreted as a particular kind of homogeneous ordered graph, namely one in which there is no induced path of length 2 ordered with its "midpoint" in the middle of the order; passing to the graph complement gives a dual class of examples.

Our classification theorem states that every countable homogeneous graph arises in one of the following ways: either in the manner studied in Dolinka and Mašulović [2012] or as the graph complement of such an ordered graph, or else as a generically ordered homogeneous tournament or a generically ordered homogeneous graph.

There are very few homogeneous countably infinite tournaments: the rational order $\mathbb{Q}$, the generic local order $\mathbb{S}$, and the random tournament $\mathbb{T}_\infty$ (Lachlan [1984]). Since the rational order falls under the classification in Dolinka and Mašulović [2012], and the generically ordered random tournament is a transform of the generically ordered random graph, the only homogeneous ordered graph that will need to be considered explicitly as a homogeneous ordered tournament here is the one associated with the generic local order $\mathbb{S}$.

But that one will indeed take some detailed attention.

3B. The catalog

The proof of Theorem 1.2 works tales up various cases which we present in a catalog which gives the following information about each the structures expected to occur.

(a) What the natural interpretation of the structure is, as a homogeneous structure;
(b) how its amalgamation class is characterized by its minimal constraints—namely, the minimal finite ordered graphs which do not embed in the given structure;
(c) which of the three broad families of homogeneous structures (partial orders, tournaments, or graphs) are the source of the structure, and how the linear order is chosen.

We have the following four types of homogeneous ordered graphs, which we claim are exhaustive (though overlapping).

EPO Generic linear **e**xtensions of homogeneous **p**artial **o**rders with strong amalgamation.

EPO$^{\mathbf{c}}$ Graph complements of graphs falling under EPO.

LT Generic **l**inear orderings of homogeneous **t**ournaments with strong amalgamation.

LG Generic **l**inear orderings of homogeneous **g**raphs with strong amalgamation.

The trivial ordered graph (on one vertex) falls under any of these classes indifferently—we will assign it to its own class **Triv**.

While these classes are not disjoint, each one contains at least one example not found in any of the others.

We will first list the possibilities explicitly, according to the division into types, before arranging them in a table according to the constraints determining them.

Triv: There is just one trivial structure.

EPO: The nontrivial homogeneous partial orders with strong amalgamation are the following.

(a) $n \cdot \mathbb{Q}$ with $1 \leq n \leq \infty$: the disjoint union of a number of copies of $\mathbb{Q}$.
(b) $\mathbb{Q}[\vec{I}_\infty]$, the result of replacing each point of $(\mathbb{Q}, <)$ by an infinite independent set.
(c) The generic partial order $\mathcal{P}$.

The generic linear extension of $n \cdot \mathbb{Q}$ will be denoted $n * \mathbb{Q}$; it can be viewed as $(\mathbb{Q}, <)$ equipped with an equivalence relation having n classes, with each class dense. For $n = 1$ this is a degenerate case of either *EPO* or *LT*.

The generic linear extension of $\mathbb{Q}[\vec{I}_\infty]$ is a homogeneous permutation, after a change of language, as described above. (Actually two homogeneous permutations, as the change of language may include reversing the order.)

The generic linear extension of the generic partial order is the Fraïssé limit of the entire class of finite linear extensions of partial orders.

EPO$^{\mathbf{c}}$: The complementary class is not very intelligible in its own right. These are the ordered graphs for which the complement becomes a homogeneous partial order when viewed as a directed graph, with the orientation given by the ordering. To classify these graphs we will simply classify their complements under the previous heading.

This will be seen more clearly in the table below.

LT: The homogeneous tournaments with strong amalgamation are the three infinite homogeneous tournaments: the rational order $(\mathbb{Q}, <)$, a tournament $\mathbb{S}$ which is the generic "local order," characterized by the condition that for any vertex v its out-neighbors v^+ and its in-neighbors v^- are linearly ordered, and the generic or random tournament.

—The generic linear extension of $(\mathbb{Q}, <)$ by a second linear order appears as the generic permutation in Cameron's classification.

—The generic linear extension of $\mathbb{S}$ occupies a distinguished position as the most complex of the "special cases" that need to be considered in our analysis, before passing to the main case;

—The generic linear extension of the random tournament, when viewed as an ordered graph, becomes the generic linear extension of the random graph, and will be treated under that heading.

So what we have here is one case which was thoroughly covered by Cameron, one case that will be handled under a different heading, and just one case that we need to deal with ourselves under the present heading.

This class is closed under reversal of the ordering, which amounts to reversal of the tournament relation. In terms of graphs, this is the operation of graph complementation.

LG: The homogeneous graphs with strong amalgamation fall into two types, along with their complements. The first class consists of equivalence relations with $n \leq \infty$ infinite classes, where the edge relation in the graph is the equivalence relation, or its complement.

The generic linear extension of such an equivalence relation appears above as $n{*}\mathbb{Q}$. Namely, view the equivalence relation as a tournament by viewing it first as a graph and then taking the orientation from the linear order. This gives $n \cdot \mathbb{Q}$, and now the order is a linear extension of this. This example is in *EPO*, and the complement is in EPO^c.

The second and main class of homogeneous ordered graphs with strong amalgamation are the Henson graphs H_n, characterized among homogeneous graphs by the minimal forbidden structure K_n (a complete graph on n vertices), together with their complements, characterized similarly; and also the random graph Γ_∞, which is the universal homogeneous graph and the Fraïssé limit of the class of all graphs. The complement of this graph is the same graph, up to isomorphism, since the complement of a graph is a graph.

The Henson graphs and their complements, with a generic linear order, appear only under the heading *LG*. The random graph appears also as the random tournament, but under the present heading we can treat Henson graphs and the random graph simultaneously.

We have encountered Table 3.1 already in §1C (p. 68). But some of the notation used remains to be explained here.

We subdivide the class *EPO* according to the type of underlying partial order $\mathcal{P}$ as follows:

$\mathbf{EPO}_0$: $\mathcal{P}$ is linear or a set of incomparable elements;
$\mathbf{EPO}_\perp$: incomparability is a nontrivial equivalence relation;
$\mathbf{EPO}_\to$: comparability is a nontrivial equivalence relation;
$\mathbf{EPO}_g$: $\mathcal{P}$ is the homogeneous universal partial order.

The labels in the table keep track of graph complementation: e.g., II.1^c is the graph complement of II.1 (or its reversal, if we use the language of ordered tournaments). Similar notation is employed in the list of forbidden substructures, given next.

Note that the definitions of Groups I, II, III given in the catalog, in terms of specific structures which are either required or forbidden, are mutually exclusive and exhaustive, and correspond fairly well to a meaningful division of the classification problem.

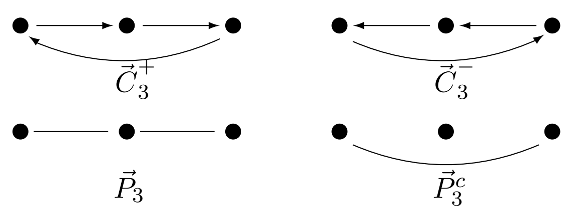
It remains to go over the notation for the constraint graphs which is used in the third column of the table.

I	*Graphs Omitting $\vec{I}_2$ or $\vec{K}_2$*		
Label	*Structure*	*Forbidden*	*Type*
I.1	$\lvert\Gamma\rvert = 1$	$\vec{K}_2$, $\vec{I}_2$	Triv
I.2	$(\mathbb{Q}, <) = \vec{K}_\infty$	$\vec{I}_2$	EPO_0, LT, LG
I.2^c	$(\mathbb{Q}, >) = \vec{I}_\infty$	$\vec{K}_2$	EPO_0, LT, LG
II	*Graphs containing $\vec{I}_2$ and $\vec{K}_2$, but not both $\vec{C}_3^+$ and $\vec{C}_3^-$*		
Label	*Structure*	*Forbidden*	*Type*
II.1	$\mathbb{Q}[\mathbb{Q}^{\mathrm{op}}] = \vec{K}_\infty[\vec{I}_\infty]$	$\vec{C}_3^+$,$\vec{I}_1 \perp \vec{K}_2$, $\vec{K}_2 \perp \vec{I}_1$, and $\vec{C}_3^-$	$\mathrm{EPO}_\perp$
II.2	Generic permutation	$\vec{C}_3^+$, $\vec{C}_3^-$	LT
II.3_n	$\vec{I}_n * \vec{K}_\infty$ dense, with each class dense ($n \cdot \mathbb{Q}$, shuffled); $n \geq 2$	$\vec{C}_3^+$, $[\vec{I}_1, \vec{I}_2]$, $[\vec{I}_2, \vec{I}_1]$ and $\vec{I}_{n+1}$ (if $n < \infty$)	$\mathrm{EPO}_\to$
II.4	$\vec{\mathcal{P}}$=Generic linear extension of generic p.o.	$\vec{C}_3^+$	EPO_g
II.1^c	$\mathbb{Q}^{\mathrm{op}}[\mathbb{Q}] = \vec{I}_\infty[\vec{K}_\infty]$	$\vec{C}_3^+$,$[\vec{I}_1, \vec{I}_2]$, $[\vec{I}_2, \vec{I}_1]$, $\vec{C}_3^-$	$\mathrm{EPO}_\perp^c$
II.3_n^c	$\vec{K}_n * \vec{I}_\infty$ dense, with each class dense	$\vec{C}_3^-$, $\vec{I}_1 \perp \vec{K}_2$, $\vec{K}_2 \perp \vec{I}_1$ and $\vec{K}_{n+1}$ (if $n < \infty$)	$\mathrm{EPO}_\to^c$
II.4^c	Reversal (complement) of II.4	$\vec{C}_3^-$	EPO_g^c
III	*Graphs containing both $\vec{C}_3^+$ and $\vec{C}_3^-$*		
Label	*Structure*	*Forbidden*	*Type*
IIIA	$\vec{\mathbb{S}}$=Generically ordered $\mathbb{S}$	$[I_1, C_3]$ and $[C_3, I_1]$ (all ordered forms)	LT
IIIB_n	$\vec{H}_n$=Generically ordered Henson graph ($n < \infty$)	$\vec{K}_{n+1}$	LG
IIIB_n^c	$\vec{H}_n^c$	$\vec{I}_{n+1}$	LG
IIIC	$\vec{\Gamma}_\infty$=Generically ordered random graph	none	LT, LG

TABLE 3.1. The Homogeneous Ordered Graphs

Notation 3.1 (CONSTRAINT GRAPHS, ORDERED AND UNORDERED).

1. $\vec{C}_3^+$ and $\vec{C}_3^-$ are the two ordered forms of the tournament C_3, which is a 3-cycle. Both may be represented as (a, b, c) with $a < b < c$, where we take $a \to b \to c \to a$ in the positive orientation, while $c \to b \to a \to c$ in the negative orientation. Viewed as graphs, $\vec{C}_3^+$ becomes an oriented path $\vec{P}_3$ with a, b adjacent to c, and $\vec{C}_3^-$ becomes its graph complement $\vec{P}_3^c$.



Ordered Tournaments and Ordered Graphs

2. $\vec{I}_n$ and $\vec{K}_n$ are an ordered independent set and an ordered clique of order n, respectively. When $n = \infty$, the order is generally assumed to be of type $(\mathbb{Q}, <)$.

3. When A, B are ordered tournaments, $[A, B]$ denotes the disjoint union of A and B with $A < B$ and $A \to B$, while $A \perp B$ denotes the disjoint union of A and B with $A < B$ and $B \to A$. In terms of ordered graphs, these conventions translate to the following: $A \perp B$ is the disjoint union of A and B with $A < B$ and no additional edges, and $[A, B]$ denotes the disjoint union of A and B with $A < B$ and all pairs in $A \times B$ or $B \times A$ added as edges.

4. In addition to $\vec{C}_3^+$ and $\vec{C}_3^-$, the graphs $\vec{I}_1 \perp \vec{K}_2$, $\vec{K}_2 \perp I_1$, and their complements $[\vec{I}_1, \vec{I}_2]$, $[\vec{I}_2, \vec{I}_1]$ are important constraints, either as forbidden or required substructures.

5. We also use the notations $[I_1, C_3]$ and $[C_3, I_1]$ for the *unordered* tournaments consisting of the disjoint union $I_1 \cup C_3$ with I_1 dominating or dominated by C_3, respectively. Each of these corresponds to 8 distinct ordered forms. Similarly, we write $I_1 + K_2$ for the *unordered* disjoint union of I_1 and K_2. This has three ordered forms, namely $\vec{I}_1 \perp \vec{K}_2$, $\vec{C}_3^-$, and $\vec{K}_2 \perp \vec{K}_1$.

6. The *reversal* of an ordered graph is the same graph with the order reversed. When we view ordered graphs as ordered tournaments, this operation corresponds to reversing both the order and the tournament relation (arcs). If we reverse only the order in an ordered tournament, this would correspond to taking the graph complement.

7. In the context of ordered tournaments, $\mathbb{Q}$ denotes $(\mathbb{Q}, <, <)$ and $\mathbb{Q}^{\mathrm{op}}$ denotes $(\mathbb{Q}, <, >)$. This is not the "reversal" of $\mathbb{Q}$ in the sense we have just introduced, and this particular notation will only be used in the catalog.

The classification corresponding to groups (I, II) is known, though not typically thought of as a class of ordered graphs. The characterization of Case $(IIIA)$ will be handled by direct means, at considerable length. Once we get completely clear of that case we will find ourselves in the generic case, where on the one hand the language of ordered graphs is natural, and on the other hand Lachlan's development of the Lachlan/Woodrow technique applies. The treatment of this case will occupy the bulk of Part I; it begins in Chapter 5.

3C. Homogeneous ordered graphs omitting $\vec{C}_3^+$ or $\vec{C}_3^-$

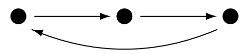
The result of Dolinka and Mašulović [2012] reads as follows, when formulated in the language of ordered tournaments, and including the complementary case.

Proposition 3.2. Let Γ be a countable homogeneous ordered tournament omitting $\vec{C}_3^+$ or $\vec{C}_3^-$. Then Γ falls into groups (I) or (II) of the catalog.

In the language of ordered graphs, $\vec{C}_3^+$ is a naturally ordered path on three vertices, denoted $\vec{P}_3$ since there are three vertices, and $\vec{C}_3^-$ is the graph complement $\vec{P}_3^c$. Since the class of homogeneous ordered graphs is closed under graph complementation, it suffices to consider the case in which Γ omits $\vec{C}_3^+$.

The ordered tournament $\vec{C}_3^+$ is as shown below, where the order goes from left to right.

We give the representation both as an ordered tournament and as an ordered graph.



$\vec{C}_3^+$

$\vec{P}_3$

To omit $\vec{C}_3^+$ is to say that the relation $\prec$ given by the intersection of the tournament relation with the ordering is transitive ($\vec{C}_3^+$ depicts a failure of transitivity). As $\prec$ is irreflexive and asymmetric, it is then a partial order. Thus Γ is a homogeneous linear extension of some partial order.

The classification of such structures is the subject of Dolinka and Mašulović [2012], giving the corresponding part of the list under Groups I and II, with the remaining entries obtained by graph complementation. The homogeneous permutations occupy Group I and positions $(II.1)$, $(II.2)$. The entries under $(II.3_n)$, and $(II.4)$ are characterized in Dolinka and Mašulović [2012].

Within these two groups, only the generic permutation needs to be viewed as an ordered expansion of a homogeneous tournament; up to complementation, all the other cases in the first two groups are expansions of homogeneous linear orderings.

With these two groups disposed of, we see that the only obstacle left before returning to the language of ordered graphs is the generically ordered local order.

CHAPTER 4

THE GENERICALLY ORDERED LOCAL ORDER

4A. Statement of the problem

The goal of the present section is the following.

Proposition 4.1. Let Γ be a countable homogeneous ordered tournament containing $\vec{C}_3^+$, $\vec{C}_3^-$, and $\vec{I}_3$, and not containing $\vec{I}_1 \perp \vec{C}_3^+$. Then Γ is the generically ordered local order $\vec{\mathbb{S}}$.

We go over the notation and terminology used here.

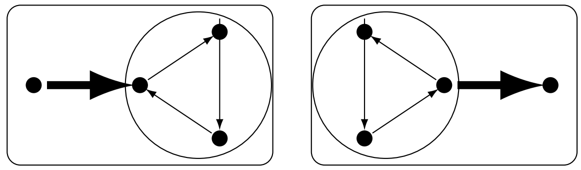
We write I_n for an independent set of n vertices (I_1 will be our usual name for the trivial graph), and $\vec{I}_n$ is our notation for the ordered version (note that this is rigid). In terms of ordered tournaments, the arc relation on $\vec{I}_n$ is the reverse of the ordering, or in other words this is the permutation in reverse order. Its graph complement $\vec{K}_n$ corresponds to the identity permutation in which the order and tournament relations coincide.

Definition 4.2. A *local order* is a tournament such that for every vertex v_*, the vertices dominating, or dominated by, v_* are linearly ordered by the arc relation. In other words, the tournament must omit the two tournaments

$$(I_1 \to C_3) \text{ and } (C_3 \to I_1)$$

of order 4 in which a vertex dominates, or is dominated by, a cyclic tournament of order 3 (Figure 4).

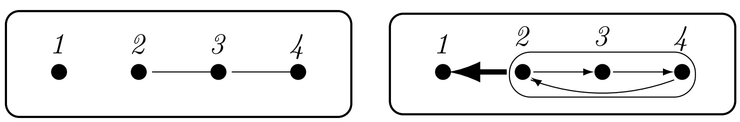
We recall that the notation $A \perp B$ denotes the ordered disjoint sum of two ordered graphs, and in terms of tournaments this means that B dominates A. When we want to emphasize the tournament structure we write $A \leftarrow B$ for this.



IC_3 C_3I

Figure 4. *Forbidden tournaments*

The structure "$\vec{I}_1 \perp \vec{C}_3^+$" has two other natural names, one when viewed as an ordered graph and one when viewed as an ordered tournament: namely, "$I_1 \perp \vec{P}_3$" as an ordered graph, and "$1 \leftarrow \vec{C}_3^+$" as an ordered tournament. Our preferred notation "$\vec{I}_1 \perp \vec{C}_3^+$" mixes these two points of view.



$I_1 \perp \vec{P}_3$ $I_1 \leftarrow \vec{C}_3$

In the statement of Proposition 4.1, the hypothesis that $\vec{I}_3$ is present is innocuous. As Γ is nontrivial, ordered, and homogeneous, it is infinite, and therefore contains either $\vec{K}_\infty$ or $\vec{I}_\infty$. By symmetry we are free to assume the latter—but we do not need to assume so much, and $\vec{I}_3$ will suffice. Without that assumption, we would have to consider the generically ordered complement of the generic triangle free graph at this point, which is irrelevant since that ordered graph will be covered later by a characterization of its complement.

We may convert the homogeneous ordered tournament $\vec{\mathbb{S}}$ to a homogeneous ordered graph by the usual process of taking as edges those pairs (a, b) for which the order and the tournament structure agree. One way of gauging the complexity of this graph is to ask for the minimal language which makes it homogeneous.

We now rephrase Proposition 4.1 in more concrete terms.

Definition 4.3. Let $\mathcal{S}$ be the set of all finite ordered tournaments which embed into every homogeneous ordered tournament which contains $\vec{C}_3^+$, $\vec{C}_3^-$, and $\vec{I}_3$, and omits $\vec{I}_1 \perp \vec{C}_3^+$.

Proposition 4.1 can be restated as follows: $\mathcal{S}$ consists of all finite ordered tournaments for which the underlying tournament is a local order—for which "$\mathcal{S}$" would be a very natural notation; at the moment our choice of notation expresses some optimism.

We will now begin to work toward the proof of Proposition 4.1 in the form just stated. That is, we will show that various ordered tournaments belong to $\mathcal{S}$.

4B. Permutations

Our first objective is the following, to be reached by stages.

LEMMA 4.4. *Every finite permutation belongs to* $\mathcal{S}$*.*

We recall that permutations are structures with two orders.

The next lemma is the first in a series of explicit amalgamation arguments. We will be moving away from a pictorial representation of such arguments to a tabulation of their combinatorial content, but here we mix the two approaches so as to illustrate both styles of argument. We will take a more pictorial approach in Part II, but in the present part the arguments are too lengthy for that style of presentation, until we reach the more substantial arguments relating to Propositions I-V below.

After working through the next lemma we will arrive at a more compact way of representing such arguments.

The pictures are most easily drawn (and understood) as ordered graphs.

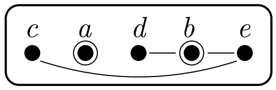
LEMMA 4.5. $\vec{I}_1 \perp [\vec{I}_2, \vec{I}_1]$ *belongs to* $\mathcal{S}$*.*

PROOF. We use a series of explicit amalgamations. We begin with the following; the notation is explained below.

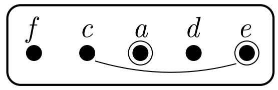
In tabular form this reads as follows.

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(adbe)$ #	$(cadb)$ $\square$	(A)	$(cade)$ $(*)_1$	$(fcae)$ $\square$
$(A1)$	$(A)_1$	$(*)_1$	(B)	#	$(*)_2$

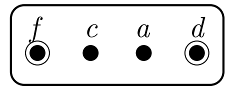
We pause to explain the graphical and tabular conventions, after which we will be in a position to continue the analysis, and fill in the details.



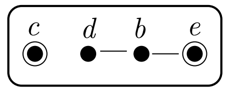
$(*)$



A



A_1



B

$A2 \ldots$

Explanation:

Each diagram shows a 2-point amalgamation problem. The two points whose type is to be determined are circled.

The order goes from left to right, e.g. $(c < a < d < b < e)$ in $(*)$; the order between a and b is one of the points to be determined in the amalgam, but in the majority of cases there will be a point in the base in between, as here, so the order is not actually in doubt.

The edges represent an ordered graph; one may recover the tournament description from the edges and the order. The graph notation is more legible, but in our context it is more natural to think of these diagrams as an efficient representation of certain ordered tournaments.

Now consider the first diagram $(*)$. The base of the amalgamation is cde to which we add the points a and b to get the factors $(acde)$ and $(bcde)$ of the amalgam, subsequently denoted by $(*)_1$ and $(*)_2$.

If we suppose that the factors $(*)_1$ and $(*)_2$ are present in Γ, then some amalgam of them is also present, as Γ is homogeneous. In this amalgam (a, b) is a non-edge or an edge, or possibly $a = b$ (though, as here, this is rarely a viable option).

In the first line of the accompanying table the entry for $(*)$ contains two pieces of information (2nd and 3rd columns).

- If (a, b) is a non-edge then $(adbe)$ gives a contradiction: namely, it has the form $\vec{I}_1 \perp \vec{C}_3^+$.
- If (a, b) is an edge then $(cadb)$ gives the configuration which is the subject of the lemma—namely, $\vec{I}_1 \perp [\vec{I}_2, \vec{I}_1]$.

The possibility $a = b$ is eliminated by inspection of the vertex d, and is not mentioned in the table.

At this point, the diagram $(*)$, or the first entry in the table, shows that it is sufficient to embed $(cade)$ and $(cdbe)$ as shown in $(*)$ into Γ to conclude the proof. These two configurations, which we call $(*)_1$

and $(*)_2$ respectively, are the subject of diagrams (A) and (B), which show two further amalgamation problems, each with its own factors.

The annotation for (A) in the table indicates that a non-edge produces the first factor of $(*)$, while an edge produces the configuration which is the subject of the lemma. Either will suffice at this point. The annotation for (B) indicates that the amalgam produces either a contradiction or the required (second) factor of $(*)$.

It remains to be checked that the factors of the diagrams (A, B) are embeddable in Γ. The factors of (B) are of order 3 and are present in Γ by hypothesis, so the table offers no further discussion of this point.

This leaves the factors of (A) for further consideration, and the analysis continues.

The diagram $(A1)$ shows the amalgam that produces the first factor of (A) (or else the first factor of $(*)$, which is even better). Since the relevant piece of $(A1)$ in each case is the whole of $(A1)$, we do not need to specify this piece in the table. Again, the factors of $(A1)$ are available by hypothesis.

This leaves us with the second factor of (A) to consider. The notation $(A2\ldots)$ means that this will be treated below.

In the vast majority of cases, but not all, the two vertices whose relation is to be determined in our diagrams are separated in the order by at least one vertex; therefore the order relation between them is known, and in particular they cannot be made equal. When this is not the case, we must consider the order of the pairs as well as the presence or absence of an edge; and also verify that some element in the base prevents them from being identified.

We have seen from these diagrams and the notes in the table that to complete the proof it will be sufficient to show either that the factor $(A_2) = (fcde)$ of (A) is present in Γ, or else that the first factor $(*)_1$ or the target configuration $\vec{I}_1 \perp [\vec{I}_2, \vec{I}_1]$ is present.

We may now proceed with the analysis, as follows, arriving again at a configuration that requires further attention $(A2.1.2)$ (see Figure 7, Table 4.1).

The table should provide adequate commentary on the pictures, in the manner described above. Diagram $A2$ will produce either the second factor A_2, or the factor $(*)_1$ itself, and the second factor of diagram $A2$ is to be produced by the simple amalgam shown as $(A2.2)$.

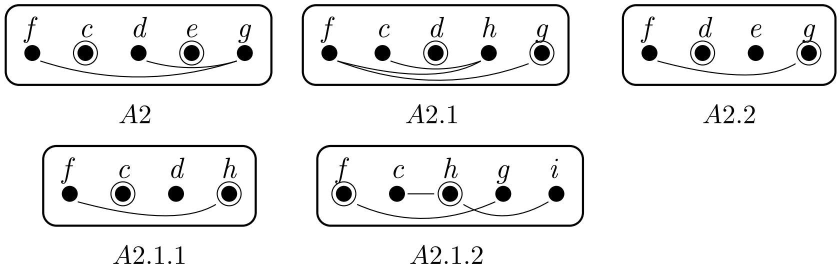


Figure 7. $A2$

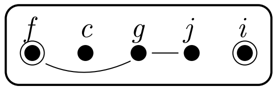
Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(A2)$	$(fceg)$ $(*)_1$	$(fcde)$ (A_2)			
$(A2.1)$	$(fcdg)$ $(*)_1$	$(fcdg)$ $(A2)_1$	$(A2.2)$	$(A2)_2$	$(*)_1$
$(A2.1.1)$	$(*)_1$	$(A2.1)_1$	$(A2.1.2)$	$(fchi)$ #	$(fchf)$ $(A2.1)_2$

Table 4.1. Amalgamation: $A2$

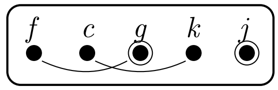
The first factor $(A2.1) = (fcdg)$ of $A2$ requires more attention. It will be obtained from the diagram $A2.1$, or else $(*)_1$ will be obtained directly. The two factors of $A2.1$ are to be produced by the amalgams shown in $(A2.1.1)$ and $(A2.1.2)$. The factor $(A2.1.1)$ requires no further discussion, but now both factors of $(A2.1.2)$ require further consideration: $(fcgi)$ and $(chfi)$.

These two configurations are dealt with by the two sets of diagrams set out below, with justifications indicated in the accompanying tables (we have abbreviated the identifying labels, which are keyed to the diagram as previously).

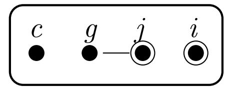
One noteworthy point is that in the amalgamation indicated in entry $(fcgi-2)$, below, the order type of the pair (i, j) is not determined, but these two elements must remain distinct in the amalgam. As a result the table has two entries for this diagram, one for the case $(i < j)$, the other for the case $(j < i)$.



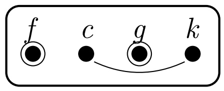
$(fcgi)$



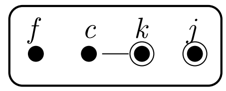
$(fcgi)_1$



$(fcgi)_2$

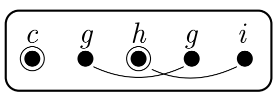


$(fcgi)_{1.1}$

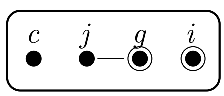


$(fcgi)_{1.2}$; see $(fcgi)_2$

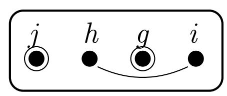
Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(fcgi)$	$(fcgi) =$ $(A2.1.2)_1$	$(fcji)$ $(*)_1$	(-1)	$(fcgj) =$ $(A2.1.2)_1$	$(fcgj) =$ $(fcgi)_1$
(-2) $(j < i)$	$(fcgi)_2$	#	(-2) $(i < j)$	$(A)_2$	□
(-1.1)	$(A)_2$	$(fcgi - 1)$			



$(chgi)$



$(chgi)_1$



$(chgi)_2$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(chfi)$	$(A)_2$	$(A2.1.2)_2$			
(-1) $(g < i)$	$(chfi)_1$	#	(-1) $(i < g)$	$(A2)_2$	□
(-2)	$(A2)_2$	$(chfi)_2$			

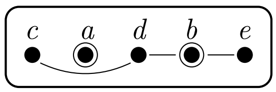
As these last cases illustrate, we are exploring a tree of possibilities, and some possible outcomes climb back toward the root rather than to the immediate parent.

This completes the analysis. □

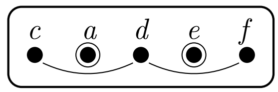
We will continue to argue in this style, mixing diagrams and some tabulated documentation, throughout most of this part.

LEMMA 4.6. *If a homogeneous ordered tournament contains both* $(\vec{C}_3^- \to 1)$ *and* $\vec{C}_3^+$*, then* Γ *contains* $\vec{I}_1 \perp \vec{C}_3^+$.

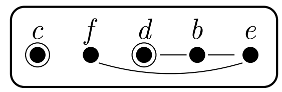
PROOF. The construction is summarized as follows.



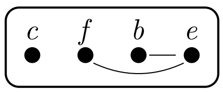
$(*)$



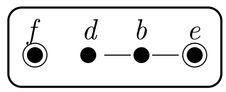
$A \ldots$



B



$B.1$



$B.2$

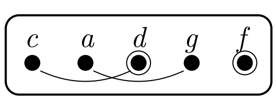
Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(adbe)$ $\square$	$(cabe)$ $\square$			
(A)	$(cade)$ $(*)_1$	$(cdef)$ $\square$	(B)	$(cdbe)$ $\square$	$(cdbe)$ $(*)_2$
$(B.2)$	$\#$	$(B)_2$			

Here we must notice that $(B.1)$ is given by Lemma 4.5; more precisely, if we suppose toward a contradiction that Γ does *not* contain $\vec{I}_1 \perp \vec{C}_3^+$, then Lemma 4.5 applies.

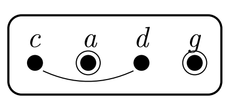
As the diagram indicates, we must still consider the two factors of (A), namely $(A)_1 = (cade)$ and $(A)_2 = (cdbe)$. And each will take some attention.

The configuration $(A)_1 = (cade)$

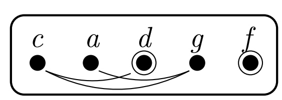
We consider two approaches to this factor, which we designate by $(A1a)$ and $(A1b)$. Both are shown below.



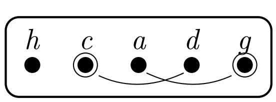
$A1a$



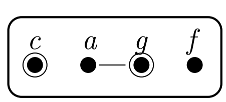
$A1a.1$



$A1b$



$A1b.1$



$A1b.2$

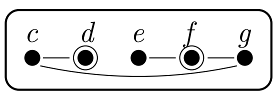
$A1a.2 \ldots$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(A1a)$	$(cadf)$ $(*)_1$	$(cadf)$ $(A)_1$	$(A1a-1)$	$(*)_1$	$(A1a)_1$
$(A1b)$	$(*)_1$	$(A)_1$			
$(A1b.1)$	$(hcdg)$ $(A1a)_2$	$(cadg)$ $(A1b)_1$	$(A1b.2)$	$(A1a)_2$	$(A1b)_2$

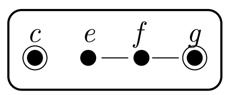
The amalgamation beginning with $(A1a)$ succeeds if the factor $(cagf)$ (not shown) is present in Γ.

On the other hand the amalgamation beginning with $(A1b)$ either succeeds, or produces that factor. So we do not need to follow up on the missing diagram $(A1a.2)$, as either way the analysis along the branch beginning at $(A)_1$ is complete.

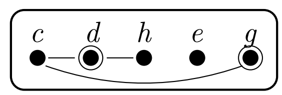
The configuration $(A)_2 = (cdef)$.



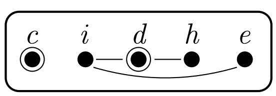
$A2$



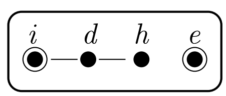
$A2.2$



$A2.1$



$A2.1.1$



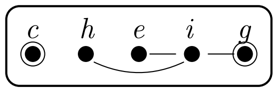
$A2.1.1.2$

$A2.1.2 \ldots$

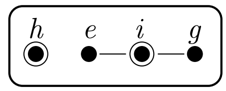
Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(A2)$	$(defg)$ $\square$	$(cdef)$ $(A)_2$	$(A2.2)$	$\square$	$(A2)_2$
$(A2.1)$	$(cdeg)$ $(A2)_1$	$(dheg)$ $(A2)_1$			
$(A2.1.1)$	$(cidh)$ $\square$	$(cdhe)$ $(A2.1)_1$	$(A2.1.1.2)$	$(A2.1)_1$	$(A2.1.1)_2$

Since diagram $(A2.2)$ has factors in Γ by hypothesis, we are concerned only with $(A2.1)$, and we will treat the second factor $(cheg)$ of $(A2.1)$ below. One factor of $(A2.1.1)$ is $(cihe) \cong \vec{C}_3^- \to 1$, which is assumed to embed into Γ. So there remains only $(A2.1.1.2)$, whose factors also embed into Γ.

Thus we need only consider the configuration $(A2.1)_2 = (cheg)$, handled as shown.



*A*2.1.2



*A*2.1.2.2

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(A2.1.2)$	$(ceig)$ □	$(cheg)$ $(A2.1)_2$	$(A2.1.2.2)$	□	$(A2.1.2)_2$

The first factor $(chei)$ of $(A2.1.2)$ is handled by Lemma 4.5 as before, hence is not shown separately.

This completes the analysis. □

LEMMA 4.7. *If the homogeneous ordered tournament* Γ *contains* $\vec{I}_3$, $\vec{C}_3^+$, *and* $\vec{C}_3^-$, *but not* $\vec{C}_3^+ \to 1$, *then* Γ *contains* $\vec{I}_2 \perp \vec{K}_2$.

PROOF. By the previous lemma Γ does not contain $(\vec{C}_3^- \to 1)$. Hence for $a \in \Gamma$, the ordered tournament induced on

$$a^{\perp +} = \{v \in \Gamma \mid v \to a, a < v\}$$

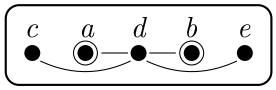
does not contain $\vec{C}_3^+$ or $\vec{C}_3^-$ and must be a homogeneous permutation.

By Lemma 4.5 the ordered tournament on $a^{\perp +}$ also contains $[\vec{I}_2, \vec{I}_1]$, which as a permutation is the pattern $(2, 1, 3)$, i.e. the first order is $1 < 2 < 3$ and the second order is $2 < 1 < 3$. By the classification of homogeneous permutations, $a^{\perp +}$ is either the permutation of type $\mathbb{Q}[\mathbb{Q}^{\mathrm{op}}]$ or the generic permutation. In particular, $a^{\perp +}$ must contain $[\vec{I}_1, \vec{I}_2]$, the pattern $(1, 3, 2)$, and thus Γ contains $\vec{I}_1 \perp [\vec{I}_1, \vec{I}_2]$.

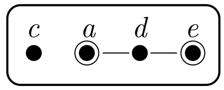
Now we perform explicit amalgamations. We begin as follows.

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(cabe)$ □	$(cabe)$ #	(A)	#	$(*)_1$
(B)	$(fdbe)$ #	$(cdbe)$ $(*)_2$	$(B1)$	$(B)_1$	$(*)_2$
$(B2)$	$(fcbe)$ $(B)_2$	$(gfce)$ #			

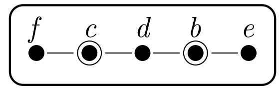
Now we must provide the factors for the diagram $(B2)$, namely $(B2)_1 = (gfcb)$ and $(B2)_2 = (gfbe)$.



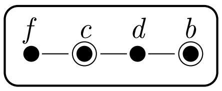
(∗)



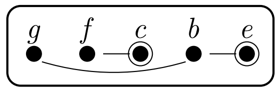
A



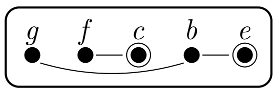
B



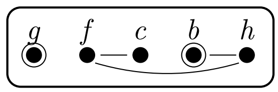
$B1$



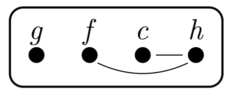
$B2 \dots$



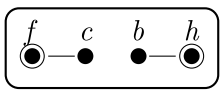
$B2$



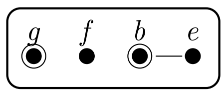
$B2.1$



$B2.1.1 \dots$



$B2.1.2$



$B2.2$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(B2)$	$(fcbe)$ $(B)_2$	$(gfce)$ #	$(B2.1)$	$(gcbh)$ □	$(gfcb)$ $(B2)_1$
$(B2.1.2)$	$(B)_2$	$(B2.1)_2$	$(B2.2)$	□	$(B2)_2$

The configuration $(B2.1.1) \cong \vec{I}_1 \perp [\vec{I}_2, \vec{I}_1]$ is given by Lemma 4.5. □

PROOF OF LEMMA 4.4. By Lemma 4.6, Γ does not contain the configuration $(\vec{C}_3^- \to 1)$.

Take $a \in \Gamma$, and consider the homogeneous ordered tournament

$$a^{\perp+} = \{v \mid a < v, a \perp v\}$$

By assumption, this ordered tournament omits $\vec{C}_3^+$ and $\vec{C}_3^-$, hence is a homogeneous permutation.

By Lemma 4.5, $a^{\perp+}$ contains $[\vec{I}_2, \vec{I}_1]$. By Lemma 4.7, $a^{\perp+}$ contains $\vec{I}_1 \perp \vec{K}_2$. By the classification of homogeneous permutations, $a^{\perp+}$ is the generic permutation, hence contains every finite permutation. So the same applies to Γ. □

4C. $I_1 \to C_3$ and $C_3 \to I_1$

The generic local order $\mathbb{S}$ may be characterized variously as the homogeneous tournament which contains C_3 and omits $I_1 \to C_3$ (or its dual $C_3 \to I_1$; or—more naturally—both of these).

The tournament C_3 has the two ordered forms $\vec{C}_3^{\pm}$. The tournaments $I_1 \to C_3$ and $C_3 \to I_1$ each have eight ordered forms. A convenient notation for these ordered tournaments, 16 in all, is

$$i \to \vec{C}_3^{\pm},\ \vec{C}_3^{\pm} \to i$$

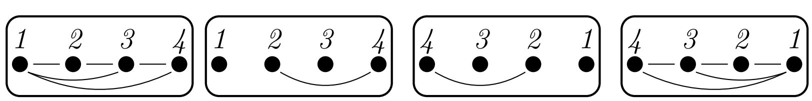
respectively, where $i \in \{1, 2, 3, 4\}$. Here i denotes the position within the four vertices of the vertex which corresponds to $\vec{I}_1$. For example, $\vec{I}_1 \perp \vec{C}_3^+$ is $(\vec{C}_3^+ \to 1)$, as we noted earlier.

Now in the language of ordered tournaments, the operations of reversal of the tournament relation or the ordering give an action of the Klein 4-group on this set of 16 ordered tournaments; in terms of ordered graphs, the group is generated by reversal and graph complement. We need to keep track of these actions in terms of our notation and also to check the translation procedure between tournaments and graphs, since for the present we will continue to use the language of ordered tournaments while displaying diagrams in the visually more compact language of ordered graphs.

We give this information in tabular form, taking the case of a tournament of the form $i \to \vec{C}_3^+$ as an example. We also write i' for the "opposite" of i, namely $i' = 5 - i$.

Transformations of $i \to \vec{C}_3^+$

Transform	Result	Ordered Graph Version
$(\to, <)$	$(i \to \vec{C}_3^+)$	(Original)
$(\to^{\mathrm{op}}, <)$	$(\vec{C}_3^- \to i)$	Graph Complement
$(\to, <^{\mathrm{op}})$	$(i' \to \vec{C}_3^-)$	Reversal of Complement
$(\to^{\mathrm{op}}, <^{\mathrm{op}})$	$(\vec{C}_3^+ \to i')$	Reversal



$1 \to \vec{C}_3^+$ $\quad$ $\vec{C}_3^- \to 1$ $\quad$ $4 \to \vec{C}_3^-$ $\quad$ $\vec{C}_3^+ \to 4$

Note also that after reversal of the order, the vertex "1" is renamed to "4," because it is in last position in the order. But in our diagrams the vertex marked 1 is the same one throughout. Also, the third diagram may look like the reversal of the second, because it is drawn as a graph rather than as a tournament (cf. the third column of the table).

The goal of this subsection is to show that if $\vec{I}_3$, $\vec{C}_3^+$, and $\vec{C}_3^-$ all occur, and $(\vec{C}_3^+ \to 1)$ does not, then none of the ordered forms of $I_1 \to C_3$ or $C_3 \to I_1$ occur. In particular this forces the underlying tournament to be a local order, and we begin to come within striking range of an explicit identification of our homogeneous ordered tournament as a generically ordered local order.

LEMMA 4.8. *Let* Γ *be a countable homogeneous ordered tournament. Then* Γ *contains* $(\vec{C}_3^+ \to 1)$ *if and only if* Γ *contains* $(4 \to \vec{C}_3^+)$.

PROOF. If Γ does not contain $\vec{C}_3^-$ then this follows from the classification in groups (I, II). So we suppose

$$\Gamma \text{ contains } \vec{C}_3^- \tag{4.1}$$

We show that if

$$\Gamma \text{ contains } (4 \to \vec{C}_3^+) \tag{4.2}$$

then Γ contains $(\vec{C}_3^+ \to 1)$. The converse then follows by duality (i.e., reversing the edge relation $\to$).

If Γ omits $(\vec{C}_3^+ \to 1)$ then Lemma 4.4 says that

$$\Gamma \text{ contains all permutations.} \tag{4.3}$$

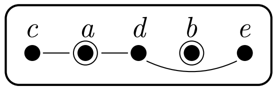
So we may assume that this holds as well.

Now argue as shown.

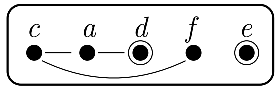
c e

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(cadb)$ □	$(cdbe)$ □	(A)	$(cade)$ □	$(cade)$ $(*)_1$
$(A1)$	□	$(A)_1$	(B)	$(dfbe)$ □	$(cdbe)$ $(*)_2$

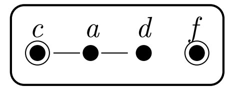
The factor $(A)_2 = (cafe)$ is the permutation (4132) and the factor $(B)_2 = (cfbe)$ is the permutation (4231), so these are afforded by Lemma 4.4. The factor $(B)_1 = (cdfb)$ is present by hypothesis. □



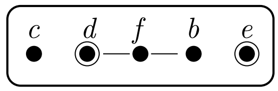
(∗)



A



$A1$



B

For the next lemma, recall that the reversal as an ordered graph is obtained by reversing both the order and the tournament relation. We will denote this operation by $X^{\mathrm{op-og}}$.

Corollary 4.8.1. Let X be a finite ordered tournament which is in $\mathcal{S}$. Then the reversal $X^{\mathrm{op-og}}$ is also in $\mathcal{S}$.

LEMMA 4.9. *Let Γ be a countable homogeneous ordered tournament which contains $\vec{I}_3$, $\vec{C}_3^+$, and $\vec{C}_3^-$, and does not contain $(\vec{C}_3^+ \to 1)$. Then Γ does not contain any of the following.*

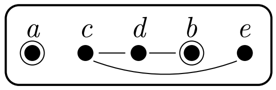
(a) $(\vec{C}_3^- \to 1)$, $(4 \to \vec{C}_3^-)$;
(b) $(3 \to \vec{C}_3^+)$, $(\vec{C}_3^+ \to 2)$;
(c) $(3 \to \vec{C}_3^-)$, $(\vec{C}_3^- \to 2)$

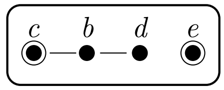
PROOF. These are listed in pairs, with the second configuration being the reverse of the first one. So it suffices to consider the first entry in each pair. We will suppose that Γ contains one of the three configurations

$$(\vec{C}_3^- \to 1),\ (3 \to \vec{C}_3^+),\ (3 \to \vec{C}_3^-)$$

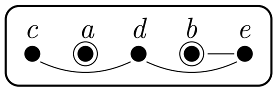
and show that Γ then contains $(\vec{C}_3^+ \to 1)$—or, equivalently, that Γ contains a copy of $(4 \to \vec{C}_3^+)$.

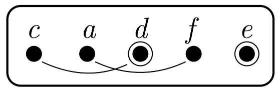
If Γ contains the first configuration shown in any of clauses (a, b, c) above, then the relevant amalgamation is shown in the corresponding line of the following.

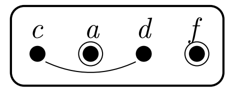




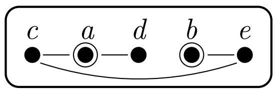
(a) aB

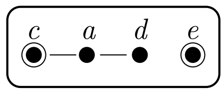






(b) (bA) $bA1$





(c) cA

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
(a)	$(abcd)$	$(abde)$	(aB)	$(4 \to \vec{C}_3^{\,+})$	$(a)_2$
		$(4 \to \vec{C}_3^{\,+})$		#	
		#			
	$(\vec{C}_3^{\,+} \to 1)$				
	#				
(b)	$(cadb)$	$(cabe)$	(bA)	$(cade)$	$(b)_1$
	$(\vec{C}_3^{\,-} \perp \vec{I}_1)$	$(\vec{C}_3^{\,+} \to 1)$			$(\vec{C}_3^{\,-} \perp \vec{I}_1)$
	(a)	#			(a)
$(bA1)$	$(4 \to \vec{C}_3^{\,+})$	$(*)_1$			
	#				
(c)	$(cadb)$	$(adbe)$	(cA)	$(4 \to \vec{C}_3^{\,+})$	$(c)_1$
				#	
	$(4 \to \vec{C}_3^{\,+})$	$(\vec{C}_3^{\,+} \to 2)$			
	#	(b)			

This requires additional commentary.

Case (a). The amalgam (a) yields $(\vec{C}_3^{\,+} \to 1)$ or $(4 \to \vec{C}_3^{\,+})$, and a contradiction. The factor $(a)_1 = (acde)$ is $(\vec{C}_3^{\,-} \to 1)$, which is assumed present in case (a).

Case (b). We assume we do not fall under Case (a) and therefore neither $(\vec{C}_3^{\,-} \to 1)$ nor $(\vec{C}_3^{\,-} \perp \vec{I}_1)$ embeds in Γ. Where one form of the amalgam leads back to case (a) that is marked in the accompanying table above.

The factor $(b)_1 = (cdbe)$ is $(3 \to \vec{C}_3^+)$ which is assumed present in Case (b). The factor $(bA)_2 = (cafe)$ is the permutation (4231), present in Γ.

Case (c). We may suppose we are not in case (a) or (b). The factor $(c)_2 = (cdbe)$ is $(3 \to \vec{C}_3^-)$, given by the case hypothesis.

This completes the proof. □

We tabulate the cases we have treated so far as follows: if Γ is a homogeneous ordered tournament containing $\vec{I}_3$, $\vec{C}_3^+$, and $\vec{C}_3^-$, but not $(\vec{C}_3^+ \to 1)$, then each of the following eight configurations is omitted.

$$(1)\ \vec{C}_3^{\pm} \to 1 \text{ or } 2; \qquad (2)\ 3 \text{ or } 4 \to \vec{C}_3^{\pm}$$

If we also had their graph complements, this would give all the configurations of type $I_1 \to C_3$ or $C_3 \to I_1$. To get closure under complementation (as for reversal) it would suffice to show that the configuration $(\vec{C}_3^+ \to 1)^c = (1 \to \vec{C}_3^-)$ can be added to this list of excluded ordered tournaments. But we do not see a very direct route to this particular claim, so we proceed much more gradually.

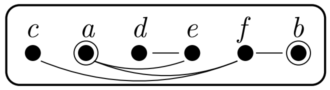
LEMMA 4.10. *Let Γ be a countable homogeneous ordered tournament which contains $\vec{I}_3$, $\vec{C}_3^+$, and $\vec{C}_3^-$, and does not contain $(\vec{C}_3^+ \to 1)$. Then Γ does not contain any of the following.*

(a) $(\vec{C}_3^- \to 3)$ *or* $(2 \to \vec{C}_3^-)$
(b) $(2 \to \vec{C}_3^+)$ *or* $(\vec{C}_3^+ \to 3)$

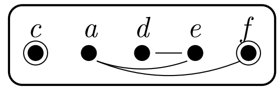
PROOF. Again we give these in pairs, with the second configuration the reverse of the first. We assume that Γ is a homogeneous ordered tournament containing $\vec{I}_3$, $\vec{C}_3^+$, and $\vec{C}_3^-$, and one of the configurations $(\vec{C}_3^- \to 3)$ or $(2 \to \vec{C}_3^+)$, and we derive one of the eight configurations $(\vec{C}_3^{\pm} \to 1, 2)$ or $(3, 4 \to \vec{C}_3^{\pm})$.

By Lemma 4.4, Γ contains every finite permutation.

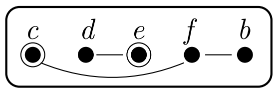
(a) : We begin by assuming $(\vec{C}_3^- \to 3)$ is in Γ.



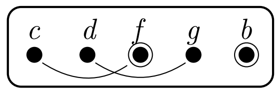
(a)



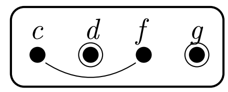
aA



aB



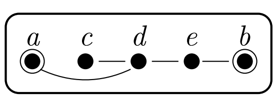
$aB2$



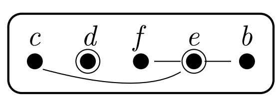
$aB2.1$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
(a)	$(aefb) = (\vec{C}_3^+ \to 2)$ #	$(cadb) = (\vec{C}_3^- \to 1)$	(aA)	$(cadf) = (\vec{C}_3^- \to 1)$ #	$(cadf) = (a)_1$ #
$(aA)_1$	Permutation: (3241)		$(aA)_2$	$(\vec{C}_3^- \to 3)$	
(aB)	$(cdef) = (a)_2$	$(cefb) = (\vec{C}_3^+ \to 2)$	$(aB)_1$	Permutation: (3412)	
$(aB2)$	$(cdfb) = (4 \to \vec{C}_3^-)$	$(cdfb) = (aB)_2$			
$(aB2.1)$	$(4 \to \vec{C}_3^-)$	$(aB2)_1$	$(aB2)_2$	Permutation (4231)	

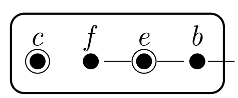
(b): Now we may suppose that $(2 \to \vec{C}_3^+)$ is present, and that we are not in case (a).



(b)



bB



$bB2$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
(b)	$(adeb)$ $(\vec{C}_3^+ \to 1)$ #	$(acdb)$ $(\vec{C}_3^- \to 3)$ #	$(b)_1$	$(2 \to \vec{C}_3^+)$	
(bB)	$(dfeb)$ $(\vec{C}_3^+ \to 1)$	$(cdeb)$ $(b)_2$	$(bB)_1$	Permutation: (4312)	
$(bB2)$	$(\vec{C}_3^+ \to 1)$	$(bB)_2$			

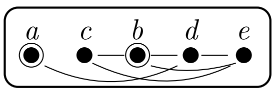
□

Our list of known forbidden configurations now runs as follows, with 12 entries.

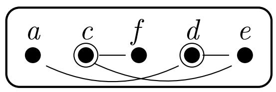
1. $\vec{C}_3^{\pm} \to 1, 2, 3$;
2. $2, 3, 4 \to \vec{C}_3^{\pm}$.

LEMMA 4.11. *Let Γ be a countable homogeneous ordered tournament which contains $\vec{I}_3$, $\vec{C}_3^+$, and $\vec{C}_3^-$, and does not contain $(\vec{C}_3^+ \to 1)$. Then Γ does not contain $(\vec{C}_3^+ \to 4)$ or $(1 \to \vec{C}_3^+)$.*

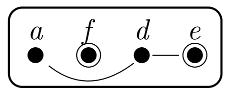
PROOF. We suppose that Γ contains $\vec{I}_3$, $\vec{C}_3^-$, and $(\vec{C}_3^+ \to 4)$. We will show that Γ contains $(\vec{C}_3^+ \to 1)$. This proves the first point and the dual follows as usual by reversal. By Lemma 4.4, Γ contains every finite permutation.



$(*)$



A



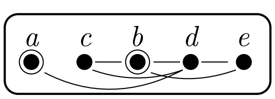
$A2$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(abde)$ $2 \to \vec{C}_3^+$ #	$(acbe)$ $(2 \to \vec{C}_3^+)$ #	(A)	$(acde)$ $(*)_1$	$(acde)$ $(2 \to \vec{C}_3^+)$ #
$(A)_1$	Permutation: (2431)		$(A2)$	$(A)_2$	$(*)_1$
$(*)_2$	$(\vec{C}_3^+ \to 4)$				

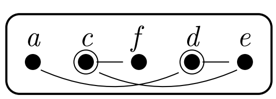
□

LEMMA 4.12. *Let Γ be a countable homogeneous ordered tournament which contains $(1 \to \vec{C}_3^-)$ and the permutation (3124). Then Γ contains $(2 \to \vec{C}_3^-)$.*

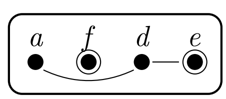
PROOF. We perform the following construction.



$(*)$



A



$A2$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(acbe)$ $(2 \to \vec{C}_3^-)$ $\#$	$(acdb)$ $(2 \to \vec{C}_3^-)$ $\#$			
$(*)_1$	Permutation: (2314)		$(*)_2$		$(1 \to \vec{C}_3^-)$

□

LEMMA 4.13. *Let Γ be a countable homogeneous ordered tournament containing $\vec{I}_3$, $\vec{C}_3^+$, and $\vec{C}_3^-$, and not containing $(\vec{C}_3^+ \to 1)$. Then Γ contains none of the ordered forms of the tournaments $I_1 \to C_3$ or $C_3 \to I_1$.*

PROOF. Putting together Lemmas 4.9, 4.10, and 4.11 excludes everything of the forms

$$(\vec{C}_3^\pm \to 1, 2, 3) \text{ or } 2, 3, 4 \to \vec{C}_3^\pm$$

as well as $(1 \to \vec{C}_3^+)$, $(\vec{C}_3^+ \to 1)$.

Since $(2 \to \vec{C}_3^-)$ is omitted, it follows from Lemma 4.12 that $(1 \to \vec{C}_3^-)$ is also omitted, hence also its graph complement $(\vec{C}_3^- \to 1)$. This accounts for all 16 ordered forms of $I_1 \to C_3$ and $C_3 \to I_1$. □

Corollary 4.13.1. Suppose that the finite ordered tournament X belongs to $\mathcal{S}$. Then the same applies to the graph complement X^c (as an ordered tournament, the order is kept the same, but the tournament relation is reversed.

PROOF. By Lemma 4.13, the constraints defining $\mathcal{S}$ may be taken to be closed under graph complementation, and the claim follows. □

4D. Characterization of the generically ordered local order

We will soon complete the proof of Proposition 4.1, characterizing Γ as the generic ordered tournament whose underlying tournament is a local order. The following lemma gives the basis for an inductive analysis via *forced amalgamations* (amalgamations with a unique admissible solution).

LEMMA 4.14. *Let Γ be a countable homogeneous ordered tournament which contains $\vec{I}_3$, $\vec{C}_3^+$, and $\vec{C}_3^-$, but does not contain $(\vec{C}_3^+ \to 1)$.*

Let A be an ordered tournament of order at most 4. Then A embeds into Γ if and only if the underlying tournament of A is a local order.

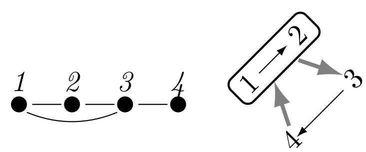
PROOF. Recall that the local orders are the tournaments not containing either of the tournaments $I_1 \to C_3$ or $C_3 \to I_1$. We have already shown that the underlying tournament of Γ is a local order. Thus it suffices to show that

any ordered local order with at most 4 *vertices embeds into* Γ

For ordered linear orders (permutations) the claim holds by Lemma 4.4. Accordingly we may restrict our attention to proper local orders A of order equal to 4, that is, those containing an oriented 3-cycle C_3.

Without the order, there is just one proper local order on 4 vertices, up to isomorphism, which may be described in "standard form" as having vertices $1, 2, 3, 4$ with $1 \to 2$ and

$$(1, 2) \to 3 \to 4 \to (1, 2)$$



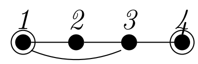
$S1234$

We will label this configuration ($S1234$), when the order is the natural one: $1 < 2 < 3 < 4$.

When there is another order on this tournament we assign the labels $(1, 2, 3, 4)$ according to the tournament structure, e.g., $(1, 2)$ denotes the edge which dominates one vertex and is dominated by the other. We thus have 24 variations $S\sigma$ labeled by the permutations σ of the vertices $1, 2, 3, 4$; that is the order of the vertices is given by the label σ.

We must show that the 24 configurations of this type all embed into Γ.

Suppose first that at least one of the pairs $(1, 4)$ or $(2, 3)$ are nonadjacent in the ordering, as is already the case in the standard form ($S1234$) with the pair $(1, 4)$. Then we claim that treating the diagram as an amalgamation problem with the ordering and tournament relations to be determined between the pair in question, there is a unique completion to an ordered local order. For example in the case of ($S1234$) we are considering the following.



$S1234$

Here the vertices $(1,4)$ are separated by a third vertex—in fact, by both of the remaining vertices. Hence their order is determined: $1 < 4$. Furthermore, taking $(1 \to 4)$ would produce the configuration $(1) \to (2,3,4)$ with $(2,3,4)$ a 3-cycle. As the amalgam must be an ordered local order, we conclude that $4 \to 1$ and the amalgam is unique.

Similarly if $(2,3)$ are nonadjacent in the order then the corresponding amalgamation is unique, as reversal of the edge $(2,3)$ would produce $(1,3,4) \to (2)$.

Thus $S\sigma$ embeds in Γ whenever at least one of the pairs $(1,4)$ or $(2,3)$ is separated in σ. This leaves only 8 of the original 24 possibilities to be considered.

σ	σ^{op}
($S2314$)	($S1423$)
($S3241$)	$S(4132)$
($S2341$)	($S1432$)
($S3214$)	($S4123$)

If these configurations are viewed as ordered graphs, then the right hand column contains the graph complements of the left hand column, and the second row contains the reversals (as ordered graphs) of the first row. The configurations in the third and fourth rows are invariant under reversal.

Therefore it suffices to prove that Γ contains the three configurations ($S2314$), ($S2341$), and ($S3214$).

We begin with ($S2314$) (Figure 24).

We proceed as shown.

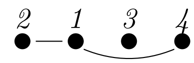
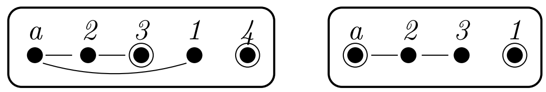


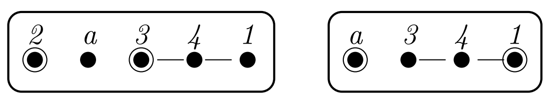
Figure 24. $S2314$



(∗) A

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
(∗)	$(a234)$ $(4 \to \vec{C}_3^+)$ #	$(S2314)$ □			
A	$(4 \to \vec{C}_3^+)$ #	$(*)_1$	$(*)_2$	Permutation (4132)	

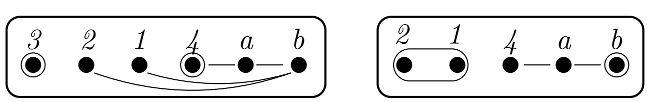
Now we consider $(S2341)$.



(∗) B

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
(∗)	(2341) $(\vec{C}_3^+ \to 1)$ #	(2341) $(S2341)$ □			
$(*)_1$	Permutation: (3241)		(B)	$(\vec{C}_3^+ \to 1)$ #	$(*)_2$

And now we deal with $(S3214)$.



(∗) B

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(34ab)$	(2341) $(S3214)$ $\square$			
	$(1 \to \vec{C}_3^+)$ $\#$				
$(*)_1$	Permutation: (43251)		(B)	$(24ab)$ or $(14ab)$: $(1 \to \vec{C}_3^+)$ $\#$	$(*)_2$

This concludes the proof. $\square$

Having disposed of all configurations with at most 4 vertices, we now treat the general case.

PROOF OF PROPOSITION 4.1. We must show that all finite ordered local orders embed into Γ (or in our earlier notation, $\mathcal{S}$ is the class of all finite ordered local orders).

We know that this holds if the ordered tournament in question has at most four vertices.

We show that any ordered local order on 5 points is the unique outcome by a forced amalgamation of tournaments on 4 points. In other words, we claim the following.

> If S is a local order on 5 points, and $<$ is a linear order of S, then there is an edge $a \to b$ in S with the following properties.
> - a, b are nonadjacent in the order;
> - Reversal of the edge $a \to b$ produces a copy of $I_1 \to C_3$ or of $C_3 \to I_1$.

Given this claim, the result then follows by induction on the order of S. Actually, we only need the claim for proper local orders, as we have already treated arbitrary permutations. But we will prove the claim as stated.

First, if S is a linear order, let its vertices be labeled $1, 2, 3, 4, 5$ in increasing order *according to the tournament order* $\to$. If one of the pairs $(1, 3)$, $(1, 4)$, or $(2, 4)$ is nonadjacent with respect to the order relation $<$ on S, then the order relation on that pair is determined, and so is the tournament relation: reversal of that edge in the tournament would produce a copy of $C_3 \to I_1$ with the vertex 5 playing the role of I_1.

On the other hand, if all of these pairs are adjacent in the order $<$ then 5 is not adjacent to both 2 and 3, and reversal of $(2,5)$ or $(3,5)$ produces $1 \to C_3$ (with some ordering).

So now suppose that S is a proper local order, that is,

$$S \text{ contains an oriented 3-cycle } C_3$$

Then either S is a composition $S = C_3[A, B, C]$, in which the points of C_3 are replaced by tournaments A, B, C which are themselves linear orders and together have a total of 5 vertices, or else S is the tournament with vertex set $\mathbb{Z}/5\mathbb{Z}$ and edge relation $\{(i,j) \mid j - i = 1 \text{ or } 2\}$, which we will denote by S_5. Suppose first that $S = C_3[A, B, C]$ with $|A| \geq 2$. Let $a_1, a_2 \in A$ with $a_1 \to a_2$. Then reversal of an edge (c, a_1) with $c \in C$ produces $[a_1, C_3]$ and reversal of an edge (a_2, b) with $b \in B$ produces $[C_3, a_2]$. So a_1 must be adjacent in the order $<$ to all elements of C, and a_2 must be adjacent to all elements of B.

If $|A| = |B| = 2$ then viewing S as $C_3[A, B, C]$ and also as $C_3[B, C, A]$ we get the following adjacencies, where we set $A_1 = \{a_1, a_2\}$, $B_1 = \{b_1, b_2\}$, $C = \{c\}$:

Vertex	*Adjacent to:*	*Vertex*	*Adjacent to:*
a_1	c	b_1	a_1, a_2
a_2	b_1, b_2	b_2	c

The adjacencies ca_1, a_1b_1, b_1a_2, a_2b_2, b_2c form a cycle, so there is no such order.

If $|A| = 3$ and $|B| = |C| = 1$ then our notation becomes $A = \{a_1, a_2, a_3\}$, $B = \{b\}$, $C = \{c\}$, and the adjacencies are

Vertex	*Adjacent to:*	*Vertex*	*Adjacent to:*	*Vertex*	*Adjacent to:*
a_1	c	a_2	c, b	a_3	b

Furthermore, reversal of the edge (a_1, a_3) turns $[A, B]$ into $[C_3, B]$, so we must also have a_1, a_3 adjacent in the order, completing the cycle $a_1ca_2ba_3a_1$.

So in all these cases our claim follows, and there remains the case

$$S = S_5$$

In this case reversal of any edge $(i, i+1)$ produces $[C_3, i]$ so all of the pairs $i, i+1$ $(i \in \mathbb{Z}/5\mathbb{Z})$ must be adjacent with respect to the ordering, and again we have a cycle.

Thus the claim holds in all cases. $\square$

The preceding lemma completes the proof of Proposition 4.1. We may now turn our attention to the generic case, which will occupy the remainder of this Part.

CHAPTER 5

ORDERED HOMOGENEOUS GRAPHS: PLAN OF THE PROOF, PROPOSITIONS (I–IX)

At this point, we adopt the language of ordered graphs rather than ordered tournaments.

The goal of the remainder of this Part is the following, which will complete the proof of Theorem 1.2.

Proposition 5.1. Let Γ be a countable homogeneous ordered graph containing $\vec{I}_k$ for all $k < \infty$, as well as $\vec{I}_1 \perp \vec{C}_3^+$ and $\vec{C}_3^-$. Then Γ is either a generically ordered K_n-free graph for some n, or the generic ordered graph.

We follow the method used to treat the unordered case in Cherlin [1998, Chap. 4].

In this chapter we lay out the plan of the proof. We proceed by induction on n. At each stage, what we must show is the following.

> *If the countable homogeneous ordered graph in question contains $\vec{K}_n$, then it contains any ordered configuration whose underlying graph is present in the corresponding Henson graph—that is, any finite $\vec{K}_{n+1}$-free ordered graph.*

The strategy at stage n is to accumulate enough special cases of the stated result to prove the general claim by formal arguments dependent on the finite Ramsey theorem (Lachlan's Ramsey theoretic method).

The goal of the present section is to present the plan of the proof and to give the formal part of the argument. It is in the formal part of the argument that most of the ideas are found. Though we will begin our treatment with this part of the argument (in §5B), it makes up the second half of the proof, as will be clear from our outline.

Later chapters will prove the five preparatory propositions on which the second half of the proof depends. These are proved by explicit amalgamation arguments. At first, these amalgamation arguments are little more than explicit computations in the manner of the previous section. Eventually they become more substantial, in ways that are visible also toward the end of Cherlin [1998, Chap. 4].

As usual, we write $\vec{P}_3$ for a naturally ordered path on three vertices, and $\vec{P}_3^c$ for its ordered graph complement.

$\vec{P}_3$ $\qquad\qquad$ $\vec{P}_3^c$

These ordered graphs correspond to the ordered tournaments $\vec{C}_3^+$ and $\vec{C}_3^-$.

On the other hand, the notation of ordered tournaments is more appropriate for most of the structures of order 4 which we have denoted by $(i \to \vec{C}_3^{\pm})$ and $(\vec{C}_3^{\pm} \to i)$; so we will retain that notation, even though we now consider these structures as ordered graphs.

In particular, we will no longer need to translate diagrams into the language of ordered tournaments: the diagrams represent the actual structures under consideration.

Definition 5.2. For $n \geq 2$, $\mathcal{A}(n)$ is the following set of finite ordered graphs.

$$\{\vec{I}_1 \perp \vec{P}_3, \vec{P}_3^c, \vec{K}_n\} \cup \{\vec{I}_k \mid k < \infty\}$$

Proposition 5.1 will be proved in the following form.

Theorem 5.3. *Let Γ be a countable homogeneous ordered graph containing copies of all the elements of $\mathcal{A}(n)$ as induced ordered subgraphs. Let X be a finite $\vec{K}_{n+1}$-free graph. Then Γ contains a copy of X.*

The proof of this theorem will be lengthy, proceeding by induction on n, with a number of auxiliary statements to be proved by simultaneous induction on the parameter n. Theorem 5.3 may be stated more explicitly as follows.

Corollary 5.3.1 (Generic Case). Let Γ be a countable homogeneous ordered graph containing $\vec{P}_3$, $\vec{P}_3^c$, and some ordered form of $\vec{I}_1 + P_3$, $\vec{I}_1 + P_3^c$, $[I_1, P_3]$ or $[I_1, P_3^c]$. Then Γ is one of the following.

(a) The generic $\vec{K}_n$-free ordered graph, for some n.
(b) The generic $\vec{I}_n$-free ordered graph, for some n.
(c) The generic ordered graph.

PROOF. It is clear that Γ is infinite. By Ramsey's theorem we may suppose that Γ contains an infinite independent set, passing to the complement if necessary. That is, we may suppose that all $\vec{I}_k$ embed into Γ for $k < \infty$.

According to Proposition 4.1, if Γ does not contain $\vec{I}_1 \perp \vec{P}_3$, then Γ embeds into the generically ordered local order, and hence contains no ordered form of $[I_1, C_3]$ or $[C_3, I_1]$, which in the notation of ordered graphs means that Γ contains none of the ordered forms of $I_1{+}P_3$, $I_1{+}P_3^c$ or their complements $[I_1, P_3^c]$, $[I_1, P_3]$, violating our hypothesis.

So in fact Γ contains $\vec{I}_1 \perp \vec{P}_3$. Thus Γ contains all elements of the set $\mathcal{A}(2)$.

If for some n, the ordered graph Γ does not contain $\vec{K}_n$, then Theorem 5.3 states that the finite structures embedding into Γ are precisely those whose underlying graph is K_n-free. In this case we have the generic $\vec{K}_n$-free ordered graph.

If the ordered graph Γ does contain $\vec{K}_n$ for all n then Theorem 5.3 states that all finite ordered graphs embed into Γ. In this case, we have the generic ordered graph. $\square$

So it remains to prove Theorem 5.3. We first lay out our general strategy for the proof.

5A. A simultaneous induction

The method used follows Lachlan's method introduced in the classification of the homogeneous tournaments in Lachlan [1984] and applied to the classification of the homogeneous directed graphs in Cherlin [1998]. More precisely, the method used is a variant of the method presented in Cherlin [1998, Chap. 4].

In Cherlin [1998, Chap. 4], the classification of the countable homogeneous *graphs* is derived by specializing the argument for *directed graphs* back to the undirected setting. The argument for directed graphs was based in turn on the method used by Lachlan to classify homogeneous *tournaments*, which added new ingredients to the method originally used by Lachlan and Woodrow for graphs.

We found that by first generalizing Lachlan's later method to directed graphs, and then specializing it back to the case of graphs, we got a second proof of the classification of homogeneous graphs. This is the proof which we are now generalizing in a different direction, to the case of *ordered graphs.*

Broadly speaking, all versions of Lachlan's scheme follow the same lines. An introduction to the method may be found in the simplified account of Lachlan's treatment of tournaments given in Cherlin [1988], where the method was implemented in a more efficient way which reduced the mass of auxiliary lemmas required.

In our introductory chapter we mentioned four technical points that occur in the present version of the proof. We now single out two of these, one of which governs the structure of our inductive approach, while the other provides the critical link between particular embedding lemmas and the general results we aim at.

For our present purposes the characteristic features of the Lachlan method are these two.

- A use of Ramsey's theorem at a critical juncture (Lachlan's Ramsey method).
- A change of categories, from structures with transitive automorphism group to structures with two orbits on vertices.

A consequence of these two points is that one is forced by the method to prove a substantial number of instances of the general embedding theorem which is our target, before inductive methods can be brought to bear. And here we find that the Ramsey argument not only reduces the problem to relatively concrete statements, but helps again in the proofs of those statements.

We require the following notations and definitions, for reasons sketched in the introductory chapters, and which will now be made completely explicit.

DEFINITION 5.4.

1. An *ordered* 2-*graph* is a structure $\mathbb{H} = (H_1, H_2)$ with H_1, H_2 a partition of the vertices of an ordered graph.[12] In other words, the vertex set of $\mathbb{H}$ is $H = H_1 \cup H_2$, and the relations considered on $\mathbb{H}$ are an order $<$, a symmetric edge relation $—$, and predicates picking out the sets H_1, H_2.

[12] As the term 2-*graph* is often used in another sense elsewhere, one might prefer the term 2-*partitioned graph* here, but it is cumbersome.

2. A 1-*type* $p = xA$ is an isomorphism type of extension of a finite ordered graph A by an additional vertex x. When A is a subset of a given homogeneous ordered graph Γ, we are interested in the set A^p of realizations of the type in Γ (that is, elements $b \in \Gamma$ for which bA is isomorphic to xA over A).

   We may speak similarly of 1-types xA in ordered 2-graphs. In this context we will generally consider only 1-types for which A lies in the second component of the ordered 2-graph, and we will have to specify which component x is to lie in. We will usually require x to lie in the first component. In this case we generally use the notation (x, A) rather than xA.

   In practice, therefore, the notation "xA" is typically reserved for 1-types in ordered graphs (occurring either on their own, or as the second component of a homogeneous ordered 2-graph).
3. When A is a finite ordered graph contained in the homogeneous ordered graph Γ and p is a 1-type over A, we view A^p as an ordered graph. By homogeneity, the isomorphism type of A^p is independent of which copy of A we take in Γ, and A^p is itself homogeneous. Sometimes we will use two distinct 1-types p, q, and get a homogeneous ordered 2-graph (A^p, A^q) as a result. We may proceed similarly with ordered 2-graphs, in which case we will always use two distinct 1-types and stay in the category of 2-graphs—which is the main virtue of this category.
4. The 1-types of interest here are usually *initial* 1-types xA, meaning $x < A$. By working with initial 1-types we will be able to keep our analysis close to the unordered case. Sometimes we switch to terminal 1-types Ax ($A < x$), but by symmetry all results proved in one case apply in the other.
5. We introduce notation for particular 1-types over a point in ordered graphs: $a^{\perp+}$, $a^{\perp-}$ denote the types of elements $x \perp a$ with $a < x$ or $x < a$ respectively. Similarly $a^{\rightarrow}$ and $a^{\leftarrow}$ denote the types of elements x with $x - a$ and $a < x$ or $x < a$ respectively. We will generalize this notation to sets A (in place of a) and to 1-types over A, as well.
6. An ordered 2-graph $\mathbb{H} = (H_1, H_2)$ is *ample* if its second component contains $\vec{I}_1 \perp \vec{P}_3$, $\vec{P}_3^c$, and all $\vec{I}_k$ for $k < \infty$, and if the first component realizes all initial 1-types (b, I) of the following form:

$I \cong I_k$ is a finite independent subset of H_2, and one of the following holds:

(i) $b \perp I$;

(ii) $b \to I$; or

(iii) $|I| = 2$.

Later we will see that these conditions force all 1-types $(b, \vec{I_k})$ over an independent set to be realized.

7. A 1-type p of the form (x, A) in a homogeneous ordered 2-graph is said to be $\mathbb{H}$-*constrained* if the restriction of p to every complete subgraph of A is realized in H_1.
8. If $\mathcal{A}$ is a set of finite ordered graphs, then $\perp \mathcal{A}$ denotes the closure of $\mathcal{A}$ under the operation $\perp$, that is $\perp \mathcal{A}$ contains all structures $\perp_{i=1}^{n} A_i$ with $A_i \in \mathcal{A}$.

In Table 5.1 below we formulate nine statements that we will prove by simultaneous induction on the parameter n, where $n \geq 2$. Entries (I) and (III) do not actually involve the parameter n, and all of the first four have a relatively concrete character).

These statements make use of the following conventions and notation.

— Γ is a homogeneous ordered graph containing all the configurations in $\mathcal{A}(2)$.

— $\mathbb{H}$ is an ample homogeneous ordered 2-graph such that all configurations in $\mathcal{A}(n)$ embed into the second component H_2 (where n is a parameter occurring in the statement of the proposition).

— Configurations denoted A or B are assumed to be finite.

We now discuss how the nine propositions shown in Table 5.1 may be proved by simultaneous induction.

The base case for our inductive argument is the case $n = 2$. The hypothesis at stage $n > 2$ is that (VIII_m) and (IX_m) hold for $m < n$, and that everything earlier on the list holds for the given value of n. We will be more explicit about these dependencies as we go along.

Propositions I–V will be proved by various amalgamation arguments. They are special cases of $(VIII, IX)$ apart from some additional "localization" of the claims to specific ordered 2-graphs $\mathbb{H}$, via the concept of $\mathbb{H}$-constraint. Thus the point of the more formal part of the proof will be to reduce the general results required to a limited number of special cases.

(I-V) Five Concrete Embedding Results

(I) If $a \in \Gamma$ then the ordered 2-graph $(a^{\perp -}, a^{\perp +})$ is ample.

(II_n) If all elements of $\mathcal{A}(n)$ embed in Γ, and $B = baK$ satisfies
— $K \cong \vec{K}_n$
— $b < a < K$
— $a \perp bK$
— B does not contain $\vec{K}_{n+1}$
then B embeds in Γ.

(III) If $p = (x, A)$ is an $\mathbb{H}$-constrained initial 1-type with $A \in \mathcal{A}(2)$, then p is realized in $\mathbb{H}$.

(IV_n) If $A \in \mathcal{A}(n)$ and $p = (x, A)$ is an initial 1-type over A which is realized in $\mathbb{H}$ with $x \in H_1$, $A \subseteq H_2$, then the ordered 2-graphs $(A^p, A^{\perp -})$ and $(A^p, A^{\perp +})$ are ample.

(V_n) If $p = (x, \vec{K}_n \perp \vec{K}_n)$ is an $\mathbb{H}$-constrained initial 1-type, then p is realized in $\mathbb{H}$.

(VI-IX) Four general embedding results

(VI_n) If $p = (x, A)$ is an $\mathbb{H}$-constrained initial 1-type with $A \in \perp \mathcal{A}(n)$, then p is realized in $\mathbb{H}$.

(VII_n) Suppose that Γ contains every configuration in $\mathcal{A}(n)$. If $B = A \cup \{b\}$ does not contain $\vec{K}_{n+1}$, and $b < A$, with $A \in \perp \mathcal{A}(n)$, then Γ contains B.

(VIII_n) Suppose Γ contains every configuration in $\mathcal{A}(n)$. If A does not contain $\vec{K}_{n+1}$ then A embeds into Γ.

(IX_n) If $p = (x, A)$ is an $\mathbb{H}$-constrained initial 1-type and A does not contain $\vec{K}_{n+1}$, then p embeds into $\mathbb{H}$.

Table 5.1. Propositions (I-IX_n)

Indeed, *Propositions VI–IX* will be seen to be formal consequences of the preparatory results (I–V), with some use of Ramsey's theorem and the general theory of amalgamation classes.

We take up the latter set of arguments first.

5B. From *(I–V) to (VI–IX)*

The formal part of the argument taking us from Propositions I–V to Propositions VI–IX contains the bulk of the conceptual ingredients of the proof.

Proposition 5.5. Let $n \geq 2$. Assume statements (I, III, II_n, IV_n, V_n). Then statements (VI_n–IX_n) follow. More precisely, we have the following.

1. (VI_n) follows from (III, IV_n, V_n)
2. (VII_n) follows from (I, II_n, VI_n)
3. (VIII_n) follows from (VII_n)
4. (IX_n) follows from (VII_n, VIII_n)

The most instructive arguments are the last two. We dispose of the first two points immediately.

LEMMA 5.6. *Let $n \geq 2$.*

1. *(VI_n) follows from (III, IV_n, V_n).*
2. *(VII_n) follows from (I, II_n, VI_n).*

PROOF.
(VI_n): $\mathbb{H}$ is an ample homogeneous ordered 2-graph, and $p = (x, A)$ is an $\mathbb{H}$-constrained initial 1-type with $A \in \perp \mathcal{A}(n)$. We must show that p is realized in $\mathbb{H}$.

We write $A = \perp_{i=1}^{k} A_i$ with $A_i \in \mathcal{A}(n)$, and proceed by induction on k. Let $p_k = p \restriction A_k$.

If $k = 1$ then (III) applies if A is in $\mathcal{A}(2)$, and otherwise A is complete and the assumption of $\mathbb{H}$-constraint applies directly.

If $k > 1$ let $\mathbb{H}(A_k) = (A^{p_k}, A_k^{\perp -})$. That is, the first component $H_1(A_k)$ consists of the realizations of the type p_k over the set A_k, which lie in the first component of $\mathbb{H}$, while the second component $H_2(A_k)$ consists of the vertices in H_2 which precede all vertices in A_k in the ordering, and are not joined to any point of A_k by an edge.

By (IV_n), the ordered 2-graph $\mathbb{H}(A_k)$ is ample.

Let $A' = \perp_{i=1}^{k-1} A_i$ and let p' be the restriction of p to A'. Our claim is that p' is realized in $\mathbb{H}(A_k)$.

By induction on k, it suffices to check that the restriction of p to A' is $\mathbb{H}(A_k)$-constrained. So let K be a complete ordered subgraph of A'. Our claim is that $\mathbb{H}$ contains a realization of the 1-type $(x, K \perp A_k)$.

Let q be $p \restriction K$, and pass to the ordered 2-graph $\mathbb{H}(K) = (K^q, K^{\perp +})$. By (IV$_n$) this is ample, and our claim is that $\mathbb{H}(K)$ realizes p_k. By (III) it suffices to check that p_k is $\mathbb{H}(K)$ constrained, or in other words we may take A_k to be complete. Then (V_n) applies.

(VII$_n$): We suppose that Γ contains every configuration in $\mathcal{A}(n)$, and that $B = A \cup \{b\}$ is a finite configuration which does not contain $\vec{K}_{n+1}$, where $b < A$ and $A \in \perp \mathcal{A}(n)$. We claim that Γ contains B.

Take $a \in \Gamma$ and consider the ordered 2-graph $\mathbb{H} = (a^{\perp -}, a^{\perp +})$. We will show that (b, A) embeds in $\mathbb{H}$, and hence B embeds in Γ.

By (I), $\mathbb{H}$ is ample.

By (II$_n$), the configuration (b, A) is $\mathbb{H}$-constrained.

By (VI_n), the configuration (b, A) embeds in $\mathbb{H}$. $\square$

The remainder of the proof of Proposition 5.5 takes some further preparation.

5C. Ramsey theoretic arguments

Turning to the second half of the proof of Proposition 5.5, we will make use of Lachlan's Ramsey theoretic argument. Some additional notation will be useful.

Notation 5.7.

1. By a 2-*type* we mean the isomorphism type of an ordered graph on two vertices a, b with $a < b$. This amounts to much the same thing as a terminal 1-type over a single element a, and we use our customary notation for the two possibilities: $\perp +$, $\to$.
2. If r is a 2-type then an r-*Ramsey* ordered graph is an ordered graph in which every pair of elements a, b with $a < b$ realizes the type r.
3. If $n \geq 2$, r is a 2-type, and $\mathcal{A}$ is an amalgamation class of finite ordered graphs, then $\mathcal{A}_n^r$ is the set of all finite ordered graphs A with the following property.

   If $R \cup A$ is a finite extension of A for which $R < A$, R is r-Ramsey, and for all $v \in R$ the configuration $\{v\} \cup A$ does not contain $\vec{K}_{n+1}$, then $R \cup A \in \mathcal{A}$.
4. We make a similar definition for amalgamation classes $\mathcal{A}$ of ordered 2-graphs, but without the parameter n. In this case $\mathcal{A}^r$ will again be a collection of finite ordered graphs, now construed

as subgraphs of the second component. We consider extensions (R, A) taking R in the first component and A in the second component, with the property that each of the configurations (b, K) with $b \in R$ and $K \subseteq A$ complete belongs to $\mathcal{A}$, and modify the previous definition accordingly.

Lemma 5.8. *Let $\mathcal{A}$ be an amalgamation class of finite ordered graphs or of finite ordered 2-graphs, let r be a 2-type, and let $n \geq 2$. Then $\mathcal{A}^r_n$ (respectively, $\mathcal{A}^r$) is an amalgamation class.*

Proof. We give this first in the notation of ordered graphs.

Supposing the contrary, we have some $A_1, A_2 \in \mathcal{A}^r_n$ such that there is no amalgam of A_1 with A_2 over their common part which lies in $\mathcal{A}^r_n$.

In that case, let $B_1, \ldots, B_N$ be a list of all possible amalgams of A_1 and A_2, and for each $i = 1, \ldots, N$ let $R_i \cup B_i$ be a finite extension of B_i by an r-Ramsey ordered graph R_i such that $R_i < B_i$, and for all $v \in R_i$ $\{v\} \cup B_i$, does not contain $\vec{K}_{n+1}$, but with

$$R_i \cup B_i \notin \mathcal{A}$$

Let R be the union of all R_i with $R_1 < \cdots < R_N$, extended to an r-Ramsey graph. Let $R \cup A_1$, $R \cup A_2$ induce $R_i \cup A_1$, $R_i \cup A_2$ for each i. Then $R \cup A_1$, $R \cup A_2$ agree on their common part, and $\{v\} \cup A_\ell$ does not contain $\vec{K}_{n+1}$ for $v \in R$, $\ell = 1, 2$. So $R \cup A_1$, $R \cup A_2$ are in $\mathcal{A}$, and they have an amalgam $R \cup B$ over their common part which lies in $\mathcal{A}$.

Necessarily $B = B_i$ for some i. Then $R \cup B$ contains $R_i \cup B_i$ and this contradicts our choice of $R_i \cup B_i$.

In the context of ordered 2-graphs the argument is the same, but one writes (R, A_ℓ) rather than $R \cup A_\ell$. □

Lemma 5.9. *Let Γ be a countable homogeneous ordered graph and suppose that Γ contains every configuration of the form $A \cup \{b\}$ for which $b < A$ and $A \in \perp \mathcal{A}(n)$, with $A \cup \{b\}$ not containing $\vec{K}_{n+1}$. Let $\mathcal{A}$ be the class of finite ordered graphs isomorphic with substructures of Γ.*

Then for one of the two possible types r of pairs (a, b) of distinct vertices with $a < b$, $\mathcal{A}^r_n$ contains $\mathcal{A}(n)$.

Proof. This has nothing to do with $\mathcal{A}(n)$ per se, so we will set $\mathcal{B} = \mathcal{A}(n)$ and merely assume that $\mathcal{B}$ is a set of finite configurations for which the corresponding assumption holds when $A \in \perp \mathcal{B}$.

Suppose our claim fails. Then for each of the relevant 2-types $r = \perp+$ or $\rightarrow$, we may select a counterexample $A_r \in \mathcal{B}$ with $A_r \notin \mathcal{A}_n^r$. So fix a finite extension $R_r \cup A_r$ of A_r by an r-Ramsey graph R_r satisfying the following three conditions.

— $R_r < A_r$
— $\{v\} \cup A_r$ does not contain $\vec{K}_{n+1}$, for any $v \in R_r$
— $R_r \cup A_r \notin \mathcal{A}$.

We let $K = \max(|R_r| \mid r \text{ is } \perp+ \text{ or } \rightarrow)$. We may suppose for convenience that $K = |R_r|$ for each of the types r.

Now we argue toward a contradiction.

Pick two large numbers N, N' whose values will be set a little later, and perform the following construction.

Construction.

$V = \{v_i \mid i = 1, \ldots, N\}$ is a set of vertices.

$A = A_{\perp+} \perp A_{\rightarrow}$

A_i is a copy of A for $i = 1, \ldots N'$ and B_j is a copy of A for $j = 1, \ldots, N$. Set $A^* = \perp_i A_i$, $B^* = \perp_j B_j$, and $C = A^* \perp B^*$. We think of this as a "stack" of copies of A.

Let $C_j = C \cup \{v_j\}$ for $j = 1, \ldots, N$, with some additional structure which will be chosen very carefully below. We will take $v_j < C$. For the rest, we need to see first where we are headed.

Once this construction is complete, we intend to amalgamate all of the configurations C_j over their common part C, and the remainder of the construction is designed to ensure the following three points.

(a) The configurations C_j are all in $\mathcal{A}$, and hence some amalgam of them is also in $\mathcal{A}$.
(b) The vertices v_j realize distinct types over B^*, and hence remain distinct in the amalgam.
(c) Some K of the vertices $v \in V$ constitute a copy of R_r, for one of the two possible choices of r, and those K vertices together with one of the copies of A_r in A^* give a copy of $R_r \cup A_r$.

Since at this point we may conclude that $R_r \cup A_r \in \mathcal{A}$, we will have the desired contradiction.

We now return to the specification of $\{v\} \cup C$ for $v \in V$, and the determination of the numbers N and N'.

First, make the types of the vertices $v \in V$ over B distinct by making the type of v_j over B_j distinct from the types of the other $v_{j'}$ over B_j. And we do so using configurations $v_j B_j$ and $v_{j'} B_j$ which do not contain $\vec{K}_{n+1}$—say, by putting in one edge to v_j and no edges to the other $v_{j'}$.

Now we look toward our third condition. We choose N large enough to guarantee that in any amalgam of the C_j over C, the N distinct vertices V contain some subset R with $|R| = K$ for which the induced structure is an r-Ramsey graph, for one of the two possible values of r. We then let $N' = \binom{N}{K}$, and for convenience relabel the A_i as A_R, with R varying over the K-subsets of V. Here A_R is $A_{\perp+,R} \perp A_{\to,R}$ with $A_{r,R}$ a copy of A_r.

To complete the construction, let vA_R be chosen for $v \in R$ so that if R is r-Ramsey, then $vA_{r,R}$ has the structure induced by RA_r on vA_r. For $v \notin R$ just take $v \perp A_R$.

As we went along, point (a) was taken care of. We also ensured at the outset that all the $v \in V$ realize distinct types over B. So we need only check point (c).

After performing our amalgamation to get a configuration VC, the value of N ensures the existence of a subset R of V such that the induced structure on R is a copy of R_r for one of the two possible values of r. Then $RA_{r,R}$ is a copy of $R_r \cup A_r$, and we have the expected contradiction. □

We now state the analogous lemma for ordered 2-graphs.

LEMMA 5.10. *Let $\mathbb{H}$ be a countable homogeneous ordered 2-graph. Suppose that $\mathbb{H}$ contains every configuration of the form (b, A) for which $b < A$ and $A \in \perp \mathcal{A}(n)$, such that (b, A_i) embeds in $\mathbb{H}$ for each summand A_i of A. Let $\mathcal{A}$ be the class of finite ordered 2-graphs associated with $\mathbb{H}$.*

Then for one of the two possible types r of pairs (a, b) of distinct vertices with $a < b$, $\mathcal{A}^r$ contains $\mathcal{A}(n)$.

The proof is the same, but now using the definitions for 2-graphs and the notation $\mathcal{A}^r$ rather than $\mathcal{A}^r_n$. In particular if $\mathbb{H}$ omits $\vec{K}_{n+1}$ in the second component, it is still possible that $\mathbb{H}$ contains a form of $\vec{K}_{n+1}$, namely $(b, \vec{K}_n)$ with $b < \vec{K}_n$.

LEMMA 5.11. *Let $n \geq 2$.*

1. ($VIII_n$) follows from (VII_n).
2. (IX_n) follows from (VI_n) and ($VIII_n$).

PROOF.
From (VII_n) to ($VIII_n$):

It is now convenient to rephrase (VIII_n) directly in terms of amalgamation classes. It then reads as follows.

> Suppose that $\mathcal{A}$ is an amalgamation class of ordered graphs containing $\mathcal{A}(n)$. Then for any finite ordered graph A which does not contain $\vec{K}_{n+1}$, A belongs to $\mathcal{A}$.

We will prove this by induction on $k = |A|$. For $k = 1$ it is clear. So suppose $|A| = k > 1$ and let $\mathcal{A}$ be any amalgamation class containing $\mathcal{A}(n)$.

If Γ is the homogeneous ordered graph associated with the class $\mathcal{A}$, then (VII_n) states that Γ contains every configuration $B = A \cup \{b\}$ satisfying the following conditions.

— $A \in \perp \mathcal{A}(n)$;
— $b < A$;
— B does not contain $\vec{K}_{n+1}$.

We apply Lemma 5.9, which says that there is then a 2-type r so that $\mathcal{A}_n^r$ contains $\mathcal{A}(n)$. Now remove the vertex $v = \min A$ from the configuration A to obtain a configuration A'. By our inductive hypothesis, A' must belong to the amalgamation class $\mathcal{A}_n^r$. Now $A = \{v\} \cup A'$ is a finite initial extension of A' by an r-Ramsey ordered graph (namely the singleton $\{v\}$), and by hypothesis the extension does not contain $\vec{K}_{n+1}$. So by the definition of $\mathcal{A}_n^r$, we have $A \in \mathcal{A}$, as claimed.

From ($VI_n, VIII_n$) to (IX_n):

We consider an ample homogeneous ordered 2-graph $\mathbb{H}$ and an $\mathbb{H}$-constrained initial 1-type $p = (x, A)$ such that A does not contain $\vec{K}_{n+1}$. We claim that p is realized in $\mathbb{H}$. We let $\mathcal{A}$ be the associated amalgamation class of finite ordered 2-graphs.

By (VI_n) the hypotheses of Lemma 5.10 are satisfied, and thus there is a 2-type r for which $\mathcal{A}^r$ contains $\mathcal{A}(n)$. By (VIII_n), the amalgamation class $\mathcal{A}^r$ must contain the configuration A. Then any $\mathbb{H}$-constrained configuration (R, A) with R r-Ramsey occurs in $\mathbb{H}$, and this applies in particular when R is a singleton. $\square$

CHAPTER 6

ORDERED HOMOGENEOUS GRAPHS: PROPOSITION I

To complete the classification of the homogeneous ordered graphs, it remains to provide the first five ingredients of our proof that the remaining countable homogeneous graphs in Group III of our catalog are generically ordered expansions of homogeneous graphs. These ingredients consist of Propositions I through V in the previous chapter. Certain challenges arise along the way, notably in connection with the proof of Proposition V.

We will take these five propositions in order, so here we deal with Proposition I. The general character of the proof in this chapter will be that of a verification by computation. We recall the statement to be proved.

Proposition (I). Let Γ be a countable homogeneous ordered graph such that all configurations in $\mathcal{A}(2)$ embed isomorphically in Γ. For any $a \in \Gamma$, the associated ordered 2-graph $(a^{\perp -}, a^{\perp +})$ is ample.

Recall that $\mathcal{A}(2)$ is

$$\{\vec{I}_1 \perp \vec{P}_3, \vec{P}_3^c, \vec{K}_2\} \cup \{\vec{I}_k \mid k < \infty\}$$

and that as $\vec{K}_2$ is superfluous we may replace this for all practical purposes by

$$\mathcal{A}(2) = \{\vec{I}_1 \perp \vec{P}_3,\ \vec{P}_3^c\} \cup \{\vec{I}_k \mid k < \infty\}.$$

Since the definition of ampleness involves two conditions that will be treated separately, let us phrase Proposition I more explicitly as follows.

Proposition (IA, IB). Let $\mathcal{A}$ be an amalgamation class that contains $\mathcal{A}(2)$. Then

(IA)**:** For any $A \in \mathcal{A}(2)$, $\vec{I}_1 \perp A$ belongs to $\mathcal{A}(2)$.

(IB)**:** For any configuration of the form $B = baI$ with $I \cong \vec{I}_k$, $k < \infty$, and $b < a < I$, $a \perp bI$, for which one of the following holds, we have $B \in \mathcal{A}$.
— $b \perp I$; or
— $b \to I$; or
— $|I| = 2$.

The conditions in (IB) are quoted from the definition of ampleness, rephrased in terms of the parameter a. In the present section we will prove along the way that the condition of ampleness implies the stronger form of (IB) in which there are no restrictions on the type of b over I, other than $b < I$. In a slightly convoluted way we will wind up verifying this stronger, and simpler, form of condition (IB).

Taken together, (IA) and (IB) will prove the ampleness of the ordered 2-graph $(a^{\perp -}, a^{\perp +})$.

6A. Proof of Proposition (IA)

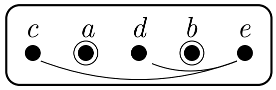
LEMMA 6.1. *Let X be a finite ordered graph and let $n \geq 2$. Suppose that every amalgamation class containing $\mathcal{A}(n)$ also contains X. Then the same applies to the ordered graph obtained from X by reversing its ordering.*

PROOF. Any amalgamation class containing $A(n)$ for fixed $n \geq 2$ also contains the ordered graph $\vec{P}_3 \perp \vec{I}_1$, by Lemma 4.8. Therefore we may replace $\mathcal{A}(n)$ by $\mathcal{A}(n) \cup \{\vec{P}_3 \perp \vec{I}_1\}$, which is closed under reversal, and the claim follows. $\square$

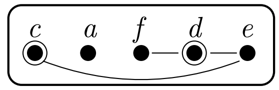
LEMMA 6.2. *Let $\mathcal{A}$ be an amalgamation class containing $\vec{P}_3^c$ and $\vec{I}_1 \perp \vec{P}_3$. Then $\mathcal{A}$ contains $\vec{I}_1 \perp \vec{P}_3^c$.*

PROOF. We perform the amalgamation $(*)$ shown, with factors $(*)_1 = (cade)$ and $(*)_2 = (cdbe)$.

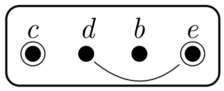
Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(adbe)$ $\square$	$(cadb)$ $\square$	(A)	$(cade)$ $(*)_1$	$(cafd)$ $(*)_1$
(B)	$\square$	$(cdbe)$ $(*)_2$			



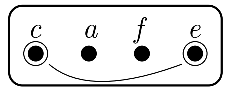
$(*)$



A



B



A_1 (S3214)

Thus it suffices to show that the factors of (A) embed in Γ, and as the factor $(A)_2 = (afde)$ is $\vec{I}_1 \perp \vec{P}_3$, this comes down to $(A)_1 = (cafe) \cong (S3214)$.

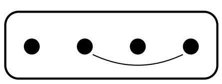
Thus if Γ is a counterexample to the lemma, then it omits the configuration

$$(*)_1 = (cade) = (3 \to \vec{C}_3^-)$$

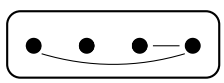
as well as $(A)_1 \cong (S3214)$. So we will suppose the following for the remainder of the argument, aiming at a contradiction.

(6.1) Γ contains $\vec{I}_1 \perp \vec{P}_3$ and $\vec{P}_3^c$;

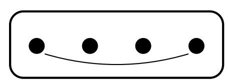
(6.2) Γ omits $\vec{I}_1 \perp \vec{P}_3^c$, $(3 \to \vec{C}_3^-)$, and $(S3214)$.



$I_1 \perp \vec{P}^c$

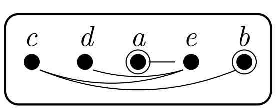


$(3 \to \vec{C}_3^-)$

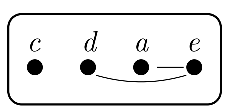


$S3214$

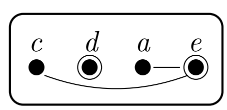
Now we try the following amalgamation, where (c, e) may be an edge or a non-edge.



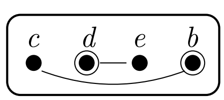
(I)



IA_1



$IB1$



$I2$

Label	*Non-edge*	*Edge*
(I)	$(cdab)$	$(cdab)$
	$(S3214)$	$(3 \to \vec{C}_3^-)$

We will refer to the variant with (c, e) a non-edge as (IA), and the variant with (c, e) an edge as (IB), and to the corresponding factors $(cdae)$ or $(cdeb)$ as $(IA)_1$, $(IB)_1$, $(IA)_2$, $(IB)_2$ correspondingly.

The factor $(IA)_1$ (shown above) is $\vec{I}_1 \perp [\vec{I}_2, \vec{I}_1]$. To see that this is realized in Γ we fix a vertex c and consider

$$\Gamma_0 = c^{\perp +} = \{v \mid c \perp e,\, c < e\}$$

where "$c \perp e$" is our notation for non-edges.

Now by hypothesis Γ_0 realizes $\vec{P}_3$ and omits $\vec{P}_3^c$, or as an ordered tournament contains $\vec{C}_3^+$ and omits $\vec{C}_3^-$. This falls under groups $(II.3_n^c)$ and $(II.4^c)$ in our catalog. In both cases the configurations $[\vec{I}_2, \vec{I}_1]$ must be realized in Γ_0 and thus $\vec{I}_1 \perp [\vec{I}_2, \vec{I}_1]$ is realized in Γ.

On the other hand in the amalgamation labelled $(IB1)$ above, if (d, e) is a non-edge then we have $(3 \to \vec{C}_3^-)$ and a contradiction, so (d, e) must be an edge and therefore Γ contains $(IB)_1$.

Thus it will suffice to have a factor of the form $(IA)_2$ or $(IB)_2$ to conclude. The diagram $(I2)$ shows an amalgamation which produces the required factor. □

LEMMA 6.3. *Let $\mathcal{A}$ be an amalgamation class containing $\vec{I}_1 \perp \vec{P}_3$ and $\vec{I}_4$. Then $\mathcal{A}$ contains $\vec{I}_2 \perp \vec{P}_3$.*

PROOF. The class $\mathcal{A}$ contains $\vec{I}_1 \perp \vec{P}_3^c$ by Lemma 6.2.

Let Γ be the corresponding homogeneous ordered graph.

Suppose toward a contradiction that

$$\Gamma \text{ does not contain } \vec{I}_2 \perp \vec{P}_3$$

Fix $a \in \Gamma$. Then $a^{\perp +}$ contains $\vec{P}_3$, $\vec{P}_3^c$, and I_3, but not $\vec{I}_1 \perp P_3$. By Proposition 4.1, $a^{\perp +}$ must be the tournament $S(2)$ equipped with a generic linear order.

So if A is any finite ordered tournament whose underlying tournament is a local order, then the corresponding ordered graph embeds into $a^{\perp +}$ and thus

$$\vec{I}_1 \perp A$$

embeds into Γ. This remark will simplify the construction of suitable amalgamation diagrams. Furthermore, if A is a finite ordered tournament which is not a local order, then the ordered graph corresponding to $\vec{I}_1 \perp A$ does not embed into Γ.

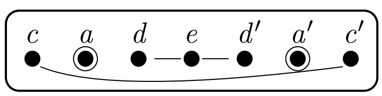
Suppose now that Γ contains $\vec{P}_3 \perp \vec{I}_2$, the reversal of the desired configuration. Then $a^{\perp -}$ contains $\vec{P}_3 \perp \vec{I}_1$ and hence contains $\vec{I}_1 \perp \vec{P}_3$ by Lemma 4.8. In other words, $a^{\perp +}$ would then contain $\vec{P}_3 \perp \vec{I}_1$, which is not a local order.

So we may also suppose

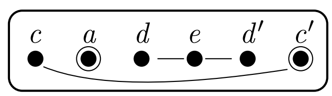
$$\Gamma \text{ does not contain } \vec{P}_3 \perp \vec{I}_2$$

By Lemma 4.8 again, Γ contains $\vec{P}_3 \perp \vec{I}_1$. Thus the hypotheses and conclusion of the lemma are both invariant under reversal. We exploit this in the following amalgamation.

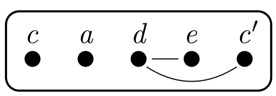
This amalgamation will produce a contradiction if the factors can be found in Γ.



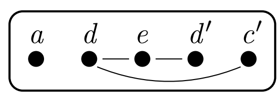
(∗)



A



1



2

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(aded'a')$	$(caea'c')$	(A)	$(caded')$ $\vec{I}_2 \perp \vec{P}_3$ #	$(*)_1$
	$\vec{I}_1 \perp \vec{P}_3^c \perp \vec{I}_1$ #	$\vec{I}_1 \perp \vec{P}_3 \perp \vec{I}_1$ #			

The reversal of (A) is an amalgamation diagram which produces either a contradiction or a copy of $(*)_2$. By our earlier remarks it suffices to show that the factors of (A) occur in Γ; then we get the same for the reversal, and thus the factors of $(*)$, giving a contradiction.

The factors of (A) are shown above: namely, (1) $\vec{I}_2 \perp [\vec{I}_1, \vec{I}_2]$ and (2) $\vec{I}_1 \perp (S4231)$. As these are of the form $\vec{I}_1 \perp \vec{S}$ with $\vec{S}$ an ordered local order, these factors are available. $\square$

Corollary 6.3.1. Let $A \in \mathcal{A}(2)$. Then any amalgamation class containing $\mathcal{A}(2)$ contains $\vec{I}_1 \perp A$ and $A \perp \vec{I}_1$. In particular, Proposition (IA) holds.

PROOF. $\mathcal{A}(2)$ consists of

$$\{\vec{I}_1 \perp \vec{P}_3, \vec{P}_3^c, \vec{K}_2\} \cup \{\vec{I}_k \mid k < \infty\}$$

and we are free to drop $\vec{K}_2$ here as it is contained in either of the first two.

We derive $\vec{I}_1 \perp A$ for $A = \vec{I}_1 \perp \vec{P}_3$ and $\vec{P}_3^c$ from Lemmas 6.3 and 6.2 so our claim follows as far as $\vec{I}_1 \perp A$ is concerned.

By Lemma 6.1, the same applies to the reversal $A \perp \vec{I}_1$. □

Before turning to Proposition (IB), we prove the analog of Corollary 6.3.1 for $\mathcal{A}(n)$ for all $n \geq 2$, within an inductive framework.

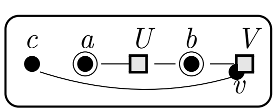
LEMMA 6.4. *Let $\mathcal{A}$ be an amalgamation class containing $\mathcal{A}(n)$. If $n > 2$, assume Proposition $(VIII_{n-1})$. Then $\vec{I}_1 \perp \vec{K}_n$ is in $\mathcal{A}$.*

PROOF. We have dealt with the case $n = 2$, so we assume here that $n \geq 3$.

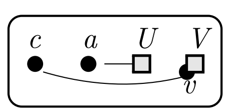
This is the first of our amalgamation arguments which goes beyond a single, small configuration. So the description will be more elaborate. We will make use of some configurations U, V isomorphic respectively to $\vec{K}_{n-2}$ and $\vec{K}_{n-1}$, and let

$$v = \min V,\ V' = V \setminus \{v\}$$

We can then present the amalgamation argument as follows.

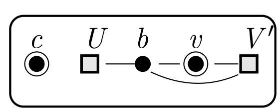


$(*)$



$U : n-2;\ V : n-1$

$(*)_1$

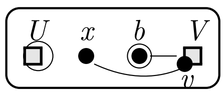


B

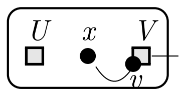
We remark that (c, v) is the only edge between c and V here.

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	(abV) □	$(caUb)$ □			
$(*)_1$	$\vec{K}_n$-free—(VIII$_{n-1}$)		(B)	(cbV) $\vec{I}_1 \perp \vec{K}_n$	$(cUvB)$ $(*)_2$

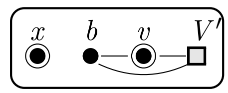
This brings us down to the factors of (B): $(B)_1 = (cUbV')$ and $(B)_2 = UbV$. Now $(cUbV')$ omits $\vec{K}_n$ so Proposition $(VIII_{n-1})$ applies. Thus we are left with $(B)_2 = (UbV)$ to consider. We treat this as follows.



$B2$



$B2$



$B2.2$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(B2)$	Contains $\vec{I}_1 \perp \vec{K}_n$	Is $(B)_2$			
$(B2)_1$ (UxV)		Omits $\vec{K}_n$	$(B2.2)$	□	$(B2)_2$

The distinction in $(B2)$ is between amalgamations with at least one non-edge (u, b), and the amalgamation with Ub a clique. □

Corollary 6.4.1. Let $n \geq 2$, and if $n > 2$ assume that $(VIII_{n-1})$ holds. Let X be a finite ordered graph which belongs to every amalgamation class containing $\mathcal{A}(n)$. Then $\vec{I}_1 \perp X$ and $X \perp \vec{I}_1$ have the same property.

Proof. By Corollary 6.3.1, the previous lemma, and Lemma 6.1, this holds if X is in $\mathcal{A}(n)$. The general case follows easily. □

6B. Some small configurations

Now we return to the business at hand, and we take up the proof of Proposition (IB).

The next step is to accumulate some consequences of the configurations in $\mathcal{A}(2)$, that is we identify a number of small configurations which can be obtained by amalgamation from $\mathcal{A}(2)$ and which are useful as ingredients in various amalgamation arguments.

We first take stock of what we know about $\mathcal{A}(2)$ and what we intend to show in the next few lemmas. We give this in tabular form.

CONSEQUENCES OF $\mathcal{A}(2)$

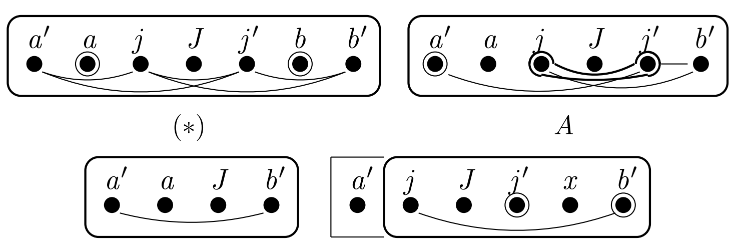
Reference	Claim
Def. 5.2 (simplified)	$\mathcal{A}(2)$: $\vec{I}_1 \perp \vec{P}_3$; $\vec{P}_3^c$; $\vec{I}_k$ $(k < \infty)$
Lemma 6.1	Closed under reversal
Corollary 6.3.1	Closed under $\vec{I}_1 \perp X$, $X \perp \vec{I}_1$
Lemma 6.5	At most one edge
Lemma 6.6	$[\vec{I}_1, \vec{I}_2]$
Lemma 6.7	$\vec{P}_4$, $(\vec{C}_3^- \to 2)$, $(S2314)$, $(S4321)$.

LEMMA 6.5. *Let $\mathcal{A}$ be an amalgamation class which contains $\mathcal{A}(2)$. Then $\mathcal{A}$ contains any finite ordered graph with at most one edge.*

PROOF. This holds by definition if there are no edges, so we suppose there is a unique edge (a, b) with $a < b$.

By repeated use of Corollary 6.3.1, we may suppose the configuration has the form aIb with $a < I < b$ and unique edge ab. We argue by induction on $k = |I|$. We may suppose $k \geq 2$.

Let J be an independent set of $(k-2)$ vertices. We analyze the following amalgamation.



$(*)$ A

$A1$ $A2'$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
j J x b (∗)	$(a'aJbb')$ □	$(ajJj'b)$ □	(A)	$(a'ajJb')$, $(a'aJj'b')$ □	$(*)_1$

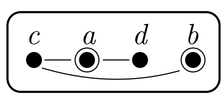
The two factors of $(*)$ are symmetric and (A) aims at giving the first; for the second we would use the reversal of (A). So it suffices to check that the factors of (A) embed into Γ. These are $(A)_1 = (a'aJb')$ and $(A)_2 = (ajJj'b')$.

The factor $(a'aJb')$ is covered by our induction hypothesis since $|aJ| < |I|$. The factor $(ajJj'b') = (a) \perp (jJj'b')$. By Corollary 6.4.1 it suffices to show that the configuration $(A)'_2 = (jJj'b')$ embeds into Γ. This is what diagram $(A2')$ aims at (with the extraneous vertex a indicated but not actually part of the diagram).

The factors of diagram $(A2')$ are $(jJj'x)$, which is an independent set, hence in Γ, and $(jJxb')$, which is covered by the induction hypothesis since $|Jx| < |I|$. □

Lemma 6.6. *If $\mathcal{A}$ is an amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$, then $\mathcal{A}$ contains $[\vec{I}_1, \vec{I}_2]$.*

Proof. Amalgamate as follows.



$(*)$

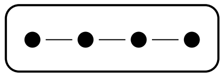
Then either (cab) or (adb) will be $[\vec{I}_1, \vec{I}_2]$. □

Lemma 6.7. *If $\mathcal{A}$ is an amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$, then $\mathcal{A}$ contains the following configurations.*

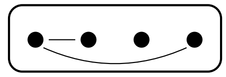
1. *The ordered path $\vec{P}_4$;*
2. *The configuration $(\vec{C}_3^- \to 2)$;*
3. *The ordered local order $(S2314)$;*
4. *The ordered local order $(S4321)$.*

Proof.

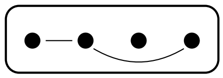
Ad 1: The path $\vec{P}_4$.



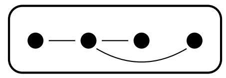
$\vec{P}_4$



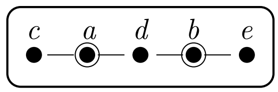
$\vec{C}_3^- \to 2$



$S2314$



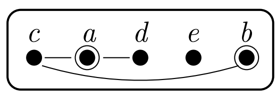
$S4321$



$(\vec{P}_4)$

Here $(cadb)$ or $(cabe)$ will be $\vec{P}_4$ and the factors are $\vec{P}_3 \perp \vec{I}_1$ and $\vec{I}_1 \perp \vec{P}_3$.

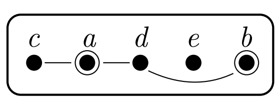
Ad 2: *The configuration* $(\vec{C}_3^- \to 2)$.



$(*)$

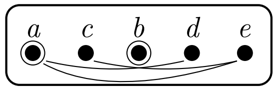
Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(caeb)$ □	$(adeb)$ □			
$(*)_1$	$\vec{P}_3 \perp \vec{P}_1$		$(*)_2$	1 Edge	

Ad 3. The ordered local order $(S2314)$.

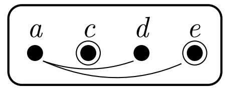


$(*)$

Here $(adeb)$ or $(caeb)$ will be $(S2314)$ and the factors are $\vec{P}_3 \perp \vec{I}_1$ and $\vec{I}_1 \perp \vec{P}_3^c$.



(∗)

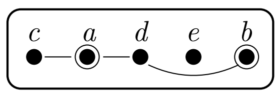


A

Ad 4: The ordered local order ($S4321$).

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
(∗)	$(abde)$ □	$(acbd)$ □			
(A)	□	$(*)_1$	$(*)_2$		1 Edge

□



(∗)

6C. Four vertices and two edges

Next we deal with $[\vec{I}_1, \vec{I}_3]$, and then all configurations of order 4 that contain at most 2 edges (Lemmas 6.9 and 6.13). We take these in the following order.

FURTHER CONSEQUENCES OF $\mathcal{A}(2)$

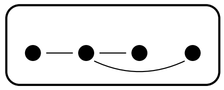
Reference	Claim
Lemma 6.9	$[\vec{I}_1, \vec{I}_3]$
Lemmas 6.10, 6.11, 6.12	$\vec{K}_2 \perp \vec{K}_2$, ($S2431$), ($S3124$)
Lemma 6.13	4 Vertices, 2 Edges

Working toward $[\vec{I}_1, \vec{I}_3]$, we begin as follows.

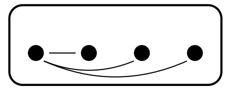
LEMMA 6.8. *If $\mathcal{A}$ is an amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$, then $\mathcal{A}$ contains at least one of the following configurations.*

(a) The ordered local order ($S3421$)*;*

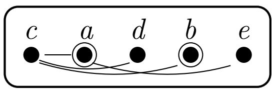
(b) $[\vec{I}_1, \vec{I}_3]$.



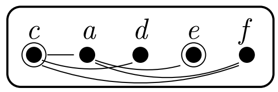
$S3421$



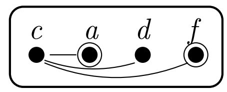
$[I_1, I_3]$



$(*)$



A



$A1$

PROOF. We use the following amalgamation.

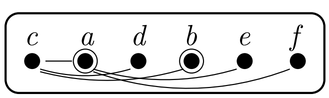
Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(cadb)$ $[\vec{I}_1, \vec{I}_3]$ □	$(cabe)$ $(S3421)$ □			
(A)	$(cade)$ $(*)_1$	$(cdeff)$ $[\vec{I}_1, \vec{I}_3]$ □	$(*)_2$	(Lemma 6.6, Cor. 6.3.1)	
$(A1)$	$[\vec{I}_1, \vec{I}_3]$ □	$(A)_1$	$(A)_2$		$(S4321)$ □

LEMMA 6.9. *If $\mathcal{A}$ is an amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$, then $\mathcal{A}$ contains $[\vec{I}_1, \vec{I}_3]$.*

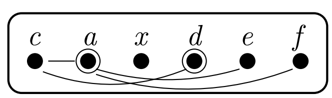
PROOF. By Lemma 6.8 we may suppose

$$\mathcal{A} \text{ contains } (S3421)$$

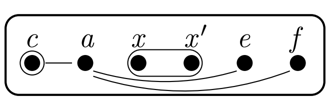
We consider the following amalgamation.



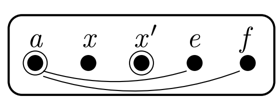
$(*)$



A



$A1$

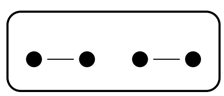


$A1.2$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(cadb)$ □	$(abef)$ □			
(A)	$(cadef)$ $(*)_1$	$(adef)$ □	$(*)_2$	Lemma 6.6, Corollary 6.3.1	
$(A1)$	$(caxef)$ or $(cax'ef)$: $(A)_1$	$(caxx')$ □	$(A)_2$	Corollary 6.3.1	
$(A1)_1$	$(S3421)$ (assumed)		$(A1.2)$	$(A1)_2$	$(ax'ef)$ □
$(A1.2)_1$	$(S4321)$		$(A1.2)_2$		$\vec{I}_4$

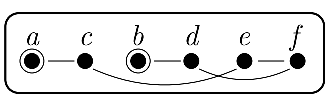
□

LEMMA 6.10. *If $\mathcal{A}$ is an amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$, then $\mathcal{A}$ contains $\vec{K}_2 \perp \vec{K}_2$.*

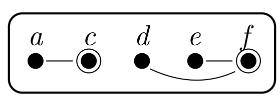


$K_2 \perp K_2$

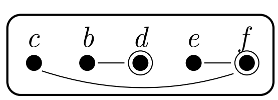
PROOF. We use the following amalgamation.



$(*)$



A

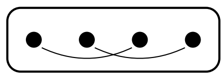


B

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(acbd)$ □	$(abef)$ □			
(A)	$(acef)$ □	$(*)_1$	(B)	$(bdef)$ □	$(*)_2$
$(A)_1, (A)_2$	Corollary 6.3.1				
$(B)_1$	Corollary 6.3.1		$(B)_2$	$(\vec{C}_3^- \to 2)$, reversed	

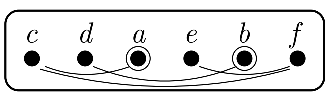
□

LEMMA 6.11. *If $\mathcal{A}$ is an amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$, then $\mathcal{A}$ contains the ordered local order $(S2431)$.*

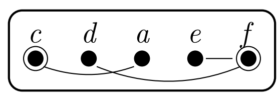


$S2431$

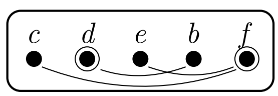
PROOF. We use the following amalgamation.



$(*)$



A

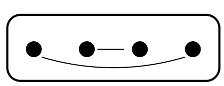


B

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(cdab)$ □	$(aebf)$ □			
(A)	$(cdaf)$ □	$(*)_1$	(B)	$(debf)$ □	$(*)_2$
$(A)_1$	Corollary 6.3.1		$(A)_2$	$(\vec{C}_3^- \to 2)$ reversed	

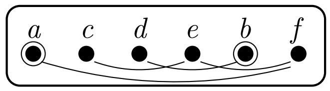
□

LEMMA 6.12. *If $\mathcal{A}$ is an amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$, then $\mathcal{A}$ contains the ordered local order $(S3124)$.*

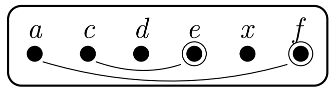


$S3124$

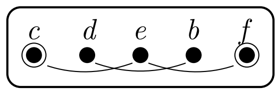
Proof. We use the following amalgamation.



$(*)$



A



B

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(adbf)$ □	$(aceb)$ □			
(A)	$(acef)$ □	$(*)_1$	(B)	$(*)_2$	$(cdbf)$ □
$(A)_{1,2}$		1 Edge	$(B)_{1,2}$		$(S2431)$

□

Lemma 6.13. *Let A be a finite ordered graph with at most four vertices and two edges. Then A is in any amalgamation class containing $\mathcal{A}(2)$.*

Proof. If there is at most one edge then this holds by Lemma 6.5. So we may suppose there are exactly two edges and four vertices $a < b < c < d$.

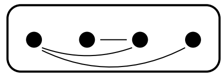
The cases in which a or d lies on no edge are covered by Corollary 6.3.1 together with Lemmas 6.6 and 6.1.

If (a, d) is an edge then up to reversal the configuration is one of $(\vec{C}_3^- \to 2)$, $(S4321)$, or $(S3124)$, which are covered by Lemmas 6.7 and 6.12.

So we may suppose that a, d lie on distinct edges. Then up to reversal the configuration is $(S2314)$, $\vec{K}_2 \perp \vec{K}_2$, or $(S2431)$, covered by Lemmas 6.7, 6.10, and 6.11. □

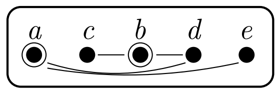
Lemma 6.14. *If $\mathcal{A}$ is an amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$, then $\mathcal{A}$ contains the ordered local order*

$$(\vec{C}_3^- \to 3).$$



$\vec{C}_3^- \to 3$

PROOF. We use the following amalgamation.



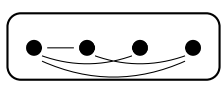
$(*)$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(abde)$ □	$(acbe)$ □	$(*)_1$, $(*)_2$		2 Edges □

6D. An ample ordered 2-graph

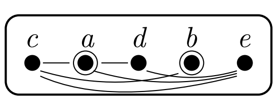
Proposition I concerns the ordered 2-graph $(a^{\perp -}, a^{\perp +})$ where $a \in \Gamma$ and Γ is a countable homogeneous ordered graph. We will first deal with the ordered 2-graph $(a^{\leftarrow}, a^{\perp -})$, in Lemma 6.17 below.

LEMMA 6.15. *Any amalgamation class which contains $\mathcal{A}(3)$ contains $[\vec{I}_1, \vec{P}_3^c]$.*

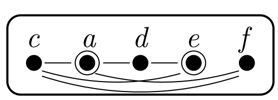


$[I_1, \vec{P}^c]$

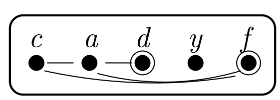
PROOF. We use the following amalgamation.



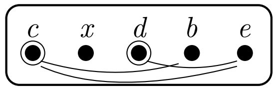
$(*)$



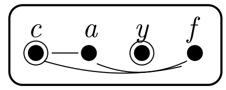
A



$A1$



B



$A1.2$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(cabe)$ $\square$	$(adbe)$ $\square$			
(A)	$(caef)$ $\square$	$(cade)$ $(*)_2$	(B)	$(cdbe)$ $(*)_2$	$(cdbe)$ $\square$
$(A1)$	$(cadf)$ $(A)_1$	$(cadf)$ $(*)_1$	$(A)_2$	$\vec{C}_3^- \to 3$	
$(A1)_1$	Corollary 6.3.1		$(A1.2)$	$(A1)_2$	$\square$
$(B)_1$	4 Vertices, 2 Edges		$(B)_2$	1 Edge	$\square$

Lemma 6.16. *Any amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$ contains the configurations $[\vec{I}_1, \vec{I}_k]$ for all $k < \infty$.*

Proof. Fix $k < \infty$.

We divide the proof into two cases.

Case I: $\mathcal{A}$ contains $\mathcal{A}(\infty) = \{\vec{I}_1 \perp \vec{P}_3, \vec{P}_3^c\} \cup \{\vec{I}_k, \vec{K}_k \mid k < \infty\}$.

By Lemma 6.4, any amalgamation class containing $\mathcal{A}(\infty)$ must contain $\vec{I}_1 \perp \vec{K}_k$.

By Lemma 6.15, any amalgamation class containing $\mathcal{A}(\infty)$ contains $[\vec{I}_1, \vec{P}_3^c] = (\vec{I}_1 \perp \vec{P}_3)^c$, and hence contains A^c for all $A \in \mathcal{A}(\infty)$. Hence for any finite configuration X which belongs to all amalgamation classes containing $\mathcal{A}(\infty)$, the same applies to X^c. In particular $(\vec{I}_1 \perp \vec{K}_k)^c = [\vec{I}_1, \vec{I}_k]$ is in every amalgamation class containing $\mathcal{A}(\infty)$.

This completes the treatment of Case I.

Case II: $\mathcal{A}$ does not contain $\vec{K}_n$ for some n.

Let Γ be the homogeneous ordered graph corresponding to $\mathcal{A}$, and fix $a \in \Gamma$. Then the ordered graph induced on $a^{\to} = \{b \mid a < b,\ a - b\}$ is homogeneous and nontrivial, hence infinite, and it does not contain $\vec{K}_n$. By Ramsey's Theorem, $a^{\to}$ contains $\vec{I}_\infty$. So Γ contains $[\vec{I}_1, \vec{I}_\infty]$ and $\mathcal{A}$ contains $[\vec{I}_1, \vec{I}_k]$ for all $k < \infty$. $\square$

Lemma 6.17. *Let Γ be a countable homogeneous ordered graph containing all configurations in $\mathcal{A}(2)$. Let $c \in \Gamma$ and set $\mathbb{H} = (c^{\leftarrow}, c^{\perp -})$. Then $\mathbb{H}$ is ample.*

PROOF. We consider initial 1-types (b, I) with $I \cong \vec{I}_k$, satisfying one of the following conditions, and we show they are realized in $\mathbb{H}$.

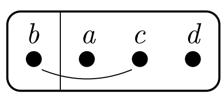
$b \perp I$: This corresponds to bIc with unique edge bc, given by Lemma 6.5.
$b \to I$: This corresponds to $bIc = [\vec{I}_1, \vec{I}_{k+1}]$, given by Lemma 6.16.
$|I| = 2$: We may suppose that bI contains exactly one edge, and thus bIc has 2 edges, and is covered by Lemma 6.13.

□

6E. 1-types over $\vec{I}_k$; Proposition (IB)

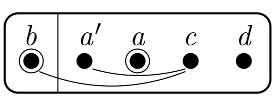
We will show shortly that in any ample homogeneous ordered graph all initial 1-types over $\vec{I}_k$ are realized ($k < \infty$).

LEMMA 6.18. *Let $\mathbb{H}$ be a countable homogeneous ordered 2-graph such that H_2 contains all configurations in $\mathcal{A}(2)$ and $\mathbb{H}$ realizes the initial 1-types (b, I) with $I \cong \vec{I}_2$ and at most one edge. Then $\mathbb{H}$ realizes the initial 1-type (b, I) with $I = acd \cong \vec{I}_3$ and with unique edge bc.*

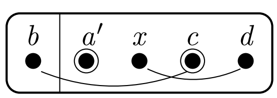


$\mathbb{H}$

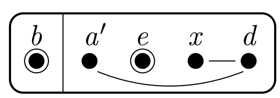
PROOF. Amalgamate as follows.



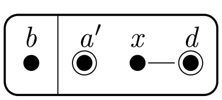
$(*)$



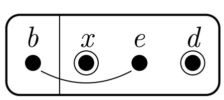
A



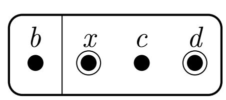
$A1$



$A1.1$



$A1.1.2$



$A2$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	(b, acd) □	$(b, a'ad)$ □			
(A)	$(b, a'cd)$ □	$(b, a'cd)$ $(*)_1$	$(*)_2$		H_2: 1 Edge
$(A1)$	(b, exd) $(A)_1$	$(b, a'ex)$ □	$(A2)$	□	$(A)_2$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(A1.1)$	$(A1)_1$	$(A)_1$	$(A1)_2$		H_2: 2 Edges
$(A1.1)_1$		Assumed	$(A1.1.2)$	□	$(A1.1)_2$
$(A1.1.2)_1$		Assumed	$(A1.1.2)_2$		Assumed
$(A2)_1$		Assumed	$(A2)_2$		H_2: $\vec{I}_3$□

LEMMA 6.19. *Let $k, \ell \geq 0$. Let $\mathbb{H}$ be a countable homogeneous ordered 2-graph satisfying the following conditions.*

(a) Every configuration in $\mathcal{A}(2)$ embeds in H_2;

(b) $\mathbb{H}$ contains every initial 1-type (b, I) with I an independent set of vertices, for which one of the following holds:

— $|I| = k$ and $b \perp I$; or

— $|I| = \ell$ and $b \to I$; or

— $|I| = 2$ and there is one edge between b and I.

Then $\mathbb{H}$ realizes every initial 1-type over $I_{k+\ell}$ for which there are exactly k edges and ℓ nonedges.

PROOF. We proceed by induction on $k + \ell$.

If k or ℓ is 0, or if $k = \ell = 1$, this holds by hypothesis.

So suppose

$$k, \ell \geq 1, \text{ and } k + \ell \geq 3$$

Set $a = \min I$ and $p = \operatorname{tp}(b/a)$. Our assumptions are symmetrical in the cross types $\perp$, $\to$, so we may suppose that

$$b \perp a$$

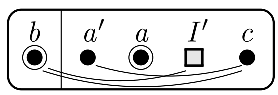
Let $\mathbb{H}'$ be the homogeneous ordered 2-graph induced on

$$(a^{\perp -} \cap H_1, a^{\perp +} \cap H_2)$$

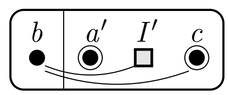
Claim. The ordered 2-graph $\mathbb{H}'$ realizes the 1-type (b, I) with $|I| = \ell$ and $b \to I$.

We claim that $\mathbb{H}$ realizes the configuration baI with $b < a < I \cong I_\ell$, $b \perp a$, $b \to I$.

Let $c = \max I$ and $I' = I \setminus \{a, c\}$. We perform the following amalgamations of ordered 2-graphs.



$(*)$



A

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(b, aI'c)$ □	$(b, a'aI')$ □			
(A)	□	$(*)_1$	$(*)_2$		H_2: 1 Edge
$(A)_1$	Induction to $(1, \ell - 1)$		$(A)_2$	Assumed	

This proves the claim.

If $k = 1$, then the configuration $(b, I_{k+\ell})$ is $b \perp a$, $b \to I'$, and our claim applies. So we may suppose from now on that

$$k \geq 2.$$

Now to conclude, it will suffice to prove the following, and apply induction.

Claim. $\mathbb{H}'$ satisfies the same hypotheses as $\mathbb{H}$, with (k, ℓ) replaced by $(k - 1, \ell)$.

By Corollary 6.4.1, every configuration in $\mathcal{A}(2)$ embeds into H_2'.

So it suffices to check the appropriate 1-types (b, I) for $\mathbb{H}$.

By hypothesis $\mathbb{H}'$ realizes the initial 1-type (b, I) with $b \perp I$ and $I \cong I_{k-1}$, and our first claim takes care of the 1-type (b, I) with $b \to I$ and $I \cong I_\ell$.

To complete the proof, it suffices to check that $\mathbb{H}'$ realizes the two configurations (b, cd) with $b < c < d$, and either

(I) $b \to c$, $b \perp d$, or

(II) $b \to d$, $b \perp c$.

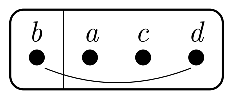
In terms of $\mathbb{H}$, we must realize the configurations (b, acd) with unique edge

(I) bc, or

(II) bd.

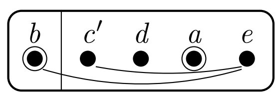
Since $k \geq 2$, Lemma 6.18 yields the configuration (b, acd) with unique edge bc (variant I).

The final configuration to be considered, namely (II), is (b, acd) with unique edge bd.

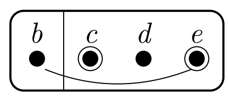


(II)

For this we amalgamate as follows.



$(*)$



A

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	(b, dae) □	(b, cda) □			
(A)	□	$(*)_1$	$(*)_2$		1 Edge
$(A)_1$, $(A)_2$		Assumed			

This completes the proof. □

Corollary 6.19.1. Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. Then $\mathbb{H}$ realizes every initial 1-type (b, I) with I an independent set of vertices.

Corollary 6.19.2. Let $\mathbb{H}$ be an ample homogeneous 2-graph, $a \in H_2$, and p an initial 1-type over a realized in H_1. Then $(a^p, a^{\perp\pm})$ is ample.

PROOF. Combine Corollaries 6.3.1 and 6.19.1. □

Now we prove the strong form of Proposition (IB) (equivalent to the original form in view of Lemma 6.19.1).

Proposition (IB). Suppose Γ is a homogeneous ordered graph which contains all configurations in $\mathcal{A}(2)$. Then Γ realizes all initial 1-types over an independent set.

PROOF. Let $A = bI$ be an initial 1-type with I an independent set. Let $k = |I|$. We argue by induction on k. Let $c = \max I$ and $I' = I \setminus \{c\}$.

If we have $b \perp c$ then by Corollary 6.3.1 this reduces to bI' with $|I'| = k - 1$, and induction applies.

If we have $b \to c$ then Lemma 6.17 applies, and (b, I') is realized in $(c^{\leftarrow}, c^{\perp -})$. $\square$

This completes the proof of Proposition I.

CHAPTER 7

ORDERED HOMOGENEOUS GRAPHS: PROPOSITION II

Before entering into the proof of Proposition II, we establish two useful lemmas. The first of these is a closure condition for the operation $\vec{K}_2 \perp$, on the class $\mathcal{A}(2)$.

7A. Closure under $\vec{K}_2 \perp X$

Our first objective is the following.

LEMMA 7.1. *Let $n \geq 2$ and let X be a finite ordered graph which belongs to every amalgamation class of finite ordered graphs containing $\mathcal{A}(2)$. Then $\vec{K}_2 \perp X$ and $X \perp \vec{K}_2$ also belong to every such amalgamation class.*

By a formal reduction, it suffices to treat the cases in which X belongs to $\mathcal{A}(2)$.

LEMMA 7.2. *Let A be a finite ordered graph of the form*

$$A_1 \perp A_2 \perp \cdots \perp A_N$$

where each A_i is either an independent set or a clique of order 2. Then any amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$ must contain A.

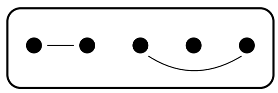
PROOF. It suffices to treat the case in which all A_i are cliques of order 2. We proceed by induction on N.

Let $\mathcal{A}$ be the amalgamation class under consideration, Γ the corresponding homogeneous ordered graph, and view A_1 as embedded in Γ. If $A_1^{\perp+}$ contains all configurations in $\mathcal{A}(2)$ then by induction $A_2 \perp \cdots \perp A_N$ embeds in $A_1^{\perp+}$ and our claim follows.

By Corollary 6.4.1 and Lemma 6.10, $A_1^{\perp+}$ contains all $\vec{K}_2 \perp \vec{I}_k$ for $k < \infty$.

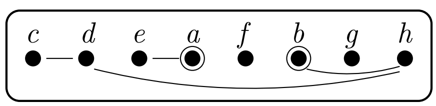
Suppose now that $A_1^{\perp+}$ does not contain $\mathcal{A}(2)$. Then either $A_1^{\perp+}$ must lie in the catalog in Group II, in which case it follows by inspection that A embeds in $A_1^{\perp+}$ and hence in Γ, or else $A_1^{\perp+}$ must omit $(\vec{I}_1 \perp \vec{P}_3)$, and then Proposition 4.1 says that $A_1^{\perp+}$ is $\vec{\mathbb{S}}(2)$, which also contains all of the desired configurations. $\square$

LEMMA 7.3. *Let $\mathcal{A}$ be an amalgamation class of finite ordered graphs containing $\mathcal{A}(2)$. Then $\vec{K}_2 \perp \vec{P}_3^c$ is in $\mathcal{A}$.*

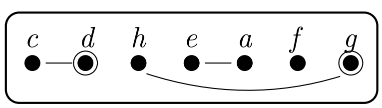


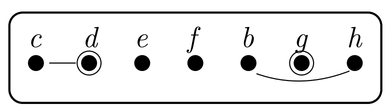
$\vec{K}_2 \perp \vec{P}_3^c$

PROOF. We use the following amalgamation.



$(*)$



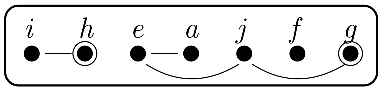


A B

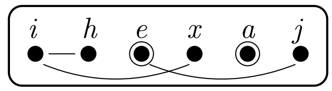
Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(eabgh)$ $\square$	$(cdafb)$ $\square$			
(A)	$(cdhfg)$ $\square$	$(cdeafg)$ $(*)_1$	(B)	$(cdbgh)$ $\square$	$(*)_2$
$(B)_1$	Corollary 6.3.1		$(B)_2$	Lemma 7.2	

So to conclude, we must show that if $\vec{K}_2 \perp \vec{P}_3^c$ is not in $\mathcal{A}$, then the factors $(A)_1 = (c\ d\ h\ e\ a\ f)$ and $(A)_2 = (c\ h\ e\ a\ f\ g)$ of (A) both embed into Γ.

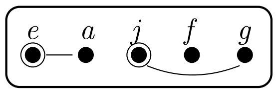
Now the factor $(A)_1$ is covered by Lemma 7.2. The factor $(A)_2 = (cheafg) = (c) \perp (heafg)$, so it suffices to treat the factor $(A)'_2 = (heafg)$. We proceed as follows.



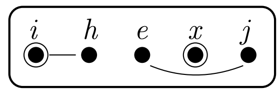
$A2'$



$A2'.1'$



$A2'.2'$



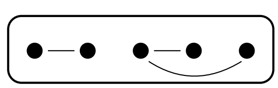
$A2'.1'.1$

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(A2')$	$(ihjfg)$ □	$(heafg)$ $(A)'_2$			
$(A2'.1')$	$(iheaj)$ □	$(iheaj)$ $(A2')'_1$	$(A2'.2')$	□	$(A2')'_2$
$(A2'.1'.1)$	□	$(A2'.1')_1$	$(A2'.1')_2$	Corollary 6.3.1	

Here the factors of $(A2')$ are $(A2')_1 = (iheajf) = (iheaj) \perp (f)$ and $(A2')_2 = (ieajfg) = (i) \perp (eajfg)$. These reduce by Corollary 6.3.1 to $(A2')'_1 = (iheaj)$ and $(A2')'_2 = (eajfg)$, dealt with by $(A2'.1')$ and $(A2'.2')$ as shown.

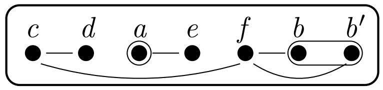
After the analysis shown it suffices to check that the factors of $(A2'.1')$ and $(A2'.2')$ are in $\mathcal{A}$. The factor $(A2'.1')_1 = (ihexaj)$ is dealt with in $(A2'.1'.1)$, whose factors are in $\mathcal{A}$ by Corollary 6.3.1 and Lemma 7.2; and similarly the other factor of $(A2'.1')$ and both factors of $(A2'.2')$ are in $\mathcal{A}$. □

LEMMA 7.4. *Let $\mathcal{A}$ be an amalgamation class of finite ordered graphs containing $\mathcal{A}(2)$. Then $\vec{K}_2 \perp [\vec{I}_1, \vec{I}_2]$ is in $\mathcal{A}$.*

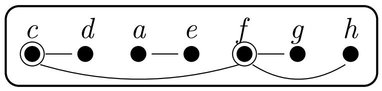


$\vec{K}_2 \perp [\vec{I}_1, \vec{I}_2]$

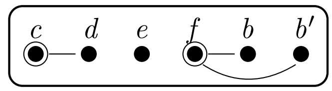
PROOF. We amalgamate as follows.



(∗)



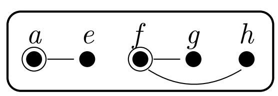
A



B

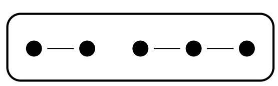
Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(aefbb')$ □	$(cdaeb)$ or $(cdaeb')$ □			
(A)	$(cdfgh)$ □	$(*)_1$	(B)	$(cdfbb')$ □	$(*)_2$
$(A)_1$	Lemma 7.2		$(B)_1, (B)_2$	Corollary 6.3.1	

This leaves us with the factor $(A)_2 = (d) \perp (aefgh)$ to consider, and by Corollary 6.3.1, this reduces to $(A)_2' = (aefgh)$. The following amalgamation gives either the target $\vec{I}_2 \perp [\vec{I}_1, \vec{I}_2]$ or the factor $(A)_2'$, and has subfactors in $\mathcal{A}$.



A2 □

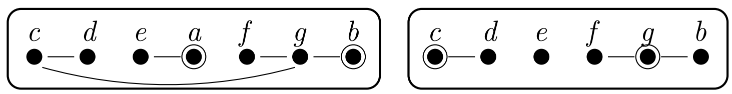
LEMMA 7.5. *Let $\mathcal{A}$ be an amalgamation class of finite ordered graphs containing $\mathcal{A}(2)$. Then $\vec{K}_2 \perp \vec{P}_3$ is in $\mathcal{A}$.*



$\vec{K}_2 \perp \vec{P}_3$

PROOF. Assuming the contrary, there is a homogeneous ordered graph Γ containing the configurations of $\mathcal{A}(2)$ but not containing $\vec{K}_2 \perp \vec{P}_3$. In this case for $K \cong \vec{K}_2$ in Γ, the ordered graph induced on $K^{\perp +}$ does not contain $\vec{P}_3$ but does contain $\vec{P}_3^c$ and $[\vec{I}_1, \vec{I}_2]$. It follows that $K^{\perp +}$ is the generic linear extension of a generic partial order, in other words generic omitting $\vec{P}_3$. So any configuration omitting $\vec{P}_3$ belongs to Γ.

We amalgamate as follows.



(∗) (∗)

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	$(eafgb)$ □	$(cdeab)$ □			
$(*)_1$	Omits $\vec{P}_3$		(B)	$(cdfgb)$ □	$(*)_2$
$(B)_1$	Omits $\vec{P}_3$		$(B)_2$	Corollary 6.3.1	

□

Now we can prove closure under $\vec{K}_2 \perp$ and $\perp \vec{K}_2$ for the structures present in every amalgamation class containing $\mathcal{A}(2)$.

Proof of Lemma 7.1. Let X be a finite ordered graph belonging to every amalgamation class containing $\mathcal{A}(2)$. Our claim is that $X \perp \vec{K}_2$ and $\vec{K}_2 \perp X$ also belong to every amalgamation class containing $\mathcal{A}(2)$. By symmetry (Lemma 6.1) it suffices to treat the case of

$$\vec{K}_2 \perp X.$$

It suffices to verify this claim in the special case where X is one of the configurations in $\mathcal{A}(2)$:

$$X = (\vec{I}_1 \perp \vec{P}_3),\ \vec{P}_3^c, \text{ or } \vec{I}_k \text{ with } k < \infty$$

By Lemma 4.8 we may take $\vec{P}_3 \perp \vec{I}_1$ in place of $\vec{I}_1 \perp \vec{P}_3$. Then the desired configuration is $\vec{K}_2 \perp \vec{P}_3 \perp \vec{I}_1$, which we have by Lemma 7.5 and Corollary 6.4.1.

We have $\vec{K}_2 \perp \vec{P}_3^c$ by Lemma 7.3, and $\vec{K}_2 \perp \vec{I}_k$ by Corollary 6.4.1.

This completes the proof. □

7B. $\mathcal{A}(n)$ and $[\vec{I}_1, \mathcal{A}(n-1)]$

In this subsection we prove a lemma relating $\mathcal{A}(n)$ and $\mathcal{A}(n-1)$ inductively, as follows.

LEMMA 7.6. *Let $n \geq 3$, and assume Proposition VIII$_2$ holds. Let X be a finite ordered graph which belongs to every amalgamation class containing $\mathcal{A}(n-1)$. Then $[\vec{I}_1, X]$ belongs to every amalgamation class containing $\mathcal{A}(n)$.*

We begin with the case $n = 3$.

LEMMA 7.7. *Assume Proposition VIII$_2$ holds, and let X be a finite ordered graph which belongs to every amalgamation class containing $\mathcal{A}(2)$. Then $[\vec{I}_1, X]$ belongs to every amalgamation class containing $\mathcal{A}(3)$.*

PROOF. Let $\mathcal{A}$ be an amalgamation class containing $\mathcal{A}(3)$, and let Γ be the homogeneous ordered graph corresponding to $\mathcal{A}$. Let $a \in \Gamma$ and let $\Gamma' = a^{\perp+}$. We must show that Γ' contains X. It suffices to show that Γ' contains each configuration in $\mathcal{A}(2)$.

By Lemma 6.16, Γ' contains all $\vec{I}_k$ for $k < \infty$. By Lemma 6.15, Γ' contains $\vec{P}_3^c$. It remains to check that Γ' contains $\vec{I}_1 \perp \vec{P}_3$.

Suppose Γ' does not contain $\vec{I}_1 \perp \vec{P}_3$. We know Γ' does contain $\vec{P}_3^c$ and all $\vec{I}_k$, by Lemma 6.15 and Corollary 6.3.1. So Γ' must be in our catalog, under group (II) or entry (IIIA). In particular Γ' contains all complete ordered graphs $\vec{K}_k$ for $k < \infty$, so Γ contains these as well.

Thus Γ^c contains all $\vec{I}_k$ for $k < \infty$, and in view of Lemma 6.15, Γ^c contains $\mathcal{A}(3)$.

By Lemma 6.15 and Corollary 6.4.1, Γ^c contains $\vec{I}_1 \perp [\vec{I}_1, \vec{P}_3^c]$ and thus Γ contains $(\vec{I}_1 \perp [\vec{I}_1, \vec{P}_3^c])^c = [\vec{I}_1, \vec{I}_1 \perp \vec{P}_3]$, as claimed. $\square$

PROOF OF LEMMA 7.6. It suffices to show that for $X \in \mathcal{A}(n-1)$, we have $[\vec{I}_1, X]$ in every amalgamation class of finite ordered graphs which contains $\mathcal{A}(n)$.

For $X \in \mathcal{A}(2)$ this holds by Lemma 7.7. For $X = K_{n-1}$ we have $[\vec{I}_1, X] = \vec{K}_n \in \mathcal{A}(n)$. Our claim follows. $\square$

Corollary 7.7.1. Assume Proposition VIII$_2$ holds. If $m+k+m' = n$ and $X \in \mathcal{A}(k)$, then any amalgamation class of finite ordered graphs containing $\mathcal{A}(n)$ contains the ordered graph $[\vec{K}_m, X, \vec{K}_{m'}]$.

PROOF. Proceed by induction on $m + m'$.

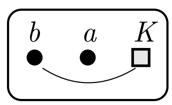
If $m = m' = 0$ the claim is vacuous.

If $m > 0$ we view X as $[\min X, X']$, apply induction to conclude that X' is in every amalgamation class containing $\mathcal{A}(n-1)$, and apply Lemma 7.6 to conclude.

If $m' > 0$ we may apply the dual (under reversal) to reduce m' and n. $\square$

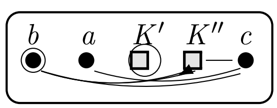
7C. Proof of Proposition II

LEMMA 7.8. *Let $n \geq 2$, and if $n \geq 3$ assume Proposition VIII$_{n-1}$ holds. Let $B = baK$ be an ordered graph with $b < a < K$, bK complete of order n, and $a \perp bK$. Then any amalgamation class of finite ordered graphs containing $\mathcal{A}(n)$ contains baK.*

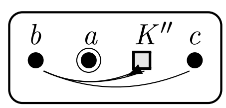


baK

PROOF. This is clear if $n = 2$. So suppose $n \geq 3$. Let $c = \max K$, and let K' be another copy of K. Set $K'' = K \setminus \{c\}$. Note that K'' is nonempty. Amalgamate as follows.



$(*)$



A

The outcome of the amalgamation either involves some non-edge (b, k) with $k \in K'$, or makes bK' a clique. This accounts for the first line below.

Label	*Non-edge*	*Edge*	*Label*	*Non-edge*	*Edge*
$(*)$	(bkK) $\square$	(baK') $\square$			
(A)	baK $\square$	$(*)_1$	$(*)_2$	Proposition VIII$_{n-1}$	

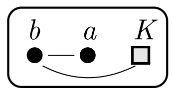
For the last line, note that $|K'| = |K| = n - 1$, and $n \geq 3$. $\square$

Formally speaking, Proposition IX$_n$ contains Proposition VIII$_n$. In the next proposition, and others of a similar character, the hypothesis "Proposition IX$_{n-1}$" is best understood as "Propositions VIII$_{n-1}$ and IX$_{n-1}$ hold."

Proposition 7.9 (II_n). Let $n \geq 2$, and if $n > 2$ assume Proposition IX_{n-1} holds. Let Γ be a homogeneous ordered graph. If all elements of $\mathcal{A}(n)$ embed in Γ, and $B = baK$ satisfies the conditions

— $K \cong \vec{K}_n$,
— $b < a < K$,
— $a \perp bK$,
— B does not contain $\vec{K}_{n+1}$,

then B embeds in Γ.



baK

PROOF. If $b \perp K$ then baK is $\vec{I}_2 \perp \vec{K}_n$ and Corollary 6.4.1 suffices. By hypothesis we do not have $b \to K$. Therefore we may suppose that b is related to K by both edges and nonedges. Take $u, v \in K$ adjacent (in the ordering) with

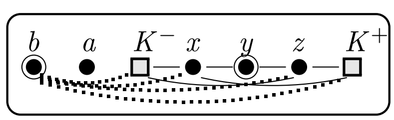
$$b \perp u \text{ and } b \to v$$

Set $x = \min(u, v)$, $z = \max(u, v)$, and let $\hat{K} = K \cup \{y\}$ with $x < y < z$.

Let $K^- = \{k \in K \mid k < x\}$ and $K^+ = \{k \in K \mid k > z\}$. Let

$$K_0 = \{k \in K^- \cup K^+ \mid (b, k) \text{ is an edge}\}$$

We form the following amalgamation, in which $\hat{K}$ is a clique with one edge removed, namely (x, z)



$(*)$

Then in the amalgam, either (baK^-xyK^+) or (b, aK^-yzK^+) will be a copy of the desired configuration (baK), depending on whether (b, y) is a non-edge or an edge (and similarly, on which applies to (b, u) and to (b, v)).

So it suffices to check that the factors of this amalgamation problem lie in Γ.

Factor $(*)_1$*:* (baK^-xzK^+)*:*

If $n = 2$ this is $baxz$ with a unique edge, and is covered by Lemma 6.5. So suppose for the present that

$$n > 2, \text{ and Proposition } \mathrm{IX}_{n-1} \text{ holds.}$$

Let $\mathbb{H} = (a^{\perp -}, a^{\perp +})$. By Proposition (I) (= Proposition 6) $\mathbb{H}$ is ample. We claim that the configuration (b, K^-xzK^+) embeds in $\mathbb{H}$. By Proposition IX_{n-1} it suffices to check that the configuration (b, K^-xzK^+) is $\mathbb{H}$-constrained. As (x, z) is not an edge, this means that (b, K^-xK^+) and (b, K^-zK^+) embed into $\mathbb{H}$.

Now K^-xK^+ and K^-zK^+ are complete graphs of order $n-1$ and unless $b \to K^-vK^+$, the corresponding configurations baK^-xK^+ and baK^-zK^+ occur in Γ by induction. And the exceptional case baK^-vK^+ with $b \to K^-vK^+$ is covered by Lemma 7.8.

Factor $(*)_2$*:* $aK^-xyzK^+ = \vec{I}_1 \perp [K^-, \vec{P}_3, K^+]$*:*

By Corollary 6.4.1 it suffices to consider the configuration $[\vec{K}^-, \vec{P}_3, \vec{K}^+]$. Here Corollary 7.7.1 applies. □

CHAPTER 8

ORDERED HOMOGENEOUS GRAPHS: PROPOSITION III

8A. Ramsey 2-types

We make use of the following terminology.

Definition 8.1. Let $\mathbb{H}$ be an ordered 2-graph and let r be one of the two possible types of pairs $a, b \in H_1$ with $a < b$ (an ordered edge or an ordered non-edge). Let X be a finite ordered graph which embeds in H_2. We say that r is a *Ramsey* 2*-type* for $\mathbb{H}$ over X if the following holds.

> For any finite r-Ramsey graph R and any finite ordered 2-graph $\mathbb{A} = (R, \perp^k X)$ with $R < X$, if every configuration of the form (a, X_i) which occurs in $\mathbb{A}$ embeds into $\mathbb{H}$, where $a \in R$ and X_i is one of the summands of $\perp^k X$, then the configuration $\mathbb{A}$ also embeds into $\mathbb{H}$.

In practice, the way we find Ramsey 2-types is by an application of Ramsey's Theorem in the following context.

Lemma 8.2. *Let $\mathbb{H}$ be a homogenous 2-graph and X a finite ordered graph. Suppose that the following holds.*

> *For any initial* 1*-type $p = (a, \perp^k X)$ over an ordered sum of copies of X for which the restrictions (a, X_i) to individual summands are realized in $\mathbb{H}$, the full 1-type p is realized in $\mathbb{H}$.*

Then there is a Ramsey 2*-type for $\mathbb{H}$ over X.*

Proof. This is a more precise formulation of Lemma 5.10, with the same proof (i.e., following the proof of Lemma 5.9). $\square$

Proposition III concerns the realization of 1-types over configurations in $\mathcal{A}(2)$. As preparation for its proof we will first need to prove the existence of a Ramsey 2-type over $\vec{K}_2$. This goal also requires extensive preparation. We begin as follows.

LEMMA 8.3. *Let* $\mathbb{H}$ *be an ample homogeneous ordered* 2*-graph. Then there is a Ramsey* 2*-type for* $\mathbb{H}$ *over* $\vec{I}_1$.

PROOF. By Lemma 8.2 it suffices to check that any $\mathbb{H}$-constrained initial 1-type over $\vec{I}_k$ is realized in $\mathbb{H}$. This is Corollary 6.19.1. □

8B. An inductive principle

We aim to treat 1-types over a base consisting of an ordered disjoint union of copies of $\vec{K}_2$ by induction on the number of copies of $\vec{K}_2$ in the base, with the case of two copies as the base case. The following result will give us a way of organizing this induction.

LEMMA 8.4. *Let* $\mathbb{H}$ *be an ample homogeneous ordered* 2*-graph and* $P = (a, K)$ *an initial 1-type over* $K \cong \vec{K}_2$ *which is realized in* $\mathbb{H}$*. Then the ordered* 2*-graph* $\mathbb{H}' = (K^P, K^{\perp\pm})$ *is ample.*

The proof of Lemma 8.4 will rely on the next two preparatory lemmas.

LEMMA 8.5. *Let* $\mathbb{H}$ *be an ample homogeneous ordered* 2*-graph. Then any* $\mathbb{H}$*-constrained initial* 1*-type* p *over* $\vec{I}_1 \perp \vec{K}_2$ *or* $\vec{K}_2 \perp \vec{I}_1$ *is realized in* $\mathbb{H}$.

As the next proof is more delicate than our earlier amalgamation arguments, and the method will remain important afterward, we will write out the argument in considerable detail.

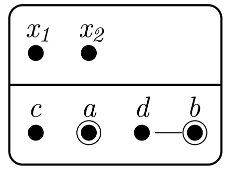
PROOF. Suppose first that

$$p = (a, bc_1c_2)$$

is a 1-type over $\vec{I}_1 \perp \vec{K}_2$ with the 1-types $p_1 = \mathrm{tp}(a/\vec{I}_1)$ and $Q = \mathrm{tp}(a/\vec{K}_2)$ realized in $\mathbb{H}$.

Take a Ramsey 2-type r over $\vec{I}_1$ for $\mathbb{H}$, and let $R = \{x_1, x_2\}$ have the type r. Amalgamate as shown below.

This picture represents an amalgamation argument which is more demanding, but also more flexible, than the ones we have been considering up to this point, and one in which the order of steps is less transparent. Our previous amalgamation arguments could be diagrammed as a search through a tree of possibilities, where the order



$x_1/adb = p$; $x_2/cab = p$

$(*)$

of the search is of no importance, but one must remember all the ancestors of a given node.

As always, the task is to ensure that both factors of the diagram are available. Usually in such cases the main difficulties are concentrated in one factor; here we must keep an eye on both factors as the argument proceeds.

For this reason, our initial description of the amalgamation diagram is not yet complete. To complete our specifications requires carrying out the steps in a definite order, and retaining parts of the solution to one amalgamation problem when setting up the next.

What the picture shows us is the set R sitting on one side of $\mathbb{H}$ (here represented as the top half), and some vertices c, a, d, b sitting on the other side. It is assumed that

$$x_1 < x_2 < c < a < d < b.$$

That is, R and $cadb$ are ordered as shown, and in the present context we are taking $R < cadb$, though this point is not shown in the diagram.

Whether R contains an edge is determined by the type r, and does not need to be indicated. Any edges present in H_2 will be indicated (of course, the type of the pair (a, b) will only be determined by completing the amalgamation diagram).

The relations between R and $(cadb)$ remain to be specified. Our annotations (under the diagram) give the relationship of x_1 to a, d, b and of x_2 to c, a, b. This should not be taken too literally, since the structure of (a, b) is unsettled at this point. What we really mean here is that the relations of x_1 to a, d, b or of x_2 to c, a, b are as specified by the type p, so that in particular if (a, b) is a non-edge then (x_1, adb) really will realize the type p, while if (a, b) is an edge then (x_2, cab) will realize the type p. In particular, any amalgamation diagram of this type will force a copy of the type p to occur in $\mathbb{H}$, as desired—once we ensure that the two factors of such a diagram embed into $\mathbb{H}$.

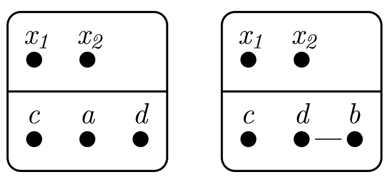
In the factors, what remains unspecified at this point is the nature of (x_1, c) and of (x_2, d):

Are (x_1, c), (x_2, d) edges or non-edges?

We will not have a definite diagram until this information is supplied. Our goal now is to fill in the missing data in such a way that the factors of the amalgamation diagram occur in $\mathbb{H}$. If we succeed in this, the proof is complete.

One proceeds by analyzing the factors further and filling in the data as needed along the way. And it is here that the order of the steps becomes important; it will be different from the natural order of analysis.

What has been said so far applies generally to constructions of this type, which will be seen again. Now let us consider the specific case at hand. We are concerned with the following factors.



(1) (2)

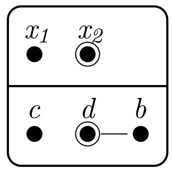
$x_1/c = ?$ $x_2/d = ?$

The question is whether we can find a suitable choice of relations on (x_1, c) and (x_2, d) (edge or non-edge), consistently, so that each factor embeds into $\mathbb{H}$.

The factor (1) is the easier of the two to deal with. Here (c, a, d) is an independent set and R is r-Ramsey over $\vec{I}_1$, so as long as the types realized by x_1 and x_2 over (c, a, d) are realized in $\mathbb{H}$, this factor will be present in $\mathbb{H}$. By Corollary 6.19.1, this is no constraint at all.

So we are left with the much simpler task of finding some form of factor (2) in $\mathbb{H}$. This argument is a little too easy: a similar argument at a crucial point in the proof of Proposition V will require more attention to what precisely needs to be copied from factor (2) into factor (1), when there are some restrictions to be observed.

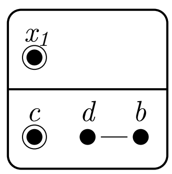
But in the present case, we need only concern ourselves with the factor (2). Once we have determined the type of (x_1, c), we will arrive at this factor by a 2-point amalgamation which determines the type of x_2, as shown below.



$x_1/c = ?$

B

Here the factors in this amalgamation diagram are $(B)_1 = (R, cb)$ and (x_1, cdb). Again, by the choice of R, either possible form of (R, cd) will be available in $\mathbb{H}$, so now we have only to choose the type of (x_1, c) so as to have (x_1, cdb) available in $\mathbb{H}$. This will be settled by the corresponding amalgamation, as shown.

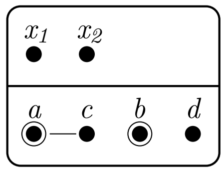


$B2$

This last diagram has a factor (x_1, db) still to be considered. But that configuration is contained in the $\mathbb{H}$-constrained type p, so by definition it is realized in $\mathbb{H}$.

This concludes our analysis, but it is worth while to recapitulate. The actual order of events in our construction begins with the last amalgamation diagram shown $(B2)$. This is to be completed in $\mathbb{H}$ to determine whether (x_1, c) should be an edge or a non-edge. Then we proceed similarly with the amalgamation represented in (B) above to determine whether (x_2, d) is an edge or not. After that one may go back to the beginning of the proof, knowing precisely what the configuration $(*)$ represents, after which the argument falls back into the line of those we carried out earlier.

We have also the second half of the claim to deal with, concerning initial 1-types over $(\vec{K}_2 \perp \vec{I}_1)$. This claim does not quite follow by symmetry on formal grounds, but in fact a symmetrical argument will work. One works with an amalgamation of the following form, and one argues as above.



(∗)

□

LEMMA 8.6. *Let* $\mathbb{H}$ *be an ample homogeneous 2-graph. Then any* $\mathbb{H}$*-constrained initial* 1*-type* $p = (a, IKJ)$ *over a configuration of the form* $\vec{I}_k \perp \vec{K}_2 \perp \vec{I}_\ell$ *is realized in* $\mathbb{H}$.

PROOF. We proceed by induction on $k + \ell$. Let P denote the restriction of p to $\vec{K}_2$. By hypothesis P is realized in $\mathbb{H}$.

If $k = \ell = 0$ there is nothing to prove. If $k > 0$ we take $b \in H_2$, let p_1 be the type of a over the first element of I, and consider the homogeneous ordered 2-graph $\mathbb{H}'$ induced on

$$(b^{p_1}, b^{\perp +})$$

where $b^{p_1} \subseteq H_1$ and $b^{\perp +} \subseteq H_2$.

By Corollary 6.19.2, $\mathbb{H}'$ is again ample, and so by induction it suffices to check the following.

> The restricted 1-type $(a, I'KJ)$ is $\mathbb{H}'$-constrained
>
> (where $I' = I \setminus \{\min I\}$).

In terms of $\mathbb{H}$, this amounts to considering restrictions of p to configurations $X \cong \vec{I}_2$ or $\vec{I}_1 \perp \vec{K}_2$. These restrictions are obtained from ampleness of $\mathbb{H}$ and Lemma 8.5.

If $k = 0$ and $\ell > 0$ we work from the other end. □

After these preparations, Lemma 8.4 may be treated as a corollary.

PROOF OF LEMMA 8.4. Recall that $\mathbb{H}' = (K^P, K^{\perp \pm})$ with $K \cong \vec{K}_2$, and our claim is that $\mathbb{H}$ is ample. Here $K^{\perp \pm}$ refers to either one of the sets $K^{\perp +}$ or $K^{\perp -}$, taken separately.

By Lemma 7.1, the second component $H_2' = K^{\perp\pm}$ contains the configurations of $\mathcal{A}(2)$.

By Corollary 6.19.1, $\mathbb{H}$ realizes all initial 1-types over all $\vec{I}_k$, and by out hypothesis $\mathbb{H}$ also realizes the type P. Thus any initial 1-type extending P over a configuration $K \perp J$ or $J \perp K$ with J an independent set will be $\mathbb{H}$-constrained. By Lemma 8.6, $\mathbb{H}$ realizes all such 1-types and $\mathbb{H}'$ realizes all 1-types over an independent set. □

8C. Ramsey 2-types over $\vec{K}_2$

We are working toward the following.

LEMMA 8.7. *Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. If $p = (a, \perp^n \vec{K}_2)$ is an $\mathbb{H}$-constrained initial 1-type over an ordered direct sum of copies of $\vec{K}_2$, then p is realized in $\mathbb{H}$.*

As we explained at the beginning of the present section, this lemma then has the following useful corollary.

Corollary 8.7.1. Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. Then there is a Ramsey 2-type for $\mathbb{H}$ over $\vec{K}_2$.

The proof of Lemma 8.7 involves something more substantial than the kind of direct amalgamation arguments used earlier. Indeed, when generalized from $\vec{K}_2$ to $\vec{K}_n$, this will become a key argument for the completion of the analysis, as discussed in detail in §2A.

The main difficulty will be the verification of our claim in the case of 1-types over

$$\vec{K}_2 \perp \vec{K}_2.$$

We will make use of the following notational conventions.

Notation 8.8.

1. We use P, Q and the like to denote initial 1-types over $\vec{K}_2$ in the ordered 2-graph setting, that is

$$(a, K) \text{ with } K \cong \vec{K}_2 \text{ and } a < K.$$

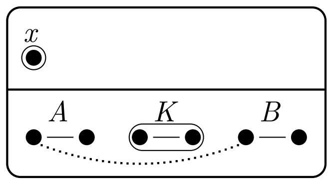
2. If P, Q are initial 1-types over $\vec{K}_2$ then

$$P \perp Q$$

is the corresponding initial 1-type over $\vec{K}_2 \perp \vec{K}_2$, with restrictions P, Q to the individual summands.

LEMMA 8.9. *Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph, and let P_1, P_2 be* 1*-types over $\vec{K}_2$ which are realized in $\mathbb{H}$. Then for some 1-type Q over $\vec{K}_2$, the* 1*-types $P_1 \perp Q$ and $Q \perp P_2$ are both realized in $\mathbb{H}$.*

PROOF. Let $A \cong B \cong K \cong \vec{K}_2$, with $A = \{a, a'\}$ and $B = \{b, b'\}$. Let AKB be $A \perp K \perp B$ with possibly one additional edge (a, b). Amalgamate as follows.



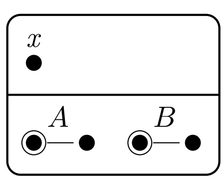
$x_1/A, B = P_1, P_2$

$(*)$

Here we show the undetermined type of the pair (a, b) as a dotted line. We also specify that the types (x/A) and (x/B) should agree with P_1 and P_2, respectively. So in the amalgam, whatever type Q is realized by x over K will be as required.

It remains to be seen whether the factors of this diagram can be obtained in $\mathbb{H}$, assuming that the type of the pair (a, b) is chosen judiciously.

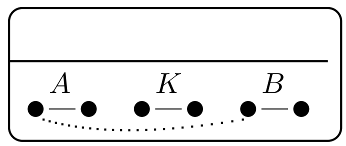
We first form the factor (x, AB) by an amalgamation determining the type of (a, b).



$x/A, B = P_1, P_2$

(1)

Here we require the subfactors (x, Ab') and $(x, a'B)$ which are afforded by Lemma 8.5. The result of the amalgamation (1) will determine whether (a, b) is an edge or not, and hence determine the form of the corresponding factor of $(*)$.



(2)

So it will suffice to obtain the two possible forms of the factor (2), with (a, b) an edge or a non-edge.

If $a \perp b$ in (2) then Lemma 7.2 applies. So we may suppose

$$a - b$$

holds in the completion of diagram (1) and the target is diagram (2) with an edge (a, b). We take an indirect approach to this.

Set

$$p = \mathrm{tp}(a/B);$$
$$\mathbb{H}' = (B^p, B^{\perp -}).$$

Claim. $\mathbb{H}'$ is ample.

Take $c \in H_2$ and let $\mathbb{H}(c) = (c^{\perp -}, c^{\perp +})$. By Proposition I

$$\mathbb{H}(c) \text{ is ample.}$$

Observe that the type p is realized in $\mathbb{H}(c)$: in terms of H_2 the required configuration is (acB) with four vertices and two edges, namely (a, b) and (b, b'), and this is covered by Lemma 6.13.

Since the type p is realized in $\mathbb{H}(c)$, the 2-graph $\mathbb{H}'(c) = (B^p, B^{\perp -})$, computed inside $\mathbb{H}(c)$, is again ample, by Lemma 8.4. But this 2-graph is contained in $\mathbb{H}'$, so $\mathbb{H}'$ is ample.

Now our claim that (2) embeds in $\mathbb{H}$ amounts to the claim that there is a realization of the initial 1-type $(a, a'K)$ in $\mathbb{H}'$, where $a'K \cong \vec{I}_1 \perp \vec{K}_2$ (Figure 74).

By Lemma 8.5 this reduces to the 1-types (a, a') and (a, K) in the ordered 2-graph $\mathbb{H}'$. Now we reinsert B and view these configurations

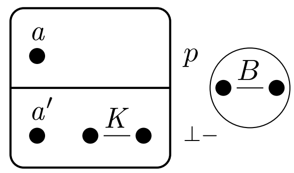
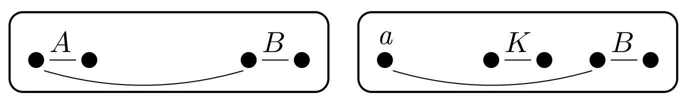


FIGURE 74. $\mathbb{H}'$

as (AB) and (aKB). So it will suffice to find those configurations in $\mathbb{H}$.



(AB) (aKB)

Now by hypothesis the configuration (AB) was already produced, with the edge (a,b) included, within the result of amalgamation (1).

For (aKB), consider the 2-graph

$$\mathbb{H}^* = (b'^{\leftarrow}, b'^{\perp -}).$$

We require the terminal 1-type (b, aK) with $aK \cong \vec{I}_1 \perp \vec{K}_2$ in $\mathbb{H}^*$. By the dual of Lemma 8.5, this reduces to (b,a) and (b,K) (in $\mathbb{H}^*$). Thus in H_2, (aKB) may be obtained from the two factors $(aB) = \vec{P}_3^+$ and $(KB) = \vec{K}_2 \perp \vec{K}_2$, both of which are available.

This completes the argument. $\square$

Now we will need to refine our inductive approach, using the following parameter.

DEFINITION 8.10. If $\mathbb{H}$ is a homogeneous ordered 2-graph and X is a finite ordered graph embedding in H_2, let $s(\mathbb{H}, X)$ be the number of initial 1-types realized by elements of H_1 over X.

LEMMA 8.11. *Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. Suppose that for every ample homogeneous ordered 2-graph $\mathbb{H}'$ for which $s(\mathbb{H}', \vec{K}_2) < s(\mathbb{H}, \vec{K}_2)$ there is a Ramsey 2-type over $\vec{K}_2$. Let P, Q be 1-types over $\vec{K}_2$ realized in $\mathbb{H}$. Suppose that $Q \perp Q$ is realized in $\mathbb{H}$. Then $P \perp Q$ is realized in $\mathbb{H}$.*

PROOF. Suppose $P\perp Q$ is not realized in $\mathbb{H}$. Let K be a copy of $\vec{K}_2$ in H_2 and consider the homogeneous ordered 2-graph

$$\mathbb{H}^Q = (K^Q, K^{\perp -}).$$

Our hypotheses are that

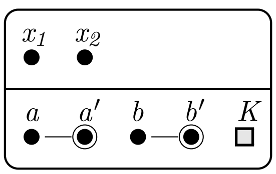
$$Q \text{ is realized in } \mathbb{H}^Q, \text{ and } P \text{ is not.}$$

Now $\mathbb{H}^Q$ is ample by Lemma 8.4. As $\mathbb{H}^Q$ does not realize the type P we have $s(\mathbb{H}^Q, \vec{K}_2) < s(\mathbb{H}, \vec{K}_2)$, so by hypothesis there is a Ramsey 2-type for $\mathbb{H}^Q$ over $\vec{K}_2$.Take a pair $R = (x_1, x_2)$ of type r.

By Lemma 8.9 there is a 1-type Q^* over $\vec{K}_2$ such that

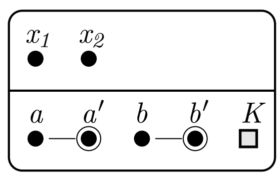
$$\text{both } P\perp Q^* \text{ and } Q^*\perp Q \text{ are realized in } \mathbb{H}.$$

We amalgamate as follows, with $A \cong B \cong K \cong \vec{K}_2$, and $A = (a, a')$, $B = (b, b')$. The type of x_2/ab is left open initially.

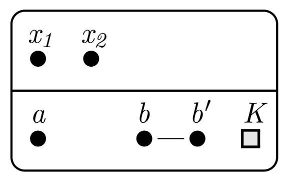


$x_1/A, B, K = P, Q, Q^*$
$x_2/ab, K = P, Q$

$(*)$



(1)



(2)

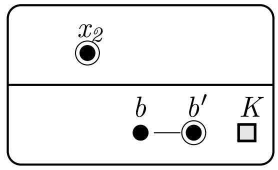
As usual, the condition "$x_2/ab = P$" should not be taken too literally: it determines the type of x_2 over a, b in such a way that if the pair (a, b) forms an edge in the amalgam, then x_2ab has the type p.

Any solution to the amalgamation problem $(*)$ will have x_1/AB or x_2/abK realizing the type $P\perp Q$, according as (a, b) is a non-edge or an edge. So as always, the problem is to build the two factors $(1, 2)$ inside $\mathbb{H}$ in such a way that they agree on the structure of $(x_2/a'b')$.

We first focus our attention on the factor (2).

We begin with the following amalgamation.

The factors here are (x_2, bK) and $(bb'K) = \vec{K}_2\perp\vec{K}_2$. To see that the factor (x_2, bK) occurs in $\mathbb{H}$, it suffices to check that it is $\mathbb{H}$-constrained, by Lemma 8.5. In other words, we need to check (x_2, b) and (x_2, K). But the type of x_2/K is Q, which by assumption occurs in $\mathbb{H}$.



B-1

Thus the diagram (B–1) can be completed in $\mathbb{H}$ and determines the type of the pair (x_2, b'). At this stage, the factor (2) is completely determined, and we must show that it occurs in $\mathbb{H}$.

To see that the factor (2) occurs in $\mathbb{H}$, it suffices to show that it occurs in $\mathbb{H}^Q$. By the choice of r and R, it suffices to check that for $i = 1, 2$ the following holds.

The types (x_i, a), (x_i, B), and (x_i, K) all occur in $\mathbb{H}^Q$.

As $\mathbb{H}^Q$ is ample the types (x_i, a) are certainly realized in $\mathbb{H}^Q$.

The types of x_1 over B and K are respectively Q and Q^*. These are both realized in $\mathbb{H}^Q$.

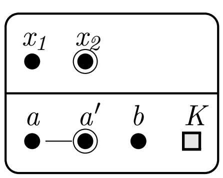
The type of x_2 over K is also Q. Finally, the type of x_2 over B is the result of the amalgamation (B–1), which puts that type into $\mathbb{H}^Q$.

Thus the factor (2) embeds in $\mathbb{H}^Q$, hence in $\mathbb{H}$.

Now we take up the factor (1).

At this point the type of (x_2, b') has been determined, and is given by the factor (2).

We amalgamate as follows.



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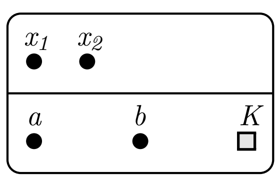
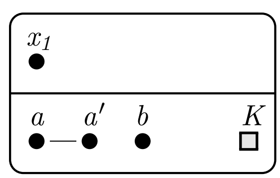


$A1$



$A2$

We claim that the factor ($A1$) can be found in $\mathbb{H}^Q$, hence in $\mathbb{H}$. For this, it suffices to check the types of x_1 and x_2 over K, and these are Q^* and Q.

The factor $(A2)$ is covered by Lemma 8.6. It suffices to check the individual types (x_1, A), (x_1, b), (x_1, K). These are all known to be realized in $\mathbb{H}$. $\square$

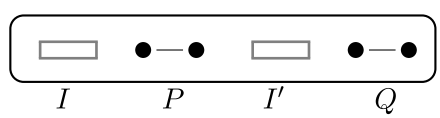
At this point, in a suitable inductive context, our concern is the realization of types of the form $P \perp P$ over $\vec{K}_2 \perp \vec{K}_2$. These are more challenging and require some further preparation.

Lemma 8.12. *Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph and P a 1-type over $\vec{K}_2$ realized in $\mathbb{H}$, such that the following holds.*

The 1-type $P \perp P$ is not realized in $\mathbb{H}$.

Then there is a cross type q_0 (the type of a pair (a, b) with $a \in H_1$ and $b \in H_2$) such that every configuration $(x, I \perp A \perp I' \perp B)$ satisfying the following conditions embeds into $\mathbb{H}$, if (x, B) does:

(a) I, I' are finite independent sets;
(b) $A, B \cong \vec{K}_2$;
(c) $(x, A) \cong P$;
(d) $(x, \min B) \cong q_0$.



Proof. While there are only two possible cross types, the main point here is that there is more than one possibility for q_0, and we have to eliminate the possibility that none of them is suitable. The argument will be written in a very general form that does not depend on the precise number of cross types involved.

Suppose the desired cross type q_0 does not exist. Then for each cross type q we may choose a counterexample consisting of the following.

- Finite independent sets I_q, I'_q, and 1-types over them: $p_q = (x, I_1)$, $p'_q = (x, I'_q)$;
- A 1-type Q_q over $B_q \cong \vec{K}_2$, with $B_q \restriction \min K = q$;

such that

$$Q_q \text{ is realized in } \mathbb{H}, \text{ but } p_q \perp P \perp p'_q \perp Q_q \text{ is not realized in } \mathbb{H}.$$

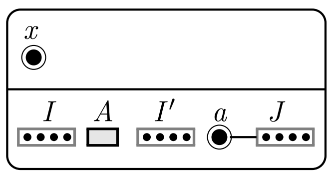
Now perform the following amalgamation. Order the cross types q in some definite order. Let

$$I = \mathop{\perp}_q I_q \qquad I' = \mathop{\perp}_q I'_q$$
$$p = \mathop{\perp}_q p_q \qquad p' = \mathop{\perp}_q p'_q$$

Thus p, p' are 1-types over the independent sets I, I' respectively.

Take $B_q = \{b, b_q\}$ a copy of $\vec{K}_2$ with $b < b_q$; here the vertex b is common to all B_q and we take an ordering on the points b_q corresponding to the ordering on the cross types. Let $J = \{b_q \mid q \text{ varies}\}$. Then $\bigcup_q B_q = [b, J]$ with J an independent set. Let $Q_J = \perp_q (Q_q \restriction b_q)$, the 1-type over J defined by the various Q_q.

Let $A \cong \vec{K}_2$ be an additional pair of points, and take $(x, A) \cong P$. Then amalgamate as follows to determine a cross type $q = (x, b)$.



$(*)$

The type q of (x, b) contradicts the choices of p_q, p'_q, and Q_q, since the type of x over $\{I_q A I'_q B_q\}$ is $p_q \perp P \perp p'_q \perp Q_q$.

So it suffices to show that the factors $(x, IAI'J)$ and $(IAI'bJ)$ of this amalgamation occur in $\mathbb{H}$.

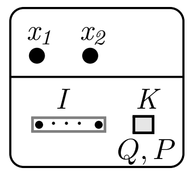
By Lemma 8.6, to get the factor $(x, IAI'J)$ it suffices to check that (x, A) is realized; and this type is P.

By Lemma 7.1, the configuration $(IAI'bJ)$ reduces to the configuration $(bJ) = [b, J]$. The latter is afforded by Lemma 6.16. $\square$

LEMMA 8.13. *Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph and let P be a 1-type over $\vec{K}_2$ realized in $\mathbb{H}$, such that the 1-type $P \perp P$ is not realized in $\mathbb{H}$. Let $\mathbb{H}^P$ be the homogeneous 2-graph $(K^P, K^{\perp +})$ where $K \cong \vec{K}_2$ is contained in H_2. Suppose that r is a Ramsey 2-type for $\mathbb{H}^P$ over $\vec{I}_1$. Let q be a cross type.*

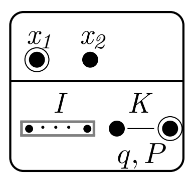
Then there is a 1-type Q over $K \cong \vec{K}_2$ such that $Q \restriction \min K$ is q, and any configuration (R, IK) with the following properties embeds into $\mathbb{H}$.

(a) $R = x_1x_2$ *is of type* r*;*
(b) I *is an independent set of vertices;*
(c) x_1, x_2 *have the types* Q *and* P *respectively, over* K



$(*)$

PROOF. We suppose the contrary, and for each extension $Q = [q, p]$ of q to a 1-type over K we select an independent set I_Q and 1-types p_Q^1, p_Q^2 over I_Q so that $\mathbb{H}$ does not realize the configuration $(R, I_Q K)$ with x_i of type p_Q^i over I_Q, and of types Q, P respectively over K. We set $I = \perp_Q I_Q$ and $p_I^i = \perp_Q p_Q^i$, and amalgamate as follows, with $K = ab$.



A

Any completion of this amalgamation problem will define a 1-type $Q = (x, K)$ whose restriction to a is q, and for which we have included a configuration which cannot be so extended in $\mathbb{H}$. So this will give a contradiction. It suffices to show that the factors

$$(R, Ia) \text{ and } (x_2, IK)$$

embed into $\mathbb{H}$.

Now R is r-Ramsey where r is a Ramsey 2-type for $\mathbb{H}^P$ over $\vec{I}_1$, and as Ia is an independent set it follows by ampleness of $\mathbb{H}^P$ that (R, Ia) embeds into $\mathbb{H}^P$ and hence into $\mathbb{H}$.

For the factor (x_2, IK), by Lemma 8.5 it suffices to check $\mathbb{H}$-constraint. The type of x_2 over K is P, so this is available. $\square$

Now we may bring this all together to prove Lemma 8.7, concerning 1-types over disjoint sums of copies of $\vec{K}_2$.

Proof of Lemma 8.7. Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. Our claim is that any $\mathbb{H}$-constrained initial 1-type $p = (a, A)$ over $A \cong \perp^k \vec{K}_2$ is realized in $\mathbb{H}$, where A is contained in H_2.

We prove this by induction on $s(\mathbb{H}, \vec{K}_2)$. So by Corollary 8.7.1 (suitably localized to an inductive setting) we know that there is a Ramsey 2-type over $\vec{K}_2$ for any ample homogeneous ordered 2-graph $\mathbb{H}'$ satisfying $s(\mathbb{H}', \vec{K}_2) < s(\mathbb{H}, \vec{K}_2)$. Thus Lemma 8.11 applies to $\mathbb{H}$.

We will prove our claim by induction over the number of summands occurring in A, denoted by k above. If $k = 1$ the claim is vacuous. Suppose

$$k > 1.$$

Let K be the first copy of $\vec{K}_2$ in A, and let P be the restriction of p to K. We consider the homogenous 2-graph $(K^P, K^{\perp +})$. This is ample by Lemma 8.4. It suffices to show that the restriction $p' = p \restriction (A \setminus K)$ is realized in $\mathbb{H}'$. If p' is $\mathbb{H}'$ constrained, this follows by induction on k.

So it remains to check that p' is $\mathbb{H}'$ constrained, which means that we may replace p' by each of its restrictions to a summand of A other than K, and effectively take

$$k = 2.$$

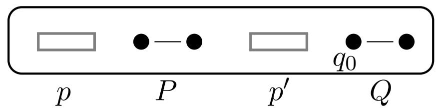
In other words, returning to $\mathbb{H}$, we are now dealing with 1-types of the form $P \perp Q$ with P, Q 1-types over $\vec{K}_2$ which are realized in $\mathbb{H}$. By Lemma 8.11 we may suppose that $P = Q$, and the type in question is

$$P \perp P$$

with P realized in $\mathbb{H}$. So suppose $\mathbb{H}$ does not realize the type $P \perp P$. Let r be a Ramsey 2-type for $\mathbb{H}^P$ over $\vec{I}_1$.

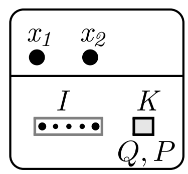
We select a cross type q in accordance with Lemma 8.12. Then for all 1-types p, p' over independent sets I, I', and for every 1-type Q over $K \cong \vec{K}_2$ whose restriction to $\min K$ is q, the 1-type $p \perp P \perp p' \perp Q$ is realized in $\mathbb{H}$.

We select a 1-type Q over $K \cong \vec{K}_2$ whose restriction to $\min K$ is q, in accordance with Lemma 8.13. Then any configuration (R, IK)



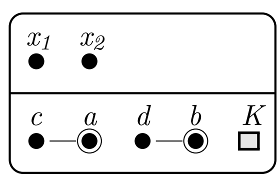
$(*)$

with $R = x_1x_2$ of type r, I independent, $K \cong \vec{K}_2$, and x_1, x_2 realizing the types Q and P respectively over K, will embed in $\mathbb{H}$.



$(*)$

Now we amalgamate as follows. We take $R = x_1x_2$ of type r, with r a Ramsey 2-type for $\mathbb{H}^P$ over $\vec{I}_1$, and $A \cong B \cong K \cong \vec{K}_2$ with $A = ca$, $B = db$. In what follows, the type of x_2 over cd will be determined as we proceed.



$x_1/ABK = PPQ$; $x_2/abK = PP$

$(*)$

If (a, b) is a non-edge then (x_1, AB) is $P \perp P$, while if (a, b) is an edge, then (x_2, abK) is $P \perp P$. So it suffices to find the factors $(R, cadK)$ and $(R, cdbK)$ in $\mathbb{H}$.

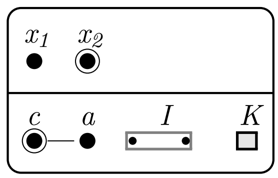
We consider the factor $(R, cadK)$.

We first define a configuration (R, I) with $I \cong \vec{I}_2$. We set $I = \{d_1, d_2\}$ and we take

$$x_1/d_i \cong P \restriction \max K.$$

We take x_2/d_1, x_2/d_2 to realize both possible cross types. We are going to replace the vertex d by the pair I.

Thus we amalgamate as follows, with (R, I) as described (Figure 78).



$x_1/AK = PQ$; $x_2/K = P$

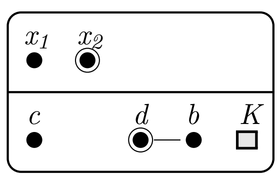
FIGURE 78. A

The factor (R, aIK) embeds into $\mathbb{H}$ by the choice of the type Q. The factor (x_1, AIK) embeds into $\mathbb{H}$ by the choice of the type q. Therefore some completion of this amalgamation problem can be found in $\mathbb{H}$.

If we restrict the result of this amalgamation to (R, AdK) with $d = d_1$ or d_2, we get two possible forms of (R, AdK) with (x_2, c) as specified by the amalgam.

We consider the factor $(R, cdbK)$.

Here the type of (x_2, c) is copied over from $(R, caIK)$, and the type of (x_2, d) remains to be determined. So now we make the corresponding amalgamation.



$x_1/BK = PQ$; $x_2/K = P$

B

Again, the factor (R, cbK) is afforded by the choice of the type Q, and the factor (x_1, cBK) is afforded by the choice of q.

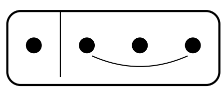
To recapitulate, the sequence of amalgamations is as follows. First, from (A), we determine the type $\text{tp}(x_2/c)$. Then from (B) we determine $\text{tp}(x_2/d)$. We then restrict the result of (A) by selecting out the vertex $d \in I$ for which x_2 realizes the correct cross type, to get a compatible factor for the diagram $(*)$. □

8D. Proof of Proposition III

Proposition (*III*). Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. If $p = (x, A)$ is an $\mathbb{H}$-constrained initial 1-type with $A \in \mathcal{A}(2)$, then p is realized in $\mathbb{H}$.

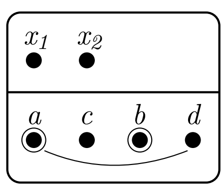
We remark that we could equally well state this for $A \in \mathcal{A}(n)$, any n, as the claim is vacuous when A is complete.

Lemma 8.14. *Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. If $p = (x, A)$ is an $\mathbb{H}$-constrained initial 1-type with $A \cong \vec{P}_3^c$, then p is realized in $\mathbb{H}$.*



Proof. Take a Ramsey 2-type r for $\mathbb{H}$ over $\vec{I}_1$. Let $R = (x_1, x_2)$ have type r.

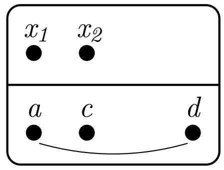
Amalgamate as follows, leaving the type of x_1c and x_2d to be determined as we proceed. For any set of three vertices $u < v < w$ we will write "$(x_i, uvw) = p$" to mean that x realizes the correct type over each vertex, whether or not the triple (uvw) is a copy of $\vec{P}_3^c$.



$(x_1, abd) = p; (x_2, acb) = p$

$(*)$

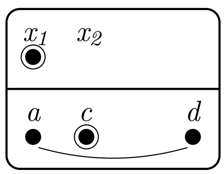
The amalgamation is designed to produce a realization of p either as (x_1, abd) or as (x_2, acb), so it suffices to show that the two factors (R, acd) and (R, cbd) embed in $\mathbb{H}$. The second factor (R, cbd) will be realized in any case by the choice of R, so it suffices to find some form of the first factor in $\mathbb{H}$.



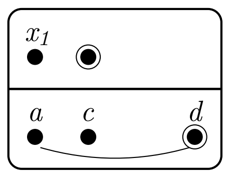
$(x_1, ad) = p \restriction (ad)$;$(x_2, ac) = p \restriction (ac)$

(1)

Since we need to determine the type of (x_1, c) and (x_2, d), we proceed as follows.



$A1$



$A2$

We complete the first diagram to find a suitable type for (x_1, c), and then take the result of this amalgamation as the second factor of our second diagram, and amalgamate to determine the type of (x_2, d). So it suffices to show that the two factors of $(A1)$ and the first factor of $(A2)$ embed into $\mathbb{H}$.

The factors of $(A1)$*:* (x_1, ad) *and* (acd).

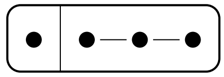
The factor (x_1, ad) exists since p is $\mathbb{H}$-constrained. The factor $(acd) \cong \vec{P}_3^c$.

The factor (R, ac) *of* $(A2)$.

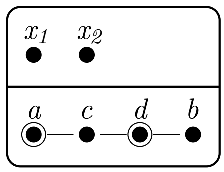
Here ac is an independent set and R is r-Ramsey with r a Ramsey 2-type for $\vec{I}_1$ in $\mathbb{H}$, so this factor is realized in $\mathbb{H}$. □

LEMMA 8.15. *Let* $\mathbb{H}$ *be an ample homogeneous ordered 2-graph. If* $p = (x, A)$ *is an* $\mathbb{H}$*-constrained initial 1-type with* $A \cong \vec{P}_3$*, then* p *is realized in* $\mathbb{H}$*.*

PROOF. By Corollary 8.7.1, there is a Ramsey 2-type r for $\mathbb{H}$ over $\vec{K}_2$.



Let $R = (x_1 x_2)$ have type r. Amalgamate as follows, leaving the types of (x_1, d) and (x_2, c) to be determined as we proceed.



$(x_1, acb) = p$; $(x_2, abd) = p$

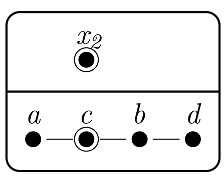
$(*)$

Since this amalgamation is designed so that any completion must satisfy either $(x_1, abd) = p$ or $(x_2, acb) = p$, it suffices to show that the two factors

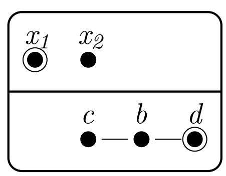
$$(R, acd) \text{ and } (R, cdb)$$

embed into $\mathbb{H}$.

The following pair of amalgamations will construct the second factor, (R, cdb).



$B1$



$B2$

First the amalgamation given as $(B1)$ determines the type of (x_2, c). The factors of this are (x_2, abd) and the path $(acbd)$. The path is given by Lemma 6.7.

The factor (x_2, abd) is a 1-type over $\vec{I}_1 \perp \vec{K}_2$ and by Lemma 8.5 it suffices to check that its type is $\mathbb{H}$-constrained, in other words that (x, bd) embeds in $\mathbb{H}$, which is part of our assumption on the type p.

Thus diagram $(B1)$ has a completion in $\mathbb{H}$. We take (x_2, cbd) from this diagram as the second factor in $(B2)$. We must show that the first factor of $(B2)$, namely (R, cb), also embeds into $\mathbb{H}$. By the choice of R it suffices to check (x_1, cb) and (x_2, cb) separately. Now (x_1, cb) is given by the hypothesis on p, and (x_2, cb) is part of the second factor just constructed. So the amalgamation $(B2)$ may be completed in $\mathbb{H}$.

Now we return to the corresponding factor of the form

$$(R, acd),$$

where the types of x_1 and x_2 over cd are taken as in the second factor. Note that

$$(acd) \cong \vec{K}_2 \perp \vec{I}_1.$$

So by the choice of R, it suffices to prove that each 1-type (x_1, acd), (x_2, acd) is realized separately in $\mathbb{H}$. By Lemma 8.5 it suffices to check that these 1-types are $\mathbb{H}$-constrained; in other words we must consider their restrictions (x_1, ac) and (x_2, ac). Now (x_1, ac) is given by the hypothesis on p, and (x_2, ac) occurred in the completion of diagram $(B1)$ in $\mathbb{H}$.

This completes the construction. $\square$

LEMMA 8.16. *Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. If $p = (x, A)$ is an $\mathbb{H}$-constrained initial 1-type with $A \cong \vec{I}_1 \perp \vec{P}_3$, then p is realized in $\mathbb{H}$.*

PROOF. Let $a = \min A$ and $p_a = p \restriction a$. We pass to $\mathbb{H}' = (a^p, a^{\perp +})$. This is ample by Corollary 6.19.2, and it suffices to prove that $p' = p \restriction A \setminus \{a\}$ is realized in $\mathbb{H}'$. By Lemma 8.15, it suffices to check that p' is $\mathbb{H}'$-constrained. Thus we may replace p' by its restrictions to complete subgraphs. In terms of $\mathbb{H}$, this means that we may replace p by its restrictions to graphs of the form $\vec{I}_1 \perp \vec{K}_2$.

By Lemma 8.5, it suffices to check that these restrictions are $\mathbb{H}$-constrained. Since the original type p was $\mathbb{H}$-constrained, this holds. $\square$

PROOF OF PROPOSITION III. $\mathbb{H}$ is an ample homogeneous ordered 2-graph and $A \in \mathcal{A}(2)$. We claim that any $\mathbb{H}$-constrained initial 1-type over A is realized in $\mathbb{H}$.

For $A \cong \vec{I}_1 \perp \vec{P}_3$, $\vec{P}_3^c$, or $\vec{I}_k$ with $k < \infty$, this is Lemma 8.16 or 8.14, or Corollary 6.19.1, respectively. $\square$

CHAPTER 9

ORDERED HOMOGENEOUS GRAPHS: PROPOSITION IV

Our next goal is the following.

Proposition (IV_n). Let $n \geq 2$, and if $n \geq 3$ assume Proposition $(VIII_{n-1})$. Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph such that each configuration in $\mathcal{A}(n)$ embeds into H_2. If $A \in \mathcal{A}(n)$ and $p = (x, A)$ is an initial 1-type over A which is realized in $\mathbb{H}$ with $x \in H_1$, $A \subseteq H_2$, then the ordered 2-graphs $(A^p, A^{\perp -})$ and $(A^p, A^{\perp +})$ are ample.

This divides into two parts.

Proposition 9.1 $(IV_n\text{–}A)$. Let $n \geq 2$, and if $n \geq 3$ assume that Proposition $(VIII_{n-1})$ holds. Let X, Y be finite ordered graphs which belong to every amalgamation class containing $\mathcal{A}(n)$. Then $X \perp Y$ has the same property.

Proposition 9.2 $(IV_n\text{–}B)$. Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph such that each configuration in $\mathcal{A}(n)$ embeds into H_2. Let $A \in \mathcal{A}(n)$, and let p be an initial 1-type over A realized in $\mathbb{H}$. Then $(A^p, A^{\perp \pm})$ realizes every 1-type over $\vec{I}_k$, for $k < \infty$.

We begin by treating the case $n = 2$.

9A. Proof of Proposition (IV_2)

We deal quickly with the proof of Proposition $IV_2\text{–}B$, then deal with Proposition $IV_2\text{–}A$ by a number of explicit amalgamation arguments.

LEMMA 9.3 (IV_2–B). *Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph and let $p = (x, IAJ)$ be an $\mathbb{H}$-constrained 1-type with $A \in \mathcal{A}(2)$, IJ an independent set, and $A \perp IJ$. Then p is realized in $\mathbb{H}$.*

PROOF. We proceed by induction on $k = |IJ|$.

If $k = 0$ this is Proposition III, proved in §8D. Suppose

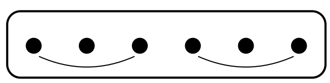
$$k > 0$$

If J is nonempty let $a = \max J$, $J' = J \setminus \{a\}$, $q = p \restriction a$, $p' = p \restriction IAJ'$, and $\mathbb{H}' = (a^p, a^{\perp -})$. If J is empty then I is nonempty, and we set $a = \min I$, $I' = I \setminus \{a\}$, $q = p \restriction a$, $p' = p \restriction I'AJ$, and $\mathbb{H}' = (a^p, a^{\perp +})$.

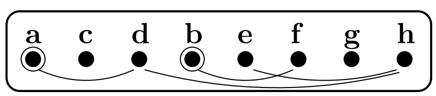
In either case $\mathbb{H}'$ is ample by Corollary 6.19.2 and our claim is that p' is realized in $\mathbb{H}'$. By induction, it suffices to show that p' is $\mathbb{H}'$-constrained.

In terms of $\mathbb{H}$, this comes down to checking that the restriction of p to any configuration $\vec{K}_2 \perp \vec{I}_1$ or $\vec{I}_1 \perp \vec{K}_2$ is realized in $\mathbb{H}$. This is covered by Lemma 8.5. $\square$

LEMMA 9.4. *Let $\mathcal{A}$ be an amalgamation class of finite ordered graphs containing $\mathcal{A}(2)$. Then $\vec{P}_3^c \perp \vec{P}_3^c$ is in $\mathcal{A}$.*



PROOF. We amalgamate as follows.



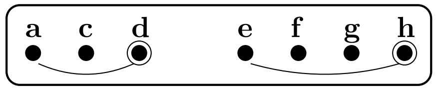
Then either $(acdbef)$ or $(acbegh)$ will be a copy of $\vec{P}_3^c \perp \vec{P}_3$. We must show that the two factors

$$(acdefgh) \text{ and } (cdbefgh)$$

are in $\mathcal{A}$.

The factor $(acdefgh)$:

We amalgamate as follows. Then either $(acdefh)$ is $\vec{P}_3^c \perp \vec{P}_3^c$, or the resulting amalgam is the desired factor of $(*)$.



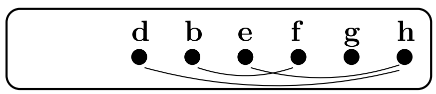
(A)

The subfactors of diagram (A) have the form

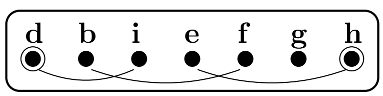
$$(acdefg) \cong (\vec{P}_3^c \perp \vec{I}_3) \qquad (acefgh) \cong (\vec{I}_2 \perp efgh)$$

By Corollary 6.3.1, these two configurations reduce to the single configuration $(efgh)$, with four vertices and two edges, which is covered by Lemma 6.13.

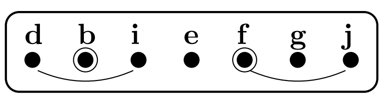
The factor $(cdbefgh) = (c) \perp (dbefgh)$*:*
By Corollary 6.3.1, the second factor reduces to $(d\ b\ e\ f\ g\ h)$.



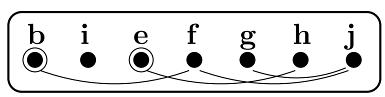
We amalgamate as follows.



(B)



$(B1)$



$(B2)$

The completion of diagram (B) gives either $(dbiegh) \cong (\vec{P}_3^c \perp \vec{P}_3^c)$, or a copy of the second factor of $(*)$. So it suffices to show that the factors

$$(B)_1 = (dbiefg) \text{ and } (B)_2 = (biefgh)$$

are in $\mathcal{A}$.

For $(B)_1 = (dbiefg)$*:* Diagram $(B1)$ completes either to $(dbifgj) \cong \vec{P}_3^c \perp \vec{P}_3^c$ or to the factor $(dbiefg)$ in (B).

The subfactors here are $(dbiegj) \cong (\vec{P}_3^c \perp \vec{I}_3)$ and $(diefgj) \cong (\vec{K}_2 \perp \vec{I}_1 \perp \vec{P}_3^c)$. These are afforded by Lemma 7.1.

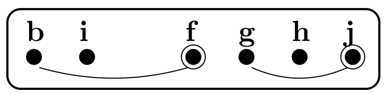
For $(B)_2 = (biefgh)$*:* Either the completion of diagram $(B2)$ completes is $(B)_2$, or else $(bieghj) \cong \vec{P}_3^c \perp \vec{P}_3^c$.

We must show that the two subfactors

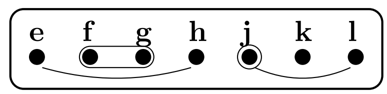
$$(B2)_1 = (bifghj) \text{ and } (B2)_2 = (i) \perp (efghj)$$

are in $\mathcal{A}$. Here $(B2)_2$ reduces to $(B2)_2' = (efghj)$, by Corollary 6.3.1.

For these subfactors we use the following amalgamations.



$(B2.1)$



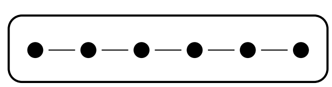
$(B2.2)$

h k l

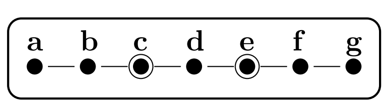
From $(B2.1)$ we get either $(bifghj)$ is $\vec{P}_3^c \perp \vec{P}_3^c$ or that the completion of the diagram is the factor $(B2)_1$. The subfactors here are $(bifgh) \cong \vec{I}_5$ and $(bighj) \cong \vec{I}_2 \perp \vec{P}_3^c$, which are in $\mathcal{A}$.

In $(B2.2)$, if the completion contains a non-edge (x, j) with $x = f$ or $x = g$, then $(exhjkl)$ is $\vec{P}_3^c \perp \vec{P}_3$, while if (f, j) and (g, j) are both edges then the required configuration $(efghj)$ results. The subfactors here are $(efghkl) \cong (efgh) \perp \vec{I}_2$ and $(ehjkl) \cong \vec{K}_2 \perp \vec{P}_3^c$. These are afforded by Corollary 6.3.1 and Lemma 7.1, and Lemma 6.13. $\square$

LEMMA 9.5. *Let* $\mathcal{A}$ *be an amalgamation class of finite ordered graphs which contains* $\mathcal{A}(2)$*, Then* $\mathcal{A}$ *contains the ordered path* $\vec{P}_6$ *with* 6 *vertices.*



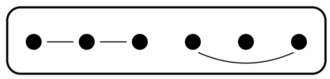
PROOF. Amalgamate as follows.



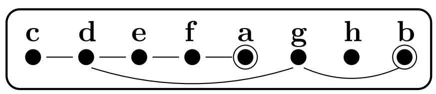
$(*)$

Then $(abcdef)$ or $(abcefg)$ will be the desired path. The factors have the form $\vec{P}_4 \perp \vec{K}_2$ and $\vec{K}_2 \perp \vec{P}_4$, which are available by Lemma 7.1. □

LEMMA 9.6. *Let $\mathcal{A}$ be an amalgamation class of finite ordered graphs containing $\mathcal{A}(2)$. Then $\vec{P}_3 \perp \vec{P}_3^c$ is in $\mathcal{A}$.*



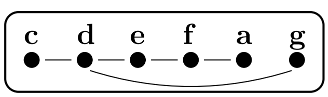
PROOF. We amalgamate as follows.



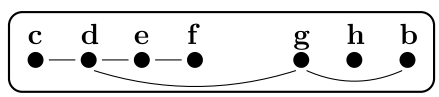
$(*)$

Then either $(efaghb)$ or $(cdeahb)$ will be a copy of $\vec{P}_3 \perp \vec{P}_3^c$. So it suffices to show that the factors $(cdefagh) = (cdefag) \perp (h)$ and $(cdefghb)$ embed into $\mathbb{H}$, and these reduce to

$$(cdefag) \text{ and } (cdefghb)$$



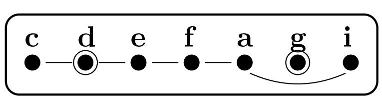
$(1')$



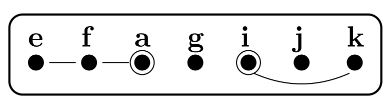
(2)

The factor $(cdefag)$:

We amalgamate as follows.



(A')



$(A'.2)$

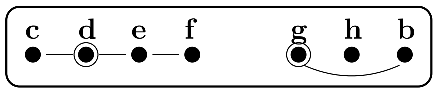
In the completion of diagram (A') either $(cdeagi)$ is $\vec{P}_3 \perp \vec{P}_3^c$, or $(cdefag)$ is the required factor. So it suffices to show that the subfactors of this diagram are in $\mathcal{A}$.

The subfactor $(cdefai)$ is $\vec{P}_6$, dealt with in Lemma 9.5.

The subfactor $(cefagi) = (cefagi)$ reduces to $(efagi)$. Here we use diagram $(A'.2)$, whose completion produces either $(efaijk) \cong \vec{P}_3 \perp \vec{P}_3^c$ or the required configuration $(efagi)$.

The factor $(cdefghb)$:

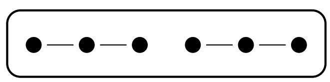
We amalgamate as follows.



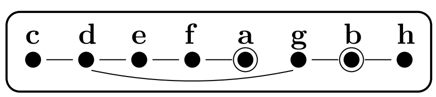
(B)

This produces either $(cdeghb) \cong \vec{P}_3 \perp \vec{P}_3^c$ or $(cdefghb)$ is the required factor. The subfactors $(cdefhb) \cong (cdef) \perp \vec{I}_2$ and $(cefghb) \cong (\vec{I}_1 \perp \vec{K}_2 \perp \vec{P}_3^c)$ are in $\mathcal{A}$, the latter by Lemma 7.1. □

Lemma 9.7. *Let $\mathcal{A}$ be an amalgamation class of finite ordered graphs containing $\mathcal{A}(2)$. Then $\vec{P}_3 \perp \vec{P}_3$ is in $\mathcal{A}$.*



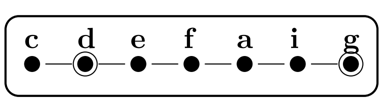
Proof. We amalgamate as follows.



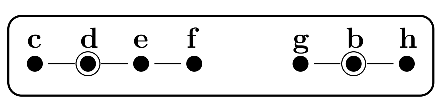
$(*)$

Then $(efagbh)$ or $(cdeabh)$ will be a copy of $\vec{P}_3 \perp \vec{P}_3$. So it suffices to show that the factors $(*)_1 = (cdefagh)$ and $(*)_2 = (cdefgbh)$ are in $\mathcal{A}$, and the former reduces to $(cdefag)$.

For these we amalgamate as follows.



(A')



(B)

Then (A') produces either $(cdeaig) \cong \vec{P}_3 \perp \vec{P}_3$, or the factor $(*)_1$, while (B) produces either $(cdegbh) \cong \vec{P}_3 \perp \vec{P}_3$, or the factor $(*)_2$.

So it suffices to show that the various subfactors involved are in $\mathcal{A}$. These are

$$\begin{array}{ll} (cdefai) \cong \vec{P}_6 & (cefaig) \cong \vec{I}_1 \perp \vec{P}_5 \\ (cdefbh) \cong \vec{P}_4 \perp \vec{K}_2 & (cefgbh) \cong \vec{I}_1 \perp \vec{K}_2 \perp \vec{P}_3 \end{array}$$

These are all available in $\mathcal{A}$ (with $\vec{P}_6$ covered by Lemma 9.5). □

Now we can assemble these ingredients to get a more general result.

Lemma 9.8. *Let X, Y be finite ordered graphs which belong to every amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$. Then $X \perp Y$ belongs to every amalgamation class of finite ordered graphs which contains $\mathcal{A}(2)$.*

Proof. It suffices to prove the claim for $X, Y \in \mathcal{A}(2)$.

If X or Y has the form $\vec{I}_k$ with $k < \infty$ then this follows from Corollary 6.3.1. So we may suppose that $X, Y \in \{\vec{I}_1 \perp \vec{P}_3, \vec{P}_3^c\}$.

If $Y = \vec{P}_3^c$ then the claim follows from Lemmas 9.6 and 9.4, and Corollary 6.3.1.

Now we take up the case $Y = \vec{I}_1 \perp \vec{P}_3$. Let Γ be a homogeneous ordered graph embedding the configurations of $\mathcal{A}(2)$, and take $Y = \vec{I}_1 \perp B \subseteq \Gamma$ with $B \cong \vec{P}_3$. We claim that all configurations in $\mathcal{A}(2)$ embed into $B^{\perp -}$; it then follows that $X \perp \vec{I}_1$ embeds into $B^{\perp -}$ and hence $X \perp (\vec{I}_1 \perp B) = X \perp Y$ embeds into Γ.

So we must show that all configurations $A \perp \vec{P}_3$ with $A \in \mathcal{A}(2)$ embed into Γ. As above, this is known for A of the form $\vec{I}_k$, and follows from Lemma 9.7 and Corollary 6.3.1 when $A = \vec{I}_1 \perp \vec{P}_3$. Suppose lastly that $A = \vec{P}_3^c$. In this case the configuration $A \perp \vec{P}_3$ is given by the dual of Lemma 9.6 under reversal.

This covers all cases. □

Proof of Proposition (IV_2). We have $\mathbb{H}$ an ample homogeneous ordered 2-graph, and $A \in \mathcal{A}(2)$ embedded into H_2, with $p = (x, A)$ an initial 1-type over A realized in $\mathbb{H}$. We claim that $\mathbb{H}' = (A^p \cap H_1, A^{\perp +} \cap H_2)$ is ample.

By Lemma 9.8, $H_2' = A^{\perp +} \cap H_2$ contains $\mathcal{A}(2)$.

By Lemma 9.3, $\mathbb{H}'$ realizes every 1-type over an independent set.

The claim is proved. □

9B. Proof of Proposition (IV_n–A): Closure under $\perp$

In this subsection we will deal with Proposition (IV_n-A) for $n \geq 3$. That is, we aim at the following.

LEMMA 9.9 (IV_n–A). *Let $n \geq 3$ and assume Proposition ($VIII_{n-1}$). Let X, Y be finite ordered graphs which belong to every amalgamation class of finite ordered graphs which contains $\mathcal{A}(n)$. Then $X \perp Y$ has the same property.*

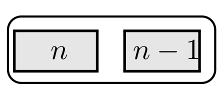
Our main goal is to show that any amalgamation class containing $\mathcal{A}(n)$ also contains $\vec{K}_n \perp \vec{K}_n$, but we will need to work up to this by stages. Namely, we deal with $\vec{K}_n \perp \vec{K}_\ell$ with $\ell \leq n$, with a big gap between the cases $\ell < n$ and $\ell = n$, as we shall see.

LEMMA 9.10. *Let $n \geq 3$. Suppose Proposition ($VIII_{n-1}$) holds, that $\ell < n$, and that $\vec{K}_n \perp \vec{K}_\ell$ belongs to every amalgamation class of finite ordered graphs which contains $\mathcal{A}(n)$. Then for any finite ordered graph X which belongs to every amalgamation class containing $\mathcal{A}(n)$, the ordered graph $X \perp \vec{K}_\ell$ has the same property.*

PROOF. We fix X, and a homogeneous ordered graph Γ such that every configuration in $\mathcal{A}(n)$ embeds into Γ. We must embed $X \perp \vec{K}_\ell$ into Γ.

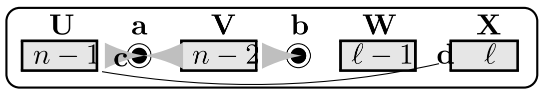
We fix a copy K of $\vec{K}_\ell$ in Γ and let $\Gamma' = K^{\perp -}$. We must show that Γ' contains X. For this it suffices to check that Γ' contains all configurations A in $\mathcal{A}(n)$. This holds for $A = \vec{K}_n$ by hypothesis. It remains to consider the configurations A in $\mathcal{A}(2)$. Then $A \perp \vec{K}_\ell$ is $\vec{K}_n$-free and the claim follows by Proposition ($VIII_{n-1}$). □

LEMMA 9.11. *Let $n \geq 3$, and assume Proposition ($VIII_{n-1}$) holds. Let $\mathcal{A}$ be an amalgamation class of finite ordered graphs which contains $\mathcal{A}(n)$. Then $\vec{K}_n \perp \vec{K}_{n-1}$ is in $\mathcal{A}$.*



Proof. We show inductively that for $\ell < n$, $\vec{K}_n \perp \vec{K}_\ell$ is in $\mathcal{A}$. The inductive hypothesis is that $\vec{K}_n \perp \vec{K}_{\ell-1}$ lies in every amalgamation class containing $\mathcal{A}(n)$, so Lemma 9.10 applies to $\ell - 1$.

We fix complete graphs U, V, W, X of orders $n-1$, $n-2$, $\ell-1$, and ℓ, and set $c = \max U$, $d = \min X$. We take vertices a, b with $U < a < V < b < W < X$ and with a adjacent to U, V, and b adjacent to V, W. We add one additional edge cd, and amalgamate as follows.



$(*)$

Then either $UabW$ or $aVbX$ will be a copy of $\vec{K}_n \perp \vec{K}_\ell$. So it suffices to show that the factors $(UaVWX)$ and $(UVbWX)$ are in $\mathcal{A}$.

The factor $(UVbWX)$ is $\vec{K}_n$-free and thus is afforded by Proposition $(VIII_{n-2})$. So it remains to discuss the factor

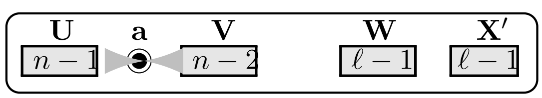
$$(UaVWX)$$

We will set $U' = U \setminus \{c\}$ and $X' = X \setminus \{d\}$. We amalgamate so as to determined the type of cd, with subfactors

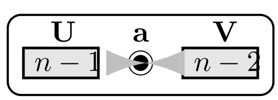
$$(UavWX') \text{ and } (U'aVWX)$$

Again, $(U'aVWX)$ is $\vec{K}_n$-free so we come down to the subfactor

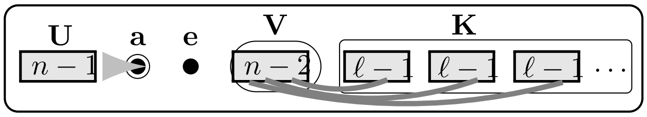
$$(UaVWX') \cong (UaV \perp W \perp X')$$



By Lemma 9.10 this configuration reduces to UaV.



For this we amalgamate as follows. For $v \in V$, let K_v be a complete ordered graph of order $\ell - 1$ and let $K = \perp_{v \in V} K_v$. We let VK be the ordered graph with $V < K$ and with additional edges vx for $v \in V$, $x \in K_v$. Thus vK_v is a complete graph of order ℓ. We add a vertex e with $a < e < V$.



So if there is some $v \in V$ with (a, v) a non-edge, then $(UavK_v) \cong (\vec{K}_n \perp \vec{K}_\ell)$, while otherwise (UaV) is the desired configuration. So it suffices to show that the configurations $(UaeK)$ and $(UeVK)$ are in $\mathcal{A}$.

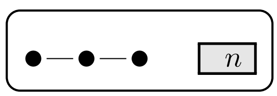
The configuration $(UeVK) \cong (\vec{K}_{n-1} \perp \vec{I}_1 \perp VK)$ is $\vec{K}_n$-free and is afforded by Proposition $(VIII_{n-2})$. The other configuration

$$UaeK \cong (\vec{K}_n \perp \vec{I}_1 \perp [\vec{K}_{\ell-1} \perp \cdots \perp \vec{K}_{\ell-1}])$$

is afforded by Lemma 9.10.

This completes the construction. $\square$

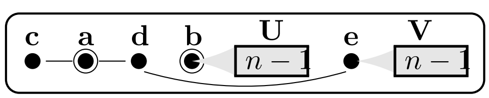
LEMMA 9.12. *Let $n \geq 3$ and assume Proposition $(VIII_{n-1})$ holds. Let $\mathcal{A}$ be an amalgamation class of finite ordered graphs which contains $\mathcal{A}(n)$. Then $\vec{P}_3 \perp \vec{K}_n$ belongs to $\mathcal{A}$.*



PROOF. Let bU and eV be complete ordered graphs of order n, with

$$b < U < e < V$$

We amalgamate as follows.

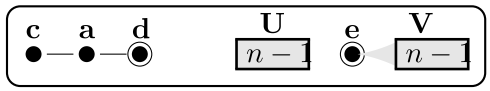


$(*)$

In the resulting amalgam, either $(cadbU)$ or $(cabeV)$ will be $\vec{P}_3 \perp \vec{K}_n$. So it suffices to show that the factors $(*)_1 = (cadUeV)$ and $(*)_2 = (cdbUeV) = (c) \perp (dbUeV)$ are in $\mathcal{A}$.

The factor $(cadUeV)$*:*

We amalgamate as follows.



(A)

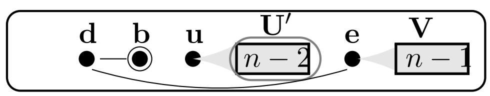
This produces either $(cadeV) \cong \vec{P}_3 \perp \vec{K}_n$ or a copy of $(*)_1$. The subfactors here are

$$(cadUV) \text{ which is } \vec{K}_n\text{-free, and } (caUeV) \cong (\vec{K}_2 \perp \vec{K}_{n-1} \perp \vec{K}_n)$$

The first factor is available by Proposition $(VIII_{n-2})$, the second by Lemmas 9.11 and 9.10.

The factor $(dbUeV)$*:*

Let $u = \min U$ and $U' = U \setminus \{u\}$. Amalgamate as follows.



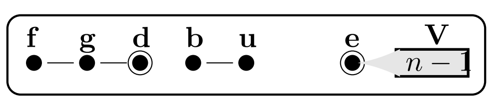
(B')

Here we determine the relations between b and U'. If there is some non-edge (b, u') with $u' \in U$" then $(buu'eV) \cong \vec{P}_3 \perp K_n$, and otherwise we have the factor $(dbUeV)$. So it suffices to show that the subfactors

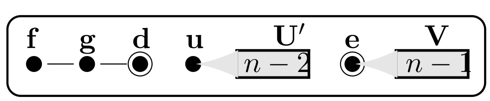
$$(B')_1 = (dbueV) \text{ and } (B')_2 = (dUeV)$$

are in $\mathcal{A}$.

For these we amalgamate as follows.



$(B'1)$



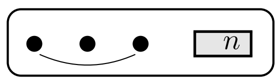
$(B'2)$

In each case we get either $\vec{P}_3 \perp \vec{K}_n$ directly, or the desired factor. So it suffices to show that the subfactors are in $\mathcal{A}$. These are

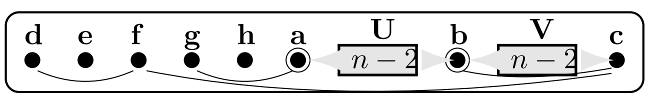
$$\begin{gathered}(fgdbuV)\\ (fgbueV) \cong (\vec{K}_2 \perp \vec{K}_2 \perp \vec{K}_n)\\ (fgdUV)\\ (fgUeV) \cong \vec{K}_2 \perp \vec{K}_{n-1} \perp \vec{K}_n\end{gathered}$$

which either omit $\vec{K}_n$ or are available by Lemma 9.10. □

Lemma 9.13. *Let $n \geq 3$ and assume Proposition $(VIII_{n-1})$ holds. Let $\mathcal{A}$ be an amalgamation class of finite ordered graphs. Then $\vec{P}_3^c \perp \vec{K}_n$ belongs to $\mathcal{A}$.*



Proof. Let $a < U < b < V < c$ with aU, Ub, bVc complete of orders $n-1$, $n-1$, and n, respectively. Amalgamate as follows.



$(*)$

This produces either $(ghabVc) \cong \vec{P}_3^c \perp \vec{K}_n$ or $(defaUb) \cong \vec{P}_3^c \perp \vec{K}_n$. So it suffices to show that the factors

$$(*)_1 = (defghaUVc) \text{ and } (*)_2 = (defghUbVc)$$

are in $\mathcal{A}$.

The factor $(defghaUvC)$ is $\vec{K}_n$-free, hence afforded by Proposition $(VIII_{n-1})$. So we consider the factor

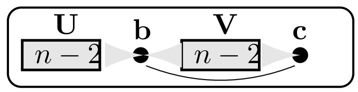
$$(defghUbVc)$$

We may view this as an amalgamation diagram with the type of (f, c) to be determined, resulting either in $(defbVc) \cong \vec{P}_3^c$ or in the desired factor. So we come down to the subfactors

$$(defghUbVc) \text{ and } (deghUbV) \cong (\vec{I}_4 \perp UbVc)$$

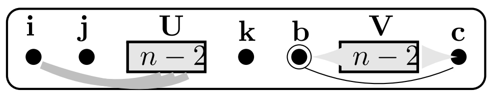
Here $(defghUbV)$ is $\vec{K}_n$-free while $(deghUbV)$ reduces to $UbVc$. So it suffices to consider

$(UbVc)$



$(B2')$

For this we amalgamate as follows.



$(B2')$

If a non-edge (u, b) results with $u \in U$ then $(ijubVc)$ is $\vec{P}_3^c \perp \vec{K}_n$, and otherwise we have the configuration $(B2')$. So it suffices to show that the factors $(ijUkVc)$ and $(ijkbVc) \cong \vec{I}_3 \perp \vec{K}_n$ are in $\mathcal{A}$. The first omits $\vec{K}_n$ and the second is known. □

Now we can put this together and get, essentially, "everything except $\vec{K}_n \perp \vec{K}_n$."

LEMMA 9.14. *Let $n \geq 3$, and assume Proposition $(VIII_{n-1})$. Let X be a $\vec{K}_n$-free finite ordered graph, and let Y be a finite ordered graph which belongs to every amalgamation class of finite ordered graphs which contains $\mathcal{A}(n)$. Then $X \perp Y$ belongs to every amalgamation class of finite ordered graphs which contains $\mathcal{A}(n)$.*

PROOF. By Proposition $(VIII_{n-1})$, X belongs to every amalgamation class of finite ordered graphs which contains $\mathcal{A}(n-1)$. So it suffices to our prove our claim in the special case

$$X \in \mathcal{A}(n-1),\ Y \in \mathcal{A}(n)$$

Now if $Y \in \mathcal{A}(2)$ then $X \perp Y$ is $\vec{K}_n$-free and our claim follows. So we suppose $Y = \vec{K}_n$.

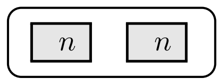
If $X = \vec{I}_k$ (some $k < \infty$), $\vec{I}_1 \perp \vec{P}_3$, or $\vec{P}_3^c$, then Corollary 6.4.1 and Lemmas 9.12, 9.13 suffice.

If $X = \vec{K}_{n-1}$, then Lemma 9.11 applies.

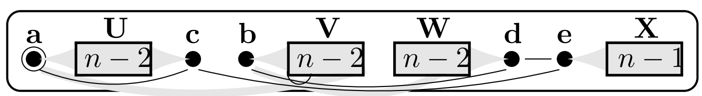
□

And after these preparations we can also get $\vec{K}_n \perp \vec{K}_n$.

LEMMA 9.15. *Let $n \geq 3$, and assume Proposition $(VIII_{n-1})$. Let $\mathcal{A}$ be an amalgamation class of finite ordered graphs which contains $\mathcal{A}(n)$. Then $\mathcal{A}$ contains $\vec{K}_n \perp \vec{K}_n$.*



PROOF. Let $a < U < c < b < W, V < d < e < X$ with U, V, W of order $n-2$, X of order $n-1$, and aUc, aV, bV, bWd, eX complete. Adjoin edges ce, de and amalgamate as follows.



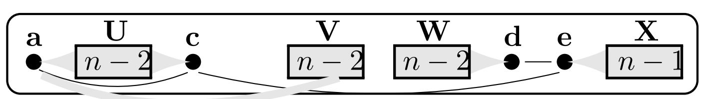
$(*)$

Then either $(aUcbWd)$ or $(abVeX)$ is $\vec{K}_n \perp \vec{K}_n$. So it suffices to show that the factors

$$(aUcVWdeX) \text{ and } (UcbVWdeX)$$

are in $\mathcal{A}$.

The factor $(aUcVWdeX)$:



(1)

We look at this as a 2-point amalgamation problem with the type of (c, e) to be determined. Then either $(aUceX)$ is $\vec{K}_n \perp \vec{K}_n$ or the desired configuration results. So it suffices to show that the subfactors

$$(aUcVWdX) = (aUcV \perp Wd \perp X) \cong ([\vec{I}_1, UcV] \perp \vec{K}_{n-1} \perp \vec{K}_{n-1})$$
$$(aUVWdeX) = (aUV \perp WdeX)$$

are in $\mathcal{A}$.

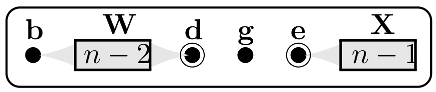
Now Lemma 9.14 reduces these factors to

$$[\vec{I}_1, UcV] \text{ and } (Wdex)$$

The configuration $[\vec{I}_1, UcV]$ is afforded by Lemma 7.6 and Proposition ($VIII_{n-1}$). Thus it will suffice now to consider the configuration

$$(Wdex)$$

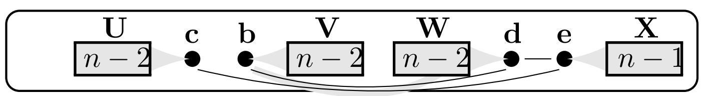
For this we amalgamate as follows, interpolating one additional vertex g, to determine the ordering in the amalgam.



(bB')

The factors are afforded by Lemma 9.10.

The factor $(UcbVWdeX)$*:*



(2)

Here we interpolate a vertex f between d and e, with no additional edges, and treat the diagram as an amalgamation problem with the type of (d, e) to be determined. This produces either $(bWdeX) \cong (\vec{K}_n \perp \vec{K}_n)$ or the desired configuration. So it suffices to show that the factors

$$(UcbVWdfX) \cong (Uc \perp bVWdf \perp X);$$
$$(UcbVWfeX)$$

are in $\mathcal{A}$.

The configuration

$$(Uc \perp bVWd \perp (f) \perp X) \cong (\vec{K}_{n-1} \perp bVWd \perp \vec{I}_1 \perp \vec{K}_{n-1})$$

reduces to

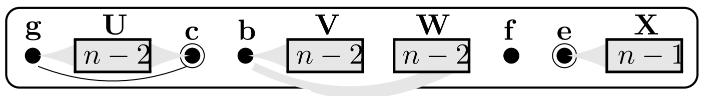
$$(bVWd) \cong [\vec{I}_1, VWd]$$

which is afforded by Lemma 7.6 and Proposition $(VIII_{n-1})$.

So we come down to

$$(UcbVWfeX)$$

For this we amalgamate as follows.



(B)

This produces either $(gUceX) \cong (\vec{K}_n \perp \vec{K}_n)$ or the desired configuration. So it suffices to show that the subfactors

$$(gUcbVWfX) \cong (gUc \perp bVW \perp (f) \perp X) \cong (\vec{K}_n \perp bVW \perp \vec{I}_1 \perp \vec{K}_{n-1});$$

$$(gUbVWfeX) \cong (gU \perp bVW \perp (f) \perp eX) \cong (\vec{K}_{n-1} \perp bVW \perp \vec{I}_1 \perp \vec{K}_n)$$

are in $\mathcal{A}$. This follows from Lemma 9.14. □

Proof of Lemma 9.9. We fix $n \geq 3$. It suffices to show that for $X, Y \in \mathcal{A}(n)$, the finite ordered graph $X \perp Y$ belongs to every amalgamation class containing $\mathcal{A}(n)$.

If X or Y is in $\mathcal{A}(n-1)$ this is covered by Lemma 9.14. So we may suppose $X, Y = \vec{K}_n$, and Lemma 9.15 applies. □

9C. Proof of Proposition $(IV_n–B)$

Before completing the proof of Proposition $(IV_n–B)$, we will prove the following special case of Proposition (V_n).

Proposition $(V_n\text{-}A)$. Let $n \geq 3$ and assume Proposition (IX_{n-1}). Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. Let $\ell < n$, $A = \vec{K}_n \perp \vec{K}_\ell$ or $\vec{K}_\ell \perp \vec{K}_n$, and let p be an $\mathbb{H}$-constrained initial 1-type over A. Then p is realized in $\mathbb{H}$.

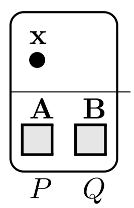
Lemma 9.16. *Suppose that Proposition (IX_{n-1}) holds, and that $\mathbb{H}$ is an ample homogeneous 2-graph for which H_2 contains a complete graph of order $n-1$. Then there is a Ramsey 2-type for $\mathbb{H}$ over the generic $\vec{K}_n$-free graph; or equivalently, over any finite $\vec{K}_n$-free graph.*

PROOF. We take the second formulation: so let X be a finite $\vec{K}_n$-free graph. We recall the criterion given in Lemma 8.2.

> For any initial 1-type $p = (a, \perp^k X)$ over an ordered sum of copies of X for which the restrictions (a, X_i) to individual summands are realized in $\mathbb{H}$, the full 1-type p is realized in $\mathbb{H}$.

The hypothesis on p implies that p is $\mathbb{H}$-constrained, and $\perp^k X$ is $\vec{K}_n$-free, so this is a special case of (IX_{n-1}). □

LEMMA 9.17. *Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph, and $n \geq 3$. Assume that (IX_{n-1}) holds (hence also $(VIII_{n-1})$). Let A, B be complete ordered graphs with $|B| < |A| = n$, and let P, Q be initial 1-types over A, B realized in $\mathbb{H}$. Suppose that some initial 1-type $Q^{\#}$ over a complete graph of order less than n is not realized in $\mathbb{H}$. Then $P \perp Q$ is realized in $\mathbb{H}$.*

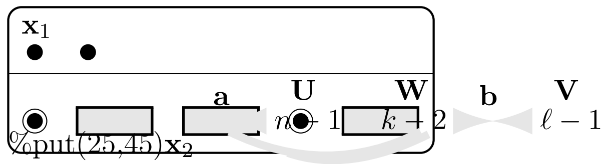


PROOF. We proceed by induction on $\ell = |B|$.

Let C be a complete graph of minimal order for which there is an initial 1-type $Q^{\#}$ over C not realized in $\mathbb{H}$. Set $k = |C|$. By Lemma 9.16 there is a Ramsey 2-type r for $\mathbb{H}$ over the generic $\vec{K}_n$-free graph.

As usual, we take $R = (x_1 x_2)$ of type r and form a suitable amalgamation. Let U, W, V be complete graphs of orders $n-1$, $k-2$, $\ell-1$ respectively, and let a, b be additional vertices with aU, aW, bV, bW complete. We take $R < a < U < W < b < V$. Denote aU, bV, aWb by $\hat{U}$, $\hat{V}$, $\hat{W}$ respectively. We will amalgamate as follows, with the precise form of the factors to be determined as we proceed.

As usual the conditions on x_1, x_2 are not intended to impose any conditions on the base sets. But they ensure that if (a, b) is a nonedge then $(x_1, \hat{U} \perp \hat{V})$ will in fact be a realization of $P \perp Q$, and if (a, b) is an edge then $(x_2, \hat{W})$ will be a realization of $Q^{\#}$; and since the second possibility is ruled out by hypothesis, this will suffice to



$x_1/\hat{U}\hat{V} = P \perp Q$; $x_2/\hat{W} = Q^{\#}$

$(*)$

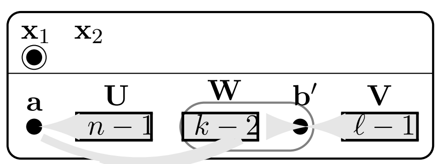
prove the lemma. So it suffices to construct suitable factors for this amalgamation. That is, we require factors of the form

(1) $(R, aUWV)$;
(2) $(R, UWbV)$

in $\mathcal{A}$ which satisfy the specified constraints, and which agree on their common part (R, UWV),

To begin with we construct a suitable form of factor (1). For this purpose we introduce a vertex b' similar to b in that $W < b' < V$ and Wb', $b'V$ are complete, but specifying additionally that (a, b') is a non-edge.

We determine x_1/Wb', and in particular x_1/W, by the following amalgamation.

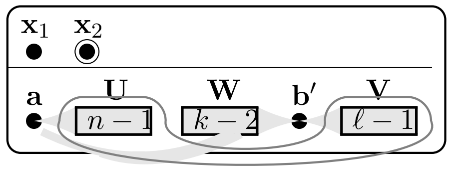


$x_1/\hat{U}V = P \perp (Q \restriction V)$

(I)

We must show that the factors (x_1, aUV) and $(\emptyset, aUWb'V)$ occur in $\mathbb{H}$. As $(aUV) \cong (\vec{K}_n \perp \vec{K}_{\ell-1})$, the factor (x_1, aUV) is available by induction on ℓ. The configuration $(aUWb'V) = ([a, UW] \perp b'V)$ is available by Lemma 9.10 and Lemma 7.6.

So we may complete the diagram shown in $\mathbb{H}$. Now we determine the type of x_2 over UV by a further amalgamation.



(II)

Here we require the factor omitting x_2 to be the one just constructed, while the structure of (x_2, aWb') is to be that of (x_2, aWb), that is agreeing with $Q^{\#}$ as far as x_2 is concerned. Furthermore in this diagram we have specified that (a, b') is not an edge.

We must show that the factor (R, aWb) embeds into $\mathbb{H}$. By the choice of R, this reduces to the two configurations (x_i, aWb) for $i = 1, 2$. For $i = 1$ this is part of the factor already constructed in $\mathbb{H}$. The configuration (x_2, aWb') is $\mathbb{H}$-constrained since k was chosen minimal and (a, b') is a non-edge. By Proposition (IX_{n-2}) this configuration occurs in $\mathbb{H}$.

Therefore we may complete this diagram to get a configuration $(R, aUWb'V)$. We take $(R, aUWV)$ as factor (1) in our amalgamation diagram $(*)$.

Now we cannot take the second factor $(R, UWbV)$ to be simply a copy of $(R, UWb'V)$ since the type of x_1 over b must be given by the type Q. So we form $(R, UWbV)$ by adjusting $(R, UWb'V)$ by correcting the type of (x_1, b). We must show that the result is in $\mathbb{H}$.

Note that $(UWbV)$ is $\vec{K}_n$-free, so by the choice of R it suffices to check that $(x_1, UWbV)$ and $(x_2, UWbV)$ occur in $\mathbb{H}$. The configuration $(x_2, UWbV)$ coincides with $(x_2, UWb'V)$ which was constructed by amalgamating in $\mathbb{H}$. There remains

$$(x_1, UWbV)$$

By Proposition (IX_{n-1}) it suffices to check that this is $\mathbb{H}$-constrained. In other words, we must check

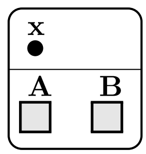
$$(x_1, U),\ (x_1, Wb),\ \text{and}\ (x_1, bV)$$

Now the first and third of these agree with P and Q, realized in $\mathbb{H}$. As $|Wb| = k - 1$, the minimality of k gives us the second as well.

This completes the construction. □

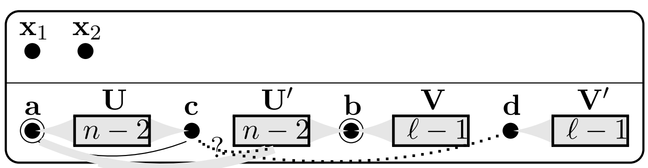
We recall once more that Proposition (IX_{n-1}) includes Proposition $(VIII_{n-1})$, which simplifies the statement of the following.

LEMMA 9.18. *Let $n \geq 3$ and assume Proposition (IX_{n-1}). Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. Let A, B be complete ordered graphs with $|B| < |A| = n$, and let P, Q be initial 1-types over A, B realized in $\mathbb{H}$. Suppose that every initial* 1*-type $Q^{\#}$ over a complete graph of order less than n is realized in $\mathbb{H}$. Then $P \perp Q$ is realized in $\mathbb{H}$.*



PROOF. We proceed by induction on $\ell = |B|$.

Take $R = x_1x_2$ realizing a Ramsey type r over the generic $\vec{K}_n$-free graph in $\mathbb{H}$. Then we amalgamate as follows.



$x_1/aUc, bV = P \perp Q,$
$x_2/aU'b, dV' = P \perp Q$

$(*)$

Then one of the two configurations involved will be $P \perp Q$, depending on the type of the pair (a, b).

Here U, U' have order $n-2$, and V, V' have order $\ell-1$. Furthermore

$$(aUc),\ (aU'b),\ (bV),\ (dV')$$

are complete (of orders n or ℓ) apart from the fact that in $(aU'b)$ the status of the pair (a, b) remains to be determined. In addition, the type of c over $U'd$ is unsettled and will be dealt with in the course of the construction.

So we must construct compatible factors $(*)_1 = (R, aUcU'VdV')$ and $(*)_2 = (R, UcU'bVdV')$ meeting our constraints, in $\mathbb{H}$.

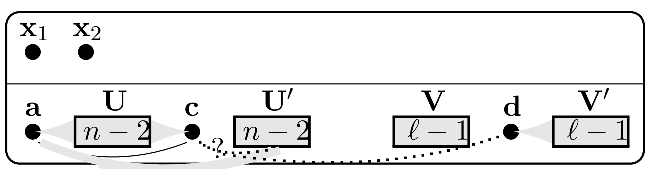
The second factor will present no difficulties. Once we settle the structure of the first factor, the structure of the second factor will be fully determined, and its second component

$$(UcU'bVdV')$$

will be $\vec{K}_n$-free. By the choice of R and Proposition (IX_{n-1}) it then suffices to check that the types $(x_i, UcU'bVdV')$ are $\mathbb{H}$-constrained. By the hypothesis of the lemma any such 1-type is $\mathbb{H}$-constrained.

So we need only concern ourselves with the construction of a suitable factor

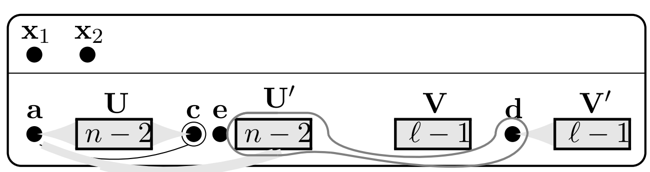
$$(R, aUcU'VdV')$$



$x_1/aUc, V = P \perp (Q{\restriction}V)$; $x_2/aU', dV' = (P{\restriction}aU') \perp Q$

(1)

We will determine the relation of c to $(U'd)$ by the following amalgamation, inserting an additional vertex e separating c from U'.



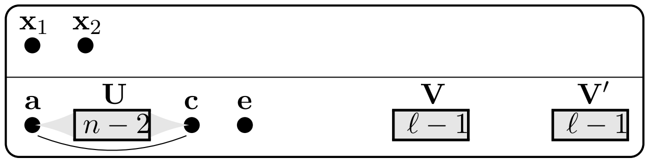
$x_1/aUc, V = P \perp (Q{\restriction}V)$
$x_2/aU', dV' = (P{\restriction}aU') \perp Q$

(A)

We must show that the factor $(R, aUceVV')$ and $(R, aUeU'VdV')$ embed into $\mathbb{H}$. In the second factor, the second component $(aUeU'VdV')$ is $\vec{K}_n$-free so again there is no difficulty.

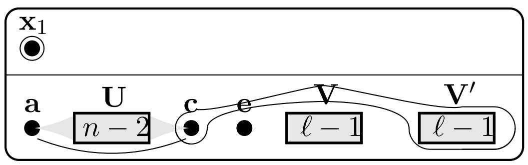
So we come down to the factor

$$(R, aUceVV') \cong (R, \vec{K}_n \perp \vec{I}_1 \perp \vec{K}_{\ell-1} \perp \vec{K}_{\ell-1})$$

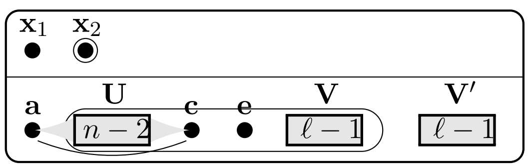


$x_1/aUc, V = P \perp (Q{\restriction}V)$
$x_2/aV' = (P{\restriction}a) \perp (Q{\restriction}V')$

For this we amalgamate in two stages as follows, first determining the type of x_1 over eV', then the type of x_2 over $UceV$.



$(A1)$



$(A2)$

So it suffices to show that the factors of these amalgamations are available in $\mathbb{H}$.

The factors in diagram $(A1)$ are

$$(A1)_1 = (x_1, aUcV) \cong (x_1, \vec{K}_n \perp \vec{K}_{\ell-1})$$
$$(A1)_2 = (\emptyset, AUceVV') \cong (\vec{K}_n \perp \vec{I}_1 \perp \vec{K}_{\ell-1} \perp \vec{K}_{\ell-1})$$

The factor $(A1)_1$ is available by induction on ℓ. The factor $(A1)_2$ is available by Lemma 9.9.

In diagram $(A2)$ the factor omitting x_2 is the one resulting from diagram $(A1)$ by amalgamation in $\mathbb{H}$. So it suffices to consider the factor

$$(R, aV')$$

By the choice of R it suffices to check that the configurations (x_i, aV') are $\mathbb{H}$-constrained for $i = 1, 2$, and as (aV') is $\vec{K}_n$-free this is immediate. $\square$

PROOF OF PROPOSITION (V_n-A). If some 1-type over a complete graph of order less than n is not realized in $\mathbb{H}$, then the result is given as Lemma 9.17, and otherwise it holds by Lemma 9.18. $\square$

We now return to Proposition (IV_n–B). As we have already dealt with the case $n = 2$, we assume $n \geq 3$ and Proposition (IX_{n-1}) holds.

PROOF OF PROPOSITION (IV_n–B), $n \geq 3$. We work in a homogeneous ordered 2-graph $\mathbb{H}$.

We fix $A \in \mathcal{A}(n)$, and p be an initial 1-type over A realized in $\mathbb{H}$. We claim that $(A^p, A^{\perp\pm})$ realizes every 1-type over $\vec{I}_k$, for $k < \infty$. It will suffice to treat $(A^p, A^{\perp+})$. If $A \in \mathcal{A}(2)$ then Proposition (IV_2–B) applies, so we may suppose $A \cong \vec{K}_n$.

In terms of $\mathbb{H}$, we are considering a configuration $(b, A \perp I)$ with $I \cong \vec{I}_k$, $k < \infty$. We proceed by induction on k. We may take $k > 0$. Set $i = \max I$, $q = \mathrm{tp}(b/i)$.

We work in $\mathbb{H}' = (i^q, i^{\perp-})$, which is ample. It suffices to show that this configuration contains $(b, AI \setminus \{i\})$. This follows by induction on k if p is realized in $\mathbb{H}'$. In other words, we have reduced to the case $k = 1$.

Now the claim is given by Proposition (V-A), since $1 < n$. $\square$

This completes the proof of Proposition (IV_n).

It is useful to derive a variant applying directly to homogeneous ordered graphs.

LEMMA 9.19. *Let $n \geq 2$ and assume Propositions (II_n, IV_n). Let Γ be a homogeneous ordered graph containing all configurations in $\mathcal{A}(n)$, let $A \in \mathcal{A}(n)$, and let P be a 1-type over A realized in Γ, other than the type $x \to \vec{K}_n$. Then $(A^P, A^{\perp\pm})$ is ample.*

PROOF. Take $c \in \Gamma$ and consider $\mathbb{H} = (c^{\perp-}, c^{\perp+})$. Then $\mathbb{H}$ is ample. It suffices to show that P is realized in H_1 over $A \subseteq H_2$, as then Proposition (IV_n) applies. By Proposition (III) (proved in §8D), if $A \in \mathcal{A}(2)$ we may reduce to complete ordered subgraphs of A. Thus we may take A to be complete.

Expressed in terms of Γ, our claim is that the type acA is realized in Γ, where $a < c < A$, A is complete of order at most n, and $c \perp aA$. As acA omits $\vec{K}_{n+1}$ this is given by Proposition (II_n). $\square$

CHAPTER 10

ORDERED HOMOGENEOUS GRAPHS: PROPOSITION V

The main result of this section is the following.

Proposition (V_n). Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph, and $n \geq 2$. If $n > 2$, assume also that Proposition IX_{n-1} holds. If $p = (x, \vec{K}_n \perp \vec{K}_n)$ is an $\mathbb{H}$-constrained initial 1-type, then p is realized in $\mathbb{H}$.

We have observed that Proposition IX_{n-1} includes Proposition VIII_{n-1}, and we have shown that it implies Proposition II_n (Proposition 7.9).

We divide the proof into two parts. We have $p = P \perp Q$ for some 1-types P, Q over $\vec{K}_n$. In the first part of the proof we show that we may take $P = Q$. In the second part of the proof we treat that case.

10A. An auxiliary type

The main result of the next subsection states that under appropriate inductive hypotheses the proof of Proposition (V_n) reduces to the consideration of 1-types of the form $P \perp P$. Here we make some preparations for that proof (Lemma 10.3 below).

LEMMA 10.1. *Let $n \geq 3$ and assume Proposition IX_{n-1} holds. Let A be an ordered graph, $b = \min A$, and suppose that $A \setminus \{b\}$ is $\vec{K}_n$-free. Then A belongs to any amalgamation class $\mathcal{A}$ containing $\mathcal{A}(n)$.*

PROOF. Let Γ be the ordered graph corresponding to the amalgamation class $\mathcal{A}$. Take $a \in \Gamma$ and let $\mathbb{H} = (a^{\perp -}, a^{\perp +})$. By Proposition I, $\mathbb{H}$ is ample.

Set $A' = A \setminus \{b\}$. It suffices to show that the 1-type (b, A') embeds into $\mathbb{H}$. Now A' is $\vec{K}_n$-free so we may apply Proposition IX_{n-1}. It suffices to show that each configuration (b, K) contained in (b, A') with K complete embeds into $\mathbb{H}$.

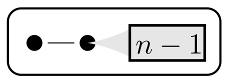
In Γ, the configuration (b, K) corresponds to baK, with

$$b < a < K \text{ and } a \perp bK, \ |K| < n$$

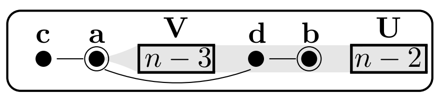
Recall that Proposition IX_{n-1} implies Proposition II_n. By Proposition II_n, baK embeds in Γ. Our claim follows. □

Now we require another very explicit configuration.

LEMMA 10.2. *Let $n \geq 3$ and assume Proposition $VIII_{n-1}$ holds. Let abK be an ordered graph with $bK \cong \vec{K}_n$, $a \perp K$, $a \to b$. Then abK belongs to every amalgamation class of finite ordered graphs containing $\mathcal{A}(n)$.*



PROOF. Amalgamate as follows, with V of order $n-3$, U of order $n-2$, and aV, Vd, and dbU complete.



(∗)

Here (dbU) is complete of order n, and $(aVdb)$ is complete of order n apart from the pair (a, b), whose type is to be determined. Thus in the amalgam either $(adbU)$ or $(caVdb)$ will be the desired configuration. So it suffices to show that the factors

$$(caVdU) \text{ and } (cVdbU)$$

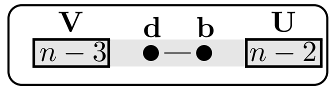
are in $\mathcal{A}$.

The factor $(caVdU)$ is $\vec{K}_n$-free, hence is in $\mathcal{A}$ by Proposition $(VIII_{n-1})$. Thus it suffices to consider $(cVdbU) = (c \perp VdbU)$, which reduces to

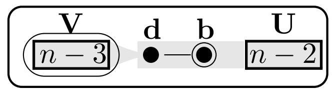
$$(VdbU)$$

Here (Vdb) is complete of order $n-1$ and (dbU) is complete of order n.

We consider this diagram as an amalgamation in which the relation of V to b remains to be determined.



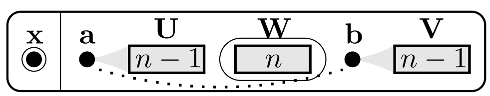
$(VdbU)$



If for some $v \in V$, the pair (v, b) is a non-edge, then the configuration $(vdbU)$ is isomorphic to the original configuration (abK) desired. On the other hand if all pairs (v, b) with $v \in V$ are edges, we have the required configuration $(VdbU)$. $\square$

LEMMA 10.3. *Let* $n \geq 3$ *and assume Propositions* (II_n) *and* (IX_{n-1}) *hold. Let* $\mathbb{H}$ *be an ample homogeneous ordered 2-graph. Let* P *and* Q *be initial 1-types over* $\vec{K}_n$ *realized in* $\mathbb{H}$*. Then there is an initial 1-type* $Q^{\#}$ *such that both* $P \perp Q^{\#}$ *and* $Q^{\#} \perp Q$ *are realized in* $\mathbb{H}$*.*

PROOF. We consider the following amalgamation, in which the point x realizes the type P over aU and Q over bV. Then $Q^{\#}$ will be the type realized by x over W.

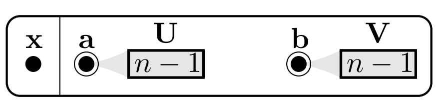


$(*)$

In this diagram, the relationship of x to W remains to be determined. We will need to decide as we go along whether (a, b) is taken to be an edge or not. It suffices to prove that there is a diagram of this type with both factors $(x, aUbV)$ and $(\emptyset, aUWbV)$ embedding in $\mathbb{H}$.

The factor $(x, aUbV)$*:*

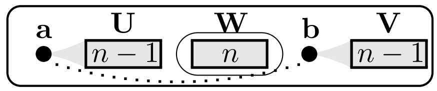
We obtain a suitable factor $(x, aUbV)$ from an amalgamation determining the type of (a, b).



(A)

So it suffices to show that the factors (x, aUV) and (x, UbV) embed into $\mathbb{H}$. This follows from Proposition (V_n)-A (§9C).

The factor $(aUWbV)$ *in* H_2*:*



(B')

Here we must deal with both possible forms of this configuration.

If (a, b) is a non-edge, then this is afforded by Lemma 9.9. So we suppose (a, b) is an edge, which means more specifically that

In the factor $(x, aUbV)$, the pair (a, b) is an edge.

We pass temporarily to consideration of the 2-ordered graph

$$\mathbb{H}(bV) = ((bV)^{p_a}, (bV)^{\perp -})$$

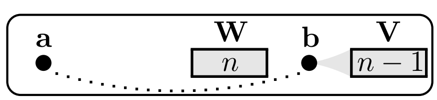
where p_a is the type realized by a (namely, $a \to b$, $a \perp V$). Then our goal is to find the configuration (a, UW) in $\mathbb{H}(bV)$.

By Lemma 9.9 and Proposition $(V_n$-$A)$ we find that $\mathbb{H}(bV)$ is ample, and its second component contains all graphs in $\mathcal{A}(n)$. Therefore by Proposition $(V_n$-$A)$ applied to $\mathbb{H}(bV)$, in order to realize the configuration (a, UW) it suffices to realize the configurations (a, U) and (a, W) in $\mathbb{H}(bV)$.

Now we return to $\mathbb{H}$; we have reduced our problem to the construction of the configurations

$$(aUbV) \text{ and } (aWbV)$$

Now $(aUbV)$ was obtained as part of the factor $(x, aUbV)$. So it suffices to consider $(aWbV)$.



$(B'.2)$

Now we pass briefly to consideration of the 2-ordered graph

$$\mathbb{H}'(V) = (V^{\leftarrow}, V^{\perp -})$$

Here $\leftarrow$ is taken because it is the type of b over V, and we are looking for the 1-type

$$(b, aW)$$

in $\mathbb{H}(V)$, which is now a *terminal* 1-type (one may also reverse the ordering on $\mathbb{H}(V)$ to stay within the framework of initial 1-types).

As above, working in $\mathbb{H}(V)$ we see that the configuration (b, aW) reduces to (b, a) and (b, W), and we return to H_2. Thus our problem now is to find

$$(abV) \text{ and } (WbV)$$

in H_2.

The configuration (abV) is afforded by Lemma 10.2, and $(WbV) \cong (\vec{K}_n \perp \vec{K}_n)$ is afforded by Lemma 9.15. □

This lemma gives us one of the essential ingredients for an amalgamation construction.

We will also make use of the Ramsey theory, as follows.

LEMMA 10.4. *Let $\mathbb{H}$ be a homogeneous ordered 2-graph, and $n \geq 2$. Suppose that for each homogeneous 2-graph $\mathbb{H}'$ contained in $\mathbb{H}$, Proposition (V_n) is valid in $\mathbb{H}'$. Then there is a Ramsey 2-type for $\mathbb{H}$ over $\mathcal{A}(n)$.*

PROOF. By Lemma 5.10 it suffices to show that for $A = \perp_i A_i$ with $A_i \in \mathcal{A}(n)$, and for $P = \perp_i P_i$ an $\mathbb{H}$-constrained initial 1-type over A, we have P realized in $\mathbb{H}$. We proceed by induction on the number of factors A_i. Working over A_1 we may reduce this number by 1, and it suffices show that the associated ordered 2-graph $\mathbb{H}' = (A_1^{P_1}, A_1^{\perp +})$ realizes the same 1-types over elements of $\mathcal{A}(n)$ as $\mathbb{H}$ does. In other words, we reduce to the case in which there are two factors, $A = A_1 \perp A_2$.

Applying Proposition III (proved in §8D) we come down to the case in which A_1 and A_2 are complete, with $\mathbb{H}$ replaced by a homogeneous ample ordered 2-subgraph, and this is given by Proposition V_n applied to such 2-subgraphs of $\mathbb{H}$. □

10B. A reduction

Now we may make the promised reduction of our problem to the "symmetric" case $p = P \perp P$.

LEMMA 10.5. *Let* $n \geq 3$ *and assume Proposition* (IX_{n-1}). *Let* $\mathbb{H}$ *be an ample homogeneous ordered 2-graph. Assume that for any homogeneous ordered 2-graph* $\mathbb{H}'$ *with* $s(\mathbb{H}', \vec{K}_n) < s(\mathbb{H}, \vec{K}_n)$, *Proposition* (V_n) *is valid for* $\mathbb{H}'$. *Let* P *and* Q *be initial 1-types over* $\vec{K}_n$ *realized in* $\mathbb{H}$. *Suppose that* $P \perp P$ *and* Q *are both realized in* $\mathbb{H}$. *Then* $P \perp Q$ *is realized in* $\mathbb{H}$.

Recall that Proposition IX_{n-1} implies Propositions VIII_{n-1} and II_n.

PROOF. Suppose the contrary.

Take $K \subseteq H_2$, $K \cong \vec{K}_n$, and form $\mathbb{H}^P = (K^P, K^{\perp +})$. Then in $\mathbb{H}^P$ the type Q over $\vec{K}_n$ is omitted and therefore

$$s(\mathbb{H}^P, \vec{K}_n) < s(\mathbb{H}, \vec{K}_n)$$

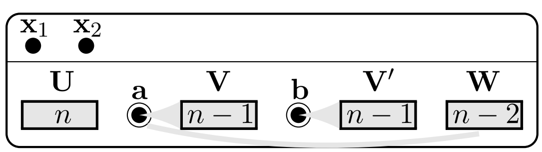
So by hypothesis Proposition (V_n) is valid for $\mathbb{H}^P$, and for any homogeneous ordered 2-graph contained in $\mathbb{H}^P$.

By Lemma 10.3 there is an initial 1-type $Q^{\#}$ so that

$$P \perp Q^{\#} \text{ and } Q^{\#} \perp Q \text{ are both realized in } \mathbb{H}$$

By Lemma 10.4 there is a Ramsey 2-type r for $\mathbb{H}^P$ over $\mathcal{A}(n)$. Let $R = (x_1 x_2)$ have type r.

We amalgamate as follows.



$$x_1/U, aV, bV' = Q^{\#}, P, Q;$$
$$x_2/U, abW = P, Q$$

$(*)$

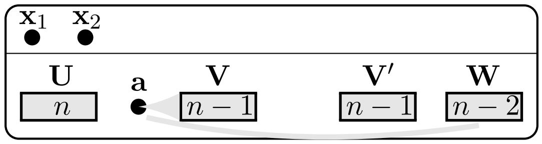
As usual the specification of the type of x_2/abW speaks only about the relation of x_2 to the points of abW, and not to the structure of abW. But when the diagram is completed these conditions ensure that either $(x_1, aVbV')$ or $(x_2, UabW)$ will be a realization of $P \perp Q$.

There remain unspecified relations between the vertices of R and those in the second component, which will be determined as we proceed.

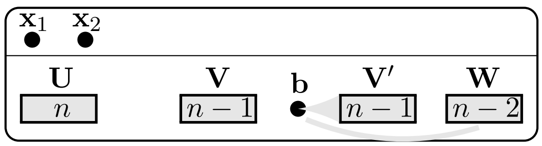
So it suffices to show that two compatible factors

$$(R, UaVV'W) \text{ and } (R, UVbV'W)$$

can be found in $\mathbb{H}$.



(1)



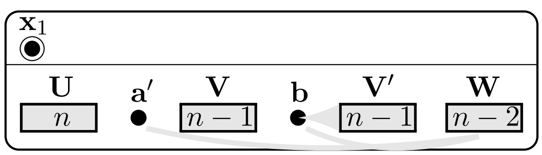
(2)

In particular we must determine the types of

$$(x_1/W) \text{ and } (x_2/VV')$$

along the way.

We begin by determining the type of x_1 over W via the following amalgamation.



(I)

Here the vertex a' is similar to the vertex a, but a' is not adjacent to the vertices of V. Still, we require that the type of x_1 over $a'V$ should be P in the usual weak sense that the type of x_1 over a' and over V should be as in P.

Claim 1. The factors $(x_1, Ua'VbV')$ and $(\emptyset, Ua'VbV'W)$ embed into $\mathbb{H}$.

We consider the first factor

$$(x_1, Ua'VbV')$$

Here we may pass to $\mathbb{H}(bV) = ((bV')^Q, (bV')^{\perp -})$ where we are looking for the configuration

$$(x_1, Ua'V)$$

Here $\mathbb{H}(bV')$ is ample and the type we wish to realize is

$$Q^\# \perp (P \upharpoonright a') \perp (P \upharpoonright V)$$

As $|\{a\}|, |V| < n$, by repeated applications of Proposition (V_n)-A in $\mathbb{H}(bV')$ it suffices to check that the configurations (x_1, U), (x_1, a'), and (x_1, V) all embed into $\mathbb{H}(bV')$. In $\mathbb{H}$ this means we must check

$$(x_1, UbV'),\ (x_1, a'bV'),\ (x_1, VbV')$$

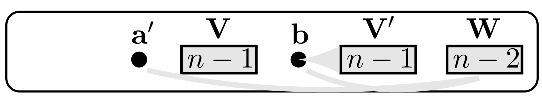
Now applying Proposition (V_n)-A in $\mathbb{H}$, the second and third configurations reduce to their constituents (x_1, a'), (x_1, V), and (x_1, bV') which are found within P and Q. As (x_1, UbV') realizes the type $Q^\# \perp P$, the choice of $Q^\#$ gives this configuration as well.

This disposes of the factor $(x_1, Ua'VbV')$. Now we must show that

$$(Ua'VbV'W)$$

belongs to any amalgamation class containing $\mathcal{A}(n)$.

By Lemma 9.9 this reduces to the configuration $(a'VbV'W)$.



Working over V, that is in $\mathbb{H}'(V) = (V^{\perp -}, V^{\perp +})$, we require the configuration $(a', bV'W)$ which reduces to (a', bV') and (a', W). In terms of H_2 this means that we require

$$(a'VbV') \cong (a' \perp V \perp bV') \text{ and } (a'VW)$$

The first is given by Lemma 9.9 so we are left with $(a'VW)$, which is $\vec{K}_n$-free.

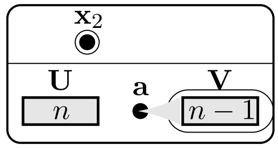
This disposes of the second factor and proves the claim.

Thus we may form the diagram I in $\mathbb{H}$, and take the amalgam to determine

$$(x_1, W)$$

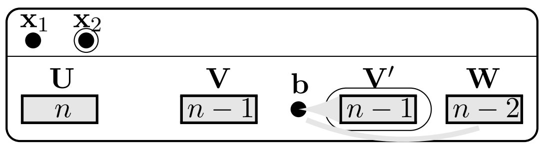
Next we determine the type of x_2 over VV'. First we determine the type of x_2 over V by the following amalgamation.

The factors here are easily seen to embed into $\mathbb{H}$.



(II)

Now we complete the determination of the factor $(R, UVbV'W)$ by the following amalgamation, which determines the type of x_2 over V'.



(III)

Claim 2. The factors

$$(R, UVbW) \text{ and } (x_1, UVbV'W)$$

of diagram (III) embed into $\mathbb{H}$.

The factor $(x_1, UVbV'W)$ is copied over from the result of the first amalgamation (diagram (I) above). So it suffices to consider

$$(R, UVbW)$$

We will show that this configuration embeds even into $\mathbb{H}^P$. By the choice of R, it suffices to check that the configurations $(x_i, UVbV')$ are $\mathbb{H}^P$-constrained. In other words, we must show that the following occur in $\mathbb{H}^P$.

$$(x_1, U) = Q^{\#},\ (x_1, V),\ (x_1, bW),$$
$$(x_2, U) = P,\ (x_2, V),\ (x_2, bW)$$

First we deal with the types over V and bW. As $|V|, |bW| < n$, Proposition (V_n)-A applies and in order to show that these embed in $\mathbb{H}^P$ it suffices to check that they embed in $\mathbb{H}$.

Now (x_1, V) is a restriction of P and (x_1, bW) is afforded by diagram I, hence occurs in $\mathbb{H}$. Similarly (x_2, V) is afforded by II and (x_2, bW) is a restriction of Q.

It remains to be checked that $Q^{\#}$ and P occur in $\mathbb{H}^P$, equivalently that

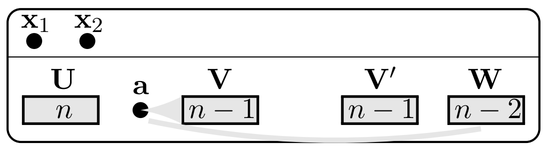
$$(P \perp Q^{\#}) \text{ and } (P \perp P)$$

occur in $\mathbb{H}$. This holds by the choice of $Q^{\#}$ and by hypothesis, respectively.

Thus our claim is proved, and we may use the diagram III to determine the factor $(R, UVbV'W)$.

Now we require a compatible factor in $\mathbb{H}$ of the form

$$(R, UaVV'W).$$



(1)

Again, we prove something stronger.

Claim 3. The factor $(R, UaVV'W)$ embeds into $\mathbb{H}^P$.

By the choice of R, this means that we must show that the types of x_1 and x_2 over U, aV, V', and aW are realized in $\mathbb{H}^P$.

We begin with the types of x_1 and x_2 over V' and aW. As $|V'|, |aW| < n$, it suffices to check that these types are realized in $\mathbb{H}$, applying Proposition (V_n)-A.

The types of x_1 over V' and aW are a restriction of Q and the result of I, respectively. Thus these occur in $\mathbb{H}$. (Note that here we finally make use of the presence of the vertex a' in I.)

The types of x_2 over V' and aW are the result of III and a restriction of Q, respectively. Thus these occur in $\mathbb{H}$.

The types of x_1 over U is $Q^{\#}$, which was chosen to be realized in $\mathbb{H}^P$.

The type of x_1 over aV or of x_2 over U is P, which is realized in $\mathbb{H}^P$ by hypothesis.

The type of x_2 over aV results from Diagram II, where x_2 realizes the type P over U, and thus this type is also realized in $\mathbb{H}^P$.

This proves our claim.

Now our analysis is complete. We have constructed compatible factors for the initial diagram $(*)$, so we conclude. $\square$

At this point, we have reduced the treatment of Proposition (V_n) to the consideration of a crucial case: 1-types of the form $P \perp P$, where P is realized in $\mathbb{H}$. We will treat this case in the next two subsections.

10C. Final preparations

We must make some further preparations for the last stage in our analysis.

An initial *cross type* in an ordered 2-graph $\mathbb{H}$ is the type of some pair (a, b) with $a \in H_1$ and $b \in H_2$, and $a < b$. Thus this is either $\perp+$ or $\to$, in our context.

Lemma 10.6. *Let $n \geq 3$, and assume Proposition (IX_{n-1}). Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph. Let p be an initial cross type. Then there is an initial cross type q with the following property.*

Assume $C = K \perp A \perp B$, $y = \min K$, and $a = \min A$ satisfy

$$K \cong \vec{K}_n,\ a \to A \setminus \{a\}, \text{ and } A \text{ omits } \vec{K}_{n+1},\ B \text{ omits } \vec{K}_n$$

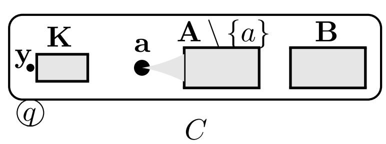
and $Q_1 \perp Q_2 \perp Q_3$ is a 1-type over C with

$$Q_1 \restriction y = q,\ Q_2 \restriction a = p$$

with Q_1, Q_2, Q_3 realized in $\mathbb{H}$.

Then

$$Q_1 \perp Q_2 \perp Q_3 \text{ is realized in } \mathbb{H}.$$

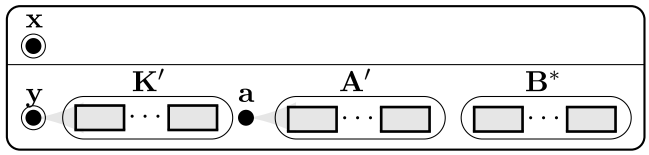


Proof. Supposing the contrary, for each initial cross type q choose a counterexample $C_q = K_q \perp A_q \perp B_q$, $Q(q) = Q_1(q) \perp Q_2(q) \perp Q_3(q)$ and form the amalgamated ordered sum

$$K^* \perp A^* \perp B^*$$

where $B^* = \perp_q B_q$ and in K^* and A^* the respective minimum elements y_Q, a_Q are identified with initial elements y, a respectively. Set $K^* = yK'$, $A^* = aA'$.

By our construction, the following amalgamation problem has no solution in $\mathbb{H}$.



$(*)$

So to get a contradiction it suffices to show the following.

Claim. The factors

$$(x, K'A^*B^*) \text{ and } (\emptyset, K^*A^*B^*)$$

of diagram $(*)$ are in $\mathbb{H}$.

We first consider $(\emptyset, K^*A^*B^*)$.

By Lemma 9.9 the configuration $(K^*A^*B^*) = ([y, K'] \perp [a, A'] \perp B^*)$ reduces to its three constituents

$$[y, K'], \ [a, A'], \text{ and } B^*$$

By Proposition $(VIII_{n-1})$ and Lemma 7.6, these constituents embed into H_2.

So we need only consider the factor

$$(x, K'A^*B^*)$$

We first work over $K'A^*$, that is we consider the associated 2-ordered graph

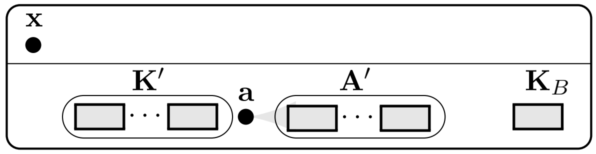
$$\mathbb{H}(K'A^*) = ((K'A^*)^{p_x}, (K'A^*)^{\perp+})$$

where p_x is the type realized by x. We look for a realization of (x, B^*).

By Proposition (IX_{n-1}) it suffices to consider the restrictions of this type of the form (x, K_B) with $K_B \subseteq B$ complete, and thus $|K_B| < n$. Thus in terms of $\mathbb{H}$, we may replace B^* by the complete graph K_B and consider

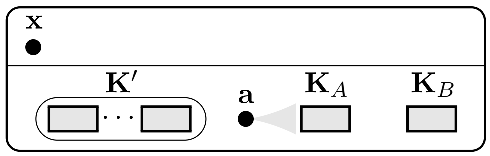
$$(x, K'A^*K_B).$$

We may work similarly over $K'aK_B$ and consider the configuration (x, A'). In this way we may reduce A' to a complete subgraph K_A of



$A'_q = A_q \setminus \{a\}$, and thus in $\mathbb{H}$ we are now considering.

$$(x, K'aK_AK_B).$$



Similarly, over K' this last configuration becomes (x, K_AK_B) which reduces to the individual constituents (x, aK_A) and (x, K_B), or in terms of $\mathbb{H}$, the following.

$$(x, K'aK_A) \text{ and } (x, K'K_B)$$

Here the components of K' are complete of order $n-1$, K_B is complete of order less than n, and aK_A is complete of order at most n.

These then reduce in a similar fashion to (x, K'), (x, K_A), and (x, K_B), all of which are present in $\mathbb{H}$. $\square$

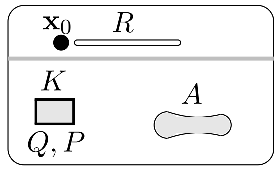
Lemma 10.7. *Let $\mathbb{H}$ be an ample homogeneous 2-graph and P a 1-type over $\vec{K}_n$ realized in $\mathbb{H}$. Let r be a Ramsey 2-type for $\mathbb{H}^P$ over $\mathcal{A}(n-1)$, and q any initial cross type. Then there is a 1-type Q over $\vec{K}_n$ whose restriction to $a = \min \vec{K}_n$ is q, with the following property.*

- For any finite configuration (R, A) realized in $\mathbb{H}$ such that R is r-Ramsey and A omits $\vec{K}_n$, if $x_0 = \min R$, then $\mathbb{H}$ contains the configuration

$$(R, K \perp A)$$

where (R, A) is as given, $K \cong \vec{K}_n$, and

$$\begin{aligned} &\mathrm{tp}(x_0/K) = Q \\ &\mathrm{tp}(x/K) = P \text{ for } x \in R,\ x > x_0 \end{aligned}$$



PROOF. Supposing the contrary, for each 1-type Q over $\vec{K}_n$ realized in $\mathbb{H}$ and extending the given cross type q, choose counterexamples

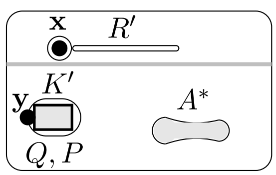
$$\mathbb{A}_Q = (R_Q, A_Q)$$

Form the ordered r-sum $R^* = \Sigma_Q R_Q$ with all copies of x_0 identified with an initial element x_0 (using the 2-type r, so that R^* is r-Ramsey), and correspondingly set

$$A^* = \underset{Q}{\perp}{}' A_Q$$

with all $a_Q = \min A_Q$ identified with an initial element a.

Let $R' = R^* \setminus \{x_0\}$. Let $K \cong \vec{K}_n$ and let $y = \min K$, $K' = K \setminus \{y\}$. Then the following amalgamation problem, in which we determine the relation of x to K', has no solution in $\mathbb{H}$.

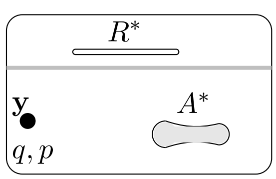


To reach a contradiction, we will show that the factors

$$(R^*, yA^*) \text{ and } (R', yK'A^*)$$

occur in $\mathbb{H}$.

The factor (R^*, yA^*)*:*

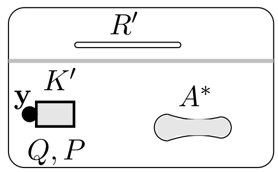


Now yA^* omits $\vec{K}_n$ and R^* is r-Ramsey for $\mathcal{A}(n-1)$, so to get this factor it suffices to check that the individual types x/K_A are realized in $\mathbb{H}^P$ for $x \in R^*$ and $K_A \subseteq yA^*$ complete. And as yA^* omits K_n, it suffices to realize these types in $\mathbb{H}$.

For $K \subseteq A^*$ this is given, and the types (x, y) for $x \in R^*$ are restrictions of P or of a type Q realized in $\mathbb{H}$.

Thus the factor (R^*, yA^*) embeds in $\mathbb{H}^P$ and hence in $\mathbb{H}$.

The factor $(R', yK'A^*)$*:*



For this factor we will work over $K = yK'$, that is we consider $\mathbb{H}^P = (K^P, K^{\perp+})$, and we claim that the configuration (R', A^*) is realized there.

Again, it suffices to check the types (x, K_A) where $K_A \subseteq A^*$ is complete. In $\mathbb{H}$ this corresponds to 1-types of the form $P \perp \mathrm{tp}(x/K_A)$ with $|K_A| < n$, which we have by Proposition (V-A).

This completes the argument. $\square$

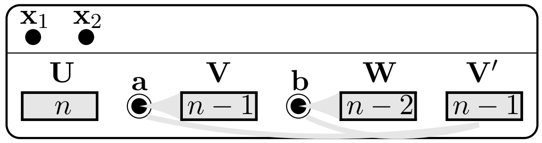
10D. Proof of Proposition V

Now we may pull the pieces together.

LEMMA 10.8. *Let $\mathbb{H}$ be an ample homogeneous ordered 2-graph and $n \geq 3$. Assume Proposition (IX_{n-1}) and let P be an initial 1-type over $\vec{K}_n$ realized in $\mathbb{H}$. Then $P \perp P$ is realized in $\mathbb{H}$.*

PROOF. We first establish our notation.

- $P = \mathrm{tp}(x/K)$, $K \cong \vec{K}_n$.
- r is a Ramsey 2-type for $\mathbb{H}^P$ over $\mathcal{A}(n-1)$.
- $R = (x_1 x_2)$ with $\mathrm{tp}(x_1 x_2) = r$.
- $U \cong \vec{K}_n$; $V_1, V_2 \cong \vec{K}_{n-1}$; $W \cong \vec{K}_{n-2}$
- $a = \min K$, $p = P \restriction a$
- q is an initial cross type given by Lemma 10.6 applied to p.
- Q is a 1-type over $\vec{K}_n$ extending q, given by Lemma 10.7



$x_1/U, aV, bV' = Q, P, P;$
$x_2/U, abW = P, P$

$(*)$

Now we amalgamate as follows.

In particular, in the amalgam we will have the following.

$$\text{Either } (x_1, aVbV') \text{ or } (x_2, UabW) \text{ is } P \perp P$$

The types

$$x_1/UW \text{ and } x_2/VV'$$

remain to be determined in the course of the construction.

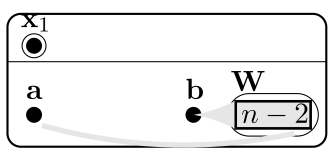
We must show that compatible factors

$$(R, UaVWV') \text{ and } (R, UVbWV')$$

satisfying our conditions can be found in $\mathbb{H}$.

Stage 1: Determination of x_1/W:

We begin with the following amalgamation to determine the type of x_1 over W.



(I)

Observe that the factor (abW) omits $\vec{K}_n$, so is available in H_2.

Stage 2: Determination of x_2/aV:

This will take some work, and depends on the specific choices of the types q and Q.

Claim 1. There is an initial 1-type P^* over aV extending $P \restriction a$ so that any configuration of the form $(R, UaVWV')$ as above, with the following properties, embeds into $\mathbb{H}$.

- $x_1/UaVW$, x_2/UaW as specified above;

- (x_2, aV_1) realizes P^*;
- (x_2, V') any type realized in $\mathbb{H}$.

Suppose the claim fails. Then for each type P^* over aV which extends $P \restriction a$ and which is realized in $\mathbb{H}$ we may choose a configuration

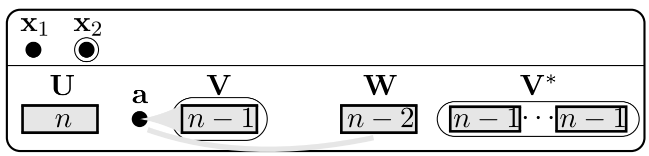
$$\mathcal{A}(P^*) = (R, UaVWV'(P^*))$$

where $V'(P^*)$ is a copy of $\vec{K}_{n-1}$, so that the type $(x_2, V'(P^*))$ is realized in $\mathbb{H}$, but the configuration $\mathcal{A}(P^*)$ is not realized in $\mathbb{H}$.

We vary P^* over all types over aV realized in $\mathbb{H}$ and form the ordered sum

$$V^* = \perp_{P^*} V'(P^*)$$

Then the following amalgamation has no solution in $\mathbb{H}$.



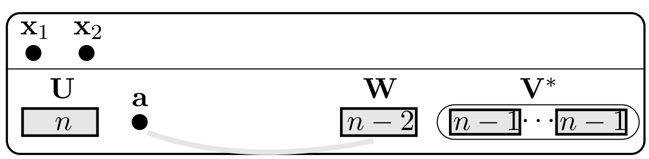
$(*)$

We must show that the factors

$$(R, UaWV^*) \text{ and } (x_1, UaVWV^*)$$

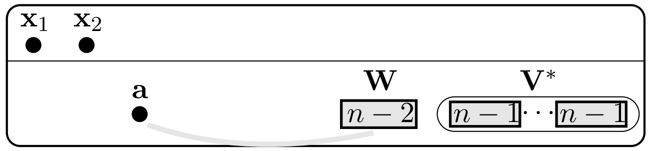
do embed into $\mathbb{H}$.

We first deal with the factor $(R, UaWV^*)$.



Recall that we require the type of x_1 over U to be Q, and the type of x_2 over U must be P. Now by the choice of the type Q, in order to embed the factor $(R, UaWV^*)$ in $\mathbb{H}$, it suffices to embed the factor (R, aWV^*).

We will show that this last configuration embeds into $\mathbb{H}^P$. Now (x_1, x_2) realizes a Ramsey 2-type for $\mathbb{H}^P$ over $\mathcal{A}(n-1)$, so it will suffice to check the types of x_1 and x_2 over aW and over the components $V'(P^*)$ which make up V^*. As the base sets involved have order less



than n, to show that these types are realized in $\mathbb{H}^P$ it suffices to show that they are realized in $\mathbb{H}$.

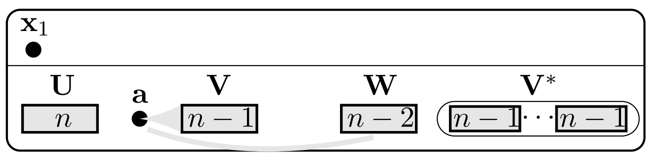
The type of x_1 over $V'(P^*)$ is the restriction of P to this set. The type of x_2 over $V'(P^*)$ is some type realized in $\mathbb{H}$, by hypothesis.

The type of x_1 over aW results from diagram I and is therefore realized in $\mathbb{H}$.

The type of x_2 over aW is the restriction of P to this set.

So (R, aWV^*) embeds in $\mathbb{H}^P$, and in particular embeds in $\mathbb{H}$.

Now we must deal with the factor $(x_1, UaVWV^*)$.



Here we will make use of the choice of the cross type q.

Recall that

$$\text{the type of } (x_1, \min U) \text{ is } q,$$

$$\text{the type of } (x_1, a) \text{ is } p = P \restriction a,$$

and that q was chosen in terms of p using Lemma 10.6.

In the notation of that lemma we are taking $A = aVW$ and $B = V^*$. So A omits $\vec{K}_{n+1}$ and B omits $\vec{K}_n$. By the choice of q, the configuration $(x_1, UaVWV^*)$ reduces to the three configurations

$$Q_1 = (x_1, U),\ Q_2 = (x_1, aVW), \text{ and } Q_3 = (x_1, V^*)$$

Now the type of Q_1 is Q, and the type of Q_3 is the sum of the 1-types of x_1 over the various sets $V'(P^*)$, which are realized in $\mathbb{H}$. Since these sets have order less than n, this suffices. So we need only consider Q_2, that is

$$(x_1, aVW)$$

We work over a, that is we pass to the 2-connected graph $\mathbb{H}(a) = (a^p, a^{\rightarrow})$. Here we require the configuration

$$(x_1, VW)$$

This configuration reduces to (x_1, V) and (x_1, W) separately, so returning to $\mathbb{H}$ we find that we require the two configurations

$$(x_1, aV) \text{ and } (x_1, aW)$$

The type of x_1 over aV is P, and the type of x_1 over aW was constructed in diagram I.

This completes the proof of the claim.

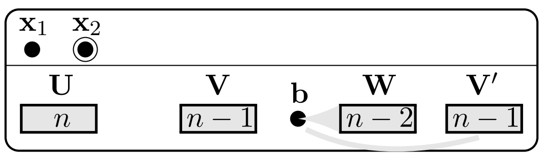
We choose the type P^* of x_2 over aV in accordance with Claim 1.

Stage 3: Determination of x_2/V':

Now we form an amalgamation diagram which produces the factor

$$(R, UVbWV')$$

and settles the type of x_2 over V'.



$$x_1/U, V, bW, bV' = Q, P, (I), P;$$
$$x_2/U, V, bW = P, (P^* \restriction V), P$$

(III)

Claim 2. The factors

$$(R, UVbW) \text{ and } (x_1, UVbWV')$$

of this amalgamation diagram embed in $\mathbb{H}$.

We begin with

$$(R, UVbW)$$

This has the form $(R, U \perp B)$ with $B = VbW$ omitting $\vec{K}_n$. Furthermore we require x_1 and x_2 to realize the types Q and P respectively over U. So by the choice of the type Q, it suffices to consider

$$(R, B) = (R, VbW)$$

We will show that this is realized in $\mathbb{H}^P$.

By the choice of R, it suffices to check that the configurations (x_i, V) and (x_i, bW) are realized in $\mathbb{H}^P$ for $i = 1, 2$. As $|V|, |bW| < n$, it suffices to check that they are realized in $\mathbb{H}$.

The type of x_1 over V, or of x_2 over bW, is a restriction of P.

The type of x_2/V was constructed in $\mathbb{H}$ at Stage 2.
The type of x_1/bW was constructed in $\mathbb{H}$ at Stage 1.
This disposes of the factor $(R, UVbW)$. Now we take up the factor

$$(x_1, UVbWV')$$

First we work over UVb, so that we are considering the configuration (x_1, WV'). This reduces to (x_1, W) and (x_1, V'), so in $\mathbb{H}$ we reduce to the configurations

$$(x_1, UVbW) \text{ and } (x_1, UVbV')$$

As $|V|, |bW| < n$, configuration $(x_1, UVbW)$ is available in $\mathbb{H}$. So we consider

$$(x_1, UVbV')$$

Working over U, we are considering the configuration (x_1, VbV') which reduces to (x_1, V) and (x_1, bV'); so in $\mathbb{H}$, this reduces to

$$(x_1, UV) \text{ and } (x_1, UbV')$$

Again $|V| < n$ so only the second configuration, (x_1, UbV'), requires attention.

Write $UbV' = U \perp B$ with $A = bV'$ and apply the choice of q. Note that x_1 realizes the type q over $\min U$ and p over b. So the configuration (x_1, UbV') reduces to (x_1, bV'), which is P.

This proves Claim 2, so we can form the diagram III and determine the type of x_2 over V'. In particular, we have constructed the second factor of the diagram $(*)$.

At this point, we should return to the first factor

$$(R, UaVWV')$$

which is now fully specified.

We have constructed the 1-type P^* in Stage 2 so that modulo the previous specification of $x_1/UaVW$, x_2/UaW, and taking x_2/aV to realize P^*, in order to embed $(R, UaVWV')$ into $\mathbb{H}$ it suffices to verify that (x_2, V') embeds into $\mathbb{H}$. But this configuration is part of the configuration $(R, UVbWV')$ just constructed. □

With this, the proof of Proposition (V_n) is complete. We repeat the statement, and review the proof.

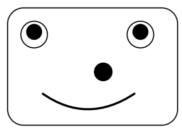
PROOF OF PROPOSITION (V_n). For $n = 2$ this is Lemma 8.7, so suppose $n > 2$.

We proceed by induction on the parameter $N = s(\mathbb{H}, \vec{K}_n)$, the number of initial 1-types realized over $\vec{K}_n$ in $\mathbb{H}$. By Lemma 10.5 it suffices to treat 1-types of the form $P \perp P$ where P is an initial 1-type over $\vec{K}_n$ realized in $\mathbb{H}$.

This is covered by Lemma 10.8.

$\square$

This completes the inductive proof of Proposition V, and thus by simultaneous induction we may prove Propositions I–IX, and thus also Proposition 5.1. With this, the proof of Theorem 1.2 is complete.



Part II. Metrically Homogeneous Graphs

CHAPTER 11

METRICALLY HOMOGENEOUS GRAPHS: PRELIMINARIES

We now take up the classification problem for metrically homogeneous graphs, that is, connected graphs which are homogeneous when viewed as metric spaces, under the path metric associated with the graph. We have described the main results to be proved in some detail in Chapters 1 and 2. In particular we have stated a conjecture which may be put in a completely explicit form as to what the final classification should be.

For ease of reference we collect some of the main points set out in our earlier discussion, including some key definitions and useful facts, in the present chapter.

11A. The main conjecture

We may summarize the conjectured classification of the (countable) metrically homogeneous graphs as follows (§1D), using the notion of *generic type* (Definition 1.17) and the notions of a *3-constrained* amalgamation class and an amalgamation class defined by *Henson constraints.*

Conjecture (Cf. Conjectures 1, 2, §1D). The metrically homogeneous graphs are the following.

(a) Non-generic type (exceptional local type or a regular tree of infinite degree);
(b) Diameter $\delta \leq 2$ (Lachlan/Woodrow classification);
(c) Generic type, $\delta \geq 3$: Fraïssé limit of

$$\mathcal{A} = \mathcal{A}_3 \cap \mathcal{A}_H$$

with $\mathcal{A}_3$ a 3-constrained amalgamation class, and $\mathcal{A}_H$ an amalgamation class given by Henson constraints.

A body of work, much of it to be presented here, gives this conjecture the form of an explicit list of a few families of examples. It is not necessary to dwell on the definition of *generic type* here (it comes back into prominence in Chapter 15). The explicit classification of the metrically homogeneous graphs of non-generic type is known, and was given in §1D as Facts 1.7 and 1.18. It is convenient to give that again here, for reference (Table 11.1).

n-gons;
antipodal, diameter 3, double covers;
tree-like ($T_{m,n}$).

Table 11.1. Metrically homogeneous graphs of non-generic type

On the other hand, the elucidation of the content of the conjecture for the case of generic type will occupy all of Chapters 12, 13, and 14. Once this work is complete, the conjecture can be viewed as proposing an explicit classification of the graphs in question (Theorems 1.19, 1.22).

We recall some terminology used in the generic type case.

An amalgamation class is *3-constrained* if the minimal structures not in the class have order at most 3. We say loosely that it is "given by forbidden triangles," which presupposes that the diameter δ is fixed in advance.

The Henson classes are given by *Henson constraints,* and these can be of two kinds—an awkward point, as there is a typical kind and a variant arising only in the antipodal case.

What we usually mean by *Henson constraint* is a $(1, \delta)$-space, with δ the diameter; this means that only the distances 1 and δ may occur in the constraint. But in the antipodal setting, an *antipodal Henson constraint* is a family of forbidden graphs consisting of an n-clique together with all of its *antipodal companions*, arising by replacing a subset of the vertices v by their antipodal vertices v'; in particular, these are $(1, \delta - 1)$-spaces.

Thus we have (ordinary) Henson constraints and antipodal Henson constraints. When we wish to specify the diameter δ these will be called *δ-Henson constraints* and *δ-antipodal Henson constraints.*

As we have said, we will be occupied for three chapters with the problem of classifying, completely and explicitly, all metrically homogeneous graphs arising as Fraïssé limits of classes $\mathcal{A}$ of the form

$$\mathcal{A}_3 \cap \mathcal{A}_H$$

with $\mathcal{A}_3$ 3-constrained and $\mathcal{A}_H$ one of the two flavors of Henson constraint. There is no trouble classifying the possibilities for $\mathcal{A}_H$; there is considerable trouble classifying the possibilities for $\mathcal{A}_3$; and one must at least pay attention to the question, which possibilities for $\mathcal{A}_3$ and $\mathcal{A}_H$ actually fit together (i.e., if $\mathcal{A}_3$ and $\mathcal{A}_H$ are amalgamation classes, does the same hold for the intersection)? But once one understands the two types separately, analyzing their interaction is straightforward.

The purpose of this chapter is to set up the framework for that analysis in detail, reviewing and completing the discussion of §2B.

Once this classification is complete, and the general classification conjecture is in a completely explicit form, in the last three chapters of this Part we will take up some first steps toward a proof of completeness of the classification in generic type, as outlined in §§2C, 2D, namely the local analysis of the induced homogeneous metric spaces Γ_i, when an edge is present, and the reduction of the bipartite and infinite diameter cases to cases of smaller diameter.

See Chapter 1 for a more detailed discussion of these results.

11B. The graphs $\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$

We review the notation introduced in §1E.

DEFINITION (Def. 1.21). The sequence of numerical parameters

$$(\delta, K_1, K_2, C_0, C_1)$$

is *acceptable* if it satisfies the following conditions.

(a) $3 \leq \delta \leq \infty$;
(b) $1 \leq K_1 \leq K_2 \leq \delta$, or $K_1 = \infty$ and $K_2 = 0$;
(c) $2\delta < C_0, C_1 \leq 3\delta + 2$; C_0 is even, and C_1 is odd;
(d) If $K_1 = \infty$ (i.e., the bipartite case) then $C_1 = 2\delta + 1$.

DEFINITION (Def. 1.20). Let $(\delta, K_1, K_2, C_0, C_1)$ be given. Then

$$\mathcal{T}(\delta, K_1, K_2, C_0, C_1)$$

is the set of triangles whose edge lengths (i, j, k) (with $1 \le i, j, k \le \delta$) satisfy one of the following conditions, where $p = i + j + k$.

$p < 2K_1 + 1$ and p is odd; $p > 2K_2 + 2\min(i, j, k)$ and p is odd;
$p \ge C_0$ and p is even; $p \ge C_1$ and p is odd.

DEFINITION 11.1. For an acceptable sequence of parameters

$$(\delta, K_1, K_2, C_0, C_1)$$

let

$$\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$$

denote the class of all finite integral metric spaces of diameter at most δ in which none of the triangles in the set $\mathcal{T}(\delta, K_1, K_2, C_0, C_1)$ embed isometrically.

When $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ is an amalgamation class, then let

$$\Gamma^{\delta}_{K_1,K_2,C_0,C_1}$$

denote its Fraïssé limit, as a metric space.

In the next definition we will require $\delta \ge 3$, though it is not necessary at this point. Since that condition will be imposed later, we build it in from the beginning.

DEFINITION 11.2 (Cf. Theorem 1.22). An acceptable sequence of parameters $(\delta, K_1, K_2, C_0, C_1)$ with $\delta \ge 3$ is *admissible* iff it satisfies one of the following three sets of conditions.

(I) $K_1 = \infty$ (the bipartite case; so $K_2 = 0$ and $C_1 = 2\delta + 1$).

(II) $K_1 < \infty$, $C \le 2\delta + K_1$ and
— $C = 2K_1 + 2K_2 + 1$;
— $K_1 + K_2 \ge \delta$;
— $K_1 + 2K_2 \le 2\delta - 1$;
(IIA) $C' = C + 1$ or
(IIB) $C' > C + 1$, $K_1 = K_2$, and $3K_2 = 2\delta - 1$.

(III) $K_1 < \infty$, $C > 2\delta + K_1$ and
— $K_1 + 2K_2 \ge 2\delta - 1$ and $3K_2 \ge 2\delta$;
— If $K_1 + 2K_2 = 2\delta - 1$ then $C \ge 2\delta + K_1 + 2$;
— If $C' > C + 1$ then $C \ge 2\delta + K_2$.

These are the main definitions, but we extend them further by considering Henson constraints in addition to constraints on triangles.

DEFINITION 11.3. The sequence of parameters

$$(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$$

is *acceptable* if

(a) $(\delta, K_1, K_2, C_0, C_1)$ is an acceptable sequence of numerical parameters.
(b) $\mathcal{S}$ is a set of Henson constraints, i.e., a set of $(1, \delta)$-spaces if $C_1 > 2\delta + 1$ or $C_0 > 2\delta + 2$, and a set of antipodal Henson constraints if $C_1 = 2\delta + 1$ and $C_0 = 2\delta + 2$.
(c) $\mathcal{S}$ is *irredundant* in the sense that no constraint in $\mathcal{S}$ contains another constraint or one of the triangles in $\mathcal{T}(\delta, K_1, K_2, C_0, C_1)$, and every constraint in $\mathcal{S}$ has order at least 4.

Acceptability has the following consequences.

— If $K_2 < \delta$ then $\mathcal{S}$ contains no space involving both distance 1 and δ; hence $\mathcal{S}$ consists of at most one clique (mutual distance 1) and one anticlique (mutual distance δ);
— If $K_1 > 1$, then the connected components (with respect to the edge relation $d(x, y) = 1$) of spaces in $\mathcal{S}$ have at most 2 vertices;
— If δ has parity $\epsilon = 0$ or 1, and $C_\epsilon \leq 3\delta$, then the spaces in $\mathcal{S}$ are unions of at most two cliques;
— If $C_1 = 2\delta + 1$ and $C_0 = C_1 + 1$, then $\mathcal{S}$ consists of cliques;
— If $K_1 = \infty$, then
  (i) $\mathcal{S}$ is empty if δ is odd, or if δ is even with $C_0 \leq 3\delta$;
  (ii) $\mathcal{S}$ is a set of anticliques (empty, or a singleton, by irredundancy) if δ is even and $C_0 = 3\delta + 2$.

DEFINITION 11.4. Let $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ be an acceptable sequence of parameters, where $\mathcal{S}$ is a set of δ-Henson constraints or δ-antipodal Henson constraints. Let $C = \min(C_0, C_1)$ and $C' = \max(C_0, C_1)$. The sequence is *admissible* if

1. The sequence $(\delta, K_1, K_2, C_0, C_1)$ is admissible.
2. If $C = 2\delta + 1$, $K_1 < \infty$, and $\mathcal{S}$ is nonempty, then $\delta \geq 4$ and $\mathcal{S}$ consists of a clique and its antipodal companions; and otherwise, $\mathcal{S}$ consists of ordinary Henson constraints. (Note here that for $C = 2\delta + 1$ we are in case (II), with $K_1 + K_2 = \delta$, and then Case (IIB) is excluded; that is, $C' = 2\delta + 2$ and this reduces to the antipodal case.)

3. If $K_1 < \infty$ and $C > 2\delta + K_1$ (case (III)), then
   — If $K_1 = \delta$ then $\mathcal{S}$ is empty;
   — If $C = 2\delta + 2$ then $\mathcal{S}$ is empty.

11C. 4-Triviality

Our intent in the next three chapters is to give an explicit classification of the 3-constrained amalgamation classes $\mathcal{A}_3$ which correspond to metrically homogeneous graphs of generic type, conducting the analysis in such a way that it also gives the classification of the classes

$$\mathcal{A}_3 \cap \mathcal{A}_H$$

with $\mathcal{A}_3$ 3-constrained and $\mathcal{A}_H$ of Henson type, and somewhat more. We claim of course that these are the classes

$$\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$$

with admissible parameters.

We show in Chapter 12 that admissibility is a sufficient condition for the amalgamation property.

Proposition 11.5. Let $(\delta, K_1, K_2, C_0, C_1)$ be an admissible sequence of parameters (in particular, $\delta \geq 3$). Then the associated class

$$\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$$

is an amalgamation class.

For the converse, we introduce the notion of 4-*triviality,* or more generally k-triviality, as follows.

DEFINITION 11.6. A metrically homogeneous graph Γ is k-*trivial* if any graph of order k which does not embed in Γ contains either a forbidden triangle, or a Henson constraint (in the antipodal case, this means an antipodal Henson constraint).

We apply similar terminology to classes of finite structures.

The point is that our classes $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ are k-trivial for all k, and that for $k = 4$ this property, which is broader than 3-constraint, is already sufficient for a portion of the analysis.

Proposition 11.7 (Strong Converse to 11.5). If $\mathcal{A}$ is a 4-trivial amalgamation class corresponding to a metrically homogeneous graph, then the set of triangles excluded by $\mathcal{A}$ is $\mathcal{T}(\delta, K_1, K_2, C_0, C_1)$ for some admissible sequence of parameters.

From this point of view, the two points that would need to be proved in order to arrive at a complete classification of the metrically homogeneous graphs are the following.

Problem (cf. §1G).

(I) Show that a countable metrically homogeneous graph of generic type is 4-trivial.
(II) Show (with the assistance of Proposition 11.7) that a 4-trivial countable metrically homogeneous graph is k-trivial for all k.

One can reduce (I) further by considering the ways in which 4-triviality is actually applied here. We do not use the full force of that hypothesis.

To conclude this section, we repeat the conjectured classification of metrically homogeneous graphs of generic type, in its most explicit form.

GENERIC TYPE (EXPLICIT FORM)

The following are the known (and conjecturally, all) graphs of generic type.

- ⋆ The antipodal graphs of Henson type $\Gamma^{\delta}_{a,n}$ with $\delta \geq 4$ and $3 \leq n < \infty$ (Definition 1.15).
- ⋆ The graphs $\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ with $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ an admissible sequence of parameters.

The antipodal graphs without antipodal Henson constraints fall into the second class, which includes the case $\delta = 3$.

And we restate the various results to be proved in Chapters 12, 13, 14, as follows.

THEOREM 11.8. *The amalgamation classes of the form*

$$\mathcal{A}_3 \cap \mathcal{A}_H$$

with $\mathcal{A}_3$ *a 3-constrained amalgamation class and* $\mathcal{A}_H$ *of Henson type are those listed above.*

The point is that any such class is 4-trivial, and we show that the triangle and Henson constraints corresponding to a 4-trivial amalgamation class are those we have described. In particular, classes with no other minimal constraints must lie in our catalog.

11D. Imprimitive metrically homogeneous graphs (Smith's Theorem)

A metric space, graph, or other type of structure is said to be *imprimitive* if it carries a nontrivial congruence (an equivalence relation invariant under automorphisms). In connection with the classification of homogeneous structures, one sometimes places the imprimitive cases on the *non-generic* side of the coin. That is not the case when we deal with metrically homogeneous graphs, since for certain values of the parameters, our graphs $\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ may be bipartite or antipodal. But the distinction between the primitive and imprimitive cases is useful, and much of Chapters 15, 16 is taken up with it, in one way or another.

As it turns out, there are only two kinds of imprimitive metrically homogeneous graphs. This holds in the finite case under the weaker hypothesis of distance transitivity, where the result is called *Smith's theorem* (Smith [1971], Alfuraidan and Hall [2006]). While much of this is not specifically tied to the finite, when we move to the infinite setting and impose the stronger hypothesis of homogeneity, along with the condition $\delta \geq 3$, and vertex degree at least 3, we get a slightly sharper conclusion. This point has already been discussed in Cherlin [2011] as well as in unpublished notes by Amato and Macpherson, a forerunner of the work reported in Amato, Cherlin, and Macpherson [2021].

To begin with, distance transitivity already suffices for the following, mentioned as Fact 7.1 in Cherlin [2011], in slightly different terms.

Fact (Theorem 1.27). Let Γ be a connected distance transitive graph of diameter $\delta \geq 3$, with vertex degrees at least 3, and let E be a nontrivial congruence of Γ.

1. E is either the relation E_2 defined by "$d(x, y)$ is even," or the relation E_δ defined by "$d(x, y)$ is a multiple of δ" (i.e., 0 or δ, with δ finite).

2. If $E = E_2$ then Γ is bipartite

The cases excluded here, namely diameter 2 or with vertex degree 2, give some obvious additional examples: complete multipartite graphs and cycles of composite order.

In the case of infinite diameter, this means that we have only the bipartite case. Graphs of finite diameter δ in which the relation E_δ defines a nontrivial congruence are called *antipodal*. In the metrically homogeneous case, in diameter $\delta \geq 3$, the equivalence classes in question have order 2 (Fact 1.28, repeated below).

The topic of Chapter 15 is local analysis: what can be said about the structure of Γ_i when Γ is metrically homogeneous of generic type and Γ_i contains at least one edge? We will show that this graph is metrically homogeneous (and in particular, connected), of generic type, and primitive, with certain obvious exceptions (Theorem 1.32, §1G). In this analysis, it is naturally very helpful to understand what the imprimitive possibilities actually are (Smith's Theorem).

In Chapter 16 we take up the reduction of the bipartite case to simpler cases (Theorem 1.30, §1F), which is in a sense half of the imprimitive case, but the easier half.

It would of course be very nice to complete the treatment of the imprimitive case, under a suitable inductive hypothesis, by dealing similarly with the antipodal case. But we do not expect a direct reduction of the kind found in the bipartite case (or in the antipodal case of finite distance transitive graphs).

Some information about each of the imprimitive cases is found in Cherlin [2011]. We will recall what is known in the bipartite case in Chapter 16. We recall for the reader's convenience the basic facts concerning the antipodal case, quoted as Fact 1.28 in Chapter 1.

Fact (Cherlin [2011, Theorem 11]). Let Γ be a metrically homogeneous and antipodal graph of diameter $\delta \geq 3$. Then for each vertex $u \in \Gamma$, there is a unique vertex $u' \in \Gamma$ at distance δ from u, and we have the "antipodal law"

$$d(u, v) = \delta - d(u', v) \text{ for } u, v \in \Gamma.$$

In particular, the map $u \mapsto u'$ is a central involution in $\mathrm{Aut}(\Gamma)$.

The classification of the antipodal graphs of diameter 3 is known, and will be useful to us later, as it diverges from the case of larger diameter. This is given both in Cherlin [2011] and in unpublished notes of Amato and Macpherson which cover a number of aspects of the case of diameter 3, including Smith's Theorem and the treatment of the antipodal case.

Fact 11.9 (Cherlin [2011, Theorem 15]). Any countable metrically homogeneous antipodal graph of diameter 3 is an antipodal double cover of one of the following.

(a) The pentagon (5-cycle);
(b) The product $K_3 \square K_3$ of two 3-cliques;
(c) An independent set I_n $(n \leq \infty)$
(d) The random graph G_∞.

We next turn to the issue of the existence of amalgamation classes with admissible parameters.

11E. Metrically homogeneous graphs: status

In view of Fact 1.18, p. 22, and the structure of the known metrically homogeneous graphs of generic type, the problem of classifying the metrically homogeneous graphs reduces to the following.

Problem 3. Complete the classification of the countable metrically homogeneous graphs of generic type: show that they are defined either by an admissible mixture of triangle and ordinary Henson constraints with $C > 2\delta + 1$, or by a mixture of triangle constraints and antipodal Henson constraints with $C = 2\delta + 1$.

This problem then breaks up into two pieces.

Problem (Problem 2, p. 33). Let $\mathcal{A}$ be an amalgamation class associated with a countable metrically homogeneous graph of generic type. Show the following.

(I) The triangle constraints associated with $\mathcal{A}$ are those of some amalgamation class $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$, and ...
(II) in fact $\mathcal{A} = \mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ for the associated class of Henson constraints $\mathcal{S}$.

We now know that the natural approach to the first part is to aim at 4-triviality, or at the instances of 4-triviality used in our arguments.

The approach to be taken to the second part is less clear. It has been worked out here in the bipartite case, and is given in Amato, Cherlin, and Macpherson [2021] in the diameter 3 case.

Recently the line of Amato, Cherlin, and Macpherson [2021] has suggested a concrete approach to the problem in general which involves not only induction on the diameter of the graph in question, but induction on the diameter of the particular configurations one seeks to embed in that graph. We will discuss our current view of this in the appendix (§18B.1).

CHAPTER 12

ADMISSIBILITY ALLOWS AMALGAMATION

In the previous chapter we gave a catalog of the known metrically homogeneous graphs. However, we have not yet proved the existence of all of the graphs in our catalog. That point will be dealt with in the present chapter.

THEOREM 12.1 (Main Thm., Part I (Existence)). *For each choice of admissible parameters $\delta, K_1, K_2, C_0, C_1, \mathcal{S}$, the corresponding family of finite metric spaces*

$$\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$$

has the amalgamation property.

In the following two chapters we will also prove a strong converse to this statement.

When we first worked out the precise definition of admissibility we began with the converse problem. More precisely, we first studied the families $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ determined by constraints on triangles, and worked out the numerical conditions which follow from the assumption of amalgamation. After finding various explicit necessary numerical conditions we then looked for an amalgamation procedure for the corresponding classes.

The notion of admissibility involves three distinct cases divided into a number of distinct subcases, and the procedure we use to amalgamate within a class $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ varies similarly, though not according to precisely the same case division.

An amalgamation strategy calculates the distance r which should be used to complete a 2-point amalgamation problem, in terms of the metric structure of the two factors.

We will give an overview of all possible amalgamation strategies for finite metric spaces generally, then single out three strategies which are particularly useful, and finally arrive at a general amalgamation strategy for the classes of interest here, specified in most cases by indicating which of the alternatives should be adopted in each case, or in some cases, using one of the parameters K_1 or K_2 (or something similar) as the appropriate value of a distance, when these parameters are particularly well placed relative to the other potential candidates. (Later combinatorial work focuses on more canonical amalgamation and completion procedures, discussed in the Appendix to this volume.)

Let us first establish our general frame of reference.

12A. 2-Point amalgamation in finite metric spaces

The amalgamation property reduces to the following, which is much easier to check in practice.

DEFINITION 12.2. A 2-*point amalgamation problem* consists of a triple of metric spaces A_0, A_1, A_2 and isometric inclusions $A_0 \subseteq A_i$ for $i = 1, 2$, where $A_i \setminus A_0$ consists of a single point a_i for $i = 1, 2$.

A class $\mathcal{A}$ of finite metric spaces has the *2-point amalgamation property* if any 2-point amalgamation problem in $\mathcal{A}$ has an amalgam in $\mathcal{A}$.

We observe that since $\mathcal{A}$ is closed under isomorphism and substructure, the 2-point amalgamation property for $\mathcal{A}$ implies the full amalgamation property, as any isometric embedding may be treated as an inclusion, and one may amalgamate in stages.

Furthermore, any solution to a 2-point amalgamation problem may be taken to consist of the union $A_1 \cup A_2$ with some distance r assigned to the pair a_1, a_2, with the proviso that this distance may be 0, in which case the points a_1, a_2 are to be identified (i.e., if A_1 and A_2 are isometric over A_0, then we may identify them).

So everything comes down to the determination of an appropriate value for $r = d(a_1, a_2)$, which in practice will need to lie between certain upper and lower bounds, and possibly satisfy a parity constraint; these conditions must be shown to be compatible. An *amalgamation strategy* is thus a formula for an appropriate value (or, more generally, a range of values) for r in terms of the structure of A_1 and A_2.

One first makes an overview of all amalgamation strategies in the category of finite metric spaces. The following is easily verified.

Lemma 12.3. *Let δ, P be fixed. Let A_0, A_1, A_2 be a 2-point amalgamation problem with all distances bounded by δ and with the perimeters of all triangles bounded by P. Set*

$$\begin{aligned} r^+ &= \min_{x \in A_0} (d(a_1, x) + d(a_2, x)); \\ r^- &= \max_{x \in A_0} (|d(a_1, x) - d(a_2, x)|); \\ \tilde{r} &= \min_{x \in A_0} (P - [d(a_1, x) + d(a_2, x)]) \end{aligned}$$

Then

1. *$r^- \le r^+$ and $\tilde{r}$; $r^- < \delta$; $r^+ > 1$.*
2. *For $r \in \mathbb{R}$, the following are equivalent.*
   (a) *The extension of the metrics to $A_1 \cup A_2$ by $d(a_1, a_2) = r$ is a pseudo-metric space with all distances bounded by δ and perimeter bounded by P*
   (b) *$r^- \le r \le \min(r^+, \tilde{r}, \delta)$.*

In particular, if we ignore the bound on δ and on P, then $d(a_1, a_2)$ can take on any value in the interval from r^- to r^+. In the presence of Henson constraints, we prefer to avoid using the values $r = 1$ or $r = \delta$. Here the relations $r^- < \delta$ and $r^+ > 1$ are helpful.

Proof. Most of this is well known and all of it is straightforward. We will confine ourselves to a discussion of the role of the parameter P.

We first show $r^- \le \tilde{r}$. So we consider $x, y \in A_0$, and we claim

$$d(a_1, x) - d(a_2, x) \le P - (d(a_1, y) + d(a_2, y))$$

Indeed, we have

$$d(a_2, y) - d(a_2, x) \le d(x, y) \le P - (d(a_1, x) + d(a_1, y))$$

which amounts to the same thing.

Now evidently the inequality $r \le \tilde{r}$ is precisely what is needed to bound the perimeters of triangles containing the pair a_1, a_2, and as far as P is concerned, the rest is clear. □

We note that a possible value of P associated with parameters (K_1, K_2, C_0, C_1) will be $\max(C_0, C_1) - 2$. In particular when C_0, C_1 differ by 1, we would take $P = \min(C_0, C_1) - 1$. On the other hand, in the bipartite case (when we forbid all triangles of odd perimeter) the relevant bound is $P = C_0 - 2$.

12B. The amalgamation strategy and the bipartite case

An amalgamation strategy for the class $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ has been presented as Table 2.2 in §2B. This will be given again explicitly in the statement of Proposition 12.14, which deals with the case $\mathcal{S} = \emptyset$. The proposed strategy is not completely sound when $\mathcal{S}$ is nonempty, so we will discuss separately the slight variations which arise in certain cases.

Recall that admissible parameter sequences come in three types: (I) bipartite; (II) $C \leq 2\delta + K_1$; (III) $C > 2\delta + K_1$. In cases (II,III) we may speak of C as "low" or "high" respectively. We tend to think of the high case as the less extreme case (the antipodal case is the most extreme of the low cases).

In showing that the amalgamation strategy is sound (at least when $\mathcal{S}$ is empty), the bipartite case is easily dealt with in full with no restriction on $\mathcal{S}$, while the other two types are best handled together, even though the details of the strategy vary according to the type.

In the bipartite case, that is, the case in which there are no triangles of odd perimeter (Type I), there is very little to check.

We assume that there are no triangles of odd perimeter. So the only parameters of interest are δ, C_0, and also $\mathcal{S}$. The upper bound P on perimeters of triangles is $C_0 - 2$. Furthermore, if $\mathcal{S}$ is nonempty, then it consists of a δ-anticlique.

We fix a 2-point amalgamation problem (A_0, A_1, A_2) with $A_i = A_0 \cup \{a_i\}$ for $i = 1, 2$. As there are no triangles of odd perimeter, it is easy to see that the parity ϵ of the expression

$$d(a_1, x) + d(a_2, x)$$

is independent of the choice of $x \in A_0$, and in particular r^- and r^+ both have this parity.

As long as r has the parity of r^- and lies in the range

$$r^- \leq r \leq \min(r^+, \delta, \tilde{r})$$

the amalgam with $d(a_1, a_2) = r$ (identifying vertices if $r = 0$) will have no triangles of odd perimeter, and will satisfy the constraints imposed by δ and C_0. If $\mathcal{S}$ is nonempty, then $\mathcal{S}$ consists of a δ-anticlique, so we impose the condition $r < \delta$ in this case.

In particular, the value $d(a_1, a_2) = r^-$ is always suitable for amalgamation in Type I.

Thus we have the following.

Lemma 12.4 (Theorem 12.1, Bipartite Case). *Suppose that*

$$(\delta, \infty, 0, C_0, C_1, \mathcal{S})$$

is an admissible sequence of parameters. Then $\mathcal{A}^\delta_{\infty,0,C_0,C_1,\mathcal{S}}$ *is an amalgamation class, so the corresponding metrically homogeneous graph* $\Gamma^\delta_{\infty,0,C_0,C_1,\mathcal{S}}$ *exists; this is the generic bipartite graph of diameter* δ *with all triangles having perimeter at most* $C_0 - 2$.

We recall that the phrase "the corresponding metrically homogeneous graph" is a little glib: what clearly exists is the Fraïssé limit, which is a metric space rather than a graph. But it is the metric space associated with the path metric in the distance 1 graph. The notation $\Gamma^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ will be used for either of these structures. In any context where the technical details matter, the metric point of view is to be preferred.

The treatment of amalgamation in Cases (II) and (III) is much more complex.

12C. Statements of the main lemmas

The essential calculations regarding the amalgamation property come into the proofs of the following nine lemmas. Here we give only the *statements* of these lemmas. In the next subsection we will make the necessary calculations which prove these lemmas, and then summarize everything in tabular form so that we can trace through our amalgamation procedure on the basis of these lemmas.

It will be convenient to write

$$\mathcal{A}^\delta_{K_1,K_2,C,C'} \text{ for } \mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$$

with $C = \min(C_0, C_1)$, $C' = \max(C_0, C_1)$). We also use the following abbreviated forms.

$$\mathcal{A}^\delta_{K_1,K_2,C} \text{ for } \mathcal{A}^\delta_{K_1,K_2,C,C+1}\text{; and}$$
$$\mathcal{A}^\delta_{K_1,K_2} \text{ for } \mathcal{A}^\delta_{K_1,K_2,3\delta+1,3\delta+2}.$$

Correspondingly we will call a parameter sequence

$$(K_1, K_2, C, C'),\ (K_1, K_2, C), \text{ or } (K_1, K_2),$$

admissible if the corresponding sequence

$$(K_1, K_2, C_0, C_1),\ (K_1, K_2, C, C+1), \text{ or } (K_1, K_2, 3\delta+1, 3\delta+2)$$

(respectively) is admissible. Note in particular that when we write

$$\mathcal{A}^{\delta}_{K_1,K_2,C,C'}$$

rather than $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$, we have $C < C'$ with C, C' of differing parity.

LEMMA 12.5. *Let $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2}$ with $1 \le K_1 \le K_2 \le \delta$. Let $A_i = A_0 \cup \{a_i\}$ $(i = 1, 2)$ be a 2-point amalgamation problem in $\mathcal{A}$, $A = A_1 \cup A_2$, and d_i the metric on A_i. Suppose that*

$$r^+ \le K_2.$$

Let d be the symmetric extension of $d_1 \cup d_2$ to A defined by

$$d(a_1, a_2) = r^+.$$

Then $(A, d) \in \mathcal{A}$.

LEMMA 12.6. *Let $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C,C'}$ with $1 \le K_1 \le K_2 \le \delta$. Let $A_i = A_0 \cup \{a_i\}$ $(i = 1, 2)$ be a 2-point amalgamation problem in $\mathcal{A}$, $A = A_1 \cup A_2$, and d_i the metric on A_i. Suppose that*

$$r^+ < K_2 \text{ and } C \ge 2\delta + K_2.$$

Let d be the symmetric extension of $d_1 \cup d_2$ to A defined by

$$d(a_1, a_2) = r^+.$$

Then $(A, d) \in \mathcal{A}$.

LEMMA 12.7. *Let $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C,C'}$, with admissible parameters δ, K_1, K_2, C, C', and $K_1 < \infty$. Let $A_i = A_0 \cup \{a_i\}$ $(i = 1, 2)$ be a 2-point amalgamation problem in $\mathcal{A}$, $A = A_1 \cup A_2$, and d_i the metric on A_i. Suppose that one of the following holds.*

A. $r^- \ge K_1$ and $C \le 2\delta + K_1$; or
B. $r^- > K_1$.

Let d be the symmetric extension of $d_1 \cup d_2$ to A defined by

$$d(a_1, a_2) = r^-.$$

Then $(A, d) \in \mathcal{A}$.

LEMMA 12.8. *Let $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C}$ with admissible parameters δ, K_1, K_2, C, C', with $K_1 < \infty$ and $C = 2K_1 + 2K_2 + 1$. Let $A_i = A_0 \cup \{a_i\}$ $(i = 1, 2)$ be a 2-point amalgamation problem in $\mathcal{A}$, $A = A_1 \cup A_2$, and d_i the metric on A_i. Suppose that*

$$\tilde{r} \le \min(K_2, r^+).$$

where $\tilde{r}$ is defined using the bound on perimeter $P = C - 1$. Let d be the symmetric extension of $d_1 \cup d_2$ to A defined by

$$d(a_1, a_2) = \tilde{r}.$$

Then $(A, d) \in \mathcal{A}$.

LEMMA 12.9. *Let $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C}$ with admissible parameters δ, K_1, K_2, C (so $C' = C + 1$). Let $A_i = A_0 \cup \{a_i\}$ $(i = 1, 2)$ be a 2-point amalgamation problem in $\mathcal{A}$, $A = A_1 \cup A_2$, and d_i the metric on A_i. Suppose that one of the following holds.*

1. *$r^+ \leq K_2$ and $C = 2K_1 + 2K_2 + 1$;*
2. *$r^+ \leq K_1$ and $C > 2\delta + K_1$;*
3. *$r^+ < K_2$ and $C \geq 2\delta + K_2$.*

Let d be the symmetric extension of $d_1 \cup d_2$ to A defined by

$$d(a_1, a_2) = min(r^+, \tilde{r}),$$

where $\tilde{r}$ is defined using the bound on perimeter $P = C - 1$.

Then $(A, d) \in \mathcal{A}$.

LEMMA 12.10. *Let $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C}$ with admissible parameters δ, K_1, K_2, C, and with $C = 2K_1 + 2K_2 + 1$. Let $A_i = A_0 \cup \{a_i\}$ $(i = 1, 2)$ be a 2-point amalgamation problem in $\mathcal{A}$, $A = A_1 \cup A_2$, and d_i the metric on A_i. Suppose that*

$$K_1 \leq \min(r^+, \tilde{r}) \text{ and } r^- \leq K_2$$

where $\tilde{r}$ is defined in terms of the bound $P = C - 1$ on perimeters.

Then

$$\max(K_1, r^-) \leq \min(r^+, \tilde{r}, 2K_2 - r^-, \delta),$$

and for any r between these two bounds, if d is the symmetric extension of $d_1 \cup d_2$ to A defined by

$$d(a_1, a_2) = r$$

then $(A, d) \in \mathcal{A}$.

LEMMA 12.11. *Let $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C,C'}$ with $K_1 = K_2$, $C = 4K_2 + 1 = 2\delta + K_2$. Let $A_i = A_0 \cup \{a_i\}$ $(i = 1, 2)$ be a 2-point amalgamation problem in $\mathcal{A}$, $A = A_1 \cup A_2$, and d_i the metric on A_i. Suppose that*

$$r^- \leq K_2 \leq r^+.$$

Let d be the symmetric extension of $d_1 \cup d_2$ to A defined by

$$d(a_1, a_2) = \begin{cases} K_2 - 1 & \text{if there is } v \in A_0 \text{ with } d(a_1, v) = d(a_2, v) = \delta; \\ K_2 & \text{otherwise.} \end{cases}$$

Then $(A, d) \in \mathcal{A}$.

LEMMA 12.12. *Let $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C}$ with admissible parameters satisfying*

$$C > 2\delta + K_1$$

Let $A_i = A_0 \cup \{a_i\}$ $(i = 1, 2)$ be a 2-point amalgamation problem in $\mathcal{A}$, $A = A_1 \cup A_2$, and d_i the metric on A_i. Suppose that

$$r^- \leq K_1 < r^+$$

Let d be the symmetric extension of $d_1 \cup d_2$ to A defined by

$$d(a_1, a_2) = \begin{cases} K_1 + 1 & \text{if } K_1 + 2K_2 = 2\delta - 1, \text{ and there is} \\ & \text{some } v \in A_0 \text{ with } d(a_1, v) = d(a_2, v) = \delta; \\ K_1 & \text{otherwise} \end{cases}$$

Then $(A, d) \in \mathcal{A}$.

LEMMA 12.13. *Let $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C,C'}$ with admissible parameters satisfying*

$$C > 2\delta + K_1,\ C \geq 2\delta + K_2, \text{ and if } 3K_2 = 2\delta \text{ then } C > 2\delta + K_2.$$

Let $A_i = A_0 \cup \{a_i\}$ $(i = 1, 2)$ be a 2-point amalgamation problem in $\mathcal{A}$, $A = A_1 \cup A_2$, and d_i the metric on A_i. Suppose that

$$\begin{aligned} & r^- \leq K_1 \text{ and} \\ & \min(K_2, C - 2\delta - 1) \leq r^+. \end{aligned}$$

Let d be the symmetric extension of $d_1 \cup d_2$ to A defined by

$$d(a_1, a_2) = \min(K_2, C - 2\delta - 1).$$

Then $(A, d) \in \mathcal{A}$.

12D. Proofs of the main lemmas

We prove Lemmas 12.5 through 12.13 here. We will briefly recall the hypotheses and the proposed amalgamation procedure in each case.

One point to observe as we go along is that the assigned distance $r = d(a_1, a_2)$ should be bounded by δ. In most cases, the choice of r specified, and the hypotheses in force, give either $r \le r^-$ or the bound $r \le K_2$, so in such cases we have $r \le \delta$.

The exceptions to this are found in Lemmas 12.10 and 12.12. In Lemma 12.10 we give a range of values for r with the upper bound δ included explicitly. In Lemma 12.12 we have $r = K_1$ or $K_1 + 1$, with the case $r = K_1 + 1$ occurring only when $K_1 + 2K_2 = 2\delta - 1$, so again $r \le \delta$.

Lemma 12.5

Hypothesis: $r^+ \le K_2$.
Amalgamation: $d(a_1, a_2) = r^+$.

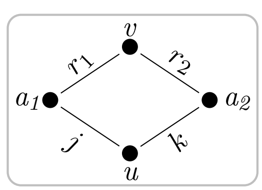
Proof of Lemma 12.5. We must check that for any u in A_0, the triangle (a_1, a_2, u) is permitted. The only constraints in force here are those associated to the parameters K_1 and K_2, which concern only triangles of *odd* perimeter.

Let $j = d(a_1, u)$, $k = d(a_2, u)$, and set $p_u = r^+ + j + k$, the perimeter of the triangle (a_1, a_2, u). We may suppose that the perimeter p_u is odd.

Fix $v \in A_0$ with

$$r^+ = d_1(a_1, v) + d_2(a_2, v)$$

Set $r_\ell = d(a_\ell, v)$ for $\ell = 1, 2$. Then $r^+ = r_1 + r_2$ and the perimeter of (a_1, u, a_2, v) is again p_u, which is odd. Therefore one of the triangles (a_1, u, v) or (a_2, u, v) has odd perimeter, at most p_u. We may suppose the triangle in question is (a_1, u, v).



As the perimeter of (a_1, u, v) is odd, it is at least $2K_1 + 1$. Hence $p_u \geq 2K_1 + 1$. So it suffices to check the inequalities corresponding to K_2, namely:

(1) $r^+ + j \leq 2K_2 + k$; (2) $r^+ + k \leq 2K_2 + j$; (3) $j + k \leq 2K_2 + r^+$.

(1,2): By assumption $r^+ \leq K_2$. Since $j \leq r^+ + k$ we have

$$r^+ + j \leq 2r^+ + k \leq 2K_2 + k,$$

and similarly $r^+ + k \leq 2K_2 + j$.

(3): As the triangle (a_1, u, v) has odd perimeter, we have the constraint

$$j + d(u, v) \leq 2K_2 + r_1$$

and therefore

$$j + k \leq j + d(u, v) + r_2 \leq 2K_2 + r_1 + r_2 = 2K_2 + r^+$$

and the final inequality holds as well. □

Lemma 12.6

Hypotheses: $C \geq 2\delta + K_2$; $r^+ < K_2$.

Amalgamation: $d(a_1, a_2) = r^+$.

PROOF OF LEMMA 12.6. By Lemma 12.5 we have $(A, d) \in \mathcal{A}^{\delta}_{K_1,K_2}$.

On the other hand for $u \in A_0$ and p_u the perimeter of (a_1, a_2, u), we have

$$p_u \leq 2\delta + r^+ < 2\delta + K_2 \leq C$$

and our claim follows. □

Lemma 12.7

Hypotheses: Either

(A) $r^- \geq K_1$ and $C \leq 2\delta + K_1$; or
(B) $r^- > K_1$.

Amalgamation: $d(a_1, a_2) = r^-$.

PROOF OF LEMMA 12.7. If $r^- = 0$ we identify a_1 and a_2. So we may suppose that $r^- > 0$ and thus d is a proper metric.

We must show that all resulting triangles (a_1, a_2, u) with $u \in A_0$ are permitted. This relies heavily on the numerical constraints (admissibility), and as there are a number of cases to consider the verification will be relatively long.

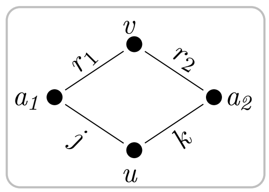
We fix $v \in A_0$ with $r^- = |d(a_1, v) - d(a_2, v)|$. We may suppose

$$r^- = d(a_1, v) - d(a_2, v)$$

We write r_ℓ for $d(a_\ell, v)$, for $\ell = 1, 2$.

Set $j = d(a_1, u)$, $k = d(a_2, u)$, $p_u = r^- + j + k$. Let p_ℓ be the perimeter of (a_ℓ, u, v) for $\ell = 1, 2$. Note that

$$p_u \equiv p_1 + p_2 \pmod 2.$$



I. The constraints corresponding to C_0, C_1.

We show first that $p_u < C_\ell$ where $\ell = 0$ or 1 is the parity of p_u. Now

$$r^- \le \tilde{r}$$

where $\tilde{r}$ is defined relative to the bound $C - 1$ if $C' = C + 1$, or the bound $C' - 1$ if $C' > C + 1$. Thus we have $p_u < C'$ in any case, and $p_u < C$ if $C' = C + 1$. So we need to show that in the case

$$\begin{aligned} &p_u \equiv C \pmod 2; \\ &C' > C + 1, \end{aligned}$$

we also have $p_u < C$.

Since the parameters are admissible, there are two possible cases in which we have $C' > C + 1$, as follows.

(12.1) $$K_1 = K_2,\ C = 4K_2 + 1 = 2\delta + K_2; \text{ or}$$

(12.2) $$C \ge 2\delta + K_2 \text{ and } C > 2\delta + K_1$$

Then

$$p_u = r_1 - r_2 + j + k \le r_1 + d(u, v) + j = p_1$$

If $p_1 < C$ then $p_u < C$. If $p_1 \ge C$, then we have $p_1 \equiv C' \not\equiv C \pmod 2$. Since $p_1 + p_2 \equiv p_u \equiv C \pmod 2$, the perimeter p_2 is odd.

Also, from $p_1 \ge C$ we get

$$2\delta + K_2 \le C \le p_1 \le 2\delta + d(u, v)$$

and thus

$$d(u,v) \geq K_2$$

Now since p_2 is odd, the triangle (a_2,u,v) satisfies the constraint correspoinding to K_2, namely

$$k + d(u,v) < 2K_2 + r_2.$$

Thus

$$k - r_2 < 2K_2 - d(u,v) \leq K_2.$$

Then

$$p_u = r_1 - r_2 + j + k < r_1 + j + K_2 \leq 2\delta + K_2 \leq C$$

and we have the required bound on the perimeter in this case as well.

This completes the verification of the constraints corresponding to C_0, C_1.

II. The constraints corresponding to K_1, K_2.

For the relevant constraints, see Definition 1.20.

Now we check that the triangle (a_1,a_2,u) is in $\mathcal{A}^{\delta}_{K_1,K_2}$. We may suppose that the perimeter p_u is odd. Then $p_u \geq 2r^- + 1$. As we assume $r^- \geq K_1$, it follows that $p_u \geq 2K_1 + 1$. So it remains to check the inequalities corresponding to K_2.

(1) $r^- + j \leq 2K_2 + k$; (2) $r^- + k \leq 2K_2 + j$; (3) $j + k \leq 2K_2 + r^-$.

We begin with the last.

(3): If $C \leq 2\delta + K_1$, then by admissibility $C = 2K_1 + 2K_2 + 1$ is odd and therefore as shown above $p_u < C$. Since $r^- \geq K_1$, we have

$$j + k \leq 2K_1 + 2K_2 - r^- \leq 2K_2 + r^-.$$

If on the other hand $C > 2\delta + K_1$, then $K_1 + 2K_2 \geq 2\delta - 1$, and in this case we have assumed $r^- > K_1$, so

$$j + k \leq 2\delta \leq K_1 + 2K_2 + 1 \leq 2K_2 + r^-.$$

We turn to the inequalities $(1,2)$. As p_u is odd, either p_1 or p_2 is odd, and then the corresponding triangle (a_ℓ,u,v) satisfies the constraints imposed by K_2. We treat these possibilities separately.

If p_1 is odd—

Then

$$r^- + j = r_1 + j - r_2 \le 2K_2 + d(u,v) - r_2 \le 2K_2 + k;$$
$$r^- + k = r_1 + k - r_2 \le r_1 + d(u,v) \le 2K_2 + j.$$

Thus $(1,2)$ both hold in this case.

If p_2 is odd—

(2): We have

$$\begin{aligned} r^- + k + d(u,v) &\le r^- + 2K_2 + r_2 \\ &= 2K_2 + r_1 \le 2K_2 + d(u,v) + j' \\ r^- + k &\le 2K_2 + j. \end{aligned}$$

(1): Here we have several combinations of admissible parameters to consider, as follows.

(A) $C = 2K_1 + 2K_2 + 1$; $C' = C + 1$; or
(B) $K_1 = K_2$, $C = 4K_2 + 1 = 2\delta + K_2$; or
(C) $C > 2\delta + K_1$ and $K_1 + 2K_2 \ge 2\delta - 1$.

Suppose first that

(A) $$C = 2K_1 + 2K_2 + 1;\ C' = C + 1$$

Then as p_2 is odd, we have $p_2 \ge 2K_1 + 1$, and as p_1 is even we have $p_1 \le 2K_1 + 2K_2$, so we take the difference, getting

$$\begin{aligned} p_1 - p_2 &= (r_1 + j) - (r_2 + k) = r^- + j - k \\ &\le (2K_1 + 2K_2) - (2K_1 + 1) = 2K_2 - 1; \\ r^- + j &< 2K_2 + k. \end{aligned}$$

Now suppose that

(B) $$K_1 = K_2,\ C = 4K_2 + 1 = 2\delta + K_2.$$

As $p_2 \ge 2K_1 + 1 = 2K_2 + 1$ we have $r_2 + k \ge p_2/2 > K_2$. Hence

$$r^- + j = r_1 + j - r_2 \le 2\delta - r_2 = 3K_2 + 1 - r_2 < 2K_2 + k.$$

Finally, suppose that

(C) $$C > 2\delta + K_1 \text{ and } K_1 + 2K_2 \ge 2\delta - 1.$$

As $p_2 \geq 2K_1 + 1$ and $r^- < \delta$ we find

$$\begin{aligned} r_2 + k &\geq p_2/2 > K_1; \\ r^- + j &\leq 2\delta - 1 \leq K_1 + 2K_2 \\ &\leq 2K_2 + (K_1 + 1) - r_2 \leq 2K_2 + (r_2 + k) - r_2 = 2K_2 + k. \end{aligned}$$

This completes the verification of inequality (1) for p_2 odd. □

Lemma 12.8

Hypotheses: $C = 2K_1 + 2K_2 + 1$, $C' = C + 1$ and $\tilde{r} \leq \min(K_2, r^+)$.
Amalgamation: $d(a_1, a_2) = \tilde{r}$.

PROOF OF LEMMA 12.8. We have

$$r^- \leq \tilde{r} \leq r^+$$

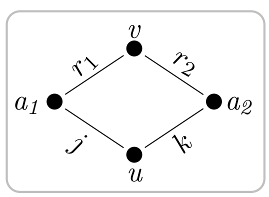
by hypothesis, and thus d is at least a pseudo-metric. As usual we may suppose that $\tilde{r} > 0$ and thus we have a metric. In view of the definition of $\tilde{r}$ and the hypothesis $C' = C + 1$, our metric respects the bound on perimeters.

So we must show that all resulting triangles (a_1, a_2, u) with $u \in A_0$ are in $\mathcal{A}^{\delta}_{K_1,K_2}$. Set $j = d(a_1, u)$, $k = d(a_2, u)$, $p_u = \tilde{r} + j + k$. We may suppose that p_u is odd.

Choose $v \in A_0$ so that

$$\tilde{r} = C - 1 - [d(a_1, v) + d(a_2, v)]$$

let $r_\ell = d(a_\ell, v)$ for $\ell = 1, 2$.



As the perimeter $C - 1$ of (a_1, a_2, v) is even, and p_u is odd, the perimeter of (a_1, u, a_2, v) is odd. Hence one of the triangles (a_1, u, v) or (a_2, u, v) has odd perimeter. We may suppose that (a_1, u, v) has odd perimeter.

We show first that

$$p_u \geq 2K_1 + 1.$$

As (a_1, u, v) has odd perimeter, we have the constraint

$$r_1 + d(u, v) \leq 2K_2 + j.$$

Hence

$$\begin{aligned}(C-1) - \tilde{r} = r_1 + r_2 \leq r_1 + d(u,v) + k &\leq 2K_2 + j + k;\\ C - 1 = 2K_1 + 2K_2 &\leq 2K_2 + \tilde{r} + j + k;\\ 2K_1 &\leq p_u,\end{aligned}$$

which suffices as p_u is odd.

Now we deal with the inequalities

$$(1)\ \tilde{r} + j \leq 2K_2 + k;\ (2)\ \tilde{r} + k \leq 2K_2 + j;\ (3)\ j + k \leq 2K_2 + \tilde{r}.$$

Since $\tilde{r} \leq K_2$, inequalities $(1, 2)$ are immediate as in the proof of Lemma 12.5.

(3): As the triangle (a_1, u, v) has odd perimeter we have

$$r_1 + j \leq 2K_2 + d(u, v).$$

The triangle (a_2, u, v) has perimeter less than C, so

$$\begin{aligned}(C - \tilde{r}) + j + k &= 1 + r_1 + r_2 + j + k\\ &\leq 1 + 2K_2 + d(u,v) + r_2 + k\\ &\leq 2K_2 + C,\end{aligned}$$

and thus $j + k \leq 2K_2 + \tilde{r}$, as required. $\square$

Lemma 12.9

Hypotheses: $C' = C + 1$, and one of the following holds.

A. $r^+ \leq K_2$ and $C = 2K_1 + 2K_2 + 1$;
B. $r^+ \leq K_1$ and $C > 2\delta + K_1$;
C. $r^+ < K_2$ and $C \geq 2\delta + K_2$.

Amalgamation: $d(a_1, a_2) = \min(r^+, \tilde{r})$.

PROOF OF LEMMA 12.9. Let $r = \min(r^+, \tilde{r})$. As $r \leq \tilde{r}$, all triangles in the amalgam $A = (A_1 \cup A_2, d)$ have perimeter less than C. So it suffices to show that A belongs to $\mathcal{A}^{\delta}_{K_1,K_2}$. If $r = r^+$, then as $r^+ \leq K_2$, this holds by Lemma 12.5.

In particular, if $C > 2\delta + r^+$ then $r^+ \leq \tilde{r}$ and our claim follows. This covers the cases (B, C), so we need only consider case (A), under the assumption

$$\tilde{r} < r^+.$$

In this case Lemma 12.8 applies. $\square$

Lemma 12.10

Hypotheses: $C = 2K_1 + 2K_2 + 1$, $K_1 \leq \min(r^+, \tilde{r})$, and $r^- \leq K_2$.

Amalgamation: $d(a_1, a_2)$ lies between

$$\max(K_1, r^-) \text{ and}$$
$$\min(r^+, \tilde{r}, 2K_2 - r^-, \delta)$$

(and the interval is in fact nonempty).

PROOF OF LEMMA 12.10. In general we have $r^- \leq \min(r^+, \tilde{r}, \delta)$ and $K_1 \leq K_2$, so the required inequality

$$\max(K_1, r^-) \leq \min(r^+, \tilde{r}, 2K_2 - r^-, \delta)$$

follows from our hypotheses.

Suppose now that $r = d(a_1, a_2)$ lies between the stated bounds. As usual we may suppose that $r > 0$ and thus d is a proper metric. As $r \leq \tilde{r}$, the metric d respects the bound on perimeters.

It remains to show that all triangles (a_1, a_2, u) with $u \in A_0$ are in $\mathcal{A}^\delta_{K_1,K_2}$. Set $j = d(a_1, u)$, $k = d(a_2, u)$, $p_u = r + j + k$. We may suppose that p_u is odd.

As $r \geq K_1$ we have $p_u \geq 2K_1 + 1$. So it suffices to check the inequalities

(1) $r + j \leq 2K_2 + k$; (2) $r + k \leq 2K_2 + j$; (3) $j + k \leq 2K_2 + r$.

We have $j - k \leq r^- \leq 2K_2 - r$ and hence

$$r + j \leq 2K_2 + k.$$

Similarly $r + k \leq 2K_2 + j$. Finally, as $r \leq \tilde{r}$ we have

$$r + j + k \leq C - 1 = 2K_2 + 2K_1 \leq 2K_2 + 2r,$$

and this concludes the proof. □

Lemma 12.11

Hypotheses: $K_1 = K_2$, $C = 4K_2 + 1 = 2\delta + K_2$, and $r^- \leq K_2 \leq r^+$.

Amalgamation: $d(a_1, a_2) = K_2 - 1$ if there is some $v \in A_0$ with $d(a_1, v) = d(a_2, v) = \delta$, and $d(a_1, a_2) = K_2$ otherwise.

PROOF OF LEMMA 12.11. Set $r = d(a_1, a_2) = K_2 - \epsilon$ with $\epsilon = 0$ or 1.

We must first verify the condition

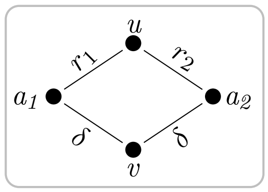
$$r^- \leq r \leq r^+$$

in the case in which $r = K_2 - 1$. So suppose $v \in A_0$ and

$$d(a_1, v) = d(a_2, v) = \delta.$$

By assumption $r^- \leq K_2$, and we claim in this case that $r^- \neq K_2$.

Assuming the contrary, we may suppose we have $u \in A_0$ with $d(a_1, u) - d(a_2, u) = K_2$. Set $r_\ell = d(a_\ell, u)$ for $\ell = 1, 2$, so that $r_1 = r_2 + K_2$. Let p_ℓ be the perimeter of the triangle (a_ℓ, u, v).



Then p_1 is

$$\begin{aligned}\delta + d(u, v) + r_1 &= \delta + d(u, v) + K_2 + r_2 \geq \delta + K_2 + d(a_2, v)\\ &= 2\delta + K_2.\end{aligned}$$

As $p_1 \geq 2\delta + K_2 = C$ and C is odd, it follows that p_1 is even.

Now $p_1 + p_2 \equiv r_1 + r_2 + 2\delta \equiv K_2 \pmod 2$ is odd since $4K_2 + 1 = 2\delta + K_2$. Hence p_2 is odd. Therefore the triangle (a_2, u, v) must satisfy the constraint

$$\delta + d(u, v) \leq 2K_2 + r_2$$

and as p_2 is odd the inequality is strict. Thus

$$\begin{aligned}\delta + d(u, v) &< 2K_2 + r_2 = K_2 + r_1;\\ 2\delta - r_2 \leq \delta + d(u, v) &< K_2 + r_1;\\ 2\delta &< r_1 + K_2 + r_2 = 2r_1,\end{aligned}$$

a contradiction.

Thus $r^- \leq r \leq r^+$ and we have a pseudo-metric d. We may suppose that $r > 0$, and thus d is a metric.

It remains to show that the amalgam $A = (A_1 \cup A_2, d)$ contains no forbidden triangles (a_1, a_2, u) with $u \in A_0$. Let $j = d(a_1, u)$, $k = d(a_2, u)$, $p_u = r + j + k$. Here

$$j + k \leq 2\delta - (1 - \epsilon).$$

We first check the bound on perimeters.

We have

$$p_u = r + j + k \leq r + (2\delta - 1 + \epsilon) < 2\delta + K_2 - 1 < C,$$

so the bound on perimeter is respected.

Next we consider the constraints associated with the parameters K_1, K_2. Recall that $K_1 = K_2$; but as we have two sets of constraints to consider, we will continue to use both notations, bearing in mind that we may also write $r = K_1 - \epsilon$. We may suppose that p_u is odd.

We claim first that $p_u \geq 2K_1 + 1$. If $r = K_1$ this is clear. If $r = K_1 - 1$, then we must consider the possibility

$$j + k = K_1.$$

We have an element $v \in A_0$ with $d(a_1, v) = d(a_2, v) = \delta$, and our assumptions imply that $K_1 = K_2$ is odd. Thus the perimeter $j + k + 2\delta = K_1 + 2\delta$ of (a_1, u, a_2, v) is odd. Hence the perimeter of one of the triangles (a_ℓ, u, v) is odd.

We may suppose that the triangle (a_1, u, v) has odd perimeter. Then

$$\begin{aligned} d(u,v) + d(a_1,v) &\geq d(a_2,v) - k + d(a_1,v) \\ &= 2\delta - k = 2\delta - K_2 + j = 2K_2 + 1 + j > 2K_2 + j, \end{aligned}$$

and the triangle (a_1, u, v) is forbidden, a contradiction.

It remains to check the inequalities

$$(1)\ r + j \leq 2K_2 + k;\ \ (2)\ r + k \leq 2K_2 + j;\ \ (3)\ j + k \leq 2K_2 + r.$$

Now

$$r + j \leq 2r + k \leq 2K_2 + k$$

and similarly $r + k \leq 2K_2 + j$.

(3): For the last inequality we have

$$j + k \leq 2\delta - 1 + \epsilon = 3K_2 - \epsilon = 2K_2 + r.$$

□

Lemma 12.12

Hypotheses: $C > 2\delta + K_1$ and $r^- \leq K_1 < r^+$.

Amalgamation: $d(a_1, a_2) = K_1 + 1$ if $K_1 + 2K_2 = 2\delta - 1$ and there is some $v \in A_0$ with $d(a_1, v) = d(a_2, v) = \delta$; and $d(a_1, a_2) = K_1$ otherwise.

PROOF OF LEMMA 12.12. We must show that for any u in A_0, the triangle (a_1, a_2, u) belongs to $\mathcal{A}^{\delta}_{K_1,K_2,C,C'}$. Let

$$r = d(a_1, a_2) = K_1 + \epsilon; \qquad j = d(a_1, u), k = d(a_2, u);$$
$$p_u = r + j + k.$$

We first check the bound on perimeters.

Now $p_u \leq 2\delta + K_1 + 1 \leq C$, so to violate the bound on perimeter would require $d(a_1, u) = d(a_2, u) = \delta$, and $r = K_1 + 1$. The latter entails $K_1 + 2K_2 = 2\delta - 1$ and then by admissibility we have

$$C > 2\delta + K_1 + 1,$$

so even in this case we have $p_u < C$.

Now we check the constraints relating to K_1 and K_2. We may suppose p_u is odd.

Since $r \geq K_1$, we have $p_u \geq 2K_1 + 1$. So it suffices to check the inequalities

$$(1)\ r + j \leq 2K_2 + k; \ (2)\ r + k \leq 2K_2 + j; \ (3)\ j + k \leq 2K_2 + r.$$

.

(1,2): We claim that $r \leq K_2$. This is clear if $r = K_1$, while if $r = K_1 + 1$ then we have $K_1 + 2K_2 = 2\delta - 1$ and by admissibility also $3K_2 \geq 2\delta$, so $K_2 > K_1$ and again $r \leq K_2$.

Thus

$$r + j \leq 2r + k \leq 2K_2 + k$$

and similarly $r + k \leq 2K_2 + j$.

(3): We have

$$j + k \leq 2\delta \leq 2K_2 + K_1 + 1 \leq 2K_2 + r + 1$$

and hence $j + k \leq 2K_2 + r$ unless

$$j + k = 2\delta = 2K_2 + K_1 + 1 = 2K_2 + r + 1.$$

In this case we have

$$j = k = \delta;$$
$$K_1 + 2K_2 = 2\delta - 1;$$
$$r = K_1,$$

but the first two conditions imply $r = K_1 + 1$. $\square$

Lemma 12.13

Hypotheses:

1. $C > 2\delta + K_1$ and $C \geq 2\delta + K_2$.
2. If $3K_2 = 2\delta$ then $C > 2\delta + K_2$.
3. $r^- \leq K_1$.
4. $\min(K_2, C - 2\delta - 1) \leq r^+$.

Amalgamation: $d(a_1, a_2) = \min(K_2, C - 2\delta - 1)$.

PROOF OF LEMMA 12.13. Set $r = \min(K_2, C - 2\delta - 1)$. We have

$$r^- \leq K_1 \leq r \leq r^+$$

and thus d is a pseudo-metric on A. We may suppose as usual that $r > 0$ and thus d is a metric.

For $u \in A_0$ we must show that the triangle (a_1, a_2, u) belongs to $\mathcal{A}^\delta_{K_1,K_2,C,C'}$. Let $j = d(a_1, u)$, $k = d(a_2, u)$, and $p_u = r + j + k$.

We have the required bound on perimeter:

$$p_u \leq 2\delta + r < C.$$

So it suffices to check the constraints imposed by K_1 and K_2. We may suppose that p_u is odd.

As $r \geq K_1$ we have $p_u \geq 2K_1 + 1$. Thus it suffices to check the inequalities

(1) $r + j \leq 2K_2 + k$; (2) $r + k \leq 2K_2 + j$; (3) $j + k \leq 2K_2 + r$.

$(1, 2)$: We have

$$r + j \leq 2r + k \leq 2K_2 + k$$

and similarly $r + k \leq 2K_2 + j$.

(3): If $2K_2 + r \geq 2\delta$ then the inequality $j + k \leq 2K_2 + r$ is immediate. So we will suppose

$$2K_2 + r < 2\delta$$

By admissibility $K_1 + 2K_2 \geq 2\delta - 1$, so

$$2\delta - 1 \leq 2K_2 + K_1 \leq 2K_2 + r < 2\delta$$

and therefore

$$2K_2 + K_1 = 2\delta - 1 \text{ and } r = K_1$$

By admissibility $3K_2 \geq 2\delta$, so the first equation implies that $K_2 > K_1$. Hence $r = C - 2\delta - 1$ and the second equation gives

$$r = C - 2\delta - 1 = K_1;$$

in other words

$$C = 2\delta + K_1 + 1.$$

But $C \geq 2\delta + K_2$, so $K_2 = K_1 + 1$ and $C = 2\delta + K_2$. Also

$$3K_2 = 2K_2 + K_1 + 1 = 2\delta,$$

and our second condition on the parameters is violated. $\square$

For ease of reference, we reproduce the gist of Lemmas 12.6 through 12.13 in tabular form (Tables 12.1).

Lemma	**Hypotheses**	**Value**
Lemma 12.6	$r^+ < K_2$, $C \geq 2\delta + K_2$	r^+
Lemma 12.7	Either (*A*) $r^- \geq K_1$ and $C \leq 2\delta + K_1$; or (*B*) $r^- > K_1$.	r^-
Lemma 12.8	$C = 2K_1 + 2K_2 + 1$, $C' = C + 1$ and $\tilde{r} \leq \min(K_2, r^+)$	$\tilde{r}$
Lemma 12.9	$C' = C + 1$, and one of the following holds. (A) $r^+ \leq K_2$ and $C = 2K_1 + 2K_2 + 1$; (B) $r^+ \leq K_1$ and $C > 2\delta + K_1$; (C) $r^+ < K_2$ and $C \geq 2\delta + K_2$.	$\min(r^+, \tilde{r})$
Lemma 12.10	$C = 2K_1 + 2K_2 + 1$, $K_1 \leq \min(r^+, \tilde{r})$, and $r^- \leq K_2$	$\max(K_1, r^-) \leq r \leq \min(r^+, \tilde{r}, 2K_2 - r^-, \delta)$
Lemma 12.11	$K_1 = K_2$, $C = 4K_2 + 1 = 2\delta + K_2$, and $r^- \leq K_2 \leq r^+$	$K_2 - \epsilon$
Lemma 12.12	$C > 2\delta + K_1$ and $r^- \leq K_1 < r^+$	$K_1 + \epsilon$
Lemma 12.13	$C > 2\delta + K_1$ and $C \geq 2\delta + K_2$; If $3K_2 = 2\delta$ then $C > 2\delta + K_2$; $r^- \leq K_1$; $\min(K_2, C - 2\delta - 1) \leq r^+$	$\min(K_2, C - 2\delta - 1)$

Table 12.1. Amalgamation Lemmas 12.6 to 12.13

12E. The Main Theorem: Part I (amalgamation)

Now we will verify the correctness of our general amalgamation strategy. We first consider the situation with no Henson constraints, setting aside Type I, which has been dealt with earlier.

Proposition 12.14. Let $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C,C'}$ with an admissible choice of parameters, and with $K_1 < \infty$. Then $\mathcal{A}$ is an amalgamation class.

Furthermore, for any 2-point amalgamation problem $A_i = A_0 \cup \{a_\ell\}$ ($\ell = 1, 2$) in $\mathcal{A}$, a suitable extension d of $d_1 \cup d_2$ to $A = A_1 \cup A_2$ is given by the following.

1. If $C \leq 2\delta + K_1$:
   (a) If $r^- \geq K_1$ then take $d(a_1, a_2) = r^-$.
   Otherwise:
   (b) If $C' = C + 1$ then:
      (i) If $r^+ \leq K_2$ then take $d(a_1, a_2) = \min(r^+, \tilde{r})$
      (ii) If $r^- < K_1$ and $K_2 < r^+$ then take $d(a_1, a_2) = \tilde{r}$ if $\tilde{r} \leq K_2$ and $d(a_1, a_2) = K_1$ otherwise.
   (c) if $C' > C + 1$ then:
      (i) If $r^+ < K_2$ then take $d(a_1, a_2) = r^+$;
      (ii) If $r^- < K_2 \leq r^+$ then take

$$d(a_1, a_2) = \begin{cases} K_2 - 1 & \text{if there is } v \in A_0 \text{ with } d(a_1, v) = d(a_2, v) = \delta \\ K_2 & \text{otherwise} \end{cases}$$

2. If $C > 2\delta + K_1$:
   (a) If $r^- > K_1$ then take $d(a_1, a_2) = r^-$;
   Otherwise:
   (b) If $C' = C + 1$ then:
      (i) If $r^+ \leq K_1$ then take $d(a_1, a_2) = \min(r^+, \tilde{r})$;
      (ii) If $r^+ > K_1$ then take

$$d(a_1, a_2) = \begin{cases} K_1 + 1 & \text{if there is } v \in A_0 \text{ with } d(a_1, v) = d(a_2, v) = \delta, \text{ and} \\ & K_1 + 2K_2 = 2\delta - 1; \\ K_1 & \text{otherwise} \end{cases}$$

   (c) If $C' > C + 1$ then:

(i) If $r^+ < K_2$ then take $d(a_1, a_2) = r^+$;
(ii) If $r^+ \geq K_2$ then take $d(a_1, a_2) = \min(K_2, C - 2\delta - 1)$.

PROOF OF PROPOSITION 12.14. We dealt previously with Type (I). We turn to types (II) and (III).

We give the proof in tabular form, with the hypotheses in effect listed first, then the relevant lemma and the value of r used to complete the amalgam. Modulo those lemmas, we argue as follows.

Type (II): suppose

$$C \leq 2\delta + K_1.$$

We subdivide this case further according to the hypotheses shown in Table 12.2.

Amalgamation, Type II: $C \leq 2\delta + K_1$ (so $C = 2K_1 + 2K_2 + 1$)

Label	Hyp. 1	Hyp. 2	Hyp. 3	Lemma	r
(IIA)	$r^- \geq K_1$			12.7	r^-
(IIB)	$r^- < K_1$, $C' = C + 1$:				
$(IIB.1)$	"	$r^+ \leq K_2$		12.9 (A)	$\min(r^+, \tilde{r})$
$(IIB.2)$	"	$r^+ > K_2$:			
$(IIB.2a)$	"	"	$\tilde{r} \leq K_2$	12.8	$\tilde{r}$
$(IIB.2b)$	"	"	$\tilde{r} > K_2$	12.10	K_1
(IIC)	$r^- < K_1$, $C' > C + 1$:				
$(IIC.1)$	"	$r^+ < K_2$		12.6	r^+
$(IIC.2)$	"	$r^+ \geq K_2$		12.11	$K_2 - \epsilon$

TABLE 12.2. Amalgamation, Case II

Here we will rely on admissibility to verify some of the hypotheses of the relevant lemmas. In case (IIB) we use the relation

$$C = 2K_1 + 2K_2 + 1.$$

In case (IIC) we use the various relations imposed by admissibility (notably in connection with Lemma 12.11).

It suffices to check that the quoted lemmas apply in each case.

Now we present Type (III), that is the case $C > 2\delta + K_1$. This will require a little more explanation.

Type III: $C > 2\delta + K_1$

Label	Hyp. 1	Hyp. 2	Lemma	r
$(IIIA)$	$r^- > K_1$		12.7 (B)	r^-
$(IIIB)$	$r^- \leq K_1$, $C' = C + 1$:			
$(IIIB.1)$	"	$r^+ \leq K_1$	12.9 (B)	$\min(r^+, \tilde{r})$
$(IIIB.2)$	"	$r^+ > K_1$	12.12	$K_1 + \epsilon$
$(IIIC)$	$r^- \leq K_1$, $C' > C + 1$:			
	$(C \geq 2\delta + K_2)$			
$(IIIC.1)$	"	$r^+ < K_2$	12.6	r^+
$(IIIC.2)$	"	$r^+ \geq K_2$	12.13	$\min(K_2, C - 2\delta - 1)$

TABLE 12.3. Amalgamation, Case III

In Case $(IIIC)$ we again apply admissibility. First, admissibility gives the inequality $C \geq 2\delta + K_2$ in this case. Additionally, to apply Lemma 12.13 we require the following condition.

$$\text{If } 3K_2 = 2\delta \text{ then } C > 2\delta + K_2.$$

But if $C = 2\delta + K_2$, then since $C > 2\delta + K_1$, we have $K_2 > K_1$. On the other hand, by admissibility we have

$$K_1 + 2K_2 \geq 2\delta - 1$$
$$\text{If } K_1 + 2K_2 = 2\delta - 1, \text{ then } C \geq 2\delta + K_1 + 2$$

Now if $3K_2 = 2\delta$ with $K_2 > K_1$, then the first inequality becomes

$$K_1 + 2K_2 = 2\delta - 1 \text{ and } K_1 = K_2 - 1.$$

Then the condition $C \geq 2\delta + K_1 + 2$ becomes $C > 2\delta + K_2$, as required. $\square$

Now we can prove Theorem 12.1.

Theorem (Theorem 12.1—Main Theorem, Part I). Suppose that

$$\delta, K_1, K_2, C, C', \mathcal{S}$$

is an admissible sequence of parameters. Then $\mathcal{A}^{\delta}_{K_1,K_2,C,C',\mathcal{S}}$ is an amalgamation class. Furthermore, if $\mathcal{S}$ is nonempty, then any amalgamation diagram can be completed without adding new pairs at distance 1 or δ.

PROOF. Recall that admissibility (as used here) includes the hypothesis

$$\delta \geq 3.$$

Furthermore, we dealt with the bipartite case in Lemma 12.4. So we will suppose that

$$K_1 < \infty.$$

In other words, of the three types requiring consideration under the definition of admissibility (Definition 11.2), we are left with types (II) and (III).

When $\mathcal{S}$ is empty, Proposition 12.14 applies. So we will also suppose that

$$\mathcal{S} \text{ is nonempty.}$$

Under this hypothesis, the definition of admissibility imposes the following additional conditions on the parameters.

Type (II) **(**$C \leq 2\delta + K_1$**):** $C \geq 2\delta+3$ (as C is odd and greater than $2\delta + 1$).

Type (III) **(**$C > 2\delta + K_1$**):** $K_1 < \delta$, $C > 2\delta + 2$.

It suffices to show that any 2-point amalgamation problem may be completed without introducing a distance which occurs in the constraint set $\mathcal{S}$.

We will use the amalgamation procedure given in Proposition 12.14, with one exception which will appear at the end, when our standard procedure gives $d(a_1, a_2) = 1$, but we substitute the value $d(a_1, a_2) = 2$ to meet our additional constraints.

We begin by summarizing once more the values for $r = d(a_1, a_2)$ given by our standard amalgamation procedure. We include the pertinent Lemma, and any relevant inequalities (or equations) imposed.

When the values shown are different from both 1 and δ, there are no difficulties.

(Type II) $C \le 2\delta + K_1$

Lemma	Value	Bound
12.6	r^+	$< K_2$
12.7	r^-	$\ge K_1$
12.8	$\tilde{r}$	$\le K_2$
12.9 (A)	$\min(r^+, \tilde{r})$	$\le K_2$
12.10	K_1	$\le K_2$
12.11	$K_2 - \epsilon$	$= \frac{(2\delta-1)}{3} - \epsilon$

(Type III) $C > 2\delta + K_1$

Lemma	Value	Bound
12.6	r^+	$< K_2$
12.7 (B)	r^-	$> K_1$
12.9 (B)	$\min(r^+, \tilde{r})$	$\le K_1$
12.12	$K_1 + \epsilon$	
12.13	min of K_2 and $C - 2\delta - 1$	

We first take up the desired upper bound

$$r < \delta.$$

We have

$$K_2 \le \delta,\ r^- < \delta,\ \text{and } (2\delta - 1)/3 < \delta.$$

In Type (II), admissibility requires $K_1 + 2K_2 \le \delta - 1$, and thus $K_2 < \delta$. Thus in all cases falling under Type (II), the bound $r < \delta$ is satisfied.

In Type (III) we have $K_1 < \delta$, so we need only consider the last two cases listed, which involve Lemmas 12.12 and 12.13.

In the case of Lemma 12.12, if $r = \delta$ then we have the following conditions.

$$\epsilon = 1$$
$$r = K_1 + 1 = \delta.$$

But the definition of ϵ entails

$$K_1 + 2K_2 = 2\delta - 1,$$

and as $K_2 \geq K_1$ this is a contradiction. So the value $r = \delta$ does not actually occur in this case.

In the case of Lemma 12.13, under the assumption $r = \delta$ we have the following conditions.

$$r = K_2 = \delta.$$

Furthermore one of the hypotheses of Lemma 12.13 is

$$C \geq 2\delta + K_2,$$

and thus $C \geq 3\delta$. It follows that $C' = C + 1$ in this case. Notice however that in the proof of Proposition 12.14, Lemma 12.13 was only applied in cases which satisfy additionally

$$C' > C + 1$$

(see the chart for the treatment of Type (III)). So in each case where this value of r was used, we have $r < \delta$.

This completes the verification of the upper bound $r < \delta$ in all instances of the standard amalgamation strategy. Now we must consider the desired lower bound

$$r > 1.$$

We claim that, with one exception, this constraint holds if we follow our standard amalgamation procedure, and that in the exceptional case, the value $r = 2$ may be substituted.

We must consider Types (II) and (III) separately, for the most part, and in detail. But we may omit the cases in which $r = r^+$ as $r^+ \geq 2$ by definition.

Type (II), Lower Bound:

If $K_1 = 1$, then as we are in Type (II) we have $C = 2\delta + 1$. But as noted above, this would force $\mathcal{S}$ to be empty. So we will suppose

$$K_1 > 1.$$

So we may set aside the cases falling under Type (II) which correspond to Lemma 12.6, 12.7, or 12.10. In the third and fourth cases listed, which correspond to Lemmas 12.8 and 12.9 (A), we must then examine the possibility

$$r = \tilde{r} = 1.$$

But as $\tilde{r} \geq (C-1) - 2\delta$, we find that

$$C \leq 2\delta + 2,$$

whereas, as noted above, this would force $\mathcal{S}$ to be empty.

The last case to consider within Type (II) corresponds to an application of Lemma 12.11. and gives the relations

$$r = 1 = K_2 - \epsilon = (2\delta - 1)/3 - \epsilon.$$

As $(2\delta - 1)/3$ is odd, it follows that $\epsilon = 0$ and $\delta = 2$, which we do not allow here.

So we have the lower bound $r > 1$ throughout Type (II), following our standard amalgamation procedure.

Type (III), Lower Bound:

Here the cases in which $r = r^+$ or $r > K_1$ may be set aside. Also, as above, the case $r = \tilde{r} = 1$ leads to $C \leq 2\delta + 2$, which would require $\mathcal{S}$ to be empty.

So within Type (III), we are left with just the last two cases, which correspond to Lemmas 12.12 and 12.13, respectively.

We first consider the last case, which corresponds to Lemma 12.13. Here we have

$$r = \min(K_2, C - 2\delta - 1) = 1.$$

As we suppose $C > 2\delta + 2$ this becomes

$$r = K_2 = 1.$$

But by admissibility $3K_2 \geq 2\delta$, so this case is excluded.

So we come down to the consideration of what turns out to be the critical case, which corresponds to Lemma 12.12. Here we may be forced to deviate from our standard amalgamation procedure.

So we suppose that we are in this case, with the following conditions applying.

$$r = K_1 + \epsilon = 1.$$

Then evidently $\epsilon = 0$ and $K_1 = 1$.

In this case we modify the choice of r:

$$r = 2.$$

Recall that in this case $r^- \leq K_1 = 1$, and thus the value $r = 2$ at least gives us a metric d' on the amalgam. As we are supposing $C > 2\delta + 2$, the bound on perimeter is respected by this choice of r.

It remains to show that the resulting amalgam $(A_1 \cup A_2, d')$ satisfies the constraints corresponding to K_1 and K_2. But $K_1 = 1$, so the only such constraints are the inequalities associated with the parameter K_2.

Now fix $u \in A_0$ and let $j = d(a_1, u)$, $k = d(a_2, u)$. Then the constraints to be checked are as follows.

$$(1)\ 2 + i \leq 2K_2 + j;\ (2)\ 2 + j \leq 2K_2 + i;\ (3)\ i + j \leq 2K_2 + 2.$$

By admissibility, we have $K_1 + 2K_2 \geq 2\delta - 1$. As $K_1 = 1$, we find

$$K_2 \geq \delta - 1.$$

So the inequality (3) is immediate.

For (1) we have

$$2 + i \leq 2 + \delta \leq 2(\delta - 1) + 1 \leq 2K_2 + j,$$

and (2) follows similarly.

This completes the proof of Part I of the main theorem: that is, the metrically homogeneous graphs $\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ exist, for admissible values of the parameters. □

CHAPTER 13

TRIANGLE CONSTRAINTS AND 4-TRIVIALITY

13A. The Main Theorem, Part II: 4-triviality

Having disposed of the existence question in the previous section, we now come to a kind of converse.

We aim at a classification theorem whose simplest form is the following.

THEOREM 13.1 (Main Theorem, Part II). *Let Γ be a countable metrically homogeneous graph of diameter $\delta \geq 3$ which is determined by a set of forbidden metric triangles. Then Γ is in our catalog, either as one of the* exceptional graphs, *or as one of the $\Gamma^{\delta}_{K_1,K_2,C_0,C_1}$ with $\delta, K_1, K_2, C_0, C_1$ admissible.*

In working out the classification of the 3-constrained metrically homogeneous graphs, one identifies the numerical constraints which are summarized in the definition of admissibility for the case in which $\mathcal{S}$ is empty. Once one has those numerical constraints, it is straightforward to tweak the definition to allow for a non-empty set of constraints of Henson type. Of course, one only knows that the analysis is complete when one can also prove the converse result, given in the previous chapter, that the relevant constraints actually do correspond to amalgamation classes.

Once the amalgamation classes $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ incorporating Henson constraints are also identified, it is more sensible to prove a stronger version of Theorem 13.1 which allows for their presence. The most direct formulation of this would then be the following (we will vary it further below).

THEOREM 13.2 (Main Theorem, Part II—Variant 1). *Let Γ be a countable metrically homogeneous graph of diameter $\delta \geq 3$ determined by a set of forbidden metric triangles and constraints of Henson* *type, i.e.,* $(1,\delta)$*-spaces if* $C > 2\delta+1$*, and antipodal Henson constraints if* $C = 2\delta + 1$*. Then* Γ *is of the form*

$$\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$$

for some admissible parameter sequence

$$\delta, K_1, K_2, C_0, C_1, \mathcal{S}.$$

One of the conjectures suggested by our catalog is that for any amalgamation class of finite metric spaces associated with a metrically homogeneous graph of generic type, if one considers only the forbidden triangles, then this gives a set of constraints which by themselves define an amalgamation class. Here we check, among other things, that this is correct for amalgamation classes of the expected form $\mathcal{A}^{\delta}_{K_1,K_2C_0,C_1,\mathcal{S}}$.

Now the result we would really want—the main conjecture—is that the same result applies to any amalgamation class corresponding to a countable metrically homogeneous graph of generic type. That claim is much sharper than anything we will prove here, but an examination of the proof of Theorem 13.2 shows that the hypothesis actually needed for the proof can be substantially restricted, in a useful way. Namely, the critical configurations to consider in our arguments never involve more than 4 points; that is, the diagrams may contain 5 points, but the factors required to set up the diagrams contain at most 4 points.[13]

So we make the following definition.

DEFINITION 13.3. Let $\mathcal{A}$ be an amalgamation class of finite metric spaces corresponding to some countable metrically homogeneous graph Γ of diameter δ. We say that $\mathcal{A}$, or Γ, is 4-*trivial* if $\mathcal{A}$ contains every metric space M on 4 vertices satisfying the following conditions.

- M contains no forbidden triangle for Γ, and
- M is not an ordinary or δ-antipodal Henson constraint (p. 215).

The proof of Part II of the Main Theorem can be interpreted as a determination of the possible classes of triangles which can be forbidden in a 4-trivial countable metrically homogeneous graphs. Of course, if we do not know that the class in question is fully determined by triangle constraints and Henson constraints, then even if

[13]For the current optimistic view of prospects for more general results, see Appendix 18B.1.

we know it is 4-trivial, this would not give us a determination of the class. But knowing what patterns of triangle constraints are possible would be a major first step toward the proof of the classification conjecture in general. .

We state our preferred variation on the Main Theorem (Part II) as follows.

THEOREM 13.4 (Main Theorem, Part II, Variant 2). *Let* Γ *be a* 4-*trivial countable metrically homogeneous graph of diameter* $\delta \geq 3$, *and let* $\mathcal{A}$ *be the associated amalgamation class of finite metric spaces. Then the following hold.*

(II-A) For some acceptable sequence of parameters $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$, *the classes* $\mathcal{A}$ *and* $\mathcal{A}^{\delta}_{K_1,K_2,C,C',\mathcal{S}}$ *contain the same triangles and the same* $(1,\delta)$*-spaces.*

(II-B) The sequence of parameters $(\delta, K_1, K_2, C_0, C_1)$ *is admissible, and the same applies to the parameters* $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ *if all constraints are either triangles or Henson constraints (normal or antipodal).*

Note that Theorem 13.4 does indeed imply Theorem 13.2. This is the formulation we will work with throughout.

The present chapter deals with the proof of part (II-A).

We will rarely refer explicitly to the non-generic/generic type distinction in what follows, but the 4-trivial amalgamation classes all fall on the side of generic type, and this is occasionally useful (as we shall see shortly). So we deal with this point formally.

LEMMA 13.5. *Any* 4*-trivial metrically homogeneous graph* Γ *of diameter* $\delta \geq 3$ *is of generic type.*

PROOF. If Γ is an n-gon with $n \geq 6$, or one of the tree-like graphs $T_{m,n}$, then let A be a 4-gon. As Γ omits A, but contains a geodesic triangle of type $(1,1,2)$, A should be a Henson constraint: but a Henson constraint cannot contain a geodesic triangle.

By the classification in non-generic type (Table 11.1, §11A), the alternative is that Γ is an antipodal double cover, of diameter 3. In particular it is finite. Let n be minimal so that Γ omits a 2-anticlique $I^{(2)}_n$ of order n. Then by 4-triviality, either $n = 3$ and $I^{(2)}_3$ is a forbidden triangle, or $n > 3$ and $I^{(2)}_n$ is a Henson constraint. But $I^{(2)}_n$ can only be a Henson constraint if $\delta = 3$ and $I^{(2)}_n$ is the antipodal companion of a clique of order n, which would force $n = 2$, for a contradiction.

So we are left with the possibility that Γ omits $I_3^{(2)}$. But if we take a geodesic (u_1, u_2, u_3) of type $(1, 1, 2)$, and replace u_2 by the antipodal vertex u_2', we get a triangle of type $(2, 2, 2)$. $\square$

13B. Recovering the parameters

We begin by associating suitable parameters $\delta, K_1, K_2, C_0, C_1, \mathcal{S}$ with any countable metrically homogeneous graph right from the start, without making any substantial assertions about these values at the outset. Then parts (*II-A*) and (*II-B*) of the Main Theorem will take on a more concrete meaning. Of course, it will be clear from the beginning that if the class in question is of the form $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$, then our particular choice of definitions does at least recover the parameters correctly.

DEFINITION 13.6. Let $\mathcal{A}$ be a class of finite integral metric spaces.

1. δ is the supremum of the distances realized by a configuration in $\mathcal{A}$ (possibly ∞).
2. K_1 is the least k such that $\mathcal{A}$ contains a triangle of type $(1, k, k)$ and K_2 is the greatest such, if there is such a triangle. Otherwise, we set $K_1 = \infty$ and $K_2 = 0$.
3. If δ is finite, then C_0 is the least even number greater than 2δ such that no triangle of perimeter C_0 is in $\mathcal{A}$; and C_1 is the least such odd number. If δ is infinite, then C_0, C_1 are undefined (they can be set equal to ∞ for more uniformity of notation).
4. $C = \min(C_0, C_1)$ and $C' = \max(C_0, C_1)$.
5. $\mathcal{S}$ is the collection of minimal Henson constraints S such that
   - S is forbidden, i.e. not a member of $\mathcal{A}$;
   - S does not contain a forbidden triangle of $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$; and
   - If $C_1 > 2\delta + 1$ or $C_0 > 2\delta + 2$, then S is a $(1, \delta)$-space; if $C_1 = 2\delta + 1$ and $C_0 = 2\delta + 2$, then S is an antipodal companion of a clique.

   In particular the spaces in $\mathcal{S}$ have order at least four, and are either $(1, \delta)$-spaces or $(1, \delta - 1)$-spaces.

When δ is finite, then C_0, C_1 are always well-defined, and are at most $3\delta + 2$.

Furthermore, with the exception of the case $K_1 = \infty$, $K_2 = 0$, and $C_1 = 2\delta + 1$, we have by definition

$$1 \leq K_1 \leq K_2 \leq \delta.$$

So with these definitions the following holds.

The parameters $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ are acceptable.

Since we make these definitions in full generality, it may be useful to keep track of their values when $\mathcal{A}$ is *not* of the form $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$, but is rather the amalgamation class of finite metric spaces associated with a metrically homogeneous graph of non-generic type. We will discuss this in §13I.

13C. The interpolation property

In what follows, we begin to use a variety of amalgamation arguments involving amalgams of two factors of order 4, and we apply 4-triviality to ensure that the factors are present in our amalgamation class. For this, we must mainly check that the triangles occurring in these diagrams are in the class. However, one must pay attention as we go along to the verification that neither factor is a Henson constraint of normal or antipodal type. For this, the following criteria are convenient.

Remark 13.7. Let A be a Henson constraint of normal or antipodal type. Then the following hold.

(a) At most two lengths occur in A.
(b) No geodesics occur in A.

DEFINITION 13.8. The *type* of a triangle (a, b, c) is the triple of lengths

$$(d(a, b), d(a, c), d(b, c)),$$

which may be taken in any order (most often, nondecreasing). Thus the type represents the isomorphism type of the metric triangle.

The type of any triangle is a triple (i, j, k) with $i, j, k \in \mathbb{N}$, $i, j, k \geq 1$, satisfying the triangle inequality, and generally we have a bound $i, j, k \leq \delta$ with the diameter δ given. When we refer to a *triangle type* (i, j, k), in principle this means that the triangle inequality and any relevant bound are assumed. If the necessary inequalities have not been established, we speak of a *triple* rather than a triangle type. For triples in general, as well as for triangle types, we attach no significance to the order of the entries.

The following general principle will be very useful throughout.

LEMMA 13.9 (Interpolation Property). *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$.*

Suppose that $\mathcal{A}$ contains triangles of the types

$$(i-1, j-1, k) \text{ and } (i-1, j+1, k)$$

where

$$2 \leq i \leq \delta,\ 2 \leq j < \delta, \text{ and } 1 \leq k \leq \delta.$$

Then $\mathcal{A}$ contains a triangle of type (i, j, k).

We prepare for the proof with a result which gives us a very small supply of specific triangles known to be realized, from the beginning.

LEMMA 13.10. *Let Γ be a metrically homogeneous graph of diameter $\delta \geq 3$, with vertex degree at least 3, and suppose*

$$i < \delta.$$

Then Γ contains a triangle of type $(2, i, i)$.

PROOF. We first run over the possibilities for Γ of non-generic type, from Table 11.1, §11A.

If Γ is finite, then it is antipodal of diameter 3, and the relevant values of i are 1 and 2. For $i = 1$ we have the geodesic of type $(1, 1, 2)$, which certainly embeds into Γ, and if we replace the midpoint v by the antipodal vertex v' we get a triangle of type $(2, 2, 2)$.

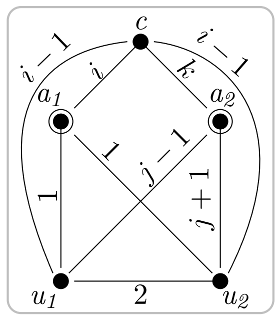
If Γ is a tree-like graph $T_{m,n}$, then it contains the regular tree $T_{m,2}$, and the claim follows.

Thus we may suppose that Γ is of generic type.

As $i < \delta$, we may take $u_1 \in \Gamma_{i-1}$, $u_2 \in \Gamma_{i+1}$ with $d(u_1, u_2) = 2$. As Γ is of generic type, u_1 and u_2 have two common neighbors v_1, v_2 with $d(v_1, v_2) = 2$. Then $v_1, v_2 \in \Gamma_i$, so the triangle (v_*, v_1, v_2), with v_* the basepoint, is of type $(i, i, 2)$. □

PROOF OF LEMMA 13.9. We will need to apply Lemma 13.10, so let us recall that 4-triviality implies generic type (Lemma 13.5), and in particular vertex degree at least 3.

We make use of the following amalgamation.



Let us check that the factors $ca_1u_1u_2$ and $ca_2u_1u_2$ are present in $\mathcal{A}$, by 4-triviality.

Each factor contains a geodesic, so neither factor is a Henson constraint (Remark 13.7).

The nongeodesic triangles involved in the factors have types

$$(2, i-1, i-1),\ (i-1, j-1, k),\ \text{and}\ (i-1, j+1, k).$$

The first of these triangles is in $\mathcal{A}$ by Lemma 13.10, and the other two are in $\mathcal{A}$ by assumption.

By 4-triviality, the two factors of the amalgamation are in $\mathcal{A}$, and thus our diagram has a completion in $\mathcal{A}$. Then the vertices u_1, u_2 force

$$d(a_1, a_2) = j$$

So the triangle (a_1, a_2, c) has type (i, j, k). $\square$

DEFINITION 13.11. An amalgamation class $\mathcal{A}$ of finite metric spaces of maximal diameter δ has the *Interpolation Property* if it satisfies the conclusion of Lemma 13.9. That is, whenever $\mathcal{A}$ contains triangles of the types

$$(i-1, j-1, k)\ \text{and}\ (i-1, j+1, k)$$

with

$$2 \leq i \leq \delta \qquad 2 \leq j < \delta \qquad 1 \leq k \leq \delta,$$

then $\mathcal{A}$ contains a triangle of type (i, j, k).

So each of the following conditions implies the next.

(a) minimal forbidden structures are triangles and Henson constraints;
(b) 4-triviality;
(c) the Interpolation Property.

13D. Small even perimeter.

LEMMA 13.12. *Let Γ be a countable metrically homogeneous graph of diameter $\delta \geq 3$ and vertex degree at least 3, and $\mathcal{A}$ the associated amalgamation class of finite metric spaces. Suppose that $\mathcal{A}$ has the Interpolation Property. Then all triangles of even perimeter $p \leq 2\delta$ are in $\mathcal{A}$.*

PROOF. We consider triangles of type (i, j, k) with perimeter $p = i + j + k$ even and bounded by 2δ, and $i \leq j \leq k \leq \delta$. We proceed by induction on i.

If the triangle is a geodesic, then it lies in $\mathcal{A}$ by hypothesis.

Suppose that the triangle is not a geodesic, so

$$k < i + j$$

As $i+j+k$ is even, it follows that $k \leq i+j-2$ and $i \geq 2$. Therefore both of the following triples satisfy the triangle inequality.

$$(i-1, j-1, k);\ (i-1, j+1, k)$$

As their perimeters are even and bounded by 2δ, and the minimal entry in each is $i-1$, by induction both triangles are in $\mathcal{A}$.

The Interpolation Property yields the claim. $\square$

13E. An inductive lemma.

We now state a general inductive "all-or-nothing" principle. For the most part our classes $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ follow the rule that all or none of the triangles of a given perimeter occur. But since the parameter K_2 breaks that rule, we must make do with a more limited version of the principle, in which we fix both the perimeter and the length of the shortest side.

LEMMA 13.13. *Let $\mathcal{A}$ be an amalgamation class of diameter δ, with the Interpolation Property. Let $m, N \in \mathbb{N}$ be fixed with $m \geq 2$.*

Suppose that

> *All triangles of perimeter $N-2$ and minimal edge length at least $m-1$ are in $\mathcal{A}$.*

Then the following hold.

(a) *If some triangle of perimeter N and minimum edge length at least m is in $\mathcal{A}$, then all such triangles are in $\mathcal{A}$;*

(b) *If some triangle of perimeter N and minimum edge length exactly $m-1$ is in $\mathcal{A}$, then any triangle of perimeter N whose minimum edge length is at least $m-1$ and with at most one edge length equal to $m-1$ also belongs to $\mathcal{A}$.*

PROOF. If $N \leq 2\delta$ is even, then by Lemma 13.12 all such triangles are in $\mathcal{A}$, and there is nothing to prove. So we may suppose that

$$N \text{ is odd, or } N > 2\delta.$$

In particular no geodesic triangles come into consideration.

As the proof takes several steps we indicate the steps which lead to part (a), given as Claim 4 below. The argument for (b) is similar, and makes use of the same ingredients.

We consider triangle types (i_1, j_1, k_1) and (i_2, j_2, k_2) satisfying our conditions, with $i_1 \leq j_1 \leq k_1$, $i_1, \leq j_2 \leq k_2$, and $i_1 \leq i_2$.

We first consider the special cases where the triangle type (i_2, j_2, k_2) is either (i_1, j_1+1, k_1-1) or (i_1+1, j_1, k_1-1), and we show that triangles of the first type occur if and only if triangles of the second type occur (Claims 1,2). The we can argue inductively to prove part (a), first considering the case $i_2 = i_1$, then reducing the case $i_2 > i_1$ to this one (Claims 3,4).

To begin with, we invoke the Interpolation property, as follows.

Suppose (i, j, k) is the type of a triangle with

$$i + j + k = N \tag{13.1}$$

$$m \leq i, j; \quad m - 1 \leq k \tag{13.2}$$

$$i, k \leq \delta; \quad j < \delta \tag{13.3}$$

$$k \leq i + j - 2 \tag{13.4}$$

Then $(i-1, j-1, k)$ is a triangle type, and by hypothesis $(i-1, j-1, k)$ is in $\mathcal{A}$. If there is also a triangle of type $(i-1, j+1, k)$ in $\mathcal{A}$ then by the Interpolation Property there is one of type (i, j, k). In symbols

$$(i-1, j+1, k) \Longrightarrow (i, j, k)$$

under the stated conditions on i, j, k.

Claim 1. Suppose

$$\begin{gathered} i+j+k=N; \\ i \leq j \leq k \leq i+j,\ j \leq i+k-2; \\ i \geq m-1,\ j \geq m,\ k>m; \\ j<\delta. \end{gathered}$$

Then there is a triangle of type (i, j, k) in $\mathcal{A}$ if and only if there is a triangle of type $(i, j+1, k-1)$ in $\mathcal{A}$. In other words,

$$(i, j, k) \iff (i, j+1, k-1)$$

under the stated conditions.

Take (i', j', k') to be $(j+1, k-1, i)$ or (k, j, i). After verifying the conditions $(1-4)$ above for (i', j', k') we will have

$$(j, k, i) \Longrightarrow (j+1, k-1, i) \text{ and } (k-1, j+1, i) \Longrightarrow (k, j, i)$$

and the claim will follow.

Condition (1) for (i', j', k') is clear in both cases. Conditions (2) and (3) come down to

$$m \leq j, k-1;\ m-1 \leq i;\ k \leq \delta; \text{ and } j<\delta$$

which were assumed.

Condition (4) comes down in both cases to

$$i \leq j+k-2$$

which holds as $i \leq j \leq j+k-2$.

Thus the Interpolation Property applies as described above.

Claim 2. Suppose

$$\begin{gathered} i+j+k=N; \\ m \leq i \leq j \leq k \leq i+j; \\ m+1 \leq k; \\ k \leq \delta;\ i<\delta. \end{gathered}$$

Then there is a triangle of type (i, j, k) in $\mathcal{A}$ if and only if there is one of type $(i+1, j, k-1)$ in $\mathcal{A}$.

Take (i',j',k') to be $(i+1,k-1,j)$ or (k,i,j) and verify conditions (1–4) above for i',j',k'. Condition (1) is clear, and conditions $(2,3)$ come down to $m \le i, k-1$ and $i, k-1 < \delta$, which all hold by hypothesis. Claim (4) states that $k' \le i'+j'-2$, which in both cases means

$$j \le i+k-2.$$

This holds since $j \le k \le (i-2)+k$.

So the Interpolation Property gives

$$(i,k,j) \Longrightarrow (i+1,k-1,j) \text{ and } (k-1,i+1,j) \Longrightarrow (k,i,j).$$

Claim 3. For $\ell = 1,2$ let (i,j_ℓ,k_ℓ) be triples satisfying

$$\begin{aligned} &i+j_\ell+k_\ell = N;\\ &i \le j_\ell \le k_\ell \le i+j_\ell;\\ &i \ge m-1,\ j_\ell \ge m,\ k_\ell \ge m+1. \end{aligned}$$

Then there is a triangle of type (i,j_1,k_1) in $\mathcal{A}$ if and only if there is one of type (i,j_2,k_2) in $\mathcal{A}$.

We may suppose that $j_1 \le j_2$ and proceed by induction on j_2-j_1. If $j_1 = j_2$ then $k_1 = k_2$ and our claim holds. So suppose that $j_1 < j_2$.

Then $k_2 < k_1$ and thus $j_1+1 \le j_2 \le k_2 < k_1$, $j_1+1 \le k_1-1$. In particular $j_1 < k_1$ and it will suffice to treat the case $j_2 = j_1+1$, $k_2 = k_1-1$, and then conclude by induction.

We invoke Claim 1 for (i,j_1,k_1). We require

$$j_1 < \delta \text{ and } j_1 \le i+k_1-2.$$

We have $j_1 < j_2 \le \delta$ and $j_1 \le k_1-2 \le i+k_1-2$.

Claim 4. For $\ell = 1,2$ let (i_ℓ,j_ℓ,k_ℓ) be triples satisfying

$$\begin{aligned} &i_\ell+j_\ell+k_\ell = N;\\ &m \le i_\ell \le j_\ell \le k_\ell \le i_\ell+j_\ell. \end{aligned}$$

Then there is a triangle of type (i_1,j_1,k_1) in $\mathcal{A}$ if and only if there is one of type (i_2,j_2,k_2) in $\mathcal{A}$.

We will suppose $i_1 \le i_2$ and proceed by induction on i_2-i_1.

If $k_1 = i_1$ then $N = 3i_1$ and hence

$$N = 3i_1 \le 3i_2 \le i_2+j_2+k_2 = N$$

forcing $i_2 = j_2 = k_2 = i_1$ and there is nothing to prove. So we will suppose

$$k_1 > i_1$$

In particular $k_1 > m$.

The case $i_1 = i_2$ is covered by Claim 3. So suppose

$$i_1 < i_2$$

Then Claim 2 applies, and a triangle of type (i_1, j_1, k_1) is in $\mathcal{A}$ if and only if one of type (i_1+1, j_1, k_1-1) is in $\mathcal{A}$. So we replace i_1 by i_1+1 and conclude by induction on $i_2 - i_1$

Now Claim (4) is part (a) of the Lemma. We turn to part (b).

We suppose that a triangle of type (i, j, k) is in $\mathcal{A}$, where

$$i + j + k = N;$$
$$i = m - 1 \leq j \leq k \leq i + j.$$

If $N < 3m - 1$ then the second part of the lemma is vacuous. So suppose

$$N \geq 3m - 1.$$

Claim 5. Some triangle of type (i, j, k) with $i = m-1$ and $j, k \geq m$, $i + j + k = N$ belongs to $\mathcal{A}$.

Supposing the contrary, our triangle of type (i, j, k) with

$$i = m - 1 \leq j \leq k, \ i + j + k = N$$

must satisfy

$$\begin{aligned} j &= m - 1; \\ k &= N - 2(m - 1); \\ &\geq (3m - 1) - 2(m - 1) = m + 1. \end{aligned}$$

We will show that a triangle of type $(m, k-1, m-1)$ is in $\mathcal{A}$, proving the claim.

Notice that conditions (13.1–13.4) above are satisfied by this triangle type, taking $i' = m$, $j' = k - 1$, $k' = m - 1$. So by the Interpolation Property it suffices to show that there is a triangle of type $(i' - 1, j' + 1, k')$ in $\mathcal{A}$. But

$$(i' - 1, j' + 1, k') = (m - 1, k, m - 1)$$

which is the type we have assumed is in $\mathcal{A}$. This proves the claim.

Now that Claim 5 gives us one triangle of the desired form we can argue as in the proof of part (a). Namely, we claim next that for any triangle types (i, j_ℓ, k_ℓ) $(\ell = 1, 2)$ with

$$\begin{aligned} i + j_\ell + k_\ell &= N; \\ i = m - 1 \le j_\ell &\le k_\ell \le i + j_\ell; \\ j_\ell, k_\ell &\ge m, \end{aligned}$$

we have a triangle of type (i, j_1, k_1) in $\mathcal{A}$ if and only if we have one of type (i, j_2, k_2). This follows as before from Claim 1, taking $j_1 \le j_2$ and applying induction on $j_2 - j_1$.

This completes the proof of part (b) as far as triangle types with minimum distance exactly $m - 1$ are concerned. In particular we are done if $N < 3m$.

Suppose

$$N \ge 3m$$

In view of part (a) of the lemma, it will now suffice to find one triangle type (i, j, k) with

$$\begin{aligned} i + j + k &= N; \\ m \le i \le j &\le k \le i + j, \end{aligned}$$

which is represented in $\mathcal{A}$.

We begin with some triangle type (i, j, k) represented in $\mathcal{A}$ with $i + j + k = N$, $i = m - 1 < j \le k \le i + j$, $i + j + k = N$. Observe that $k \ge (N - i)/2 > m$.

We consider the triple $(i', j', k') = (m, j, k - 1)$. We apply the Interpolation Property.

Note first that $(m, j, k-1)$ satisfies the triangle inequality. It suffices therefore to check that $\mathcal{A}$ contains triangles of the types $(m - 1, k - 2, j)$ and $(m - 1, k, j)$; but we already have the second one.

So consider the triple $(m - 1, k - 2, j)$. This satisfies the triangle inequality and has perimeter $N - 2$, and minimal entry $m - 1$. So such a triangle is in $\mathcal{A}$ by hypothesis. $\square$

13F. Triangles of even perimeter

Lemma 13.14. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of diameter δ. Then a triangle of type (i, j, k) with even perimeter $p = i + j + k$ is in $\mathcal{A}$ if and only if $p < C_0$.*

PROOF. Here C_0 is minimal such that

C_0 is even, $C_0 > 2\delta$, and there is no triangle of perimeter C_0 in $\mathcal{A}$.

Let C_0' be minimal even such that there is *some* triangle of perimeter C_0' which does not occur in $\mathcal{A}$. Then $C_0' \leq C_0$, and by Lemma 13.12, $C_0' > 2\delta$.

If $p < C_0'$ is even, then by definition all triangles of perimeter p are in $\mathcal{A}$.

Now suppose $p = C_0'$, and let $m = p - 2\delta \geq 2$. Every triangle of perimeter p has minimal length at least m. Apply Lemma 13.13. Either all such triangles of perimeter p are in $\mathcal{A}$, or none are. As $p = C_0'$ the conclusion is that none are, and thus $C_0' = C_0$. So we have the following.

For even p less that C_0, all triangles of perimeter p are in $\mathcal{A}$.
For $p = C_0$, no triangles of perimeter p are in $\mathcal{A}$.

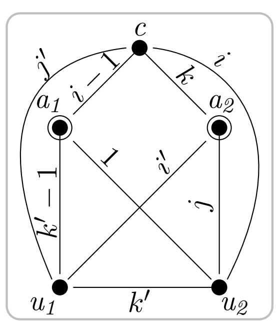
It remains to show that for even p greater than C_0, no triangles of perimeter p are in $\mathcal{A}$.

We proceed inductively. So suppose that $p > C_0$, that no triangle of perimeter $p-2$ is in $\mathcal{A}$, but that some triangle of type (i, j, k) with $i + j + k = p$ even, $i \leq j \leq k$ is in $\mathcal{A}$.

Define

$$\begin{aligned} i' &= (-i + j + k)/2; \\ j' &= (i - j + k)/2; \\ k' &= (i + j - k)/2. \end{aligned}$$

As $p > C_0 \geq 2\delta + 2$ we have $p \geq 2\delta + 4$ and hence $k' \geq 2$ and $i \geq 4$. It follows that all lengths in the following 2-point amalgamation problem lie in the interval $[1, \delta]$.



This diagram could be suggested by the following considerations. If we fill in the distance $d(a_1, a_2) = j - 1$, then the resulting configuration embeds in a tree with root u_1, and with a_2, c, u_2 points at distances i', j', k' respectively lying on paths from the root which meet only at u_1. Then a_1 represents a vertex adjacent to u_2, and lying on the path from the root to u_2.

In particular, all triangles other than (a_2, c, u_2) and (a_2, c, a_1) are geodesics. And of course (a_2, c, u_2) is a triangle of type (i, j, k). Thus neither factor of the amalgamation involves a forbidden triangle. Furthermore as (c, u_1, u_2) is a geodesic and $\delta \geq 4$ here, neither factor is a Henson constraint of normal or antipodal type.

By 4-triviality, it follows that the amalgamation diagram may be completed in $\mathcal{A}$, and as $i' + k' = j$, the vertices u_1, u_2 force $d(a_1, a_2) = j - 1$. The triangle (a_1, a_2, c) then has type $(i-1, j-1, k)$ and perimeter $p - 2$, a contradiction. Thus no triangle of even perimeter greater than C_0 is found in $\mathcal{A}$. □

Corollary 13.14.1. Let $\mathcal{A}$ be a 4-trivial amalgamation class of diameter δ. Suppose $K_1 = \infty$, that is the corresponding metrically homogeneous graph is bipartite. Then $\mathcal{A}$ and $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1,\mathcal{S}}$ contain the same triangles and the same δ-Henson constraints.

Of course, we have only addressed the triangles in $\mathcal{A}$, but once they match up with $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ then our original definition of $\mathcal{S}$ will in fact give the δ-Henson constraints of $\mathcal{A}$.

13G. Triangles of odd perimeter

LEMMA 13.15. *Let $\mathcal{A}$ be an amalgamation class of diameter δ, and Γ the Fraïssé limit of $\mathcal{A}$. Assume that some triangle of odd perimeter occurs in $\mathcal{A}$, and let p be the least odd number which is the perimeter of a triangle in $\mathcal{A}$. Then the following hold.*

1. *A p-cycle embeds isometrically in* Γ.
2. $p \leq 2\delta + 1$.
3. $p = 2K_1 + 1$.

PROOF. We introduce some metric space terminology which differs noticeably from graph theoretic terminology. We will call any sequence of vertices $(a_0, \ldots, a_n)$ in Γ a *path*, and the *length* of a path is the sum of successive distances $d(a_i, a_{i+1})$. A path is *full* if all successive distances are equal to 1. A path is a *circuit* if $a_n = a_0$.

By assumption Γ contains a circuit of odd length. Let C be a circuit of minimal odd length p_0. Then $p_0 \leq p$. Any path extends to a full path of the same length, so we may suppose that C is a full path. Then

$$C = (a_0, \ldots, a_{p_0})$$

with $a_{p_0} = a_0$.

As the circuit C has minimal odd length, the vertices a_i must be distinct for $i < p_0$. We claim that the embedding of C into Γ is isometric. If not, we may label the vertices of C so that for some pair (a_0, a_m) we have

$$d_\Gamma(a_0, a_m) < d_C(a_0, a_m) \tag{13.5}$$

In particular $m < p_0$. Consider the circuits induced in Γ on

$$C' = (a_0, a_1, \ldots, a_m, a_0); \qquad C'' = (a_0, a_m, a_{m+1}, \ldots, a_{p_0}).$$

Let the lengths of C' and C'' be ℓ' and ℓ'' respectively.

In view of (13.5), we have $\ell', \ell'' < p_0$. Furthermore $\ell' + \ell'' = p_0 + 2d(a_0, a_m)$ and thus one of ℓ', ℓ'' is odd. This contradicts the minimality of p_0. Thus the embedding of C into Γ is an isometry.

With $m = (p_0 - 1)/2$, it follows that the triangle (a_0, a_m, a_{m+1}) is of type $(1, m, m)$ and perimeter p_0. Thus $p_0 = p$. Our first point is proved.

Since $p = 2m + 1 \leq 2\delta + 1$ our second point follows. And since we have a triangle of type $(1, m, m)$, we have $K_1 \leq m$, $2K_1 + 1 \leq p$, hence $2K_1 + 1 = p$ and $m = K_1$. □

LEMMA 13.16. *Let $\mathcal{A}$ be an amalgamation class of diameter δ and associated parameters K_1, K_2. Then*

$$\mathcal{A} \subseteq \mathcal{A}^\delta_{K_1,K_2}$$

PROOF. It suffices to show that any triangle occurring in $\mathcal{A}$ belongs to $\mathcal{A}^\delta_{K_1,K_2}$. In other words, we must show that no triangles of type (i, j, k) with $p = i+j+k$ odd and satisfying either one of the following two constraints can occur in $\mathcal{A}$ (Definition 1.20).

$$p < 2K_1 + 1;$$
$$p > 2K_2 + 2i.$$

By Lemma 13.15 (3) we have $p \geq 2K_1 + 1$. So we turn to the second point.

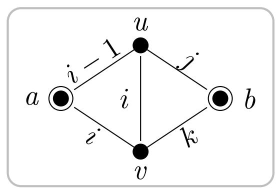
Suppose we have a triangle of type (i, j, k) in $\mathcal{A}$ with $p = i + j + k$ odd and

$$p > 2K_2 + 2i.$$

Suppose here that i is minimized. If $i = 1$ then as p is odd we have $j = k > K_2$ and this contradicts the definition of K_2. So

$$i > 1.$$

Consider the following amalgamation.



This has a completion in $\mathcal{A}$. As $i \le j \le k$ and $p > 2K_2 + i$, we have $k > K_2$ and hence $d(a, b) \ne k$. So $d(a, b) = k + \epsilon$ with $\epsilon = \pm 1$. Then $\mathcal{A}$ contains a triangle of type

$$(i - 1, j, k + \epsilon)$$

of odd perimeter, and minimal entry $i-1$, and the perimeter is greater than $2K_2 + 2(i - 1)$. This contradicts the choice of i. □

Lemma 13.17. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter δ. If*

$$K_1 \le k \le K_2,$$

then a triangle of type $(1, k, k)$ belongs to $\mathcal{A}$.

Proof. We may suppose that

$$K_1 < k < K_2.$$

Consider the following amalgamation diagram (Figure 117).

The factor $(cu_1a_1u_2)$ is a geodesic.

In the factor $(ca_2u_1u_2)$, there are at least three distinct lengths occurring, so this factor is not a Henson constraint of normal or antipodal type. In this factor, the nongeodesic triangles occurring have types $(1, K_1, K_1)$ and $(1, K_2, K_2)$, both assumed present in $\mathcal{A}$. By 4-triviality, the diagram has a completion in $\mathcal{A}$.

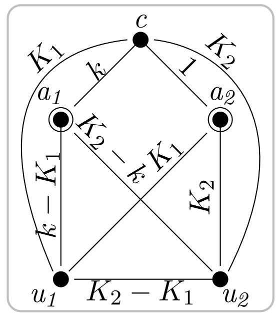


FIGURE 117

In the amalgam, the points u_1, u_2 ensure that $d(a_1, a_2) = k$, and thus the triangle (a_1, a_2, c) has type $(1, k, k)$. □

LEMMA 13.18. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter δ. Suppose that (i, j, k) is the type of a triangle satisfying the following conditions.*

1. $K_1 \leq i \leq K_2$;
2. $p = i + j + k \leq C_0 - 2$;
3. *If* $C_0 = 2\delta + 2$ *then* $i < \delta$;
4. *If* p *is odd, then* $\min(j, k) < \delta$.

Then there is a triangle of type (i, j, k) in $\mathcal{A}$.

PROOF. By Lemma 13.14 we may suppose that

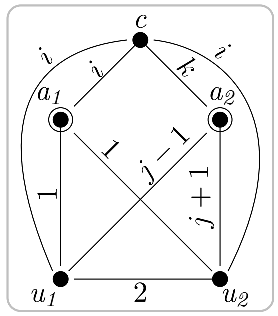
$$p \text{ is odd.}$$

Our assumptions then rule out the case $j = k = \delta$, so we may suppose $j < \delta$. If $j = 1$ then as p is odd the triangle inequality gives $i = k$ and a triangle of type (i, j, k) belongs to $\mathcal{A}$ by Lemma 13.17. So we suppose

$$1 < j < \delta$$

Consider the amalgamation shown below. As each factor contains a geodesic, neither factor is a Henson constraint of normal or antipodal type (Remark 13.7).

So we consider the triangles present.



Leaving aside the geodesic triangles (a_1, u_1, u_2) and (a_2, u_1, a_2), the remaining triangles in this diagram have types

$$(2, i, i),\ (1, i, i),\ (i, j - 1, k), \text{ and } (i, j + 1, k).$$

Now $2i + 2 < C_0$ by our hypothesis, so a triangle of type $(2, i, i)$ belongs to $\mathcal{A}$ by Lemma 13.14.

As $K_1 \leq i \leq K_2$, a triangle of type $(1, i, i)$ belongs to $\mathcal{A}$ by Lemma 13.17.

We must consider the remaining two triples

$$(i, j - 1, k), \text{ and } (i, j + 1, k)$$

These satisfy the triangle inequality since (i, j, k) does and p is odd. Thus they represent triangle types, and their perimeters are $p \pm 1 < C_0$, even. Thus triangles of these types occur in $\mathcal{A}$.

Therefore the diagram has a completion in $\mathcal{A}$, and then the triangle (a_1, a_2, c) has type (i, j, k). □

LEMMA 13.19. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a metrically homogeneous graph of diameter δ, with associated parameters K_1, K_2, C_0, C_1. Then*

$$C_1 \geq \min(2\delta + K_2, C_0 - 1).$$

PROOF. If $C_0 = 2\delta + 2$ then $C_1 \geq C_0 - 1$ and there is nothing to show. So we suppose

$$C_0 > 2\delta + 2.$$

Suppose toward a contradiction that

$$C_1 < 2\delta + K_2,\ C_0 - 1.$$

. Let $K_2 \equiv \epsilon \pmod 2$ with $\epsilon = 0$ or 1. Let $K_2' = K_2 + 1 - \epsilon$. Then K_2' is odd. Set

$$\begin{aligned} i &= K_2; \\ j &= \frac{C_1 - K_2'}{2}; \\ k &= j + (1 - \epsilon). \end{aligned}$$

Then

$$\begin{aligned} j &< (2\delta + K_2 - K_2')/2 \leq \delta; \\ i + j + k &= i + 2j + (1 - \epsilon) = K_2 + (C_1 - K_2') + (1 - \epsilon) = C_1. \end{aligned}$$

We claim that (i, j, k) is the type of a triangle.

As $|j - k| \leq 1$ we have

$$j \leq i + k \text{ and } k \leq i + j.$$

Furthermore $C_1 \geq 2\delta + 1$, so

$$i < C_1 - i = j + k.$$

Thus (i, j, k) is the type of a triangle of perimeter C_1.

Now Lemma 13.18 applies since $C_1 \leq C_0 - 2$, $C_0 > 2\delta + 2$, and $j < \delta$. Thus there is a triangle of this type in $\mathcal{A}$, contradicting the definition of C_1. □

LEMMA 13.20. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter δ, with associated parameter C_1. Then no triangle of odd perimeter $p \geq C_1$ belongs to $\mathcal{A}$.*

PROOF. Suppose on the contrary that there is some odd $p \geq C_1$ and some triangle of type (i, j, k) with $i + j + k = p$ which belongs to $\mathcal{A}$. Take i to be minimal. We have $p > C_1$, so also $p - 2 \geq C_1$.

Fix a triangle (a, b, c) in $\mathcal{A}$ with $d(b, c) = i$, $d(a, b) = j$, $d(a, c) = k$. Take u on a geodesic from b to c with

$$d(b, u) = i - 1,\ d(u, c) = 1 \text{ (Figure 118).}$$

If $d(a, u) = k \pm 1$ then (a, b, u) is a triangle of perimeter p or $p - 2$ and type $(i - 1, j, k \pm 1)$, violating the minimality of i. Thus

$$d(a, u) = k.$$

So (a, u, c) is a triangle of type $(1, k, k)$. Thus $k \leq K_2$, and we have $p \leq 2\delta + K_2$.

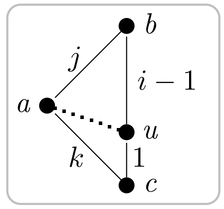


FIGURE 118

Furthermore (a, b, u) is a triangle of type $(i-1, j, k)$ and perimeter $p-1$, so $p-1 < C_0$. As p is odd it follows that $p \leq C_0 - 1$. Then $p \leq \min(2\delta + K_2, C_0 - 1)$ and thus $C_1 < \min(2\delta + K_2, C_0 - 1)$, contradicting Lemma 13.19. □

Corollary 13.20.1. Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter δ, with associated parameters K_1, K_2, C_0, C_1. Then

$$\mathcal{A} \subseteq \mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}.$$

PROOF. Lemmas 13.14, 13.16 and 13.20. □

LEMMA 13.21. *Let $\mathcal{A}$ be an amalgamation class of diameter δ determined by constraints on triangles and δ-Henson constraints, with associated parameters K_1, K_2. Then for any triangle type (i, j, k) with $p = i + j + k$ odd, if*

$$2K_1 + 1 \leq p \leq 2K_2 + 1$$

then there is a triangle of type (i, j, k) in $\mathcal{A}$.

PROOF. For $p = 2K_1 + 1$ apply Lemma 13.15 (1). So we assume

$$p > 2K_1 + 1$$

and we proceed inductively.

Taking $N = p$ and $m = 2$, Lemma 13.13 implies that any triangle of type (i', j', k') for which

$$i' + j' + k' = p,\ i' \geq 1, \text{ and } j', k' \geq 2$$

belongs to $\mathcal{A}$, since our induction hypothesis applies to $p - 2$.

This applies in particular to the triple (i, j, k), taking $i \leq j \leq k$, unless $j = 1$, in which case $i = 1$, $p = 3$, and $2K_1 + 1 < 3$, which is impossible. □

13H. Identification of $\mathcal{A}$

Proposition 13.22. Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter δ, with associated parameters K_1, K_2, C_0, C_1, and $\mathcal{S}$. Then $\mathcal{A}$ and $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ contain the same triangles, and the same Henson constraints.

Proof. We first verify the claim for triangles.

We have $\mathcal{A} \subseteq \mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ by Corollary 13.20.1. So it suffices to show conversely that each triangle type (i,j,k) realized in $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ is realized in $\mathcal{A}$.

Let $p = i+j+k$. If p is even, then Lemma 13.14 gives the desired result. So we suppose

$$p \text{ is odd, and } i \le j \le k$$

We have the following conditions.

1. $p \ge 2K_1 + 1$;
2. $p < 2K_2 + 2i$;
3. $p < C_1$.

If $p \le 2K_2 + 1$ then a triangle of type (i,j,k) belongs to $\mathcal{A}$ by Lemma 13.21. So we will suppose

$$p > 2K_2 + 1$$

and proceed by induction on p (odd).

Set

$$m = (p - 2K_2 + 1)/2$$

The condition $p < 2K_2 + 2i$ may be expressed as: $i \ge m$. Thus our claim may be expressed as follows.

$$\begin{array}{l}\text{If } i+j+k = p,\ \min(i,j,k) \ge m,\\ \text{then the type } (i,j,k) \text{ is represented in } \mathcal{A}\end{array}$$

The corresponding claim for $p-2$ involves $m-1$, and holds by our inductive hypothesis. By Lemma 13.13, if there is some triangle in $\mathcal{A}$ whose type (i,j,k) satisfies the conditions $i+j+k = p$, $\min(i,j,k) \ge m$, then all such types are represented by triangles in $\mathcal{A}$. Furthermore, since $\mathcal{A} \subseteq \mathcal{A}^\delta_{K_1,K_2}$, it suffices to find a triangle of type (i,j,k) in $\mathcal{A}$ with $i+j+k = p$, as the other condition then necessarily holds.

If $p > 2\delta$, then since $p < C_1$ there is such a triangle by the definition of C_1. So suppose

$$p < 2\delta.$$

Consider the triple (i, j, k) with

$$\begin{aligned}
i &= K_2;\\
j &= \lfloor \frac{p - K_2}{2} \rfloor;\\
k &= (p - K_2) - j = p - (i + j).
\end{aligned}$$

Then $j \le k \le j + 1$ and $i < 2j \le j + k$. Thus (i, j, k) is the type of a triangle with $i + j + k = p$. Furthermore

$$j \le \frac{p - K_2}{2} < \frac{2\delta - K_2}{2} < \delta$$

and thus also $k \le \delta$.

Apply Lemma 13.18. Since we have $i = K_2$, $p < 2\delta \le C_0 - 2$, and $j < \delta$, a triangle of type (i, j, k) is in $\mathcal{A}$. This concludes the proof.

The claim for Henson constraints follows from the definition of $\mathcal{S}$ and the definition of $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$. □

This proves the first half of Part II of the Main Theorem (i.e., part $(II\text{–}A)$). The second part concerns the admissibility of the parameter sequence $K_1, K_2, C_0, C_1, \mathcal{S}$.

13I. Parameters for metrically homogeneous graphs of non-generic type

We append here a discussion of the parameters assigned to metrically homogeneous graphs of non-generic type by our conventions, for $\delta \ge 3$. Of course, δ will be the diameter of Γ in any case.

The metrically homogeneous graphs of non-generic type are either finite or of infinite diameter, and the finite ones are either of degree 2 or antipodal of diameter 3 and degree at least 3.

(I) Infinite diameter: The tree-like graphs $T_{m,n}$.

— $\delta = \infty$, C_0, C_1 undefined (or ∞, if this is unsatisfactory).

— $K_1 = 1$, $K_2 = \infty$ $(n \ge 3)$ or $K_1 = \infty, K_2 = 0$ $(n = 2)$.

— If $\mathcal{S}$ is nonempty, then $3 \le n < \infty$ and $\mathcal{S}$ consists of a clique of order $n + 1$.

(II) Finite of degree 2: an n-gon for some $n \geq 6$
If n is even:
— $\delta = n/2$.
— $K_1 = \infty, K_2 = 0$.
— $C_1 = 2\delta + 1$, $C_0 = 2\delta + 2$
— $\mathcal{S}$ empty.

If n is odd:
— $\delta = (n-1)/2$.
— $K_1 = K_2 = \delta$.
— $C_1 = 2\delta + 3$, $C_0 = 2\delta + 2$
— $\mathcal{S}$ empty.

(III) Finite, antipodal of diameter 3, degree at least 3 (double cover of an independent set, C_5, or $K_3 \,\square\, K_3$).
— $\delta = 3$, $C_1 = 7$, $C_2 = 8$;
— $K_1 = 1$, $K_2 = 2$ or $K_1 = \infty$, $K_2 = 0$.
— $C_1 = 2\delta + 1$, $C_0 = 2\delta + 2$.
— $\mathcal{S}$ consists of a clique of order 4 or 5, and its antipodal companions. or is empty.

Alternatively, sorting these out in our usual manner according to the three types (for the admissible case), we arrive at the following.

(I) $K_1 = \infty$, $K_2 = 0$
— $T_{m,2}$ (regular tree): parameters of the generic bipartite graph.
— n-gon, $n \geq 6$ even: parameters of the generic antipodal bipartite graph.
— Antipodal double cover of an independent set (complement of a perfect matching): parameters of the generic antipodal bipartite graph of diameter 3 (overlaps with n-gon case for $n = 6$).

(II) $C \leq 2\delta + K_1$
— n-gon, $n \geq 6$ odd: the values $K_1 = K_2 = \delta$ violate various clauses in the definition of admissibility.
— antipodal of diameter 3, double cover of C_5 or $K_3 \,\square\, K_3$: similar to the parameters of $\mathcal{A}^{\delta}_{a,n}$, the generic K_n-free antipodal graph of diameter δ, with $n = 4$ or 5; but that would require $\delta \geq 4$.

(III) $C > 2\delta + K_1$
— $T_{m,n}$ with $n \geq 3$: The parameters K_1, K_2, C_0, C_1 impose no constraints; these parameters agree with the parameters

of the generic metrically homogeneous graph of unbounded diameter with Γ_1 a Henson graph, or random.

We remark that the antipodal double covers of C_5 and $K_3 \,\square\, K_3$ are not 4-trivial. By Lemma 13.10 or inspection, they contain triangles of type $(2,2,2)$, and then 4-triviality would imply the existence of an infinite 2-anticlique $I_\infty^{(2)}$.

CHAPTER 14

AMALGAMATION REQUIRES ADMISSIBILITY

To complete the proof of Theorem 13.4, we must deal with part (II–B) of our more explicit formulation.

Theorem (Main Theorem, Part (II–B)). Let $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ be the sequence of parameters associated to a 4-trivial amalgamation class $\mathcal{A}$ with $\delta \geq 3$. Then the sequence

$$(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$$

is admissible.

We are interested in applying this mainly in the case

$$\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$$

itself; that is, if we define a class of finite structures in this particular way, then we get an amalgamation class if and only if the parameter sequence is admissible (keeping Theorem 12.1 in mind).

Our more general formulation, or the details of its proof, may be useful also as a part of an approach to the general problem of classifying the countable metrically homogeneous graphs.

We must prove that the numerical criterion given as the definition of admissibility as far as $\delta, K_1, K_2, C_0, C_1$ are concerned is necessary for the amalgamation property, and that the mild additional conditions imposed on the set $\mathcal{S}$ of Henson constraints are also necessary, under the assumption that we have a 4-trivial amalgamation class.

As usual, we set $C = \min(C_0, C_1)$ and $C' = \max(C_0, C_1)$. We know that the parameter sequence is acceptable, so as far as the numerical parameters are concerned, we need only check the further conditions imposed for admissibility.

If $K_1 = \infty$, then there are no triangles of odd perimeter, and hence $K_2 = 0$, $C_1 = 2\delta + 1$. There are no further conditions on C_0 or $\mathcal{S}$ to be checked in this case.

So we will assume throughout that

$$K_1 < \infty.$$

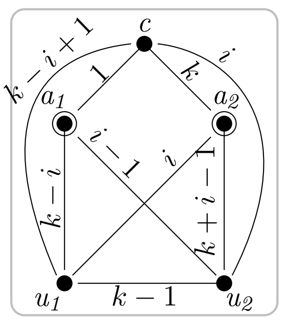
14A. K_1 and K_2

LEMMA 14.1. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$. Suppose we have*

$$1 < i < k \text{ and } i + k \leq \delta + 1,$$

and triangles of type $(i, k - i + 1, k)$ and $(i, i + k - 1, k)$ are in $\mathcal{A}$. Then a triangle of type $(1, k, k)$ is in $\mathcal{A}$.

PROOF. Consider the following amalgamation. Each factor contains a geodesic, hence is not a Henson constraint.



The triangles (a_1, u_1, c), (a_1, u_1, u_2), (a_1, c, u_2), and (a_2, u_1, u_2) are geodesic (of lengths $k-i+1$, $k-1$, i, $k+i-1$, respectively). The other triangles (c, a_2, u_1), (c, a_2, u_2), (c, u_1, u_2) involved in the diagram have types

$$(i, k - i + 1, k),\ (i, k + i - 1, k), \text{ and } (i, k - i + 1, k - 1).$$

Of these, all but the last have been assumed to be in $\mathcal{A}$.

For the type $(i, k - i + 1, k - 1)$, one first checks the triangle inequality, which is immediate. The perimeter is $2k \leq 2\delta$, so this type is represented in $\mathcal{A}$ by Lemma 13.12.

By 4-triviality, it follows that the two factors of this amalgamation diagram occur in $\mathcal{A}$. Therefore the diagram has a completion in $\mathcal{A}$, and in this completion the triangle (a_1, a_2, c) has type $(1, k, k)$, since the points u_1, u_2 ensure $d(a_1, a_2) \leq k$ and $d(a_1, a_2) \geq k$, respectively. □

LEMMA 14.2. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C_0, C_1. Suppose that i, k satisfy*

1. $K_1 \leq i \leq K_2$;
2. $1 < i < k$;
3. $i + k \leq \min(\delta + 1, (C_0 - 2)/2)$.

Then $k \leq K_2$.

PROOF. We show that there is a triangle of type $(1, k, k)$ in $\mathcal{A}$.

By Lemma 14.1 it suffices to check that $\mathcal{A}$ contains triangles of the types

$$(i, k - i + 1, k) \text{ and } (i, i + k - 1, k).$$

It is clear that these triples satisfy the triangle inequality.

We apply Lemma 13.18. Notice first that $i, k < \delta$. Therefore it suffices to check that the perimeters are at most $C_0 - 2$. The two perimeters in question are

$$2k + 1 \leq 2(i + k) - 1.$$

We have $2(i + k) - 1 < C_0 - 2$ by hypothesis. This completes the proof. □

LEMMA 14.3. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2. Then one of the following holds.*

1. $K_1 + K_2 \geq \delta + 1$;
2. $K_1 = 1$, $K_2 = \delta - 1$;
3. $K_1 > 1$, $K_1 + K_2 = \delta$, *and* $C_0 = 2\delta + 2$.

PROOF. Suppose first that

$$K_1 = 1.$$

Then by Lemma 13.18 a triangle of type $(1, \delta - 1, \delta - 1)$ belongs to $\mathcal{A}$, and so $K_2 \geq \delta - 1$. This leads to Case 1 or 2.

Now suppose that

$$K_1 > 1.$$

If $K_1 > \delta/2$ then Case 1 applies, so suppose

$$K_1 \leq \delta/2.$$

Set $i = K_1$, $k = \delta + 1 - i$. If $C_0 > 2\delta + 2$, then by Lemma 14.2 we have $k \leq K_2$ and $K_1 + K_2 \geq \delta + 1$.

If $C_0 = 2\delta + 2$, then we arrive similarly at $K_1 + K_2 \geq \delta$, and thus Case 1 or 3. If $K_1 = \delta/2$, this is clear, while if $K_1 < \delta/2$, then we apply Lemma 14.2 with $k = \delta - i$. □

The major case division among admissible values of parameters is between the case $C \leq 2\delta + K_1$ and $C > 2\delta + K_1$. The following is the first step in this direction.

14B. The low case: $C \leq 2\delta + K_1$

Lemma 14.4. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C', where $K_1 < \infty$. If*

$$C \leq 2\delta + K_1$$

then

$$C > 2K_1 + 2K_2.$$

It is noteworthy that the conclusion fails in the case of the class $\mathcal{A}$ associated with an n-gon, where n is odd. But this class is not 4-trivial.

Proof. Suppose first that

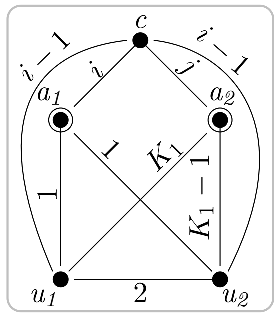
$$K_1 = 1.$$

Then our hypothesis is that $C = 2\delta + 1$. So there is no triangle of type $(1, \delta, \delta)$, and therefore $K_2 \leq \delta - 1$. Thus the desired inequality holds.

Now assume that

$$K_1 > 1.$$

Set $j = \lfloor \frac{C-K_1}{2} \rfloor$, and $i = (C - K_1) - j$. Then $1 < j \leq i \leq \delta$. Consider the following amalgamation.



The factor omitting a_2 contains a geodesic, so is not a Henson constraint. The factor omitting a_1 contains an edge of length 2, so is not a $(1,\delta)$-space, but might be a Henson constraint of antipodal type if $\delta = 3$ and $K_1 = j = 2$. In this case consideration of (c, a_2, u_2) shows $i - 1 = 2$, while consideration of (c, u_1, u_2) shows $i - 1 = 1$, a contradiction. So neither factor is a Henson constraint.

The triangles (a_1, u_1, u_2), (a_1, u_1, c), (a_1, u_2, c) are geodesic, and the other triangles occurring here are of the types

$$(2, K_1 - 1, K_1),\ (2, i-1, i-1),\ (i-1, j, K_1 - 1), \text{ and } (i-1, j, K_1).$$

A triangle of type $(2, K_1 - 1, K_1)$ has perimeter $2K_1 + 1$, and belongs to $\mathcal{A}$ by Lemma 13.21. A triangle of type $(2, i-1, i-1)$ belongs to $\mathcal{A}$ by Lemma 13.12. If there are also triangles of the other two types

$$(i-1, j, K_1 - 1) \text{ and } (i-1, j, K_1)$$

then the amalgamation diagram has a completion in $\mathcal{A}$. The vertices u_1, u_2 then force $d(a_1, a_2) = K_1$ and thus $\mathcal{A}$ contains a triangle of type (K_1, i, j) and perimeter C, a contradiction.

Next we check that the remaining types $(i-1, j, K_1 - 1)$ and $(i-1, j, K_1)$ represent valid triangle types, that is, that the triangle inequality is satisfied.

As $j \leq i \leq j + 1$ and $K_1 > 1$, the only inequality that needs to be checked is

$$K_1 \leq i - 1 + j.$$

But $i-1+j = C - K_1$ and $C > 2\delta$, so

$$K_1 < C - K_1 = i - 1 + j.$$

So we conclude from our analysis so far that a triangle of one of the two types $(i-1, j, K_1 - 1)$ or $(i-1, j, K_1)$ must be forbidden. Let us write the forbidden type as

$$(i-1, j, K_1 - \epsilon)$$

with $\epsilon = 0$ or 1.

The perimeters here are

$$p = C - 1 - \epsilon$$

with $\epsilon = 0$ or 1. One of these perimeters is even, and the corresponding triangle is in $\mathcal{A}$ by Lemma 13.14. So it is the triangle of odd perimeter which is forbidden.

Now we claim

$$C - 2 \geq 2K_1.$$

We have $C - 2 \geq 2\delta - 1$, so this can fail only if

$$K_1 = \delta,\ C = 2\delta + 1.$$

But if $K_1 = \delta$, then $\mathcal{A}$ contains a triangle of perimeter $2\delta + 1$, and thus $C > 2\delta + 1$.

Since $C - 2 \geq 2K_1$ and $p \geq C - 2$ is odd, we have $p \geq 2K_1 + 1$. As $p < C$, it follows that the forbidden triangle of type $(i-1, j, K_1 - \epsilon)$ and odd perimeter p must satisfy the conditions

$$(*) \qquad \begin{aligned} p &> 2K_2 + 2\min(i-1, j, K_1 - \epsilon) \\ &= 2K_2 + 2\min(i-1, K_1 - \epsilon). \end{aligned}$$

Next we show

$$C - 1 \leq 2K_2 + 2(i-1).$$

We have

$$j \leq (C - K_1)/2,\ i \geq (C - K_1)/2.$$

and thus

$$\begin{aligned} 2K_2 + 2(i-1) &\geq 2K_2 + (C - K_1) - 2 \\ &= C + (2K_2 - K_1 - 2) \\ &\geq C + K_2 - 2 \geq C - 1. \end{aligned}$$

Hence

$$p \leq 2K_2 + 2(i-1).$$

So condition $(*)$ becomes

$$(*') \qquad p > 2K_2 + 2(K_1 - \epsilon).$$

If $\epsilon = 0$ then $C > p > 2K_2 + 2K_1$. If $\epsilon = 1$, then C is odd and $C \geq p + 2 > 2K_2 + 2K_1$. So we have $C > 2K_1 + 2K_2$ in either case. $\square$

In the remainder of our analysis of the case $C \leq 2\delta + K_1$, we consider separately the cases

(a) $C_0 \leq 2\delta + K_1$;
(b) $C_1 \leq 2\delta + K_1 < C_0$.

Furthermore, case (a) will be subdivided further according as the parameter K_1 is less than, or equal to, K_2.

LEMMA 14.5. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose*

$$K_1 < \infty,\ C_0 \leq 2\delta + K_1.$$

Then $C_0 = 2K_1 + 2K_2 + 2$ and $C_1 \geq C_0 - 1$.

PROOF. We have $C \leq C_0 \leq 2\delta + K_1$, so by the previous lemma we find

$$C > 2K_1 + 2K_2.$$

In other words,

$$C_0 \geq 2K_1 + 2K_2 + 2;$$
$$C_1 \geq 2K_1 + 2K_2 + 1.$$

Set $i = K_1$, $j = \lfloor \frac{C_0 - K_1 - 3}{2} \rfloor$, $k = (C_0 - K_1 - 3) - j$. Then we have $j \leq k \leq j+1$. We claim

$$i \leq j \leq k < \delta.$$

We have

$$j \geq (C_0 - K_1 - 4)/2 \geq (K_1 + 2K_2 - 2)/2.$$

As $C_0 \leq 2\delta + K_1$ we find $K_1 \geq 2$ and thus $j \geq K_2 \geq i$. Also

$$k \leq j + 1 \leq (C_0 - K_1 - 2)/2 \leq \delta - 1.$$

Since $i \le j \le k \le j+1$ the triple (i,j,k) satisfies the triangle inequality. The perimeter $p = i+j+k$ is $C_0 - 3$. Lemma 13.18 applies, and there is a triangle of type (i,j,k) in $\mathcal{A}$.

As $C_0 - 3$ is odd, and $\mathcal{A} \subseteq \mathcal{A}^\delta_{K_1,K_2}$, we find

$$C_0 - 3 < 2K_2 + 2i = 2K_1 + 2K_2.$$

Thus

$$C_0 = 2K_1 + 2K_2 + 2.$$

Hence $C_1 \ge C_0 - 1$ as well. □

In order to control C_1 further in the context of the preceding lemma, we insert the following.

LEMMA 14.6. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \ge 3$ with associated parameters K_1, K_2, C, C'. Suppose*

$$C \ge 2K_1 + 2K_2 \text{ and } C \text{ is even.}$$

Then

$$C > \min(4K_2, 4\delta - 2K_2 - 2).$$

PROOF. Suppose on the contrary that

$$C \le 4K_2, 4\delta - 2K_2 - 2.$$

If $K_1 = 1$ then $K_2 \ge \delta - 1$ by Lemma 14.3. In this case we have $C \le 4\delta - 2(\delta-1) - 2 \le 2\delta$, which is impossible. So we suppose

$$K_1 > 1.$$

Set $k = \frac{C-2K_2}{2}$, $i = \lfloor \frac{C-k}{2} \rfloor$, and $j = (C-k) - i$. Then by our assumptions we have

$$K_1 \le k \le K_2.$$

We claim also that

$$1 < k < i \le j \le \delta.$$

The inequality $k < i$ can be written as

$$2k \le C - k - 2,$$

which reduces to

$$C \le 6K_2 - 4.$$

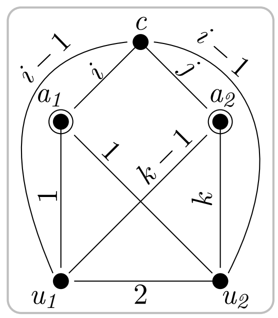
This holds since $C \leq 4K_2$ and $K_2 \geq 2$. Furthermore

$$2i \leq C - k = (C + 2K_2)/2 \leq (4\delta - 2)/2 = 2\delta - 1,$$

and thus

$$i \leq \delta - 1 \text{ and } j \leq i + 1 \leq \delta.$$

Now consider the following amalgamation.



The factor omitting a_2 contains a geodesic, so is not a Henson constraint. The factor omitting a_1 contains the distinct lengths $j, k > 1$, so is not a Henson constraint.

The triangles (a_1, u_1, c), (a_1, u_2, c), and (a_1, u_1, u_2) are geodesic, and the other triangles involved here have types

$$(2, k-1, k),\ (2, i-1, i-1),\ (i-1, j, k-1), \text{ and } (i-1, j, k).$$

These all satisfy the triangle inequality, so they are indeed triangle types.

As $K_1 \leq k \leq K_2$ and $k < \delta$, by Lemma 13.18 a triangle of type $(2, k-1, k)$ belongs to $\mathcal{A}$. By Lemma 13.12 a triangle of type $(2, i-1, i-1)$ belongs to $\mathcal{A}$. The third triangle has even perimeter $C - 2$, so belongs to $\mathcal{A}$ by Lemma 13.14.

We claim finally that there is also a triangle of type

$$(i-1, j, k)$$

in $\mathcal{A}$. As the perimeter of this triangle is $C-1$, it suffices to check that this triangle belongs to $\mathcal{A}^{\delta}_{K_1,K_2}$. This comes down to the inequality

$$C - 1 \leq 2K_2 + 2k,$$

which corresponds to the definition of k.

Therefore the amalgamation diagram can be completed in $\mathcal{A}$. The distance $d(a_1, a_2)$ must be $k-1$ or k, and as $i+j+k=C$ we find

$$d(a_1, a_2) = k-1.$$

So the triangle (a_1, a_2, c) has type $(i, j, k-1)$. As $i+j+(k-1) = C-1$ is odd, we must then have the inequality

$$C-1 < 2K_2 + 2(k-1) = C-2$$

and so we arrive at a contradiction. □

LEMMA 14.7. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that*

$$C_0 \leq 2\delta + K_1 \text{ and } K_1 < K_2.$$

Then $C_1 = 2K_1 + 2K_2 + 1$ and $C_0 = 2K_1 + 2K_2 + 2$.

PROOF. By Lemma 14.5 we have

$$C_0 = 2K_1 + 2K_2 + 2,$$

and $C_1 \geq C_0 - 1$. We claim $C_1 = C_0 - 1$.

Suppose $C_1 > C_0$. Then $C = C_0$ is even, and by Lemma 14.6 we have

$$2K_1 + 2K_2 + 2 > \min(4K_2, 4\delta - 2K_2 - 2).$$

As $K_2 > K_1$ we have

$$2K_1 + 2K_2 + 2 \leq 4K_2,$$

so our inequality becomes

$$\begin{aligned} 2K_1 + 2K_2 + 2 &> 4\delta - 2K_2 - 2; \\ K_1 + 2K_2 &\geq 2\delta - 1. \end{aligned}$$

Then

$$C_0 = (K_1 + 2K_2) + (K_1 + 2) \geq 2\delta + K_1 + 1,$$

contradicting our initial hypothesis. □

LEMMA 14.8. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph* *of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that K_2 is odd and that*

$$3K_2 \leq 2\delta - 1.$$

Then $C_1 \leq 4K_2 + 1$.

PROOF. This will require an amalgamation argument.

Suppose on the contrary that

$$C_1 > 4K_2 + 1.$$

Our assumptions imply $K_2 < \delta - 1$, and hence by Lemma 14.3, we have $K_1 > 1$. As K_2 is odd, we have

$$3 \leq K_2 \leq \delta - 2 \text{ and } \delta \geq 5.$$

If $C_0 \leq 4K_2$, then $C = C_0$, and by Lemma 14.4 we find

$$C_0 > \min(2\delta + K_1, 2K_1 + 2K_2),$$

while $2\delta + K_1 \geq (3K_2 + 1) + K_1 \geq 2K_1 + 2K_2$, so we have

$$C_0 > 2K_1 + 2K_2.$$

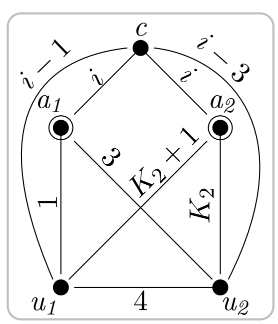
Then by Lemma 14.6 we find

$$C_0 > \min(4K_2, 4\delta - 2K_2 - 2) = 4K_2,$$

and we have a contradiction. Thus in fact

$$C_0 > 4K_2.$$

Now let $i = (3K_2 + 1)/2$. Then $1 < i \leq \delta$. Consider the amalgamation



Note that $K_2 + 2 \leq K_2 + (K_2 + 1)/2 = (3K_2 + 1)/2 \leq \delta$. Thus all lengths in this diagram are in the range $[1, \delta]$.

Each factor contains at least two lengths greater than 1, so is not a Henson constraint.

There are four nongeodesic triangles involved in this diagram. Two of them have even perimeter:

$$\begin{array}{lll} cu_1u_2 & (4, i-3, i-1) & p = 3K_2 + 1 \leq 2\delta; \\ ca_2u_2 & (K_2+2, i-3, i) & p = 4K_2 < C_0. \end{array}$$

Thus these types are represented in $\mathcal{A}$.

The other two nongeodesic triangles involved have odd perimeter:

$$\begin{array}{lll} a_2u_1u_2 & (4, K_2+1, K_2+2) & p = 2K_2 + 7 \leq 4K_2 + 1 < C_1; \\ cu_1a_2 & (K_2+1, i-1, i) & p = 4K_2 + 1 < C_1. \end{array}$$

In order to see that triangles of these types are in $\mathcal{A}$, we must show that they are in $\mathcal{A}^{\delta}_{K_1,K_2}$.

Both perimeters are at least $2K_2 + 1$, so it suffices to check the inequalities corresponding to the bound K_2:

$$2K_2 + 7 < 2K_2 + 2 \cdot 4;$$
$$4K_2 + 1 < 2K_2 + 2(K_2 + 1).$$

As these inequalities are satisfied, our amalgamation diagram has a completion in $\mathcal{A}$. In this completion, the vertex u_1 forces

$$d(a_1, a_2) = K_2 \text{ or } K_2 + 2.$$

Therefore $\mathcal{A}$ must contain a triangle of one of the following types:

$$(K_2, i, i) \text{ or } (K_2 + 2, K_2 + 2, 3).$$

These have odd perimeter, respectively $4K_2 + 1$ and $2K_2 + 7$, so we must then have one of the corresponding inequalities

$$4K_2 + 1 < 2K_2 + 2 \cdot K_2;$$
$$2K_2 + 7 < 2K_2 + 2 \cdot 3,$$

and as both fail, we reach a contradiction. □

LEMMA 14.9. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that K_2 is even and*

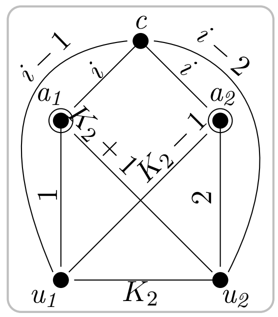
$$3K_2 \leq 2\delta - 2,\ C_0 = 4K_2 + 2.$$

Then $C_1 = 4K_2 + 1$.

PROOF. We have $C_1 \geq 4K_2 + 1$ by Lemma 13.19. Suppose

$$C_1 > 4K_2 + 1.$$

Let $i = (3/2)K_2 + 1$. Consider the amalgamation



Here $K_1 \neq 1, \delta - 1, \delta$, so neither factor is a Henson constraint.

The nongeodesic triangles occurring in this diagram have types

$(2, K_2 - 1, K_2)$, $(K_2, i - 2, i - 1)$, $(K_2 + 1, i - 2, i)$, and $(K_2 - 1, i - 1, i)$.

These triangles have perimeters respectively

$$2K_2 + 1,\ 4K_2 - 1,\ 4K_2 + 1, \text{ and } 4K_2.$$

As $4K_2 < C_0$ the last is realized.

For the other three triangles, we apply Proposition 13.22. As the three perimeters are odd, at least $2K_1+1$, and less than C_1, it suffices to check the inequalities corresponding to K_2, namely

$$\begin{aligned}
&2K_2 + 1 < 2K_2 + 2 \cdot 2;\\
&4K_2 - 1 < 2K_2 + 2 \cdot K_2;\\
&4K_2 + 1 < 2K_2 + 2 \cdot (K_2 + 1).
\end{aligned}$$

Since these all hold, it follows that the amalgamation diagram can be completed in $\mathcal{A}$.

The possible values of $d(a_1, a_2)$ are $K_2 - 1$ and K_2. So the triangle (a_1, a_2, c) has type

$$(i, i, K_2 - \epsilon)$$

with $\epsilon = 0$ or 1, and perimeter $4K_2 + 2 - \epsilon$. If $\epsilon = 0$ then this is C_0, which is impossible. So $d(a_1, a_2) = K_2 - 1$ and the triangle

(a_1, a_2, c) has odd perimeter $4K_2 + 1$. Its type must therefore satisfy the inequality

$$4K_2 + 1 \leq 2K_2 + 2(K_1 - 1).$$

But this fails, so we have a contradiction. □

LEMMA 14.10. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that $K_1 < \infty$ and*

$$C_0 \leq 2\delta + K_1.$$

Then $C_1 = 2K_1 + 2K_2 + 1$ and $C_0 = 2K_1 + 2K_2 + 2$.

PROOF. If $K_1 < K_2$ then Lemma 14.7 applies. So suppose that

$$K_1 = K_2.$$

By Lemma 14.5 we have

$$C_0 = 4K_2 + 2;$$
$$C_1 \geq 4K_2 + 1.$$

Since $C_0 = 4K_2 + 2 \leq 2\delta + K_1 = 2\delta + K_2$ we have

$$3K_2 \leq 2\delta - 2,$$

and one of Lemmas 14.8, 14.9 applies □

LEMMA 14.11. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Then one of the following holds.*

1. $K_1 + K_2 \geq \delta + 1$;
2. $K_1 = 1$, $K_2 = \delta - 1$;
3. $K_1 > 1$, $K_1 + K_2 = \delta$, $C_1 = 2\delta + 1$, *and* $C_0 = 2\delta + 2$.

PROOF. This differs from Lemma 14.3 only in Case 3, where we specify the value of C_1. So we may suppose

$$K_1 > 1,\ K_1 + K_2 = \delta, \text{ and } C_0 = 2\delta + 2.$$

Then Lemma 14.10 determines C_1. □

As a slight variation we have the following.

LEMMA 14.12. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that $K_1 < \infty$ and $C_1 = 2\delta + 1$. Then*

$$K_1 + K_2 = \delta \text{ and } C_0 = 2\delta + 2.$$

PROOF. By Lemma 13.19 we have

$$C_1 \geq \min(2\delta + K_2, C_0 - 1).$$

As $C_1 < 2\delta + K_2$ it follows that $C_1 \geq C_0 - 1$ and thus $C_0 = 2\delta + 2$. We still need to check that $K_1 + K_2 = \delta$.

As $C_1 = 2\delta + 1$ we have $K_2 < \delta$. By Lemma 14.11 if $K_1 = 1$ we have our result. So suppose

$$K_1 \geq 2.$$

By Lemma 14.10 we have

$$C_1 = 2K_1 + 2K_2 + 1,$$

and thus $K_1 + K_2 = \delta$. □

LEMMA 14.13. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that*

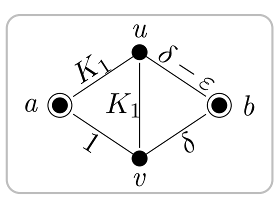
$$C_0 > 2\delta + K_1.$$

Then

$$K_1 + 2K_2 \geq 2\delta - 1.$$

PROOF. We may suppose that $K_2 < \delta$.

Let $K_1 \equiv \epsilon \pmod 2$ with $\epsilon = 0$ or 1. Consider the following amalgamation.



Of course the triangle (a, u, v) belongs to $\mathcal{A}$. As the perimeter p of the triangle (b, u, v) is $K_1 + 2\delta - \epsilon$, the perimeter is even. As $p < C_0$, the triangle (b, u, v) also belongs to $\mathcal{A}$. Thus this diagram can be completed in $\mathcal{A}$.

As $K_2 < \delta$, in the completed diagram we must have $d(a, b) = \delta - 1$, so that the triangle (a, b, u) has type

$$(K_1, \delta - 1, \delta - \epsilon).$$

In particular, the triangle (a, b, u) has odd perimeter. Since this triangle is in $\mathcal{A}$, we must have the inequality

$$(\delta - 1) + (\delta - \epsilon) \leq 2K_2 + K_1.$$

If $\epsilon = 0$ this is our claim. If $\epsilon = 1$, then K_1 is odd and $2K_2 + K_1 \geq 2\delta - 2$, so again our claim follows. □

Proposition 14.14. Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that

$$K_1 < \infty, \; C \leq 2\delta + K_1.$$

Then one of the following holds.

1. $C' = C + 1$ and:

$$C = 2K_1 + 2K_2 + 1 \geq 2\delta + 1 \text{ and } K_1 + 2K_2 \leq 2\delta - 1.$$

2. $C' > C + 1$ and:

$$K_1 = K_2 \text{ and } C = 4K_2 + 1 = 2\delta + K_2.$$

PROOF. If $C = 2\delta + 1$, then by Lemma 14.12 we have $K_1 + K_2 = \delta$ and $C_0 = 2\delta + 2$, so $C' = C + 1$ and $C = 2K_1 + 2K_2 + 1$. Also as $C = 2\delta + 1$ we have $K_2 < \delta$, and thus

$$K_1 + 2K_2 = K_2 + \delta \leq 2\delta - 1.$$

So we arrive at Case 1.

Suppose therefore that

$$C > 2\delta + 1.$$

If $C_0 \leq 2\delta + K_1$, then by Lemma 14.10 we have

$$C = 2K_1 + 2K_2 + 1 \text{ and } C' = C + 1.$$

In particular, the condition $C_0 \leq 2\delta + K_1$ becomes

$$K_1 + 2K_2 + 1 \leq 2\delta.$$

and so we have the situation described by (1).

So we may suppose

$$C_0 > 2\delta + K_1.$$

Then our initial hypothesis gives

$$C_1 \leq 2\delta + K_1.$$

Case 1. Suppose $C_1 = C_0 - 1$.

Then we have

$$C_1 = 2\delta + K_1,\ C_0 = 2\delta + K_1 + 1.$$

By Lemma 14.4 we have

$$2\delta + K_1 > 2K_1 + 2K_2,\ \ K_1 + 2K_2 \leq 2\delta - 1$$

Then by Lemma 14.13 we have

$$K_1 + 2K_2 = 2\delta - 1.$$

Thus $C_1 = 2K_1 + 2K_2 + 1$, and we have Case 1.

Case 2. Suppose that $C_0 > C_1 + 1$.

Then by Lemma 13.19 we have

$$C_1 \geq 2\delta + K_2.$$

As $C_1 \leq 2\delta + K_1$, we have $C_1 = 2\delta + K_2$ and $K_1 = K_2$. Again, by Lemma 14.4 we have

$$2\delta + K_2 > 4K_2; \qquad\qquad 3K_2 \leq 2\delta - 1$$

and by Lemma 14.13 we find

$$3K_2 = 2\delta - 1,$$

arriving at the situation envisioned in (2). □

14C. The high case: $C > 2\delta + K_1$

Lemma 14.15. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that*

$$C > 2\delta + K_1.$$

Then $3K_2 \geq 2\delta$.

PROOF. By Lemma 14.13 we have

$$K_1 + 2K_2 \geq 2\delta - 1.$$

If $3K_2 < 2\delta$, then we find $K_1 = K_2$ and $3K_2 = 2\delta - 1$. Then by Lemma 14.8, we have

$$C_1 \leq 4K_2 + 1 = 2\delta + K_2 = 2\delta + K_1,$$

contradicting our hypothesis. □

LEMMA 14.16. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that $C_1 > C_0 > 2\delta + K_1$ and $3K_2 = 2\delta$. Then*

$$C_0 > 2\delta + K_2.$$

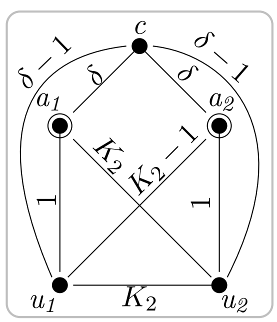
PROOF. Suppose on the contrary that $C_0 \leq 2\delta + K_2$. Then $K_1 < K_2$. By Lemma 14.13, we find

$$3K_2 \geq 2K_2 + (K_1 + 1) \geq 2\delta.$$

As $3K_2 = 2\delta$, we conclude

$$K_2 = K_1 + 1,\ C_0 = 2\delta + K_2.$$

Consider the following amalgamation.



Each factor contains two distinct lengths K_1, δ greater than 1, so is not a Henson constraint.

The nongeodesic triangles involved in this diagram have the types

$$(1, K_2, K_2),\ (K_2, \delta - 1, \delta - 1),\ (K_2 - 1, \delta - 1, \delta),\ \text{and}\ (K_2, \delta - 1, \delta).$$

The first of these is in $\mathcal{A}$ by definition. The second and third triangles have perimeter equal to $C_0 - 2$, and are in $\mathcal{A}$ by Lemma 13.14. So consider the last type:

$$(K_2, \delta - 1, \delta),$$

of perimeter $p = C_0 - 1 < C_1$. It suffices to check that triangles of this type are in $\mathcal{A}^{\delta}_{K_1,K_2}$.

As $p \geq 2K_1 + 1$, it suffices to check the inequality

$$C_0 - 1 \leq 2K_2 + 2 \cdot K_2 = 2\delta + K_2,$$

which holds.

The amalgamation diagram can be completed in $\mathcal{A}$ with $d(a_1, a_2) = K_2 - 1$ or K_2. Thus the triangle (a_1, a_2, c) has type $(K_2 - 1, \delta, \delta)$ or (K_2, δ, δ), and in the second case the perimeter is C_0, which is impossible. So $\mathcal{A}$ must contain a triangle of type $(K_2 - 1, \delta, \delta)$ and odd perimeter $C_0 - 1$, yielding the inequality

$$C_0 - 1 \leq 2K_2 + 2(K_2 - 1) = 2\delta + K_2 - 2,$$

which is a contradiction. $\square$

LEMMA 14.17. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. If $K_2 \geq \delta - 1$, then either $C' = C + 1$ or $C = 3\delta - 1$.*

PROOF. If $C \geq 3\delta - 1$, then our claim follows easily. So suppose

$$C < 3\delta - 1 \text{ and } C' > C + 1.$$

If C is odd then by Lemma 13.19 we find

$$C \geq 2\delta + K_2 \geq 3\delta - 1,$$

a contradiction. So

$$C = C_0 \text{ is even.}$$

Now let (i, j, k) be the type of a triangle in $\mathcal{A}$ with perimeter

$$p = i + j + k = C + 1$$

and with $i \leq j \leq k$. Then

$$i < \delta \text{ and } k \geq 3.$$

If $C \leq 2\delta + K_1$, then by Proposition 14.14 we have

$$C = 2\delta + K_2 \geq 3\delta - 1,$$

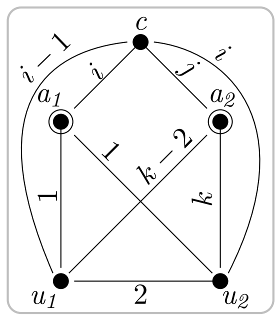
a contradiction. Hence

$$C > 2\delta + K_1.$$

Therefore

$$i > K_1.$$

Now consider the following amalgamation.



Each factor contains a geodesic, so is not a Henson constraint.

The nongeodesic triangles involved in this diagram have types

$$(2, i-1, i),\ (1, i, i),\ (i-1, j, k-2), \text{ and } (i, j, k).$$

The last of these is in $\mathcal{A}$ by hypothesis. As $K_1 \le i \le K_2$, there is a triangle of type $(1, i, i)$ in $\mathcal{A}$. Lemma 13.18 shows that a triangle of type $(2, i-1, i)$ is in $\mathcal{A}$. We claim that there is also a triangle of type

$$(i-1, j, k-2)$$

in $\mathcal{A}$.

First, we check the triangle inequality for this triple. As $i \le j \le k \le i + j$, this comes down to the inequality

$$j \le (i-1) + (k-2).$$

As $i + j + k = C_0 + 1 \ge 2\delta + 3$, we have $i \ge 3$, and the inequality follows. Now the perimeter $p = (i-1) + j + (k-2)$ is $C_0 - 2$, so there is a triangle of this type in $\mathcal{A}$ by Lemma 13.14.

Thus the amalgamation diagram has a completion in $\mathcal{A}$. Then $d(a_1, a_2) = k - 1$, so the triangle (a_1, a_2, c) has type $(i, j, k-1)$ and perimeter C, a contradiction. $\square$

LEMMA 14.18. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. If*

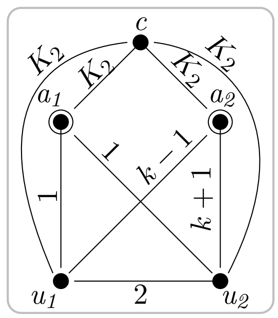
$$2K_2 + 2 < C < \delta + 2K_2$$

then $C' = C + 1$.

PROOF. Let $k = C - 2K_2$. Then

$$3 \leq k < \delta.$$

Consider the following amalgamation.



Each factor contains a geodesic, so is not a Henson constraint.

If this diagram has a completion in $\mathcal{A}$, then we have $d(a_1, a_2) = k$, and the triangle (a_1, a_2, c) has type (k, K_2, K_2) and perimeter C, which is a contradiction. Since neither factor is a Henson constraint, one of the nongeodesic triangles involved must be a forbidden triangle.

The triangle types in question are

$$(2, K_2, K_2),\ (1, K_2, K_2),\ (k \pm 1, K_2, K_2).$$

As $2K_2 + 2 < C$ is even, there is a triangle of the first type in $\mathcal{A}$, and by definition there is one of the second type. So one of the triangle types

$$(k \pm 1, K_2, K_2)$$

must be forbidden.

Let us check that these are in fact triangle types. The triangle inequality takes the form

$$k + 1 \leq 2K_2,\ C \leq 4K_2 - 1.$$

Now $2K_2 \geq K_1 + K_2 \geq \delta$ by Lemma 14.3, and $C < \delta + 2K_2 \leq 4K_2$, so this holds.

Triangles of type $(k\pm 1, K_2, K_2)$ have perimeter $C\pm 1$. Now suppose toward a contradiction that $C' > C+1$. If C is odd, then both triangle types are realized, so suppose that C is even. We claim that these triangle types are in $\mathcal{A}$ in this case as well.

By assumption

$$C - 1 > 2K_2 + 1 \geq 2K_1 + 1,$$

so the perimeters satisfy the required lower bound $p \geq 2K_1 + 1$. It remains to check the inequalities corresponding to the parameter K_2. We require the following inequalities.

$$C + 1 \leq 2K_2 + 2 \cdot K_2; \qquad C \pm 1 < 2K_2 + 2(k \pm 1).$$

We showed above that $C \leq 4K_2 - 1$. Furthermore

$$C \pm 1 = 2K_2 + k \pm 1 < 2K_2 + 2(k \pm 1),$$

since $k > 1$. So both triangle types $(k \pm 1, K_2, K_2)$ are in $\mathcal{A}$, and we arrive at a contradiction.

Thus $C' = C + 1$. ☐

Lemma 14.19. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. If $K_2 = \delta$ and $C > 2\delta + K_1$, then $C' = C + 1$.*

Proof. If $C > 2\delta + 2$, then by Lemma 14.18, either $C \geq 3\delta$ or $C' = C + 1$. If $C \geq 3\delta$ then in any case $C' = C + 1$. So we may suppose

$$C \leq 2\delta + 2.$$

As $C > 2\delta + K_1$, we find

$$K_1 = 1,\ C = 2\delta + 2.$$

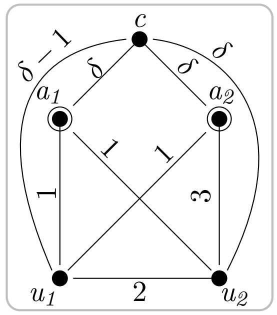
Consider the following amalgamation.

Each factor contains a geodesic, so is not a Henson constraint.

The nongeodesic triangles involved here are (u_1, c, u_2), (a_1, c, u_2), and (a_2, c, u_2), of types

$$(2, \delta - 1, \delta),\ (1, \delta, \delta), \text{ and } (3, \delta, \delta).$$

The first two types are certainly in $\mathcal{A}$. If we suppose $C' > C + 1$, then a triangle of the third type is also in $\mathcal{A}$, and the diagram can



be completed in $\mathcal{A}$, with $d(a_1, a_2) = 2$. So the triangle (a_1, a_2, c) has type $(2, \delta, \delta)$, contradicting $C = 2\delta + 2$. $\square$

Lemma 14.20. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. If $C > 2\delta + K_1$ and $C' > 2\delta + K_2$, then $C \geq 2\delta + K_2$.*

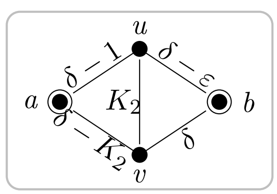
Proof. Suppose on the contrary

$$C < 2\delta + K_2 < C'.$$

Then by the previous lemma we have

$$K_2 < \delta.$$

Let $C' - K_2 \equiv \epsilon \pmod 2$ with $\epsilon = 0$ or 1. Consider the following amalgamation.



The triangle (a, u, v) has type $(\delta - K_2, K_2, \delta - 1)$ and perimeter $2\delta - 1 \geq 2K_1 + 1$. So to see that this type is realized in $\mathcal{A}$, it suffices to check the inequality

$$(\delta - 1) + K_2 \leq 2K_2 + (\delta - K_2),$$

which is evident.

The triangle (b, u, v) has type $(K_2, \delta, \delta - \epsilon)$ and perimeter

$$p = 2\delta + K_2 - \epsilon \equiv C' \pmod 2.$$

As $p < C'$, to see that this is represented in $\mathcal{A}$ it suffices to check the inequalities

$$p \geq 2K_1 + 1; \qquad\qquad p \leq 2K_2 + 2K_2.$$

The first holds since $K_1 < \delta$.

By Lemma 14.15 we have $3K_2 \geq 2\delta$ and the second inequality holds.

Therefore the diagram has a completion in $\mathcal{A}$. Let i be the value of $d(a, b)$ in this completion. Then $i \geq K_2$.

We claim that the perimeter of (a, b, u) is at least C. If $\epsilon = 0$ then $i+(\delta-1)+(\delta-\epsilon) \geq 2\delta+K_2-1 \geq C$. If $\epsilon = 1$ then $C \equiv K_2 \pmod 2$ and $C < 2\delta + K_2$, so $C \leq 2\delta + K_2 - 2$ and the inequality still holds.

Since the perimeter $2\delta + i - 1 - \epsilon \geq C$, we must have

$$i + \epsilon + 1 \equiv C' \pmod 2; \qquad\qquad i \equiv K_2 + 1.$$

So the perimeter of the triangle (a, b, v) is odd. This gives the inequality

$$\begin{aligned} \delta + i &\leq 2K_2 + (\delta - K_2) \\ i &\leq K_2, \end{aligned}$$

which contradicts the conditions

$$i \geq K_2 \text{ and } i \equiv K_2 + 1 \pmod 2.$$

□

LEMMA 14.21. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that the following hold.*

1. *$C > 2\delta + K_1$.*
2. *$K_1 + 2K_2 = 2\delta - 1$.*

Then $C \geq 2\delta + K_1 + 2$.

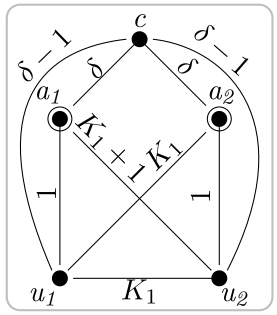
PROOF. Notice that K_1 is odd.

Consider the following amalgamation.

Each factor contains the lengths δ and $\delta - 1$, so is not a Henson constraint.

The nongeodesic triangles involved in this diagram have the types

$(1, K_1, K_1)$, $(K_1, \delta - 1, \delta - 1)$, $(K_1 + 1, \delta - 1, \delta)$, and $(K_1, \delta - 1, \delta)$.



A triangle of the first type is in $\mathcal{A}$ by definition. The last type has even perimeter

$$2\delta + K_1 - 1 < C,$$

and thus is also in $\mathcal{A}$.

There remain the types

$$(K_1, \delta - 1, \delta - 1) \text{ and } (K_1 + 1, \delta - 1, \delta).$$

with odd perimeters $2\delta+K_1-2$ and $2\delta+K_1 < C$. As these perimeters are at least $2K_1 + 1$, to see that triangles of these types are in $\mathcal{A}$ it suffices to check the following inequalities.

$$2\delta - 2 \leq 2K_2 + K_1; \qquad 2\delta - 1 \leq 2K_2 + K_1 + 1,$$

both of which hold by hypothesis.

Therefore the amalgamation diagram has a completion in $\mathcal{A}$, with

$$d(a_1, a_2) = K_1 \text{ or } K_1 + 1.$$

A triangle of type (K_1, δ, δ) has odd perimeter, and

$$\delta + \delta = K_1 + 2K_2 + 1 > 2K_2 + K_1,$$

so this triangle type violates the constraint afforded by K_2, hence does not occur in $\mathcal{A}$. Thus we have

$$d(a_1, a_2) = K_1 + 1.$$

Then the triangle (a_1, a_2, c) has perimeter $2\delta + K_1 + 1$, and thus $C \neq 2\delta + K_1 + 1$. □

LEMMA 14.22. *Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that*

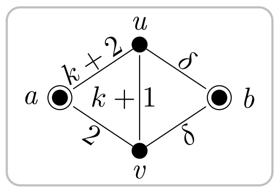
1. $2\delta + K_1 < C < 2\delta + K_2$.
2. $K_2 \le \delta - 2$.

Then $C' = C + 1$.

PROOF. Suppose toward a contradiction that

$$C + 1 < C'.$$

Let $k = C - 2\delta$. Then $k < \delta - 2$. Consider the following amalgamation.



The triangle (a, u, v) has type $(2, k+1, k+2)$ and perimeter $2k+5$. We claim that $2k + 5 < C$, or equivalently $C < 4\delta - 5$. Indeed $C < 2\delta + K_2 \le 3\delta - 2 \le 4\delta - 5$.

To see that the triangle (a, u, v) is in $\mathcal{A}$ it suffices to check the following inequalities.

$$2k + 5 \ge 2K_1 + 1; \qquad 2k + 5 \le 2K_2 + 2 \cdot 2.$$

As $K_1 < k < K_2$, both inequalities hold. Thus the triangle (a, u, v) belongs to $\mathcal{A}$.

The triangle (b, u, v) has type $(k+1, \delta, \delta)$ and perimeter $C+1$. We claim that the triangle (b, u, v) is in $\mathcal{A}$.

Since we have assumed $C+1 < C'$, it suffices to check the inequalities

$$C + 1 \ge 2K_1 + 1; \qquad C + 1 \le 2K_2 + 2(k + 1).$$

As $C > 2\delta$ the first inequality is clear. The second inequality reduces to

$$C + 2K_2 + 1 \ge 4\delta.$$

Indeed, we have

$$C + 2K_2 + 1 > (2\delta + K_1) + (2K_2 + 1) \ge 2\delta + (2\delta - 1) + 1 = 4\delta,$$

by Lemma 14.13.

Thus the diagram can be completed in $\mathcal{A}$, with $d(a,b) \geq \delta - 2$. If $d(a,b) = \delta - 2$ or δ, then the triangle (a,b,u) has perimeter C or $C+2$, a contradiction. Thus

$$d(a,b) = \delta - 1.$$

So the type of the triangle (a,b,v) is $(2,\delta-1,\delta)$, which yields the inequality

$$(\delta - 1) + \delta \leq 2K_2 + 2; \qquad K_2 \geq \delta - 1,$$

contrary to our hypothesis. □

Proposition 14.23. Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter $\delta \geq 3$ with associated parameters K_1, K_2, C, C'. Suppose that

$$K_1 < \infty, \ C > 2\delta + K_1.$$

Then the following hold.

1. $K_1 + 2K_2 \geq 2\delta - 1$.
2. $3K_2 \geq 2\delta$.
3. If $K_1 + 2K_2 = 2\delta - 1$, then $C \geq 2\delta + K_1 + 2$.
4. If $C' > C + 1$ then $C \geq 2\delta + K_2$.

PROOF. The first three points hold by Lemmas 14.13, 14.15, and 14.21 respectively. Now suppose

$$C' > C + 1.$$

We must show that $C \geq 2\delta + K_2$.

If $C = C_1$, then $C_1 < C_0 - 1$ and by Lemma 13.19 we find $C \geq 2\delta + K_2$. So suppose

$$C = C_0.$$

If $K_2 \geq \delta - 1$, then by Lemmas 14.17 and 14.19, we find

$$K_2 = \delta - 1 \text{ and } C = 3\delta - 1 = 2\delta + K_2,$$

and our claim holds.

On the other hand if $K_2 \leq \delta - 2$ then Lemma 14.22 gives the result. □

We now have enough to check the admissibility of the parameter sequence $(\delta, K_1, K_2, C_0, C_1)$.

Proposition 14.24. Let $\mathcal{A}$ be a 4-trivial amalgamation class of finite metric spaces associated with a countable metrically homogeneous graph. Then the associated numerical parameter sequence

$$(\delta, K_1, K_2, C_0, C_1)$$

is admissible.

PROOF. We have previously noted that the parameter sequence is acceptable, and we began the present chapter by observing that there is nothing to prove if $K_1 = \infty$. So we suppose

$$K_1 < \infty.$$

If $C \leq 2\delta + K_1$, then the conditions for admissibility are given by Proposition 14.14.

If $C > 2\delta + K_1$ then the conditions for admissibility are given by Proposition 14.23. □

This gives us the bulk of Part $(II\text{–}B)$ of our main theorem. It remains to pin down the interactions between the numerical parameters and the set $\mathcal{S}$ of Henson constraints.

Theorem (Main Theorem, Part II, Variant 2—Subdivided). Let Γ be a 4-trivial countable metrically homogeneous graph of diameter $\delta \geq 3$, and let $\mathcal{A}$ be the associated amalgamation class of finite metric spaces. Then the sequence of parameters

$$(\delta, K_1, K_2, C_0, C_1)$$

is admissible.

If all minimal constraints for $\mathcal{A}$ are either triangles or Henson constraints, then the sequence of parameters

$$(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$$

is also admissible.

PROOF. We dealt with the numerical parameter sequence

$$(\delta, K_1, K_2, C_0, C_1)$$

in Proposition 14.24.

Accordingly, we now suppose that $\mathcal{A}$ is determined by triangle constraints and Henson constraints. We must check that the additional constraints on $\mathcal{S}$ which are required for admissibility are satisfied.

These constraints are found in Definition 11.4. We may assume that

$$K_1 < \infty.$$

Then the required conditions may be stated as follows.

— If $C = 2\delta + 1$ and $\mathcal{S}$ is nonempty, then $\delta \geq 4$ and $\mathcal{S}$ consists of a clique and its antipodal companions.
— If $K_1 = \delta$, then $\mathcal{S}$ is empty.
— If $K_1 = 1$ and $C = 2\delta + 2$, then $\mathcal{S}$ is empty.

The case $C = 2\delta + 1$.

This falls under Type (II), with $C = 2K_1 + 2K_2 + 1$, so $K_1 + K_2 = \delta$. We first check that $C' = C + 1$ (Type (IIA)). Otherwise, we have the conditions

$$K_1 = K_2,\ 3K_2 = 2\delta - 1,$$

and thus $3(\delta/2) = 2\delta - 1$, or $\delta = 2$.

Since $C' = C + 1$, we are speaking of the antipodal case. So the distance δ will not occur in a Henson constraint of order greater than 3, and any Henson constraint in $\mathcal{S}$ will be an antipodal companion of a clique. Furthermore, $\mathcal{S}$ is closed under the operation of forming antipodal companions, so if $\mathcal{S}$ is non-empty it must reduce to a single clique and all of its antipodal companions.

In this case, we must eliminate the possibility

$$\delta = 3.$$

By Fact 11.9, the possibilities for antipodal Γ are explicitly known when $\delta = 3$, and are antipodal double covers of one of the following graphs.

(a) the pentagon (5-cycle);
(b) The product $K_3 \square K_3$ of two 3-cliques;
(c) an independent set I_n $(n \leq \infty)$;
(d) The random graph G_∞.

The first two graphs mentioned are not 4-trivial, as noted in §13I. And in the third and fourth cases, $\mathcal{S}$ is empty.

The case $K_1 = \delta$ and $C = 3\delta + 1$.

In particular $K_1 > 1$, and the configurations in $\mathcal{S}$ must be of the form

$$(A_1, \ldots, A_k)$$

where the A_i are vertices or pairs of adjacent vertices, and the distance between elements of distinct sets A_i, A_j is δ.

Our claim is that under the stated hypothesis, all such configurations already occur in $\mathcal{A}$, and therefore $\mathcal{S}$ is empty.

Claim 1. Γ_δ is connected of diameter $\delta \geq 3$ with associated parameter $K_1 = \delta$.

We denote the parameters of Γ_δ corresponding to δ and K_1 by $\tilde{\delta}$ and $\tilde{K}_1$.

Let us be more explicit about the conditions on Γ. As $K_1 = \delta$, by the admissibility conditions the possibility $C \leq 2\delta + K_1$ is ruled out. Therefore we have

$$\begin{aligned} &K_1 = K_2 = \delta; \\ &C = 3\delta + 1,\ C' = C + 1. \end{aligned}$$

We claim that every distance d in the range $1 \leq d \leq \delta$ occurs in Γ_δ. That is, we must check that the triangle types (d, δ, δ) are realized. As $C > 3\delta$ we need only check for d odd that we have the inequalities

$$2\delta + d \geq 2K_1 + 1; \qquad\qquad 2\delta \leq 2K_2 + d,$$

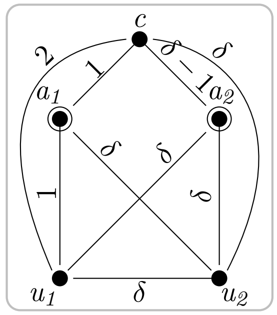
and as $K_1 = K_2 = \delta$ this is clear.

So Γ_δ, viewed as a graph with edge relation $d(x, y) = 1$, is connected of diameter $\tilde{\delta} = \delta$, and is countable metrically homogeneous.

Furthermore we claim that Γ_δ is not of exceptional type. This follows easily from our assumptions on the constraints of Γ.

Finally, we claim that Γ_δ contains a triangle of type $(1, \delta, \delta)$, so that $\tilde{K}_1 = \delta$.

For this, we make use of an explicit amalgamation, as follows.



Neither factor is a $(1, \delta)$-space or a $(1, \delta - 1)$-space, so to see that the factors occur in $\mathcal{A}$ it suffices to check the triangles. These are of types

$$(1, 1, 2),\ (1, \delta, \delta),\ (\delta, \delta, \delta),\ (\delta - 1, \delta - 1, \delta),\ (\delta - 1, \delta, \delta).\ (2, \delta - 1, \delta)$$

As the only constraints on triangles apply to perimeter less than $2\delta + 1$ these are all realized in $\mathcal{A}$.

In the amalgam, the vertex u_1 forces $d(a_1, a_2) \geq \delta - 1$ and the vertex c prevents $d(a_1, a_2) = \delta - 1$, so the configuration $a_1a_2u_1u_2$ is as required.

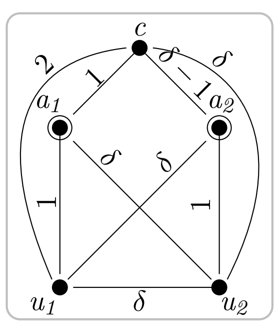
This proves the claim.

Claim 2. For a, b a pair of adjacent vertices of Γ, the graph

$$\Gamma_\delta(a, b) = \Gamma_\delta(a) \cap \Gamma_\delta(b)$$

is again a countable metrically homogeneous graph of diameter $\delta \geq 3$, not of exceptional type, with associated parameter $K_1 = \delta$.

We first check that $\Gamma_\delta(a, b)$ contains an edge. This involves an amalgamation much like the previous one.



All that has changed is that the distance $d(a_2, u_2)$ is now 1. As the triangles containing a_2, u_2 are of types

$$(1, \delta, \delta) \text{ and } (1, \delta - 1, \delta),$$

the factors are realized in $\mathcal{A}$, and in the amalgam we have the desired configuration $(a_1 a_2 u_1 u_2)$.

Now the lengths in the range $(2, \ldots, \delta - 1)$ are clearly realized in $\Gamma_\delta(a, b)$ as the corresponding configurations are not forbidden. The distance δ is also realized because if we write $(abcc')$ for the desired configuration, it suffices to find (abc) in $\Gamma_\delta(c')$, and this is covered by Claim 1.

Thus $\Gamma_\delta(a, b)$ is connected and countable metrically homogeneous. One checks that it is not of exceptional type directly, by checking that the relevant configurations are not forbidden (and noting that the configuration corresponding to an edge in $\Gamma_\delta(a, b)$ was already checked).

Finally, we claim that $\Gamma_\delta(a, b)$ contains a triangle (c_1, c_2, d) of type $(1, \delta, \delta)$, where we choose notation so that $d(c_1, c_2) = 1$. Then this corresponds to a configuration (abc_1c_2) in $\Gamma_\delta(d)$. By Claim 1 $\Gamma_\delta(d)$ satisfies the same conditions as Γ, and the configuration (abc_1c_2) is simply an edge in $\Gamma_\delta(a, b)$, so this follows.

This proves the second claim.

Now it follows inductively that any configuration consisting of n pairs (a_i, b_i) with $d(a_i, b_i) = 1$ and all other distances equal to δ will be realized in Γ. As any forbidden configuration compatible with $K_1 > 1$ would embed in one of this type, it follows that $\mathcal{S}$ is empty.

The case $K_1 = 1$ and $C = 2\delta + 2$.

In this case, Γ_δ is a complete graph. As Γ is not antipodal, Γ_δ is infinite. Thus all cliques K_n embed into Γ.

On the other hand $C' = C + 1$: the alternative (in Type (III)) is $C \geq 2\delta + K_2$ and $K_2 \leq 2$. This gives $2\delta - 1 \leq K_1 + 2K_2 \leq 5$ and $\delta = 3$, $K_2 = 2$. Thus $K_1 + 2K_2 = 2\delta - 1$ and in this case we have the constraint $C \geq 2\delta + K_1 + 2$, which is a contradiction.

Since $C' = C + 1$, we have no triangle of perimeter 3δ, so any Henson constraint would necessarily consist of exactly two cliques at distance δ, say $S = (K_m, K_n)$ with $m \leq n$. Assuming that n and $m + n$ are minimized, we fix a vertex $a \in K_n$ and adjoin a vertex b

with

$$d(b,a) = 1;$$
$$d(b,x) = \delta - 1 \ (x \in K_m);$$
$$d(b,y) = 2 \ (y \in K_n \setminus \{a\}),$$

and view the resulting configuration as an amalgam with the distances between a and $K_n \setminus \{a\}$ to be determined. Again the bound $C = 2\delta+2$, $C' = C+1$ forces these distances all to be equal to 1, and we are left to consider the factors of the amalgam. The $(1,\delta)$-spaces involved are available by induction, and the triangles involved are all of suitably bounded perimeter. $\square$

Next, we turn our attention in the remainder of this Part to what can be said without assuming 4-triviality. We first take up local analysis, meaning we consider the structure of Γ_i in those cases in which an edge is present (presumably, whenever $K_1 \leq i \leq K_2$, but this point has not actually been proved in general).

CHAPTER 15

LOCAL ANALYSIS

15A. Introduction

At this point, we have completed the analysis of amalgamation classes determined by constraints on triangles and constraints of Henson type (normal or antipodal). This completes the detailed presentation of our catalog, and provides some initial justification for its contents.

What follows provides some essential tools for analyzing potential counterexamples to the classification conjecture. We consider the following explicit form of the main conjecture.

Conjecture (Conjecture 2, explicit form). Any countable metrically homogeneous graph of generic type is of one of the following two forms

1. $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ with admissible parameters
$$(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$$
where $\mathcal{S}$ is a set of $(1,\delta)$-spaces.
2. $\mathcal{A}^{\delta}_{a,n}$ (antipodal).

Since the countable metrically homogeneous graphs of non-generic type were previously classified, a proof of this conjecture would complete the classification of countable metrically homogeneous graphs. Given a countable metrically homogeneous graph of generic type, we have already associated to it an acceptable sequence of parameters $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$, and the next goal would be to show that this sequence of parameters is admissible, so that the corresponding class $\mathcal{A}^* = \mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ exists, We could then take as our target the claim that the given class $\mathcal{A}$ coincides with the class $\mathcal{A}^*$.

It is natural to approach this problem via *local analysis.* By this we mean the study of the induced metric spaces Γ_i. Given a countable metrically homogeneous graph of diameter δ, and a fixed value $i \le \delta$, we recall that Γ_i consists of the points at distance i from a fixed basepoint, with the induced metric. We are mainly interested in the case in which Γ_i contains a pair of points at distance 1, and can hence be usefully viewed also as a graph. In this case we aim to show that Γ_i is itself a metrically homogeneous graph, and then by some form of induction one would argue either that Γ_i is already in our catalog, or else that Γ_i has precisely the same associated parameters as Γ does, which is also helpful, though in a more limited way.

Our conjectured classification includes detailed predictions concerning the structure of Γ_i, in terms of the parameters associated with Γ, but at this point we aim at something more qualitative, as expressed by Theorem 1.32 and Proposition 1.33, recalled below.

The first of these results justifies our remarks about the use of induction to identify or constrain Γ_i in the presence of an edge.

Theorem (1.32). Let Γ be a countable metrically homogeneous graph of generic type and of diameter δ, and suppose $i \le \delta$. Suppose that Γ_i contains an edge. Then Γ_i is a countable metrically homogeneous graph (and, in particular, is connected).

Furthermore, Γ_i is primitive and of generic type, apart from the following two cases.

1. $i = \delta$;
   $K_1 = 1$; $\{C_0, C_1\} = \{2\delta + 2, 2\delta + 3\}$;
   Γ_δ is an infinite complete graph (hence not of generic type).
2. $\delta = 2i$;
   Γ is antipodal (hence Γ_i is imprimitive, namely antipodal).

This leaves aside the question of the relationship of the parameters for Γ_i to the parameters for Γ. We will return to this point in our analysis of the case of infinite diameter in Chapter 16.

The second result of this chapter states that when $K_1 \le 2$, the first result can be applied. While this condition is very restrictive, this will be a useful tool in the remainder of this part. This result is also very useful when $\delta \le 3$, the case studied in Amato, Cherlin, and Macpherson [2021].

Proposition (1.33). Let Γ be a countable metrically homogeneous graph of diameter δ. Suppose

$$K_1 \le 2.$$

Then for $2 \leq i \leq \delta - 1$, Γ_i contains an edge.

The proofs of these two results depend on one major case division: $K_1 = 1$ or $K_1 > 1$. A more expressive statement of this case distinction is the following: Γ_1 contains an edge, or Γ_1 is an independent set.

At a more technical level, some of the results achieved along the way to our two main results have some independent value for the further development of the classification theory in their own right.

First, we have the following structural results which are related to the main results but are not subsumed under them.

Lemma 15.2 can be rephrased as a characterization of generic type by a stronger and more readily applicable pair of conditions.

- Γ_1 is an independent set, a Henson graph, or a random graph.
- For u_1, u_2 vertices at distance 2, the graph induced on their common neighbors is isomorphic to Γ_1.

Lemma 15.3 isolates the antipodal case: if some Γ_i is finite then $i = \delta$ and we are in the antipodal case.

Lemma 15.4 has a rather technical look to it: namely, setting aside the case $i = \delta$, $K_1 = 1$, the metric space Γ_i is connected with respect to the relation "$d(x, y) = 2$." This helps with the proof of Theorem 1.32, and unlike the latter, it can be proved by induction on i. In addition this result is occasionally useful in its own right.

Lemma 15.5 states mainly that each vertex in $\Gamma_{\delta-1}$ has infinitely many neighbors in Γ_δ (a one-sided variation on Lemma 15.2).

Last in this vein: the technical Lemma 15.15 concerns only the case $K_1 = 2$: if u_1, u_2 are at distance 2, then the intersection $\Gamma_2(u_1) \cap \Gamma_2(u_2)$ contains an edge.

Then we have some structural results which are evident when the graphs in question are of known type, and which are helpful in laying the foundations of local analysis.

Lemma 15.6 relates the perimeters of triangles to the diameter of Γ_δ.

Applying our main results, we get the very useful *Lemma 15.16* concerning the realization of certain configurations of order 4 containing geodesics, which we expect to make extensive use of in the future in place of the hypothesis of 4-triviality. In the present monograph this lemma is invoked only in §15H.

15B. The type of Γ_i

We aim here at the following statement, to be proved at the end of the section.

LEMMA 15.1. *Let Γ be a countable metrically homogeneous graph of generic type and diameter $\delta \geq 3$. Let $i \leq \delta$, and suppose that the graph Γ_i is connected.*

Then Γ_i is metrically homogeneous, and one of the following holds.

(a) Γ_i is of generic type, and not bipartite of diameter 2.
(b) $i = \delta$, $C \leq 2\delta + 2$ and $C' = C + 1$, Γ_δ is complete, and either
- *$K_1 = 1$, or*
- *Γ is antipodal;*

We remark that complete bipartite graphs actually meet our definition of generic type, so must be explicitly excluded in the statement of our lemma.

We begin our analysis with some useful technical points which will be worth bearing in mind throughout.

LEMMA 15.2. *Let Γ be a countable metrically homogeneous graph of generic type. Let u_1, u_2 be vertices of Γ with $d(u_1, u_2) = 2$, and let $\Gamma' = \Gamma_1(u_1, u_2)$ be the graph induced on their common neighbors*

$$\Gamma_1(u_1, u_2) = \Gamma_1(u_1) \cap \Gamma_1(u_2).$$

Then $\Gamma' \cong \Gamma_1$.

PROOF. If $K_1 > 1$ (Γ_1 is an independent set), then this is given by the definition of generic type (Definition 1.17). So we suppose

$$K_1 = 1.$$

By the definition of generic type and the Lachlan/Woodrow classification, Γ_1 must then be a Henson graph H_n with $3 \leq n < \infty$, or a random graph.

We may take the vertices u_1, u_2 in question to lie in Γ_1. In this case Γ' contains the chosen basepoint v_* for Γ, along with $\Gamma' \cap \Gamma_1$, and in addition Γ' is a homogeneous graph. If Γ_1 is a random graph then $\Gamma' \cap \Gamma_1$ is also a random graph, and it follows at once that Γ' is a random graph, hence isomorphic to Γ_1. So we suppose

$$\Gamma_1 \cong H_n,$$

a Henson graph with

$$3 \leq n < \infty.$$

Inspection of $\{v_*\} \cup (\Gamma' \cap \Gamma_1)$ gives us the following properties for Γ'.

1. Γ' contains an infinite independent set;
2. Γ' contains K_{n-1};
3. Γ' contains a geodesic path (a_1, a_2, a_3) of length 2.

Since Γ' embeds in Γ_1, it will suffice now to check that Γ' is primitive. This will follow once we show the following.

(4) Γ' contains a configuration $(v_1 v_2 v_3)$ with

$$\begin{aligned} d(v_1, v_2) &= 1 \\ d(v_1, v_3) = d(v_2, v_3) &= 2 \end{aligned}$$

The configuration $(u_1 u_2 v_1 v_2 v_3)$ can be found in Γ_1 if $n \geq 4$, and hence in Γ, so we now assume

$$n = 3.$$

Thus H_n is the generic triangle-free graph.

In this case, we use a direct amalgamation argument (Figure 119).

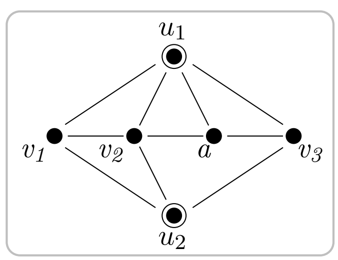


FIGURE 119. $u_1 u_2 v_1 v_2 v_3 a$

We first adjoin an additional vertex a to the configuration $(u_1 u_2 v_1 v_2 v_3)$ satisfying the following.

$$\begin{aligned} d(a, u_1) = d(a, v_2) = d(a, v_3) &= 1 \\ d(a, v_1) = d(a, u_2) &= 2 \end{aligned}$$

Set $V = \{a, v_1, v_2, v_3\}$. We view $V \cup \{u_1, u_2\}$ as a 2-point amalgamation problem with the distance $d(u_1, u_2)$ to be determined. This is depicted as a graph in Figure 119, with non-edges representing distance 2.

We will show that if the desired configuration $(u_1u_2v_1v_2v_3)$ does not embed in Γ then the factors $V \cup \{u_1\}$ and $V \cup \{u_2\}$ embed in Γ; and then we show that in the resulting amalgam, $(u_1u_2v_1v_2v_3)$ will be the desired configuration.

In the factor $(u_1v_1v_2v_3a)$, we have $v_1, v_2, v_3, a \in \Gamma_1(u_1)$ and as $(v_1v_2v_3a)$ is triangle free this is realized.

For the factor $(u_2v_1v_2v_3a)$, we view this as a 2-point amalgamation problem with the distance $d(a, v_1)$ to be determined. It is easy to see that the subfactors of this configuration occur in Γ. In the resulting amalgam, if $d(a, v_1) = 1$ then we have an isomorphic copy $(au_2v_1v_2v_3)$ of the desired configuration $(u_1u_2v_1v_2v_3)$ and we conclude. Otherwise, if $d(a, v_1) = 2$, we have the second factor of the amalgamation diagram on display.

So we may suppose that this amalgamation diagram occurs in Γ. In its completion, since $\Gamma_1(u_1)$ must be triangle free, we will have $d(u_1, u_2) = 2$, as required.

Thus condition (4) above is verified, and the proof is complete. □

Corollary 15.2.1. Let Γ be a countable metrically homogeneous graph of generic type and diameter δ. Let $i < \delta$. Then Γ_1 embeds into Γ_i.

Proof. Apply Lemma 15.2 to a pair of vertices in $\Gamma_{i\pm1}$ at distance 2. □

The following simple principle is useful for isolating the antipodal case.

Lemma 15.3. *Let Γ be a countable metrically homogeneous graph of generic type and diameter δ with Γ_i finite for some i, $1 \le i \le \delta$.*

Then Γ is of antipodal type and $i = \delta$.

Proof. Let u be the vertex taken as a basepoint for Γ. We consider the model theoretic algebraic closure of u, denoted $\mathrm{acl}(u)$; this is the union of the finite orbits of the stabilizer of u in the automorphism group. By homogeneity, $\mathrm{acl}(u)$ is the union of $\{u\}$ together with the sets of the form Γ_i for which Γ_i is finite.

But by Corollary 15.2.1, if $i < \delta$ then Γ_i contains a copy of Γ_1. Thus only Γ_δ can be finite, and if this occurs, then $\mathrm{acl}(u) = \{u\} \cup \Gamma_\delta$.

But the relation

$$\mathrm{acl}(u) = \mathrm{acl}(v).$$

is an equivalence relation on Γ, and we conclude that this relation must coincide with the relation

$$d(u, v) = 0 \text{ or } \delta.$$

By definition, this is the antipodal case. $\square$

Lemma 15.4. *Let Γ be a countable metrically homogeneous graph of generic type and diameter δ. Suppose $i \le \delta$, and suppose also that if $i = \delta$ then $K_1 > 1$. Then the metric space Γ_i is connected with respect to the edge relation defined by*

$$d(x, y) = 2.$$

Corollary 15.4.1. Let Γ be a countable metrically homogeneous graph of generic type and diameter δ. Suppose $i \le \delta$, that Γ_i is nontrivial, and suppose also that if $i = \delta$ then $K_1 > 1$. Then Γ_i is not bipartite, and is not complete multipartite with more than one component.

Proof of Lemma 15.4. We proceed by induction on i.

For $u \in \Gamma_{i-1}$, let $I_u = \Gamma_1(u) \cap \Gamma_i$. We claim that I_u is contained in a single connected component C_u of Γ_i with respect to the relation "$d(x, y) = 2$."

If $K_1 > 1$, then this is immediate. If $K_1 = 1$, then we are assuming $i < \delta$, and therefore by Corollary 15.2.1, Γ_i contains a copy of Γ_1. As $K_1 = 1$ and Γ_1 is neither complete nor imprimitive, it follows that the connected component in Γ_i of a vertex, with respect to the relation "$d(x, y) = 2$," is closed under the relation "$d(x, y) \le 2$". So whatever the value of K_1, the set I_u is contained in a single such component C_u.

Take $v \in \Gamma_i$. By Lemma 15.2 v has two neighbors u_1, u_2 in Γ_{i-1} with $d(u_1, u_2) = 2$. We have $C_{u_1} = C_{u_2}$ and $d(u_1, u_2) = 2$, and Γ_{i-1} is assumed connected with respect to the relation "$d(x, y) = 2$" by our induction hypothesis. It follows that C_u is independent of the choice of $u \in \Gamma_{i-1}$, and thus Γ_i consists of a single such component. $\square$

LEMMA 15.5. *Let Γ be a countable metrically homogeneous graph of generic type. Suppose $1 \le i \le \delta - 1$. Then for $u \in \Gamma_{i\pm 1}$, $\Gamma_1(u) \cap \Gamma_i$ is infinite. The same holds for $i = \delta$ and $u \in \Gamma_{i-1}$, unless Γ is of antipodal type.*

PROOF. Suppose first that

$$i < \delta.$$

Then our claim follows from Corollary 15.2.1.

So suppose for the remainder of the argument that

$$i = \delta.$$

For $u \in \Gamma_{\delta-1}$, let I_u be the set of neighbors of u in Γ_δ. Our claim is that I_u is infinite.

Suppose the contrary. Then I_u is finite of some fixed order k, for $u \in \Gamma_{\delta-1}$.

For $w \in \Gamma_{\delta-2}$ let J_w be the set of neighbors of w in $\Gamma_{\delta-1}$. We also fix $v \in \Gamma_\delta$ with $d(v, w) = 2$ and consider $\Gamma' = \Gamma_1(v) \cap \Gamma_1(w)$. Then

$$\Gamma' \subseteq J_w \subseteq \Gamma_1(w).$$

By Lemma 15.2, Γ_1 embeds into Γ'. It follows that

$$\Gamma' \cong J_w.$$

since these are homogeneous graphs which embed into each other. Since Γ' and J_w are both contained in $\Gamma_{\delta-1}$, they are isomorphic also over the basepoint.

Now the intersection of all I_u for $u \in \Gamma'$ is nonempty, as it contains v. Since the I_u are finite, it follows by homogeneity that the intersection

$$J_w^* = \bigcap_{u \in J_w} I_u.$$

is also nonempty. Furthermore, the elements of J_w^* are at distance 2 from w. By homogeneity, J_w^* consists of precisely the vertices of Γ_δ at distance 2 from w. In particular, the set J_w^* can also be expressed as

$$J_w^* = \bigcup_{u \in J_w} I_u.$$

and thus this union is finite.

By Lemma 15.2, J_w contains an infinite independent set of vertices J'_w. Then J'_w contains an infinite subset J''_w such that the sets I_u for

$u \in J''_w$ form a Δ-system. On the other hand each of these sets I_u is contained in the finite set J^*_w. Therefore all of these sets are equal. It follows by homogeneity that for $u_1, u_2 \in \Gamma_{\delta-1}$ with $d(u_1, u_2) = 2$, we have

$$I_{u_1} = I_{u_2}.$$

Now $\Gamma_{\delta-1}$ is connected with respect to the relation "$d(x, y) = 2$," by Lemma 15.4. It follows that I_u is independent of the choice of $u \in \Gamma_{\delta-1}$, and therefore Γ_δ is finite. But then Γ is antipodal by Lemma 15.3. □

We are dealing here with countable metrically homogeneous graphs Γ which are not necessarily in our catalog, but which do have well-defined associated parameters

$$\delta, K_1, K_2, C, C', \mathcal{S}.$$

We must be careful not to assume however that these parameters are admissible, or that they give much information about Γ—at least, not until some of the relevant issues have actually been addressed. In particular, it will be convenient to have the following now (and more later on).

LEMMA 15.6. *Suppose that Γ is a countable metrically homogeneous graph of diameter δ which contains a triangle of perimeter $p = 2\delta + d$. Then Γ_δ has diameter at least d.*

PROOF. Take a triangle (a, b, c) of maximal perimeter and of type (i, j, k) with $i + j$ maximized, and $i \le j$. Let us take the notation as follows.

$$d(a, b) = i; \qquad d(a, c) = j; \qquad d(b, c) = k.$$

If $i = j = \delta$ then our claim follows.

So suppose

$$i < \delta.$$

and take b' so that (abb') is a geodesic with $d(a, b') = i + 1$ (Figure 120).

Then $d(b', c) \ge k - 1$, so the perimeter of the triangle $(ab'c)$ is at least $i + j + k$. As $(i + 1) + j > i + j$ this contradicts the choice of (a, b, c). □

Now we may take up the proof of Lemma 15.1.

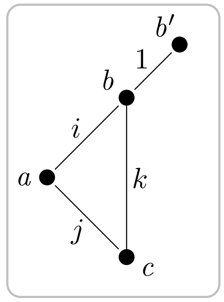


Figure 120

Proof of Lemma 15.1. We are supposing that Γ_i is connected. We show first that the induced metric on Γ_i is the graph metric (a point also covered in Cameron [1998]).

Suppose there is a pair of points u, v whose distance r in the induced metric does not agree with the distance r' in the graph metric, and take r minimal. Take a path P of minimal length r' from u to v. Then P is a geodesic in the graph metric. Take w on P at distance r from u along P. Then in the induced metric, $d(u, w) \leq r$. By minimality of r, $d(u, w) = r$. As $d(u, w)$ agrees with the graph metric, by homogeneity $r = r'$, and we have a contradiction.

Now since Γ_i is homogeneous in the induced metric, and this is the graph metric on Γ_i, it follows that Γ_i is metrically homogeneous.

Suppose Γ_i is not of generic type. Consider the classification of countable metrically homogeneous graphs of non-generic type, given explicitly by Fact 1.18. These fall under the following headings.

1. Complete;
2. Complete multipartite with parts of order greater than 1, and at least two parts;
3. The complement H_n^c of a Henson graph, with $3 \leq n < \infty$;
4. Finite;
5. Infinite diameter.

Certainly Γ_i has finite diameter, so we discard the last possibility.

If Γ_i is finite, then Γ is antipodal and $i = \delta$ by Lemma 15.3. This is one of our exceptional cases:

$$i = \delta,\ C = 2\delta + 1,\ C' = C + 1.$$

So we can set aside the last two possibilities, and we are left with cases (1–3) to consider.

Suppose first that

$$i < \delta.$$

Then by Lemma 15.2, the graph Γ_i contains a copy of Γ_1, and in particular Γ_i contains an infinite independent set. This excludes cases $(1, 3)$, so Γ_i is complete multipartite. But Γ_i should be connected with respect to the relation $d(x, y) = 2$, and so we have a contradiction.

So we suppose henceforth

$$i = \delta.$$

We consider cases (1–3) individually.

Case I. Suppose Γ_δ *is complete.*

If Γ_δ is trivial, then Γ is antipodal and $C = 2\delta + 1$, $C' = C + 1$. So we suppose

$$\Gamma_\delta \text{ is nontrivial.}$$

Then Γ_δ is infinite, by Lemma 15.3. In particular

$$K_1 = 1.$$

Also, Γ_δ has diameter 1, and it follows from Lemma 15.6 that there are no triangles of perimeter greater than $2\delta + 1$. So by definition of C_0, C_1 we find

$$C_0 = 2\delta + 2 \text{ and } C_1 = 2\delta + 3.$$

This of course is one of the stated exceptions.

Case II. Suppose Γ_δ *is complete multipartite with nontrivial parts.*

Take a pair of vertices (v_1, v_2) in Γ_δ at distance 2. Extend the pair (v_1, v_2) to a geodesic (uv_1v_2) with $d(u, v_2) = 3$. Then necessarily $u \in \Gamma_{\delta-1}$.

For $u \in \Gamma_{\delta-1}$ let I_u denote $\Gamma_1(u) \cap \Gamma_\delta$.

Suppose there is some $v \in I_u$ with $d(v, v_1) = 1$. Let A_v be the part of Γ_δ containing v. For $a \in A_v$, we have $d(a, v_1) = 1$ and hence $d(a, u) \leq 2$. But by homogeneity, comparing v and v_1, there should be v' in A_v with $d(u, v') = 3$. This contradiction shows that I_u is contained in a unique part $\hat{I}_u$ of Γ_δ.

Take $v \in \Gamma_\delta$. Then there is a pair $u, u' \in \Gamma_{\delta-1}$ of neighbors of v with $d(u, u') = 2$ (Lemma 15.2). Then $\hat{I}_u = \hat{I}_{u'}$. By homogeneity, the same applies to any pair $u, u' \in \Gamma_{\delta-1}$ at distance 2. As $\Gamma_{\delta-1}$ is connected with respect to the relation $d(x, y) = 2$, it follows that $\hat{I}_u$

is independent of the choice of $u \in \Gamma_{\delta-1}$, and hence Γ_δ has only one part, a contradiction.

Case III. Suppose Γ_δ is the complement H_n^c of a Henson graph, with $3 \leq n < \infty$.

Then Γ contains an infinite clique, so Γ_1 contains an infinite clique, and must be a random graph. In particular, for any finite subset A of Γ_1, the induced graph on the set of common neighbors of A is a homogeneous graph containing the random graph, and is therefore the random graph.

Let I be a 2-anticlique of order $n-1$ in Γ_δ ($I \cong I_n^{(2)}$). The induced graph on the set of common neighbors of I is the random graph, hence is not contained in Γ_δ. Thus we can find a vertex $v \in \Gamma_{\delta-1}$ adjacent to all vertices of I.

Now beginning with a pair of vertices u_1, u_2 in Γ_δ with $d(u_1, u_2) = 2$, and extending to a geodesic $(u_1 u_2 v_1)$ with $d(u_1, v_1) = 3$, we find $v_1 \in \Gamma_{\delta-1}$. By homogeneity we can find $u \in \Gamma_\delta$ so that $d(u, v) = d(u_2, v_1) = 3$.

Then for $a \in I$ we must have $d(a, u) \geq 2$. But Γ_δ has diameter 2, so $I \cup \{a\}$ is a 2-anticlique of order n in Γ_δ, a contradiction. □

15C. Local connectivity

In the present subsection we aim at the following result.

LEMMA 15.7. *Let Γ be a countable metrically homogeneous graph of generic type and of diameter $\delta \geq 3$, and let $i \leq \delta$. Suppose that Γ_i contains an edge. Then Γ_i is connected.*

We first treat the more typical cases in which either $i < \delta$ or $K_1 > 1$.

LEMMA 15.8. *Let Γ be a countable metrically homogeneous graph of generic type and of diameter δ, and let $i \leq \delta$. Suppose that Γ_i contains an edge, and one of the following holds.*
(a) $i < \delta$; or
(b) $K_1 > 1$.
Then Γ_i is connected.

PROOF. We may suppose $i > 1$. We proceed by induction on i. Thus if Γ_{i-1} has an edge, we suppose that Γ_{i-1} is connected.

For $u \in \Gamma_{i-1}$, we let I_u be the set of neighbors of u in Γ_i.

Suppose first that

(∗) The connected components of Γ_i have diameter at least 2.

Then each set I_u is contained in a unique connected component C_u of Γ_i.

For $v \in \Gamma_i$ we have a pair of neighbors u_1, u_2 of v in Γ_{i-1} with $d(u_1, u_2) = 2$, by Lemma 15.2. It follows that

$$C_{u_1} = C_{u_2}$$

whenever $d(u_1, u_2) = 2$. But by Lemma 15.4, Γ_{i-1} is connected with respect to the relation "$d(x, y) = 2$," and thus the component C_u is independent of $u \in \Gamma_{i-1}$, and Γ_i must be connected.

Now we suppose the following.

(∗∗) The connected components of Γ_i are complete.

If $K_1 = 1$, then by hypothesis $i < \delta$, so Γ_i contains a copy of Γ_1 by Lemma 15.2, and this contradicts (∗∗).

So in our present case we will have

$$K_1 > 1.$$

But then Γ contains no cliques of order 3, and hence the connected components of Γ_i have order precisely 2. That is, there is a definable pairing

$$v \leftrightarrow v'$$

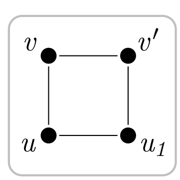
on Γ_i defined by

$$d(v, v') = 1.$$

Take $u \in \Gamma_{i-1}$ and consider as usual the set I_u of neighbors of u in Γ_i. For $v \in I_u$ we have

$$d(u, v') = 2.$$

By Lemma 15.2 there is a vertex u_1 adjacent to both u and v', and not equal to v. So u_1 cannot be in Γ_i, and must be in Γ_{i-1}.



Thus Γ_{i-1} contains the edge (u, u_1), and so by induction, we suppose Γ_{i-1} is connected. In particular any pair of vertices at distance 2 in Γ_{i-1} have a common neighbor in Γ_{i-1}.

By homogeneity, for any edge (u_1, u_2) in Γ_{i-1}, and for any $v \in I_{u_1}$, we have $d(u_2, v') = 1$. This may be expressed as follows.

$$\{v' \mid v \in I_{u_1}\} \subseteq I_{u_2}.$$

Switching u_1 and u_2, we may conclude that

$$\{v' \mid v \in I_{u_1}\} = I_{u_2} \text{ when } d(u_1, u_2) = 1.$$

When $u_1, u_2 \in \Gamma_{i-1}$ satisfy $d(u_1, u_2) = 2$, as u_1, u_2 have a common neighbor in Γ_{i-1}, we find $I_{u_1} = I_{u_2}$. But Γ_{i-1} is connected with respect to the relation "$d(x, y) = 2$" by Lemma 15.4, and thus I_u is independent of u. But then $\Gamma_i = I_u$ is an independent set, and we have assumed it contains an edge. □

Now we deal with the case left over from the previous lemma.

LEMMA 15.9. *Let Γ be a countable metrically homogeneous graph of generic type and of diameter $\delta \geq 3$, with $K_1 = 1$. Suppose that Γ_δ contains an edge. Then Γ_δ is connected.*

PROOF. The graph $\Gamma_{\delta-1}$ contains a copy of Γ_1 by Corollary 15.2.1, and in particular $\Gamma_{\delta-1}$ contains an edge. By Lemma 15.8, we have

$$\Gamma_{\delta-1} \text{ is connected.}$$

For $u \in I_{\delta-1}$, we consider the set I_u of neighbors of u in Γ_δ.

Suppose first that

(†) The connected components of Γ_δ have diameter greater than 1.

Then for $u \in \Gamma_{\delta-1}$, the set I_u is contained in a unique connected component C_u of Γ_δ. Then as usual, there are pairs u_1, u_2 in $\Gamma_{\delta-1}$ with $d(u_1, u_2) = 2$ and with $C_{u_1} = C_{u_2}$, and it follows from Lemma 15.4, that Γ_δ consists of a single connected component. So the case of interest is the remaining one.

(‡) The connected components of Γ_δ are complete

Suppose first that for $u \in \Gamma_{\delta-1}$, the set I_u is a union of certain connected components of Γ_δ. This means that any pair v_1, v_2 of adjacent vertices in Γ_δ have the same neighbors in $\Gamma_{\delta-1}$, and it follows that they have the same neighbors in Γ, apart from each other. But then

Γ carries a nontrivial definable equivalence relation. So Γ is bipartite or antipodal; but in either case, there would be no edge in Γ_δ.

So I_u will not be a union of connected components of Γ_δ. We extend I_u to the set $\hat{I}_u$ which is the union of all connected components of Γ_δ which meet I_u. At this point we know the following.

$$\hat{I}_u \text{ contains a vertex at distance 2 from } u.$$

By homogeneity, we may conclude

$$\hat{I}_u = \{v \in \Gamma_\delta \mid d(u,v) \leq 2\}.$$

Let $u_1, u_2 \in \Gamma_{\delta-1}$ be adjacent. We will show that

$$\hat{I}_{u_1} = \hat{I}_{u_2}.$$

For $v \in I_{u_1}$, we have $d(u_2, v) \leq 2$, and hence $v \in \hat{I}_{u_2}$. Thus

$$I_{u_1} \subseteq \hat{I}_{u_2},$$

and hence

$$\hat{I}_{u_1} \subseteq \hat{I}_{u_2}.$$

Then by symmetry $\hat{I}_{u_1} = \hat{I}_{u_2}$.

Now as $\Gamma_{\delta-1}$ is connected, it follows that $\hat{I}_u$ is independent of u, and thus $\Gamma_\delta = \hat{I}_u$. Thus all distances occurring between $\Gamma_{\delta-1}$ and Γ_δ are at most 2, and I_u meets every connected component of Γ_δ.

Suppose Γ_δ is not connected. Let $u_1, u_2 \in \Gamma_\delta$ be chosen to maximize the distance

$$d = d(u_1, u_2),$$

and let C_1, C_2 be the connected components in Γ_δ of u_1, u_2, respectively. Then we have $C_1 \neq C_2$ and $d \geq 2$.

If $d < \delta$, let v be adjacent to u_2 with $d(u_1, v) = d+1$. By the choice of d, we have $v \in \Gamma_{\delta-1}$. Hence $d(u_1, v) \leq 2$, a contradiction. Thus we conclude that

$$d = \delta \geq 3.$$

As $\hat{I}_u = \Gamma_\delta$ for $u \in \Gamma_{\delta-1}$, we have $d \leq 3$. We conclude

$$d = \delta = 3.$$

If v_* is our chosen basepoint for Γ, then (v_*, u_1, u_2) is a triangle of type $(3,3,3)$. The basepoint v_* belongs to $\Gamma_3(u_1)$. The connected component of $\Gamma_3(u_1)$ containing u_2 is contained in Γ_3, because $\Gamma_3(u_1)$

does not meet Γ_2. Thus in $\Gamma_3(u_1)$ we see the following configuration: a connected component, and a vertex lying at distance 3 from each point in that connected component.

But in Γ_3, there is no vertex v which lies at distance 3 from all vertices in one of the connected components, because v has a neighbor $u \in \Gamma_2$ and I_u meets each connected component.

This is a contradiction. Thus Γ_δ is connected. $\square$

Now the proof of Lemma 15.7 is complete, with the first two cases covered by Lemma 15.8, and the last case covered by Lemma 15.9.

15D. Local primitivity

Now we aim at the following.

LEMMA 15.10. *Let Γ be a countable metrically homogeneous graph of generic type and of diameter $\delta \geq 3$. Suppose that $i \leq \delta$, and Γ_i contains an edge. Then Γ_i is primitive, unless one of the following occurs.*

1. *$i = \delta$, $K_1 = 1$, and Γ_δ is bipartite of diameter at least 3; or*
2. *Γ is antipodal, δ is even, and $i = \delta/2$.*

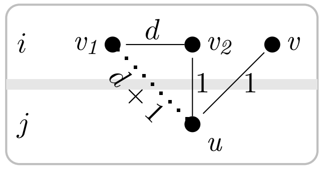
This will require some preliminary analysis. By Smith's theorem (Fact 1.27), in a counterexample Γ_i would be either bipartite or antipodal.

We already showed in Corollary 15.4.1 that if Γ_i is bipartite, then $i = \delta$ and $K_1 = 1$. In Lemma 15.1 we disposed of the case in which Γ_δ is bipartite in diameter 2. So if Γ_δ is bipartite, we arrive at (1). Thus our analysis will take as its starting point the assumption that Γ_i is antipodal.

LEMMA 15.11. *Let Γ be a countable metrically homogeneous graph of generic type and of diameter δ. Let $i \leq \delta$. If Γ_i is antipodal, then either the diameter of Γ_i is δ, or Γ is antipodal and $i = \delta$.*

We remark that when Γ is antipodal and $i = \delta$, one may consider Γ_δ as a trivial antipodal graph of diameter 0. For that reason we chose to include this possibility explicitly in the statement of the lemma.

PROOF. Let d be the diameter of Γ_i, and suppose that $d < \delta$. Let $v_1, v_2 \in \Gamma_i$ with $d(v_1, v_2) = d$. Take u adjacent to v_2 with $d(u, v_1) = d + 1$. Then $u \in \Gamma_j$ with $j = i \pm 1$.



If $v \in \Gamma_i$ is adjacent to u, then $d(v_1, v) = d$, so $v = v_2$. Thus any $u \in \Gamma_j$ has a unique neighbor in Γ_i.

But this contradicts Lemma 15.5. □

LEMMA 15.12. *Let Γ be a countable metrically homogeneous graph of generic type and of diameter δ. Let $i \leq \delta$. If Γ_i contains an edge and is antipodal, then for any pair of vertices $v, v' \in \Gamma$ with*

$$d(v, v') = \delta,$$

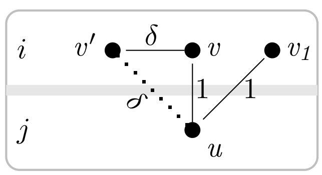
and for any u adjacent to v, we have

$$d(v', u) = \delta - 1.$$

PROOF. By Lemma 15.11, we may take $v, v' \in \Gamma_i$ with $d(v, v') = \delta$, and then for $u \in \Gamma_i$ this is just an instance of the antipodal law in Γ_i. But we must also consider $u \in \Gamma_{i\pm1}$.

Suppose then that we have u adjacent to v with $d(u, v') \neq \delta - 1$. Then $u \in \Gamma_j$ for some $j = i \pm 1$, and we have

$$d(u, v') = \delta.$$



Suppose $v_1 \in \Gamma_i$ is adjacent to u and distinct from v. Then $d(v', v_1) \geq \delta - 1$ and $d(v', v_1) \neq \delta$, so

$$d(v', v_1) = \delta - 1.$$

By the antipodal law in Γ_i, $d(v, v_1) = 1$. Thus

For $u \in \Gamma_j$, the set I_u of neighbors of u in Γ_i forms a clique.

If $i < \delta$, then Lemma 15.2 shows that the neighbors of u in Γ_i contain a copy of Γ_1, and we have a contradiction. So we conclude

$$i = \delta,\ j = \delta - 1.$$

Thus Γ_δ has diameter δ.

Consider a pair of points $v_1, v_2 \in \Gamma_\delta$ with $d(v_1, v_2) = 2$, and any common neighbor v of v_1, v_2. Then v cannot lie in $\Gamma_{\delta-1}$, and therefore v lies in Γ_δ.

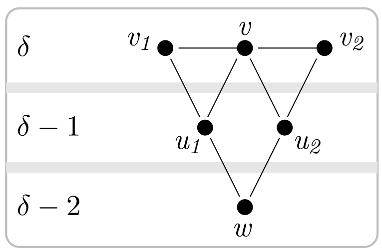
We consider two cases. First suppose the following (illustrated below).

There is a common neighbor u_1 of v, v_1 in $\Gamma_{\delta-1}$.

Then there is also a common neighbor u_2 of v, v_2 in $\Gamma_{\delta-1}$. Note that

$$d(u_1, u_2) \leq 2.$$

Then by Lemma 15.2, u_1, u_2 have a common neighbor w in $\Gamma_{\delta-2}$.



Now v_1, v_2, w is a triple of points at mutual distance 2. Therefore there is a vertex u adjacent to all three vertices v_1, v_2, w. Then $u \in \Gamma_{\delta-1}$ and I_u is not a clique, giving a contradiction.

Now we consider the remaining case.

There is no common neighbor of v, v_1 in $\Gamma_{\delta-1}$.

Take $u \in \Gamma_{\delta-1}$. As I_u is a clique, and the elements of I_u have the common neighbor $u \in \Gamma_{\delta-1}$, it follows that I_u consists of a single point. But this contradicts Lemma 15.5. $\square$

LEMMA 15.13. *Let Γ be a countable metrically homogeneous graph of generic type and of diameter $\delta \geq 3$. Let $i \leq \delta$. If Γ_i contains an edge and is antipodal, then Γ is antipodal, δ is even, and $i = \delta/2$.*

PROOF. By Lemmas 15.11 and 15.12, the graph Γ_i has diameter δ, and Γ satisfies the following special case of the antipodal law.

If $d(v, v') = \delta$ and $d(v, u) = 1$ then $d(v', u) = \delta - 1$.

In particular

Γ_δ contains no edge.

Hence

$$i < \delta.$$

As the diameter of Γ_i is δ, we also have $i \geq \delta/2$. If Γ is antipodal, then we have $\Gamma_i \cong \Gamma_{\delta-i}$, and hence $\delta - i \geq \delta/2$. Thus $i = \delta/2$, and in particular δ is even. So everything we aim at holds in this case. Therefore we suppose toward a contradiction that

$$\Gamma \text{ is not antipodal.}$$

Claim 1. For v, v' in Γ_i with $d(v, v') = \delta$, there are

$$u_- \in \Gamma_{i-1} \cup \Gamma_{i-2},\ u_+ \in \Gamma_{i+1} \cup \Gamma_{i+2},$$

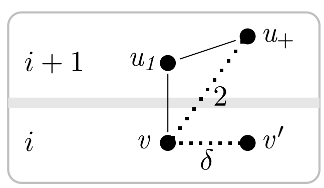
with

$$d(v, u_\pm) = 2,\ d(v', u_\pm) = \delta.$$

Take $u_1 \in \Gamma_{i+1}$ adjacent to v. Then

$$d(v', u_1) = \delta - 1.$$

Now we wish to extend this configuration by a vertex u_+ adjacent to u_1, at distance 2 from v, and at distance δ from v'.



To see that a suitable vertex u_+ exists, we work for a moment relative to the basepoint v'. We have $u_1 \in \Gamma_{\delta-1}(v')$ and we consider the set I_{u_1} of neighbors of u_1 in $\Gamma_\delta(v)$. By Lemma 15.5 this set is infinite, so we take $u_+ \in I_{u_1}$ with $u_+ \neq v$.

At this point we have

$$d(u_1, u_+) = 1,\ d(v', u_+) = \delta,\ u_+ \neq v.$$

Since $u_+ \neq v$ and Γ_i is antipodal, it follows that $u_+ \notin \Gamma_i$. Thus u_+ is in Γ_{i+1} or Γ_{i+2}.

Now $d(v, u_+) \leq 2$. If $d(v, u_+) = 1$, then $d(v', u_+) = \delta - 1$, a contradiction. So $d(v, u_+) = 2$.

This completes the construction of u_+, and the construction of u_- is similar. Thus the claim is proved.

Now we introduce an auxiliary graph G. The vertex set of G will be $\Gamma_\delta(v')$, and the edge relation is given by

$$d(x, y) = 2.$$

The vertices v, u_-, u_+ are in G, with u_- and u_+ adjacent to v in G.

Claim 2. The vertex v is a cut vertex for its connected component in G.

We divide $G \setminus \{v\}$ into the two parts

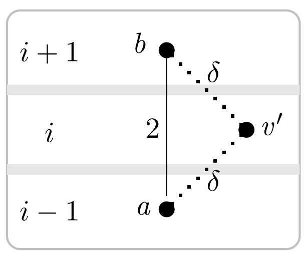
$$A = \bigcup_{j<i} (\Gamma_j \cap G)$$

$$B = \bigcup_{j>i} (\Gamma_j \cap G)$$

It suffices to show that there are no edges of G joining A to B, or in other words no $a \in A$, $b \in B$ with $d(a, b) = 2$.

If (a, b) were such a pair then we would have

$$a \in \Gamma_{i-1},\ b \in \Gamma_{i+1},\ d(a, v') = d(b, v') = \delta, \text{ and } d(a, b) = 2.$$



In this case we consider a common neighbor u of a, b. Then $u \in \Gamma_i$. As $d(u, a) = 1$, we have

$$d(u, v') = \delta - 1.$$

By the antipodal law in Γ_i we find

$$d(u, v) = 1.$$

Thus $d(v, a), d(v, b) \leq 2$ and as $\Gamma_\delta(v')$ contains no edge we find

$$d(v, a) = d(v, b) = d(a, b) = 2.$$

Now we have just seen that any common neighbor u of a, b is a neighbor of v. By homogeneity if we take two vertices b', c' in Γ_2 at distance 2, and let a' be the basepoint of Γ, we find that any neighbor of b' in Γ_1 is a neighbor of c', or by symmetry that b', c' have the same neighbors in Γ_1.

By Lemma 15.4, Γ_2 is connected with respect to the relation given by "$d(x, y) = 2$" and we conclude that all vertices of Γ_2 have the same neighbors in Γ_1. It follows that every vertex of Γ_2 is adjacent to every vertex of Γ_1, or in other words $\Gamma_1 \subseteq \Gamma_1(u)$ for $u \in \Gamma_2$.

Then again by symmetry, $\Gamma_1(u_1) = \Gamma_2(u_2)$ whenever $d(u_1, u_2) = 2$. Now taking u_1 as the basepoint and $u_2 \in \Gamma_2$, we find that Γ_3 is empty and $\delta = 2$, a contradiction.

This proves the claim: v is a cut vertex in G.

It now follows by metric homogeneity that every vertex of the graph G is a cut vertex for its connected component. In particular G is a forest. But as G has finite diameter, it would then have leaves, giving a final contradiction. □

PROOF OF LEMMA 15.10. We suppose that Γ is metrically homogeneous of diameter δ, and that Γ_i contains an edge, with $i \leq \delta$.

We assume further that Γ_i is imprimitive. In this case by Smith's Theorem (Fact 1.27), the graph Γ_i is bipartite or antipodal (possibly both).

We consider the two possibilities separately.

Case I. Γ_i *is bipartite.*

By Corollary 15.4.1 this forces

$$\begin{aligned} i &= \delta; \\ K_1 &= 1. \end{aligned}$$

This is our exceptional case (1).

Case II. Γ_i *is antipodal.*

Then Lemma 15.13 applies and gives us our exceptional case (2). □

15E. The case of Γ_δ bipartite

We eliminate the exceptional case left over in Lemma 15.10.

LEMMA 15.14. *Let Γ be a countable metrically homogeneous graph of generic type and of diameter $\delta \geq 3$. Suppose that Γ_δ contains an edge. Then Γ_δ is not bipartite*

PROOF. Assuming the contrary, we know

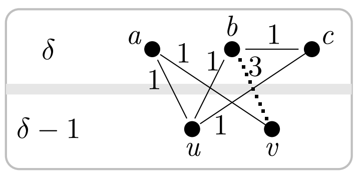
$$K_1 = 1.$$

We make an amalgamation argument to embed a triangle of type $(2,2,1)$ into Γ_δ.

We consider the configuration $(abcuv)$ where a, b, c is a triangle of type $(2,2,1)$ with

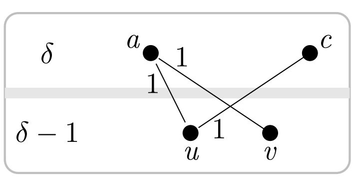
$$\begin{array}{llll}
d(a,b) = d(a,c) = 2 & d(b,c) = 1 & & \\
d(u,a) = 1 & d(u,b) = 1 & d(v,a) = 1 & d(v,b) = 3 \\
d(u,v) = 2 & & & \\
d(u,c) = 1 & d(v,c) = 2 & & \\
a, b, c \in \Gamma_\delta & u, v \in \Gamma_{\delta-1} & &
\end{array}$$

As the last line indicates, the basepoint is also included in this configuration.

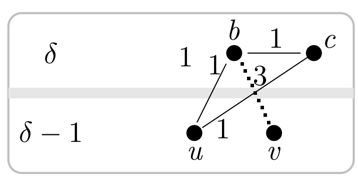


(Distances not shown explicitly equal 2.)

The existence of a, b, c in Γ_δ suffices to give our claim. We may view this configuration as a two point amalgamation problem in which the distance $d(a,b)$ is to be determined, with the value controlled by the vertices u, v. It therefore suffices to show that the two factors of the diagram embed isometrically into Γ.



(I)



(II)

Factor (I): (a, c, u, v)

We view the first configuration as a two point amalgamation problem with the distance from v to the basepoint to be determined. As (a, c, v) is a triangle of type $(2, 2, 1)$, if this distance is δ we have the desired contradiction.

As $d(a, v) = 1$, the only alternative is that $v \in \Gamma_{\delta-1}$, as desired. So the configuration $(acuv)$ will be embedded as shown if and only the following conditions are satisfied.

1. The configuration (auc) embeds as shown over the basepoint.
2. The configuration $(acuv)$ embeds into Γ, but not necessarily respecting the distances to the basepoint.

The configuration (acu) consists of a vertex $u \in \Gamma_{\delta-1}$ adjacent to two vertices of Γ_δ at distance 2. By Lemma 15.5, u has infinitely many neighbors in Γ_δ, and as we assume Γ_δ is bipartite, this configuration is realized.

The configuration $(acuv)$ is already found in Γ_1.

Factor (II): (b, c, u, v)

We view this factor as a two point amalgamation problem determining the distance from u to the basepoint. As (ubc) is a triangle of type $(1, 1, 1)$, that distance is forced to be $\delta - 1$.

So the configuration $(bcuv)$ will be embedded as shown if and only the following conditions are satisfied.

1. The configuration (bcv) embeds as shown over the basepoint.
2. The configuration $(bcuv)$ embeds into Γ, but not necessarily respecting the distances to the basepoint.

Now the configuration (bcv), over the basepoint, represents a vertex $v \in \Gamma_{\delta-1}$ at distances $2, 3$ from a pair of adjacent vertices in Γ_δ.

Fix $v \in \Gamma_{\delta-1}$ and let J_v be the set of vertices in Γ_δ at distance at most 2 from v. If $J_v \neq \Gamma_\delta$, then by the connectivity of Γ_δ, there is a pair of adjacent vertices b, c with $c \in J_v$ and $b \notin J_v$. This is the desired configuration.

We must show that $J_v \neq \Gamma_\delta$.

Assume the contrary: $J_v = \Gamma_\delta$. Let d be the diameter of Γ_δ, and take $w, w' \in \Gamma_\delta$ at distance d. If $d < \delta$ then extending to w'' adjacent to w' with $d(w, w'') = d + 1$ gives a contradiction. Therefore $d = \delta$. As $\Gamma_\delta(w)$ is connected, and contains the basepoint as well as a point in Γ_δ, it must contain a point of $\Gamma_{\delta-1}$. This contradicts $J_v = \Gamma_\delta$.

At this point we have dealt with the configuration (bcv) over the basepoint, and we turn to $(bcuv)$, with the basepoint discarded.

Treating u as the basepoint for this configuration, we require $v \in \Gamma_2$ to lie at distances 2,3 from a pair of adjacent points in Γ_1.

So we now let J_v denote the set of vertices in Γ_1 at distance at most 2 from v, and if $J_v \neq \Gamma_1$ conclude as above.

So it suffices to show that the distance 3 occurs between vertices in Γ_1 and Γ_2: or in other words, that Γ contains a triangle of type $(1, 2, 3)$, which is simply a geodesic.

This completes the analysis. $\square$

15F. Proofs of Theorem 1.32 and Proposition 1.33

Now we bring together the results proved piecemeal above.

PROOF OF THEOREM 1.32. Our assumptions are that Γ is a countable metrically homogeneous graph of generic type and diameter δ, and that Γ_i contains an edge, where $i \leq \delta$.

In Lemma 15.7 we saw that Γ_i is connected. By Lemma 15.1, Γ_i is metrically homogeneous, and one of the following conditions hold.

(a) Γ_i is of generic type, or
(b) $i = \delta$, $C \leq 2\delta + 2$ and $C' = C + 1$, Γ_δ is complete, and either
 — $K_1 = 1$, or
 — Γ is antipodal;

By Lemmas 15.10 and 15.14 we also have one of the following.

(c) Γ_i is primitive; or
(d) Γ is antipodal, δ is even, and $i = \delta/2$.

So the exceptional cases, when Γ_i is either not of generic type, or is imprimitive (but contains an edge) are as given in the statement of Theorem 1.32. $\square$

Now we take up Proposition 1.33, relating to the case $K_1 \leq 2$. We have already assembled the necessary technical ingredients in the proof of Theorem 1.32.

But we will treat one further point separately, as it has value for its own sake.

LEMMA 15.15. *Let Γ be a countable metrically homogeneous graph of generic type, with $K_1 = 2$. Let $u_1, u_2 \in \Gamma$ with $d(u_1, u_2) = 2$. Then $\Gamma_2(u_1) \cap \Gamma_2(u_2)$ contains an edge.*

PROOF. Take $v \in \Gamma_1$, and $u \in \Gamma_2$ adjacent to v. As Γ_2 is connected and metrically homogeneous of generic type, we may find distinct neighbors u_1', u_2' of u in Γ_2. As $K_1 > 1$ and v, u_1', u_2' are all adjacent to u, we have

$$d(u_1', u_2') = 2;$$
$$d(v, u_1') = d(v, u_2') = 2.$$

If we denote by v_* the basepoint for Γ, then (v, v_*) is an edge lying in $\Gamma_2(u_1') \cap \Gamma_2(u_2')$. By homogeneity the same configuration occurs with u_1, u_2 playing the role of u_1', u_2'. □

PROOF OF PROPOSITION 1.33. We suppose that Γ is a countable metrically homogeneous graph of generic type and diameter $\delta \geq 3$. We assume in addition that $K_1 \leq 2$, which means that there is an edge in either Γ_1 or Γ_2. We claim that all Γ_i contain edges, for $2 \leq i \leq \delta - 1$.

So fix i with

$$2 \leq i \leq \delta - 1.$$

By Corollary 15.2.1, Γ_i contains a copy of Γ_1. Therefore, if Γ_1 contains an edge then so does Γ_i. So we need only consider the case

$$K_1 = 2,$$

that is, Γ_1 is an infinite independent set, and Γ_2 contains an edge.

Claim 1. Let $u_- \in \Gamma_{i-1}$ and $u_+ \in \Gamma_{i+1}$ with $d(u_-, u_+) = 2$. Then $\Gamma_2(u_-) \cap \Gamma_2(u_+)$ meets Γ_{i-1}, and also meets Γ_{i+1} unless Γ is antipodal and $i = \delta - 1$.

Take $v \in \Gamma_i$. By Lemma 15.5, v has distinct neighbors $u_-, u_-' \in \Gamma_{i-1}$ and $u_+, u_+' \in \Gamma_{i+1}$. The claim follows.

Claim 2. Let $u_1, u_2 \in \Gamma$ with $d(u_1, u_2) = 2$. Then

$$\Gamma_2(u_1) \cap \Gamma_2(u_2) \text{ is connected.}$$

Let $\Gamma' = \Gamma_2(u_1)$. By Theorem 1.32, Γ' is metrically homogeneous, and of generic type. If we take u_2 as the basepoint for Γ', then $\Gamma_2' = \Gamma_2(u_1) \cap \Gamma_2(u_2)$, and by a second application of Theorem 1.32, it suffices to show that $\Gamma_2(u_1) \cap \Gamma_2(u_2)$ contains an edge. But this is given by Lemma 15.15.

This proves our second claim.

Now we put the two claims together. We take $u_- \in \Gamma_{i-1}$ and $u_+ \in \Gamma_{i+1}$ with $d(u_-, u_+) = 2$. We consider the connected graph $\Gamma_2(u_-) \cap \Gamma_2(u_+)$, which meets both Γ_{i-1} and Γ_{i+1}, and therefore meets Γ_i. Thus we get a triangle u_-, u_+, u with $u \in \Gamma_i$ and all distances equal to 2.

By homogeneity there is a vertex v adjacent to u_-, u_+, and u. Then as $u_\pm \in \Gamma_{i\pm 1}$, we have $v \in \Gamma_i$ and (v, u) is an edge in Γ_i, as required. □

15G. An application

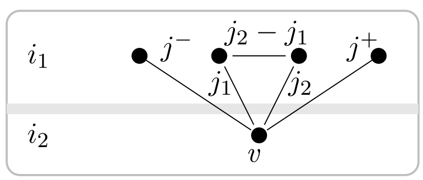
We deduce the following more specialized point.

LEMMA 15.16. *Let Γ be a countable metrically homogeneous graph of diameter δ, not of exceptional local type, and let $1 \leq i_1, i_2 \leq \delta$ with i_1, i_2 distinct. Suppose that Γ_{i_1} contains an edge. Set*

$$j^- = \min(d(u,v) \mid u \in \Gamma_{i_1}, v \in \Gamma_{i_2}),\ j^+ = \max(d(u,v) \mid u \in \Gamma_{i_1}, v \in \Gamma_{i_2}).$$

Suppose $j^- \leq j_1 \leq j_2 \leq j^+$. Then there are $u, u' \in \Gamma_{i_1}$ and $v \in \Gamma_{i_2}$ with

$$d(v,u) = j_1,\ d(v,u') = j_2,\ d(u_1,u_2) = j_2 - j_1.$$



PROOF. Take $v \in \Gamma_{i_2}$ and j with $j^- \leq j < j^+$. Let X_v^j be the set

$$\{u \in \Gamma_{i_1} \mid d(u,v) \leq j\}.$$

Then X_v^j is a proper non-empty subset of the graph Γ_{i_1}, which is connected by Theorem 1.32.

Take $u_0 \in X_v^j$ and $u_1 \in \Gamma_{i_1} \setminus X_v$ so that $d(u_0, u_1) = 1$. Since $d(v, u_0) \leq j$ and $d(v, u_1) > j$ it follows that we have

$$d(v,u_0) = j,\ d(v,u_1) = j+1.$$

Beginning with $j = j^-$ and with a pair $u_0 \in \Gamma_{i_1}$, $v \in \Gamma_{i_2}$ at distance j^-, iterate this procedure to define u_i with $d(u_{i-1}, u_i) = 1$

and $d(v, u_i) = j^- + i$ for $0 \le i \le j^+ - j^-$. It follows that $d(u_{j_1}, u_{j_2}) = j_2 - j_1$. □

15H. The interpolation property revisited

In our proof of the interpolation property (Lemma 13.9) we made heavy use of the hypothesis of 4-triviality. Some of this can be eliminated at the cost of additional effort. In particular we have the following variant of the main lemma.

LEMMA 15.17. *Let $\mathcal{A}$ be an amalgamation class of finite metric spaces corresponding to a countable metrically homogeneous graph of diameter δ. Suppose that*

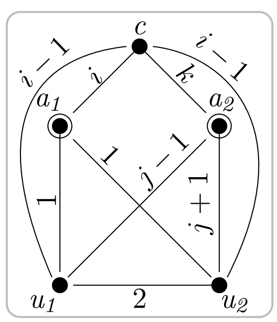
$$2 \le i \le \delta,\ 2 \le j < \delta,\ \textit{and}\ 1 \le k \le \delta.$$

If $\mathcal{A}$ contains triangles of the types

$$(1, i-1, i-1),\ (i-1, j-1, k)\ \textit{and}\ (i-1, j+1, k).$$

then $\mathcal{A}$ contains a triangle of type (i, j, k).

PROOF. We use the same amalgam considered in the previous version.



Our first claim does not require a triangle of type $(1, i-1, i-1)$.

Claim 1. The factor $A_1 = (a_1 u_1 u_2 c)$ lies in $\mathcal{A}$.

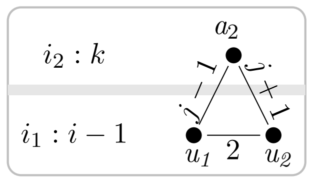
Taking a_1 as the basepoint for Γ, and $c \in \Gamma_i(a_1)$, we require two points u_1, u_2 adjacent to c in $\Gamma_{i-1}(a_1)$, with $d(u_1, u_2) = 2$. By Lemma 15.2, the neighbors of c in $\Gamma_{i-1}(a_1)$ contain a copy of Γ_1, so this is possible.

Now we consider the factor $A_2 = (a_2 u_1 u_2 c)$, and here we will indeed assume that there is a triangle of type $(1, i-1, i-1)$ in $\mathcal{A}$, or in other words, that Γ_{i-1} contains an edge.

Claim 2. The factor $A_2 = (a_2 u_1 u_2 c)$ lies in $\mathcal{A}$.

Taking c as the basepoint, we require the following configuration.

$$\begin{array}{ccc} u_1, u_2 \in \Gamma_{i-1} & a_2 \in \Gamma_k & \\ d(u_1, u_2) = 2 & d(a_2, u_1) = j-1 & d(a_2, u_2) = j+1 \end{array}$$



By our hypotheses the distances $j-1$ and $j+1$ occur between points in Γ_{i-1} and Γ_k. We apply Lemma 15.16 with $i_1 = i-1$, $i_2 = k$, $j_1 = j-1$, $j_2 = j+1$ and our claim follows. $\square$

An inductive argument gives the following case of the Interpolation Property.

LEMMA 15.18. *Let Γ be a countable metrically homogeneous graph of generic type and diameter $\delta \geq 3$. Let $\mathcal{A}$ be the associated amalgamation class of finite metric spaces. Suppose that $K_1 \leq 2$. Then Γ has the Interpolation Property.*

PROOF. Let (i, j, k) be the triangle type in question. If Γ_{i-1} contains an edge, then Lemma 15.17 applies.

By Theorem 1.32 the hypothesis that Γ_{i-1} contains an edge holds unless $i = 2$.

If $i = 2$, then $\mathcal{A}$ is assumed to contain triangles of types $(1, j \pm 1, k)$, and it follows that $k = j$ and the triangle type in question is

$$(2, j, j)$$

with $j < \delta$.

Then by Corollary 15.2.1, Γ_j contains a copy of Γ_1, and thus contains a pair of points at distance 2. $\square$

This then gives the following.

LEMMA 15.19. *Let* Γ *be a countable metrically homogeneous graph of generic type and diameter* $\delta \geq 3$. *Let* $\mathcal{A}$ *be the associated amalgamation class of finite metric spaces. Suppose that* $K_1 \leq 2$. *Then* $\mathcal{A}$ *contains all triangles whose perimeter is even and bounded by* 2δ.

PROOF. By Lemma 15.18, Γ has the Interpolation Property. By Lemma 13.12 the result follows. $\square$

CHAPTER 16

THE BIPARTITE CASE

If Γ is a metrically homogeneous bipartite graph, recall that $B\Gamma$ denotes the graph induced on either half of a bipartition by the edge relation

$$d(x,y) = 2.$$

This has the effect of rescaling the graph metric on $B\Gamma$ by a factor of $1/2$, and then $B\Gamma$ is again a metrically homogeneous graph.

Our aim now is to prove the following.

Theorem (1.30). Suppose that Γ is a countable bipartite metrically homogeneous graph, and that $B\Gamma$ is one of the graphs in our catalog. Then Γ is also in our catalog. In particular, if Γ is of generic type, then Γ has the form

$$\Gamma^{\delta}_{\infty,0,C_0,2\delta+1,\mathcal{S}}$$

with admissible parameters

$$(\delta, K_1 = \infty, K_2 = 0, C_0, C_1 = 2\delta+1, \mathcal{S}).$$

Here the parameters C_0, C_1 are to be omitted if $\delta = \infty$.

This result is intended to function, in a suitable inductive framework, as a satisfactory reduction of the bipartite case. But as noted in §1F, this reduction is more satisfactory in the case of finite diameter than in the case of infinite diameter.

When Γ has infinite diameter, then $B\Gamma$ will also have infinite diameter, but as we shall see in Lemma 16.3, $B\Gamma$ also has $K_1 = 1$. In this case, we will see in Chapter 17 that the determination of $B\Gamma$ can itself be reduced to the case of finite diameter. Thus in all cases, Theorem 1.30 leads eventually to a reduction in the diameter.

The proof of Theorem 1.30 begins with a lemma proving that the relationship between the parameters of Γ and $B\Gamma$ is what one would expect (Lemma 16.3).

16A. Preliminary analysis

Since we have the full classification of metrically homogeneous graphs of non-generic type, it suffices to prove Theorem 1.30 in the generic case.

Another class of bipartite examples was already classified in Cherlin [2011, Theorem 13]. Namely, if Γ is of generic type then we can show that, typically, $B\Gamma$ should be a random graph, $B\Gamma \cong G_\infty$. There are exceptions when the diameter of Γ is at most 5, and these exceptions were treated in Cherlin [2011], as follows.

Fact 16.1 (Cherlin [2011, Theorem 13]). Let Γ be a bipartite metrically homogeneous graph, of diameter at least 3, and degree at least 3, and with Γ_1 infinite. Then either

$$(B\Gamma)_1 \text{ is isomorphic to the random graph}$$

or $B\Gamma$ and Γ are in the catalog under one of the following headings given in Table 16.1.

Accordingly, Theorem 1.30 reduces to the following proposition. In the statement, a metrically homogeneous graph is said to be *of known type* if it is listed in the explicit version of our catalog.

Proposition 16.2. Let Γ be a countable bipartite metrically homogeneous graph for which

$$(B\Gamma)_1 \text{ is the random graph } G_\infty.$$

Suppose that $B\Gamma$ is of known type. Then Γ is generic bipartite subject to a bound on perimeter and some δ-Henson constraints, i.e.

$$\Gamma \cong \Gamma^\delta_{\infty,0,C_0,2\delta+1,\mathcal{S}}$$

for some even C_0 with $2\delta + 2 \leq C_0 \leq 3\delta + 2$.

Here the parameters C_0, C_1 are to be omitted if $\delta = \infty$.

The next lemma clarifies the relationship between $B\Gamma$ and Γ. Inspection of our catalog indicates how the parameters defining $B\Gamma$ should relate to the parameters defining Γ. Recall that if Γ is of

$B\Gamma$	$(B\Gamma)_1$	δ	Γ	Catalog entry
$T_{\infty,\infty}$	$\infty \cdot K_\infty$	∞	infinitely branching tree $T_{2,\infty}$	listed as such
K_∞	K_∞	3	the complement of a perfect matching, or the generic bipartite graph	$\Gamma^3_{\infty,0,8,7,\emptyset}$ or $\Gamma^3_{\infty,0,10,7,\emptyset}$
$K_\infty[I_2]$	$K_\infty[I_2]$	4	generic antipodal bipartite graph of diameter 4	$\Gamma^4_{\infty,0;10,9;\emptyset}$
H_n^c ($n \geq 3$)	H_n^c	4	the generic bipartite graph with no $K_n^{(4)}$ (anticlique)	$\Gamma^4_{\infty,0;14,9;\{I_n^{(4)}\}}$
H_3^c	H_3^c	5	generic antipodal bipartite of diameter 5	$\Gamma^5_{\infty,0;12,11;\emptyset}$

TABLE 16.1

generic type, but not necessarily of known type, we have a reasonable way to specify the associated parameters $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$, even if they do not provide an immediate characterization of the graph Γ. We are able to relate the parameters of the graph $B\Gamma$ to those of the unknown graph Γ, as follows.

LEMMA 16.3. *Let Γ be a countable bipartite metrically homogeneous graph of generic type with*

$$(B\Gamma)_1 \cong G_\infty,$$

the random graph.

Suppose that the parameters associated with Γ are

$$(\delta, \infty, 0, C_0, 2\delta + 1, \mathcal{S})$$

with $\delta \geq 3$. Suppose that $B\Gamma$ is of known type. Then

$$B\Gamma \cong \Gamma^{\tilde{\delta}}_{\tilde{K}_1,\tilde{K}_2,\tilde{C},\tilde{C}',\tilde{\mathcal{S}}}$$

where $B\Gamma$ has parameters

$$(\tilde{\delta}, 1, \tilde{K}_2, \tilde{C}, \tilde{C}', \tilde{\mathcal{S}})$$

satisfying the following conditions.

— $\tilde{\delta} = \lfloor \delta/2 \rfloor$;
— $\tilde{K}_1 = 1$;
— $\tilde{C}' = \tilde{C} + 1$;
— *If* $\tilde{\delta} = 2$*, then* $B\Gamma$ *is a random graph* $G_\infty \cong \Gamma^2_{1,2,7}$;
— *If* $\delta \geq 3$*, then one of the following holds.*
  (a) $\tilde{K}_2 = \tilde{\delta} - 1$ *and* Γ *is antipodal of even diameter;* $\tilde{C} = 2\tilde{\delta} + 1$ *and* $\tilde{\mathcal{S}} = \emptyset$*; or*
  (b) $\tilde{K}_2 = \tilde{\delta}$, $\tilde{C} \geq 2\tilde{\delta} + 2$.

In particular, when $\tilde{\delta} \geq 3$ *we have*

$$B\Gamma \cong \Gamma^{\tilde{\delta}}_{1,\tilde{\delta}-1,2\tilde{\delta}+1} \text{ or } \Gamma^{\tilde{\delta}}_{1,\tilde{\delta},\tilde{C},\mathcal{S}}$$

PROOF. As we suppose $(B\Gamma)_1 \cong G_\infty$, the random graph, inspection of the catalog gives

$$B\Gamma \cong \Gamma^{\tilde{\delta}}_{\tilde{K}_1,\tilde{K}_2,\tilde{C}_0,\tilde{C}_1,\tilde{\mathcal{S}}}$$

for some admissible choice of parameters $\tilde{\delta}, \tilde{K}_1, \tilde{K}_2, \tilde{C}_0, \tilde{C}_1, \tilde{\mathcal{S}}$. By definition

$$\tilde{\delta} = \lfloor \delta/2 \rfloor$$

As $B\Gamma$ contains the random graph, its diameter $\tilde{\delta}$ is at least 2, and also

$$\tilde{K}_1 = 1.$$

If $\tilde{\delta} = 2$, then as $B\Gamma$ is homogeneous and contains a random graph, it follows that $B\Gamma$ is a random graph. In this case we have nothing more to prove.

So we may now suppose

$$\tilde{\delta} \geq 3.$$

Then by admissibility we have $\tilde{K}_2 \geq \tilde{\delta} - 1$. Now we treat the two possible values for $\tilde{K}_2$ separately.

Case 1. $\tilde{K}_2 = \tilde{\delta} - 1$.

Then $B\Gamma$ contains no triangle of type $(1, \tilde{\delta}, \tilde{\delta})$, and therefore Γ contains no triangle of type $(2, 2\tilde{\delta}, 2\tilde{\delta})$. Set $i = 2\tilde{\delta}$. As Γ_i is connected with respect to the relation $d(x, y) = 2$ (Lemma 15.4), it follows that Γ_i reduces to a single point. This forces $2\tilde{\delta} = \delta$, and Γ is antipodal of even diameter.

Hence $B\Gamma$ is also antipodal, and in particular $\tilde{C}' = \tilde{C} + 1$. Thus

$$B\Gamma \cong \Gamma^{\tilde{\delta}}_{1,\tilde{\delta}-1,2\tilde{\delta}+1,\tilde{\mathcal{S}}}.$$

By irredundance, $\tilde{\mathcal{S}}$ consists at most of cliques, and there are no forbidden cliques, so $\tilde{\mathcal{S}} = \emptyset$.

Case 2. $\tilde{K}_2 = \tilde{\delta}$.

In this case, as there is a triangle in $B\Gamma$ of (rescaled) type $(1, \tilde{\delta}, \tilde{\delta})$, we have

$$\tilde{C} \geq 2\tilde{\delta} + 2 > 2\tilde{\delta} + K_1$$

and $B\Gamma$ falls under clause (III) of the definition of admissibility.

If $\tilde{C}' > \tilde{C} + 1$, then by admissibility we have $\tilde{C} \geq 3\delta$ and thus by definition $\tilde{C}' = \tilde{C} + 1$ ($\tilde{C}'$ imposes a vacuous constraint). Thus $\tilde{C}' = \tilde{C} + 1$. $\square$

16B. Realization of triangles

The next lemma will provide a robust interpretation of the parameter C_0 in our graph Γ, and determine which triangles embed in Γ; as we will see explicitly in Corollary 16.4.1.

Lemma 16.4. *Let Γ be a countable bipartite metrically homogeneous graph with $(B\Gamma)_1$ a random graph, and*

$$B\Gamma \cong \Gamma^{\tilde{\delta}}_{1,\tilde{K}_2,\tilde{C},\tilde{S}}.$$

If (i, j, k) is a triangle type with $p = i + j + k$ even and $p < 3\delta - 1$, then there is a triangle of type (i, j, k) in Γ if and only if $p < 2\tilde{C}$.

Proof. If the distances i, j, k are all even, then this holds by inspection of $B\Gamma$. So we may suppose

$$i \text{ is even, } j \text{ and } k \text{ are odd, with } j \leq k.$$

Suppose first that

(∗) There is a triangle (a, b, c) of type (i, j, k) in Γ,

where

$$d(a, b) = i,\ d(a, c) = j,\ d(b, c) = k.$$

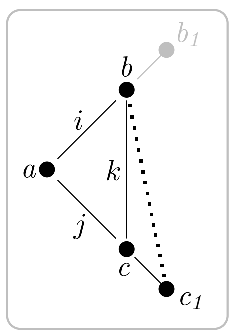
We must show $p = i + j + k < 2\tilde{C}$.

If

$$j < \delta$$

then take c_1 with

$$d(c_1, a) = j + 1,\ d(c_1, c) = 1.$$



Then $d(b, c_1) = k \pm 1$ and (a, b, c_1) is a triangle of type $(i, j+1, k\pm 1)$, with all distances even. It follows that

$$p \leq i + (j + 1) + (k \pm 1) < 2\tilde{C}.$$

If on the other hand

$$j = \delta,$$

then also $k = \delta$. Since $p < 3\delta - 1$, we have $i < \delta - 1$.

In this case, choose b_1 with $d(b_1, a) = i + 1$ and $d(b_1, b) = 1$. Then the triangle (a, b_1, c) has type $(i+1, \delta-1, \delta)$. Here $\delta - 1 = j - 1$ is even and $i + 1, \delta$ are odd. As $i + 1 < \delta$, the triple $(\delta - 1, i + 1, \delta)$ falls under the case just treated, and gives $i + j + k = (\delta - 1) + (i + 1) + \delta < 2\tilde{C}$.

Now suppose conversely that

$$(**) \qquad p < 2\tilde{C}, 3\delta - 1.$$

We will show that the triangle type is realized.

We may suppose that

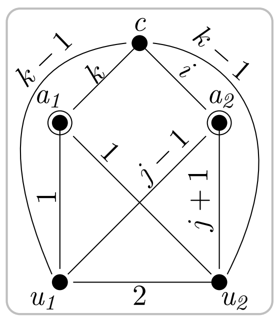
$$(i, j, k) \text{ is not a geodesic.}$$

We consider separately the cases in which $j < \delta$ and $j = \delta$.

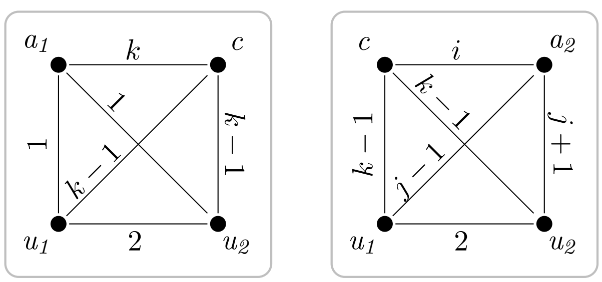
Suppose first that

$$j < \delta.$$

In this case, we consider the following 2-point amalgamation diagram.



The factors are as follows.



(I) (II)

We must show that these factors occur in Γ.

The factor $(I) = (a_1 u_1 u_2 c)$*:*

Taking a_1 as the basepoint in Γ, we require a vertex $c \in \Gamma_k$ such that there are two vertices $u_1, u_2 \in \Gamma_1$ lying at distance $k-1$ from u.

If this is not possible, then the relation

$$d(x, y) = k - 1.$$

defines a function

$$f : \Gamma_k \to \Gamma_1.$$

In particular, if Γ is antipodal then $k < \delta$.

Let $x_1, x_2 \in \Gamma_k$ have $d(x_1, x_2) = 2$. Since any $u \in \Gamma_{k-1}$ has two distinct neighbors in Γ_k, it follows that x_1, x_2 have a common neighbor $v \in \Gamma_{k-1}$. Then taking $u \in \Gamma_1$ with $d(u, v) = k - 2$ we find $f(x_1) = f(x_2) = u$. As Γ_k is connected with respect to the relation $d(x, y) = 2$ (Lemma 15.4), it then follows that the function f is constant, a contradiction.

The factor $(II) = (a_2 u_1 u_2 c)$*:*

We should first check that the configuration shown is indeed a metric space. As $i < j + k$ and $i + j + k$ is even, we have

$$i \leq (j - 1) + (k - 1),$$

and the triangle inequality is easily verified throughout the diagram.

So the diagram (II) represents a metric space in which all distances are even. After rescaling by a factor of $(1/2)$, we claim that the result embeds in $B\Gamma$. Let us call the rescaled diagram $(\widetilde{II})$.

Now antipodal Henson constraints do not come into play in the bipartite antipodal case. We check now that diagram $(\widetilde{II})$ does not correspond to a $\tilde{\delta}$-Henson constraint in $\tilde{\mathcal{S}}$.

The lengths involved in $(\widetilde{II})$ are 1, $(j \pm 1)/2$, $(k-1)/2$, and $i/2$. As $j < \delta$ we have $(j-1)/2 < \tilde{\delta}$. So if $(\widetilde{II})$ is a $(1, \tilde{\delta})$-space then $j = 3$ and $\tilde{\delta} = (j+1)/2 = 2$. In this case $B\Gamma$ is a random graph and hence $\tilde{\mathcal{S}} = \emptyset$.

Now let us check that diagram $(\widetilde{II})$ involves no forbidden triangles. We claim that the only constraints to be met are bounds on perimeter. If $\tilde{K}_2 = \delta$ this is true by definition, while if $\tilde{K}_2 < \tilde{\delta}$ then $\tilde{K}_2 = \tilde{\delta} - 1$ and $\tilde{C} = 2\tilde{\delta} + 1$, so the constraint on perimeter suffices.

The perimeters of the four triangles occurring in diagram (II) are $p-2$, $2k$, and $2(j+1)$. Here $2k \leq p$. As (i, j, k) is not a geodesic, and the perimeter is even, we also have $2(j+1) \leq p$. Thus after rescaling all perimeters are bounded by $p/2 < \tilde{C}$.

This disposes of the case in which $j < \delta$. There remains the possibility

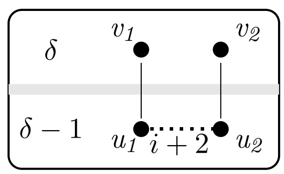
$$j = k = \delta.$$

Thus the triangle in question has type

$$(i, \delta, \delta)$$

with δ odd and i even, and $i < \delta - 1$.

Then by the first case treated, a triangle of type $(i+2, \delta-1, \delta-1)$ is realized in Γ. So in $\Gamma_{\delta-1}$ we can find a pair of vertices u_1, u_2 at distance $i+2$.



Take v_1, v_2 in Γ_δ adjacent to u_1, u_2 respectively. Then $d(v_1, v_2) \geq i$ and

$$d(v_1, v_2) \equiv i \equiv 0 \pmod 2.$$

Now in Γ_δ all distances are even, and Γ_δ is connected with respect to the relation $d(x, y) = 2$ by Lemma 15.4. Therefore we have a pair of vertices v_1', v_2' in Γ_δ at distance i, as required. □

Corollary 16.4.1. Under the hypotheses of the previous lemma, for some even

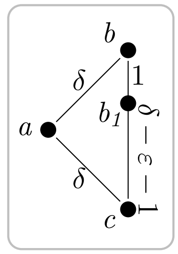
$$C_0 \geq 2\delta + 2$$

the triangles which embed in Γ are those of even perimeter less than C_0.

Proof. All triangles in Γ have even perimeter.

If all triangles in Γ have perimeter less than $3\delta - 1$, then by Lemma 16.4 we may take $C_0 = 2\tilde{C}$.

Now suppose there is a triangle (a, b, c) of perimeter $3\delta - \epsilon$ in Γ, where $\epsilon = 0$ or 1. Then this triangle has type $(\delta - \epsilon, \delta, \delta)$ and $\delta \equiv \epsilon \pmod 2$. We will suppose that $d(a, b) = d(a, c) = \delta$, $d(b, c) = \delta - \epsilon$.



Take b_1 with $d(b_1, b) = 1$ and $d(b_1, c) = \delta - \epsilon - 1$. Then the triangle (a, b_1, c) has type $(\delta - 1, \delta, \delta - \epsilon - 1)$ and perimeter $3\delta - 2 - \epsilon$, so by Lemma 16.4 we have $3\delta - 2 - \epsilon < 2\tilde{C}$. Therefore every triangle of even perimeter embeds into Γ. We set $C_0 = 3\delta + 2 - \epsilon$ in this case. □

16C. Embedding lemma: δ even or infinite

Having determined the triangles embedding in our bipartite graph Γ, we now pass to general configurations.

Definition 16.5. Let Γ be a metrically homogeneous graph of diameter δ, and A a finite metric space. We say that A is Γ-*constrained* if every triangle in A and every $(1, \delta)$-space contained in A embeds isometrically into Γ.

Alternatively, if Γ^* is the bipartite graph of generic type in the catalog, with the same parameters as Γ, our condition means that A embeds into Γ^*. This is a useful point, as it shows that the class of Γ-constrained finite metric spaces is an amalgamation class.

When Γ is bipartite and the diameter δ is odd, our definition simplifies, as there are no $(1, \delta)$-spaces with more than two points contained in Γ. In addition, there will be no vertices whose distances lie at the extremes (1 or δ) in either half of the bipartition in this case.

So we will treat the cases of even and odd diameter slightly differently.

First we discuss the case of even diameter. We will include the case of infinite diameter under this heading.

Lemma 16.6. *Let Γ be a countable bipartite metrically homogeneous graph of even diameter δ. Then $\tilde{S}$ consists of $\tilde{\delta}$-anticliques (a set of points mutually at distance $\tilde{\delta} = \delta/2$). For δ infinite, this means that $\tilde{S}$ is empty.*

PROOF. Let $\tilde{A}$ be a minimal Henson constraint for $B\Gamma$, not containing a forbidden triangle, and suppose that $\tilde{A}$ contains a nontrivial maximal clique $\tilde{A}_1$. But $\tilde{A}$ must contain a pair of vertices at distance $\tilde{\delta}$ as well, as $B\Gamma$ contains an infinite clique. So we have

$$C_0 > 2\delta + 2.$$

If $\delta = 4$, then as $B\Gamma$ contains a random graph, there are no Henson constraints on $B\Gamma$. So we may suppose

$$\delta \geq 6.$$

Let N be the sum of the sizes of the nontrivial maximal cliques in $\tilde{A}$. We proceed by induction on N.

We work in Γ with A and A_1, having the same vertices as $\tilde{A}$ and $\tilde{A}_1$, but with distances doubled to 2 and δ.

Fix $a \in A_1$ and set $A' = A_1 \setminus \{a\}$, $B = A \setminus A_1$. Adjoin vertices $C = \{c_1, c_2\}$ with $d(c_1, c_2) = 2$ and with distances $d(c_i, x)$ for $x \in A$ given by the following. the following conditions.

	$x = a$	$x \in A'$	$x \in B$
$d(c_1, x)$	1	1	$\delta - 1$
$d(c_2, x)$	1	3	$\delta - 1$

View $A \cup \{c_1, c_2\}$ as an amalgamation problem in which the distances between a and A' remain to be determined, with c_1, c_2 forcing the value 2.

It remains to be checked that the factors aBC and $A'BC$ both embed into Γ. Write both these factors as $A''BC$ with A'' equal to $\{a\}$ or to A'.

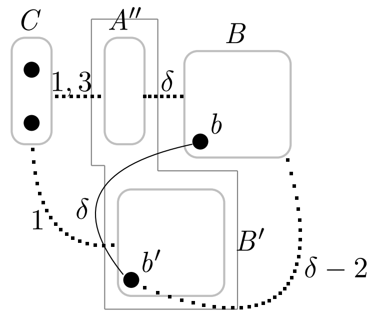
For $b \in B$ adjoin a new vertex b' and set $B' = \{b' \mid b \in B\}$. We put a metric on $A''BCB'$ as follows.

On $A'' \cup B'$, all distances equal 2. Between B' and C all distances equal 1. Between B and B' we use the rule

$$d(b_1, b_2') = \begin{cases} \delta & \text{if } b_1 = b_2; \\ \delta - 2 & \text{otherwise.} \end{cases}$$

We view the resulting configuration as an amalgamation problem in which distances between C and B are to be determined. The vertices of B' force all these distances to be $\delta - 1$.

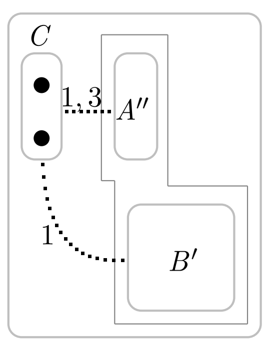
The factors of this configuration are $CA''B'$ and $A''B'B$.



In $A''BB'$ all distances are even so this factor can be considered (rescaled) in $B\Gamma$. Therefore it suffices to check the triangles and Henson constraints resulting. The triangles are of even perimeter at most $2\delta + 2$ when viewed in Γ, hence are realized in $B\Gamma$. The Henson constraints for $B\Gamma$, when viewed in Γ, will not involve the distance $\delta - 2$. So these either lie in $A''B$ or in $A''bb'$ for some $b \in B$. In either case, the parameter N is decreased.

Thus the factor $A''B'B$ is realized, rescaled, in $B\Gamma$, and the factor as it stands is realized in Γ.

There remains the factor $CA''B'$.

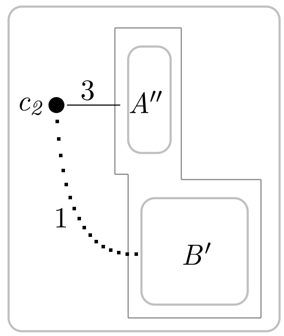


Now it is possible that the distance 3 does not actually occur here, and that $A''B'$ represents an independent set of vertices adjacent to the two vertices c_1, c_2 of C, in which case the configuration is afforded by Lemma 15.2.

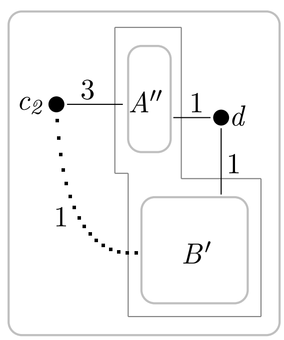
On the other hand, if the distance 3 does occur, then we may view this configuration as an amalgamation problem with distance $d(c_1, c_2)$ to be determined, and with the value 2 forced.

So we may consider the two factors $c_1A''B'$ and $c_2A'''B'$ separately. Again, c_1 is adjacent to all vertices of $A''B'$ so the configuration $c_1A''B'$ is realized.

There remains $c_2A''B'$.



We can simplify this further by adjoining a vertex d at distance 1 from everything in $A''B'$, and at distance 2 from c_2, as shown below. Then the distances between A'' and B' are forced to be 2, and we come down to the factors c_2dA'' and c_2dB'. Here c_2dB' again represents an independent set of common neighbors of two vertices at distance 2, so we need only consider c_2dA''.



For c_2dA'', we view c_2 as the basepoint, and we require $d \in \Gamma_2$ adjacent to A'' in Γ_3, which we have. $\square$

Proposition 16.7. Let Γ be a countable bipartite metrically homogeneous graph of diameter δ, with δ either even or infinite. Suppose that $(B\Gamma)_1$ is a random graph and $B\Gamma$ is of known type. Let (A, B)

be a finite bipartite Γ-constrained metric space. Then (A, B) embeds into Γ.

PROOF. We proceed by induction on $|B|$, and for $|B|$ fixed by induction on

$$\min_{b \in B} |\{a \in A \mid d(a, b) \neq 1\}|.$$

First suppose

$$B \text{ is empty.}$$

Then it suffices to consider $B\Gamma$. That is, we write $\tilde{A}$ for the metric space A rescaled by a factor of $1/2$, and we claim that $\tilde{A}$ embeds into $B\Gamma$.

We must check that any triangles or $\tilde{\delta}$-Henson constraints that embed into $\tilde{A}$ also embed into $B\Gamma$.

First we consider a triangle T embedding into A and the corresponding triangle $\tilde{T}$ in $\tilde{A}$. As A is Γ-constrained, T embeds into Γ, and thus $\tilde{T}$ embeds into $B\Gamma$.

Now we consider $\tilde{\delta}$-Henson constraints in $B\Gamma$. By Lemma 16.6 we are dealing with an anticlique in $B\Gamma$, and this is also an anticlique in Γ, that is, a set of vertices at mutual distance δ (this case falls away if δ is infinite). So if A is Γ-constrained it embeds into Γ.

Now suppose

$$|B| \geq 1.$$

Case I. There is a pair (b, b') with $d(b, b') \neq \delta$.

In this case we will reduce the size of $|B|$.

Set $k = d(b, b')$ and add witnesses a_1, a_2 to A satisfying

$$d(a_1, b) = 1; \qquad d(a_1, b') = k - 1;$$
$$d(a_2, b) = 1; \qquad d(a_2, b') = k + 1;$$
$$d(a_1, a_2) = 2.$$

Note that $k \leq \delta - 2$, so the triangles involved have perimeters bounded by 2δ.

We must extend the configuration $A \cup B \cup \{a_1, a_2\}$ to a Γ-constrained configuration. This may be done by amalgamating $\{a_1, a_2, b, b'\}$ with $A \cup B$ over $\{b, b'\}$.

View the resulting configuration $A \cup B \cup \{a_1, a_2\}$ as a 2-point amalgamation problem with the distance $d(b, b')$ to be determined. The factors embed in Γ by induction on $|B|$, and the result of the amalgamation is uniquely determined as the parameters a_1, a_2 serve to determine $d(b, b')$.

If δ is infinite then there is only Case (I). So suppose that δ is finite and even for the remainder of the argument.

Case II. B is an anticlique; that is, $d(b, b') = \delta$ for $b, b' \in B$.

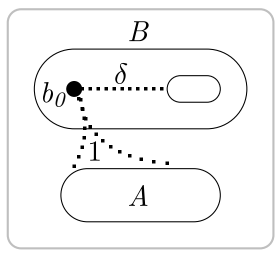
Fix $b_0 \in B$ minimizing the quantity

$$\nu(b_0) = |\{a \in A \mid d(a, b_0) \neq 1\}|.$$

Suppose first that

$$\nu(b_0) = 0.$$

That is, b_0 is adjacent to all vertices of A.



Observe that all distances $d(a, b)$ with $a \in A$, $b \in B$, and $b \neq b_0$ must be equal to $\delta - 1$. So we may view $A \cup B$ as an amalgamation diagram with these distances to be determined. It suffices to show that the two factors

$$A \cup \{b_0\} \text{ and } B$$

occur in Γ.

The factor $(A, \{b_0\})$ embeds in Γ since Γ_1 is infinite. And the anticlique B embeds in Γ by the hypothesis of Γ-constraint.

So now we suppose

$$\nu(b_0) > 0,$$

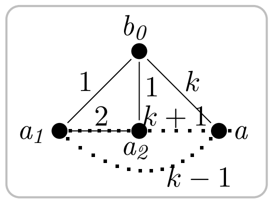
and we pick some $a \in A$ with $d(a, b_0) \neq 1$.

Set

$$k = d(a, b_0)$$

and adjoin witnesses a_1, a_2 to A satisfying

$$d(a_1, b_0) = 1; \qquad d(a_1, a) = k - 1;$$
$$d(a_2, b_0) = 1; \qquad d(a_2, a) = k + 1;$$
$$d(a_1, a_2) = 2.$$



Since this configuration is actually a geodesic $(a_2 b_0 a_1 a)$ of length $k+1$, it occurs in Γ. We may extend to a Γ-constrained configuration $A \cup \{a_1, a_2\} \cup B$ by amalgamating with $A \cup B$.

View the resulting configuration $A \cup \{a_1, a_2\} \cup B$ as a 2-point amalgamation problem with the distance $d(a, b_0)$ to be determined. The parameters a_1, a_2 determined this distance uniquely. So it suffices to check that the factors omitting b_0 or a are in Γ.

The factor omitting b_0 has smaller $|B|$ and is therefore realized, by induction.

The factor omitting a has fewer non-neighbors of b_0, and the same set B, so this is also realized, by induction.

This completes the argument. □

16D. Henson constraints: δ finite and odd

Now we turn to the case of odd finite diameter. Our goal is the following.

Proposition 16.8. Let Γ be a countable bipartite metrically homogeneous graph of odd finite diameter δ, with $(B\Gamma)_1$ a random graph and $B\Gamma$ of known type. Let (A, B) be a finite Γ-constrained bipartite metric space. Then (A, B) embeds into Γ.

With Γ bipartite, any Γ-constrained metric space is automatically bipartite, in the sense that there are no triangles of odd perimeter, but in the statement of the proposition we also specify a particular bipartition, to be respected by the embedding.

The special case of this proposition for which the graph Γ is also antipodal was given in Cherlin [2011] as follows.

Theorem (Cherlin [2011, Theorem 12]). Let Γ be a metrically homogeneous graph of odd diameter $\delta = 2\delta' + 1$ which is both antipodal and bipartite. Then $B\Gamma$ is connected, and Γ is the bipartite double cover of $B\Gamma$. The graph $B\Gamma$ is a metrically homogeneous graph with the following properties:

1. $B\Gamma$ has diameter δ';
2. No triangle in $B\Gamma$ has perimeter greater than $2\delta' + 1$;
3. $B\Gamma$ is not antipodal.

Conversely, for any metrically homogeneous graph G of diameter with the three stated properties, there is a unique antipodal bipartite graph of diameter $2\delta' + 1$ such that $B\Gamma \cong G$.

This allows us to assume for the remainder of the analysis that

$$C_0 > 2\delta + 2.$$

Our first goal is to show that in this case there are no Henson constraints on $B\Gamma$, other than constraints on triangles.

LEMMA 16.9. *Let Γ be a countable bipartite metrically homogeneous graph of odd finite diameter δ with $(B\Gamma)_1$ a random graph and $B\Gamma$ of known type.*

Then any $(1, \tilde{\delta})$-space which does not contain a forbidden triangle embeds into $B\Gamma$.

PROOF. In terms of Γ, the statement refers to $(2, \delta - 1)$-spaces. We will work with the unscaled metric on Γ.

If $\delta \leq 5$, then $\delta = 5$ and $B\Gamma$ is the random graph G_∞, which has no constraints. So we may suppose

$$\delta \geq 7.$$

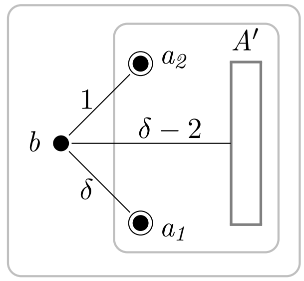
We deal first with the case of an anticlique A (all distances equal to $\delta - 1$). We may suppose A has order at least 4 and that

$$C_0 \geq 3\delta - 1.$$

We proceed by induction on the order of A.

We take $a_1, a_2 \in A$ and let $A' = A \setminus \{a_1, a_2\}$. Adjoin b with $d(b, a_1) = 1$, $d(b, a_2) = \delta$, and $d(b, a) = \delta - 2$ for $a \in A'$.

View $A \cup \{b\}$ as an amalgamation problem determining $d(a_1, a_2)$. It suffices to consider the factors $A'a_1b$, $A'a_2b$.



The factor ba_1A'

We adjoin a vertex c at distance 1 from b, 2 from a_1, and $\delta - 3$ from the vertices of A', and view $ba_1A'c$ as an amalgamation determining the distances from b to A'.

The factors are the geodesic a_1bc and the configuration $a_1A'c$ in which all distances are even. As $B\Gamma$ is of known type it suffices to check that the triangles and Henson constraints for $B\Gamma$ embedding in $a_1A'c$ embed into Γ (when written in terms of the metric on Γ).

The triangles pose no problems as $C_0 \geq 3\delta - 2$. In the Henson constraints the distance $\delta - 3$ will not occur since $\delta > 5$.

So we come down to the configuration a_1A', and we conclude by induction on $|A|$.

The factor ba_2A'

We adjoin vertices c_a for $a \in A'$ at distance 1 from b, $\delta - 1$ from a_2 and a, and at distance $\delta - 3$ from the rest of A'. Let $C = \{c_a \mid a \in A'\}$. We take $d(c_1, c_2) = 2$ on C.

We view $ba_2A'C$ as an amalgamation problem with the distances from b to A' to be determined, and forced to be $\delta - 2$. So it suffices to consider the factors ba_2C and $a_2A'C$.

If we view ba_2C as an amalgamation problem in which the distances between a_2 and C are to be determined, it suffices to check that the factors ba_2 and bC embed into Γ. But this is clear.

So we consider $a_2A'C$. Here all distances are even and it suffices to check that the triangle and Henson constraints for $B\Gamma$ are respected. The triangle constraints present no issues.

For the Henson constraints, since we may set aside the distance $\delta - 3$, we are either dealing with triangles a_2ac_a, which we have, or the configuration a_2C. For the latter, it suffices to take v with $d(a_2, v) = \delta$, and then a set of neighbors of v.

This disposes of the case in which the Henson constraint is an anticlique. Now we suppose

A contains a nontrivial clique.

We proceed by induction on the sum of the sizes of the nontrivial maximal cliques in A.

Let A_1 be a nontrivial maximal clique in A, and take $a \in A_1$, $A' = A_1 \setminus \{a\}$, $B = A \setminus A_1$. Adjoin vertices $C = \{c_1, c_2\}$ with $d(c_1, c_2) = 2$, $d(c_i, b) = \delta - 2$ for $b \in B$, $d(c_1, x) = 1$ for $x \in A_1$, and $d(c_2, a) = 3$, $d(c_2, x) = 3$ for $x \in A'$.

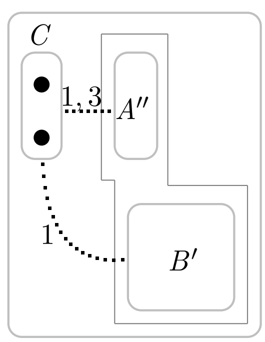
View the configuration AC as an amalgamation determining the distances between a and A'. It suffices to show the factors aBC, $A'BC$ are in Γ. Set $A'' = a$ or A' correspondingly and consider $A''BC$.

Adjoin vertices $B' = \{b' \mid b \in B\}$ with $d(x, y) = 2$ on B', $d(x, c) = 1$ for $x \in B'$, $c \in C$, $d(x, y) = 2$ for $x \in B'$, $y \in A''$, $d(b', b) = \delta - 1$, and $d(b'_1, b_2) = \delta - 3$ for $b_1, b_2 \in B$ distinct.

View $A''BCB'$ as an amalgamation determining the distances between C and B, with factors $A''CB'$ and $A'''BB'$.

In $A''BB'$ all distances are even so it suffices to consider the triangle constraints and Henson constraints for $B\Gamma$. There are no triangles of large perimeter unless B contains vertices at distance $\delta - 1$, in which case A contained a triangle of perimeter $3(\delta - 1)$ and $C_0 > 3(\delta - 1)$. So it suffices to consider Henson constraints, and for this we set aside the distance $\delta - 3$. Thus we are left either with $A''B$, to which induction applies, or with $A''bb'$ for some $b \in B$, which consists of a vertex b at distance $\delta - 1$ from $A''b'$, a set of points mutually at distance 2. For this we take v with $d(b, v) = \delta$ and then a set of neighbors of v.

So we come down to the factor $A''CB'$.



If the distance 3 does not occur then Lemma 15.2 applies.

If the distance 3 does occur then $A'' = \{a\}$ and we view this configuration as an amalgamation determining $d(a, b')$ for $b' \in B'$. One factor aC is a geodesic and the other factor CB' is covered by Lemma 15.2. $\square$

16E. A reduction lemma for δ finite and odd

We now reduce the general embedding lemma to a special case.

LEMMA 16.10. *Let Γ be a countable bipartite metrically homogeneous graph of odd finite diameter δ with $(B\Gamma)_1$ a random graph and $B\Gamma$ of known type. Suppose that every Γ-constrained bipartite configuration (A, B) with $|B| = 1$ and $d(a, b) = \delta$ for $a \in A$, $b \in B$ embeds into Γ. Then every Γ-constrained configuration embeds into Γ.*

PROOF. We proceed by induction on $|B|$. If B is empty, we work in $B\Gamma$. There are no Henson constraints by Lemma 16.9 and any forbidden triangles would correspond to forbidden triangles in Γ.

So we suppose

$$|B| \geq 1.$$

Let us first deal with the case

$$|B| = 1.$$

Let $B = \{b\}$.

Define

$$A_i = \{a \in A \mid d(a, b) = i\}.$$

Note that A_i is empty for i even. We proceed by induction on

$$|A \setminus A_1|.$$

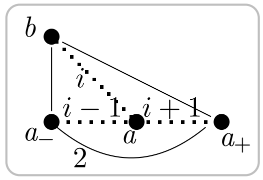
Suppose first that for some i with $1 < i < \delta$ there is $a \in A$ with

$$d(a, b) = i.$$

Then adjoin a_-, a_+ with

$$\begin{aligned} d(a_-, b) = d(a_+, b) &= 1; \\ d(a_-, a) &= i - 1; \\ d(a_+, a) &= i + 1. \end{aligned}$$

Extend to a Γ-constrained configuration $A \cup \{a_\pm\} \cup B$. View the resulting configuration as a 2-point amalgamation with the distance



$d(a, b)$ to be determined. The points a_1, a_2 force $d(a, b) = i$. So it suffices now to find the factors $A \cup \{a_-, a_+\}$ and $(A \setminus \{a\}) \cup \{a_-, a_+, b\}$ in Γ.

For the first factor, we return to the case $|B| = 0$ and conclude by inspection of $B\Gamma$.

For the second factor, since we have reduced the parameter $|A \setminus A_1|$ by removing a, we may conclude by induction.

Now we suppose there is no $a \in A$ at distance i from b for $1 < i < \delta$, or in other words

$$A = A_1 \cup A_\delta.$$

If both A_1 and A_δ are nonempty then we view $A \cup B$ as an amalgamation problem with the distances between A_1 and A_δ to be determined. These distances are necessarily equal to $\delta - 1$. So we may suppose now

$$A = A_1 \text{ or } A_\delta.$$

If $A = A_1$ we conclude since Γ_1 contains an infinite independent set.

On the other hand, if $A = A_\delta$ then we have a configuration satisfying the stated condition that $|B| = 1$ and $d(a, b) = \delta$ for $a \in A$, and this is in Γ by hypothesis.

So now suppose

$$|B| > 1.$$

Take $b_1, b_2 \in B$ distinct. Let $k = d(b_1, b_2)$. As k is even we have $1 < k < \delta$.

Adjoin vertices a_-, a_+ with

$$\begin{aligned} d(a_-, b_1) &= 1; & d(a_-, b_2) &= k - 1; \\ d(a_+, b_1) &= 1; & d(a_+, b_2) &= k + 1; \end{aligned}$$
$$d(a_1, a_2) = 2.$$

Complete $(A \cup \{a_1, a_2\}, B)$ to a Γ-constrained configuration.

We can view the resulting configuration as an amalgamation problem in which the distance $d(b_1, b_2)$ is to be determined, with a unique value forced by a_1 and a_2. In the factors, the cardinality of the B side is reduced, so by induction these are available in Γ. $\square$

16F. The embedding lemma for δ odd, finite

LEMMA 16.11. *Let Γ be a countable bipartite metrically homogeneous graph of odd finite diameter δ with $(B\Gamma)_1$ a random graph and $B\Gamma$ of known type. Let $N \geq 0$ be fixed. Suppose that every Γ-constrained bipartite configuration (A, B) with*

$$|B| = 1; \qquad |A| \leq N;$$
$$A = A_\delta.$$

embeds into Γ, where $A_\delta = \{a \in A \mid d(a, b) = \delta\}$, $B = \{b\}$.

Then every Γ-constrained bipartite configuration (A, B) with

$$|B| = 1; \qquad |A_\delta| \leq N$$

embeds into Γ.

PROOF. We set $B = \{b\}$ and

$$A_i = \{a \in A \mid d(a, b) = i\}.$$

We proceed by induction on $|A \setminus (A_1 \cup A_\delta)|$.

If there is $a \in A_k$ for some k with $1 < k < \delta$, then

Adjoin vertices a_-, a_+ satisfying the following conditions.

$$d(a_-, b) = d(a_+, b) = 1;$$
$$d(a_-, a) = k - 1;$$
$$d(a_+, a) = k + 1.$$

We extend to a Γ-constrained configuration $A \cup \{b, b_1, b_2\}$ by amalgamating with $A \cup B$. Recall that according to our amalgamation strategy in such cases, this does not introduce any new pairs at distance δ, beyond those occurring in the factors. As $k < \delta - 1$ all such pairs will in fact occur in $A \cup B$. Note also that we necessarily have $d(b, b_\pm) = \delta - 1$ here.

We extend this to a Γ-constrained configuration $A \cup \{a_-, a_+\} \cup B$ by amalgamating with $A \cup B$.

We view the resulting configuration as a 2-point amalgamation problem with the distance $d(a, b)$ to be determined, and controlled by the parameters a_-, a_+. So it suffices to show that the factors $A \cup \{a_-, a_+\}$ and $(A \setminus \{a\}) \cup \{a_-, a_+, b\}$ embed into Γ.

The factor $A \cup \{a_-, a_+\}$ is realized as usual by embedding into the known graph $B\Gamma$.

In the second factor

$$A^* \cup \{b\}$$

with $A^* = (A \setminus \{a\}) \cup \{a_-, a_+\}$, we have at most N points at distance δ from b, and removal of a decreases the parameter $|A \setminus (A_1 \cup A_\delta)|$. So in this case we conclude by induction.

This leaves the case in which

$$A = A_1 \cup A_\delta.$$

Then in an amalgam of (A_1, B) with (A_δ, B) we must have $d(a, a') = \delta - 1$ for $a \in A_1$ and $a' \in A_\delta$. Thus it suffices to check that the factors (A_1, B) and (A_δ, B) embed into Γ. The factor (A_1, B) is available since Γ_1 is an infinite independent set, and the factor (A_δ, B) is available by hypothesis.

This completes the analysis. □

LEMMA 16.12. *Let Γ be a countable bipartite metrically homogeneous graph of odd finite diameter δ with $(B\Gamma)_1$ a random graph and $B\Gamma$ of known type. Let $N \geq 2$ be fixed. Suppose that every Γ-constrained bipartite configuration (A', B') with*

$$|B'| = 1; \qquad A' = A_\delta;$$
$$|A'| < N$$

embeds into Γ. Let (A, B) be a Γ-constrained bipartite configuration with

$$|B| = 1; \qquad A = A_\delta;$$
$$|A| = N.$$

Suppose there is a pair $a_1, a_2 \in A$ with $d(a_1, a_2) < \delta - 1$. Then (A, B) embeds into Γ.

PROOF. We let $B = \{b\}$ and

$$k = d(a_1, a_2) < \delta - 1.$$

As k is even, we have $k > 1$.

We adjoin elements b_-, b_+ satisfying the following.

$$
\begin{aligned}
d(b_-, a_1) = d(b_+, a_1) &= 1;\\
d(b_-, a_2) &= k-1;\\
d(b_+, a_2) &= k+1.
\end{aligned}
$$

We extend to a Γ-constrained configurations $A \cup \{b, b_-, b_+\}$. Recall that our amalgamation strategy in such cases does not introduce any pairs at distance δ beyond those already in one of the factors. As $k < \delta - 1$ this means that the only pairs at distance δ in this configuration are those in (A, B).

We view the resulting configuration

$$A \cup \{b, b_-, b_+\}.$$

as a 2-point amalgamation problem with the distance $d(a_1, a_2)$ to be determined, and controlled by the parameters $b_\pm$. It suffices to show that the factors of this amalgamation diagram embed into Γ.

These factors have the form

$$A^* \cup \{b, b_-, b_+\}.$$

with $A^* = (A \setminus \{a_i\}) \cup \{b, b_-, b_+\}$ and $i = 1$ or 2. Now we need to return to a situation in which the three points b, b_-, b_+ are reduced to one.

Consideration of a_1, b, b_-, b_+ shows that

$$d(b, b_-) = d(b, b_+) = \delta - 1.$$

We adjoin a parameter a_3 with

$$
\begin{aligned}
d(a_3, b) &= 1;\\
d(a_3, b_-) = d(a_3, b_+) &= \delta.
\end{aligned}
$$

Then we extend to a Γ-constrained configuration $A^* \cup \{a_3\} \cup \{b, b_-, b_+\}$.

This configuration may be viewed as an amalgamation diagram in which the distances $d(b, b_-)$ and $d(b, b_+)$ are to be determined, and are controlled by the parameter a_3. So it suffices to embed the corresponding subfactors

$$A^* \cup \{a_3, b\} \text{ and } A^* \cup \{a_3, b_-, b_+\}.$$

into Γ.

Now in $A^* \cup \{a_3, b\}$, we again have $|B| = 1$, and now the number of points of $A^* \cup \{a_3\}$ at distance δ from b is less than N. So by our hypothesis and Lemma 16.11, this configuration embeds into Γ.

Now we consider the factor

$$A^* \cup \{a_3, b_-, b_+\}.$$

Here we introduce additional parameters a_4, a_5 satisfying

$$d(a_4, b_-) = d(a_4, b_+) = 1;$$
$$d(a_5, b_-) = 1;$$
$$d(a_5, b_+) = 3.$$

We extend to a Γ-constrained configuration $A^* \cup \{a_3, a_4, a_5\} \cup B$. We may view this as a 2-point amalgamation with the distance $d(b_-, b_+)$ to be determined, and controlled by the parameters a_4, a_5. So we come down to the individual factors

$$(A^{**}, \{b'\})$$

with $b' = b_-$ or b_+ and $A^{**} = A^* \cup \{a_3, a_4, a_5\}$. Recall that there are no points $a \in A^*$ with $d(b', a) = \delta$.

Furthermore since $B\Gamma$ is not complete, we have

$$\delta > 3,$$

and thus the only point $a \in A^{**}$ at distance δ from b' is a_3.

Since $N \geq 2$, Lemma 16.11 applies here. $\square$

LEMMA 16.13. *Let Γ be a countable bipartite metrically homogeneous graph of odd finite diameter δ with $(B\Gamma)_1$ a random graph and $B\Gamma$ of known type. Then every Γ-constrained bipartite configuration (A, B) with $|B| = 1$, $d(a, b) = \delta$ for $a \in A$ and $b \in B$ embeds into Γ.*

PROOF. We proceed by induction on $N = |A|$. If $N \leq 2$, then the hypothesis of Γ-constraint gives the required embedding. So we suppose

$$N \geq 3.$$

If there is a pair of vertices $a_1, a_2 \in A$ with $d(a_1, a_2) < \delta - 1$, then Lemma 16.12 applies. So we suppose

$$d(a, a') = \delta - 1 \text{ for } a, a' \in A.$$

Fix two vertices a_1, a_2 in A and set $A_0 = A \setminus \{a_1, a_2\}$. Adjoin a vertex b_1 satisfying the following conditions.

$$d(b_1, a_1) = 1, \quad d(b_1, a_2) = \delta, \quad d(b_1, a) = \delta - 2 \text{ for } a \in A_0,$$
$$d(b_1, b) = \delta - 1.$$

View the resulting configuration as an amalgamation problem with the distance

$$d(a_1, a_2)$$

to be determined. The parameter b_1 ensures that this distances is equal to $\delta - 1$. So it suffices to show that the two factors $(A_0 \cup \{a_i, b, b_1\})$ embed into Γ for $i = 1, 2$.

We adjoin additional vertices a_-, a_+ satisfying the following conditions.

$$d(a_-, b_1) = d(a_+, b_1) = 1,$$
$$d(a_-, a) = \delta - 3 \text{ for } a \in A_0,$$
$$d(a_+, a) = \delta - 1 \text{ for } a \in A_0.$$

Extend to a Γ-constrained configuration $A_0 \cup \{a_i, a_-, a_+, b, b_1\}$. Here as usual we avoid introducing further pairs at distance δ.

View the resulting configuration as an amalgamation diagram in which the distances

$$d(b_1, a) \text{ for } a \in A_0$$

are to be determined, and are controlled by the parameters a_-, a_+.

So it suffices to show that the corresponding subfactors

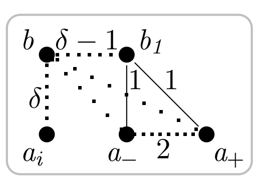
$$\{a_i, a_-, a_+, b, b_1\} \text{ and } A_0 \cup \{a_i, a_-, a_+, b\}$$

embed into Γ.

In the second of these factors we have $|B| = 1$ and the vertex b is at distance δ only from the points of $A_0 \cup \{a_i\}$. So Lemma 16.11 and the induction hypothesis ensure that this configuration embeds into Γ.

So we come down to the configuration

$$\{a_i, a_-, a_+, b, b_1\}.$$



Here to simplify further we introduce a parameter a_3 with

$$d(a_3, b) = 1, \qquad d(a_3, b_1) = \delta,$$

and extend to a Γ-constrained configuration $\{a_i, a_-, a_+, a_3\} \cup \{b, b_1\}$.

Setting $A' = \{a_i, a_-, a_+, a_3\}$, we then reduce to the configurations $A' \cup \{b\}$ and $A' \cup \{b_1\}$ in which $|B| = 1$ and the number of points in A' at distance δ from b or from b_1 is at most 2. As $N > 2$, Lemma 16.11 applies. □

Now we may assemble these ingredients.

Proof of Proposition 16.8. Let (A, B) be an arbitrary Γ-constrained bipartite finite metric space. We need to show that (A, B) embeds into Γ isometrically. But Lemma 16.10 reduces the problem to the case treated in Lemma 16.12. □

16G. Conclusion of the proof

At this point the proof of Theorem 1.30 is complete. Corollary 16.4.1 gives us an appropriate value for C_0. If δ is odd and finite, then Proposition 16.8 shows that

$$\Gamma \cong \Gamma^{\delta}_{\infty,0;C_0,2\delta+1}.$$

If δ is even or infinite, then Proposition 16.7 shows that

$$\Gamma \cong \Gamma^{\delta}_{\infty,0;C_0,2\delta+1;\mathcal{S}},$$

where if $\mathcal{S}$ is nonempty, it consists of a single δ-clique of order at least 4 (by irredundance).

In the second case, we note that the role of $\mathcal{S}$ depends on the value of C_0.

1. If $C_0 \leq 3\delta$, then $\mathcal{S}$ will be empty (by irredundancy).
2. If $C_0 = 3\delta + 2$, then $\mathcal{S}$ is either empty or consists of a single δ-anticlique $\{I_n^{\delta}\}$ with $n \geq 4$,

CHAPTER 17

INFINITE DIAMETER

Our next target is the following reduction of the classification problem in infinite diameter to the finite diameter problem. Of course, if our catalog at some point needs to be enlarged, then this result will need to be revisited.

Theorem (1.26). Suppose that every countable metrically homogeneous graph of finite diameter is of known type. Then every countable metrically homogeneous graph is of known type.

Let Γ be a countable metrically homogeneous graph of infinite diameter. Since the non-generic case has been fully classified, we may suppose

$$(\star) \qquad \Gamma \text{ is of generic type.}$$

In particular, the parameter sequence

$$\delta, K_1, K_2, \mathcal{S}$$

associated with Γ is well-defined. As the diameter is infinite, the parameters C_0, C_1 have been dropped. In this context, Henson constraints are cliques. One hopes that the parameter K_2 is also superfluous, but this must still be shown.

In view of Theorem 1.30 and the reduction of that statement to Proposition 16.2, we have the following.

Remark 17.1. If every countable metrically homogeneous graph of generic type with $K_1 = 1$ is of known type, then every countable metrically homogeneous bipartite graph is of known type.

So in what follows we may suppose

$$(\star\star) \qquad K_1 < \infty.$$

17A. Toward Theorem 1.26

We aim next at the following lemma, which reduces Theorem 1.26 to something much more concrete.

LEMMA 17.2. *Suppose that every countable metrically homogeneous graph of finite diameter is of known type. Let* Γ *be a countable metrically homogeneous graph of infinite diameter, with* $K_1 < \infty$*. Let* $K = \max(K_1, 2)$*. Suppose that* Γ_K *contains a triangle of type* $(K_1, K_1, 1)$*. Then* Γ *is of known type, specifically of the form*

$$\Gamma^{\infty}_{K_1,\mathcal{S}}$$

with $\mathcal{S}$ *either empty or consisting of one clique.*

We prepare for this as follows.

LEMMA 17.3. *Suppose that every countable metrically homogeneous graph of finite diameter is of known type. Let* Γ *be a countable metrically homogeneous graph of infinite diameter, with* $K_1 < \infty$*. Let* $K = \max(K_1, 2)$*. Suppose that* Γ_K *contains a triangle of type* $(K_1, K_1, 1)$*. Then for* $i \geq K$*, we have the following.*

Γ_i *contains triangles of type* $(j, j, 1)$ *for* $K_1 \leq j \leq i$*.*

PROOF. We proceed by induction on i with

$$i \geq K.$$

We first check the claim for $i = K$. If $K_1 > 1$, this is our assumption. If $K_1 = 1$, then $K = 2$ and we need to check that Γ_2 contains triangles of types $(1, 1, 1)$ and $(2, 2, 1)$. This can be seen by considering Γ_1.

Now we pass from i to $i + 1$. Let us write $\tilde{\Gamma}$ for Γ_i, and then write

$$(\tilde{\delta}, \tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{C}', \tilde{\mathcal{S}})$$

for the associated parameters.

By our inductive hypothesis, we know in particular that $\tilde{\Gamma}$ contains an edge. By Theorem 1.32, $\tilde{\Gamma}$ is of generic type, and primitive. We are assuming that $\tilde{\Gamma}$ is of known type.

Our induction hypothesis amounts to

$$\tilde{K}_1 = K_1; \qquad\qquad \tilde{K}_2 \geq i.$$

Observe also that

$$\tilde{\delta} = 2i.$$

Suppose first that

$$\tilde{K}_2 > i.$$

Then $\tilde{\Gamma}_{i+1}$ contains triangles of types $(j, j, 1)$ for $\tilde{K}_1 \le j \le \tilde{K}_2$, and in particular for $K_1 \le j \le i+1$. Therefore the same applies to Γ_{i+1}, and the inductive step is complete.

So we must exclude the alternative

$$\tilde{K}_2 = i.$$

Now by the admissibility conditions on the parameters associated to $\tilde{\Gamma}$, we have the following possibilities.

Type (II): $\tilde{K}_1 + \tilde{K}_2 \ge \tilde{\delta}$; $\tilde{C} = 2(\tilde{K}_1 + \tilde{K}_2) + 1$;
Type (III): $\tilde{K}_2 \ge (2/3)\tilde{\delta} = (4/3)i$.

If $\tilde{\Gamma}$ is of Type (III), then as $(4/3)i > i$, we have $\tilde{K}_2 > i$, and we arrive at a contradiction.

So suppose $\tilde{\Gamma}$ is of Type (II). Then we get

$$\tilde{K}_1 + \tilde{K}_2 \ge \tilde{\delta} = 2i = 2\tilde{K}_2,$$

and therefore

$$\tilde{K}_1 = \tilde{K}_2 = i.$$

Hence

$$\tilde{C} = 4i + 1 = 2\tilde{\delta} + 1.$$

If $C' = C + 1$, then $\tilde{\Gamma}$ is antipodal; but as $\tilde{\Gamma}$ is primitive, this is a contradiction.

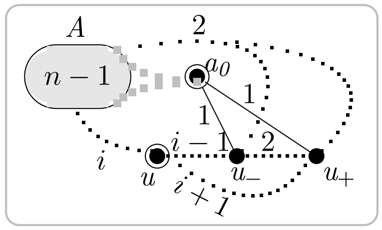
If $C' > C + 1$, then in Type (II) we have

$$3\tilde{K}_2 = 2\tilde{\delta} - 1; \qquad 3i = 4i - 1; \qquad i = 1.$$

But $i \ge K \ge 2$, a contradiction. □

LEMMA 17.4. *Suppose that every countable metrically homogeneous graph of finite diameter is of known type. Let Γ be a countable metrically homogeneous graph of infinite diameter and generic type containing a clique of order $n \ge 3$ (in particular, $K_1 = 1$). Then for $i \ge 2$, Γ_i contains a clique of order n.*

PROOF. We proceed by induction on n, with the base case $n = 3$. We use an explicit amalgamation, as shown.

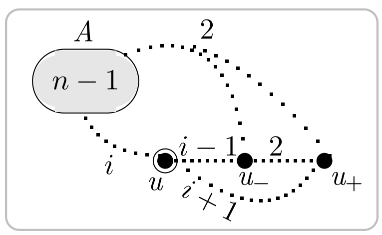


Here the auxiliary parameters u_-, u_+ serve to force $d(a_0, u) = i$ and the desired configuration is the clique $A \cup \{a_0\}$ extended by u. We also take $d(u_\pm, a) = 2$ for $a \in A$. If the factors of this configuration embed into Γ then in the amalgam, $A \cup \{a_0\}$ is contained in $\Gamma_i(u)$.

The factor omitting u can be viewed as $A \cup \{u_-, u_+\}$ inside $\Gamma_1(a_0)$. Since $A \cup \{u_-, u_+\}$ embeds into Γ_1, this is available.

So we come down to the factor

$$A \cup \{u, u_-, u_+\}.$$



We claim that this factor embeds into Γ_i and hence into Γ.

By Theorem 1.32, $\tilde{\Gamma} = \Gamma_i$ is a countable metrically homogeneous graph of generic type. We write $(\tilde{\delta}, \tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{C}', \tilde{\mathcal{S}})$ for the associated parameters.

The diameter is $\tilde{\delta} = 2i$, so all of the distances shown are less than the diameter. Therefore it suffices to show that this configuration contains no forbidden triangles or $\tilde{\delta}$-Henson constraints for Γ_i.

We first consider the triangles involved. The maximal perimeter involved is $2i + 3 < 2\tilde{\delta}$, so there is no forbidden triangle of large perimeter. The triangle types of odd perimeter represented here are

$$(i, i, 1), (i, i \pm 1, 2), (2, 2, 1),$$

and these occur in Γ_i by Lemma 17.3.

As there are no pairs at distance $\tilde{\delta} = 2i$ here, the only $\tilde{\delta}$-Henson constraints to be checked are cliques. But the cliques involved have order at most $n-1$; if $n > 3$ then these are available by induction, and otherwise it suffices to note that Γ_i contains edges. □

PROOF OF LEMMA 17.2. We consider a finite configuration A which contains no triangle of odd perimeter less than $2K_1 + 1$, and no forbidden clique. We must show that A embeds into Γ.

We take

$$i \geq \max(K_1, \max_{a,a' \in A} d(a,a')).$$

We claim that A embeds into $\tilde{\Gamma} = \Gamma_i$. As usual we write $(\tilde{\delta}, \tilde{K}_1, \tilde{K}_2, \tilde{C}, \tilde{C}', \tilde{\mathcal{S}})$ for the parameters associated with $\tilde{\Gamma}$. We must check that A contains no forbidden triangle or $\tilde{\delta}$-Henson constraint for Γ_i.

We deal first with triangles. By Lemma 17.3 we have

$$\tilde{K}_1 = K_1;$$
$$\tilde{K}_2 \geq i.$$

Therefore any triangle of odd perimeter p and type (j,k,ℓ) in A satisfies

$$p \geq 2\tilde{K}_1 + 1;$$
$$p \leq 2i + 2\min(j,k,\ell) \leq 2K_2 + 2\min(j,k,\ell).$$

We also have $p \leq 3i < 2\tilde{\delta} < \tilde{C}$.

Thus all triangles occurring in A occur in Γ_i.

Now we deal with the $\tilde{\delta}$-Henson constraints. No pair in A lie at distance $\tilde{\delta}$, so we concern ourselves only with cliques K_n occurring in Γ. Then Lemma 17.4 applies. □

17B. The cases $K_1 \leq 2$

If we apply Lemma 17.2 to the cases considered in Proposition 1.33, then we arrive at the following.

LEMMA 17.5. *Set $K_0 = 1$ or 2. Suppose the following.*

> *Every countable metrically homogeneous graph of generic type and finite diameter for which $K_1 = K_0$ is of known type.*

Then every countable metrically homogeneous graph of generic type and infinite diameter for which $K_1 = K_0$ is also of known type.

Furthermore, if the hypothesis holds with $K_0 = 1$, then every countable metrically homogeneous bipartite graph is of known type.

PROOF. For the first part, we apply Lemma 17.2. Then it suffices to check that Γ_2 contains a triangle of type $(K_1, K_1, 1)$.

If $K_1 = 1$ then this follows from Lemma 17.4.

If $K_1 = 2$ this follows from Lemma 15.15.

The final point was discussed already in Remark 17.1. □

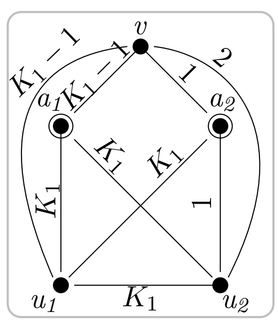
In particular, there is something to be said for taking up the general classification problem in the case $K_1 = 1$, since in any case one must sometimes treat the case $K_1 = 1$ separately from the case $K_1 > 1$ (finite); and this would have the further merit of disposing of the bipartite case unconditionally.

17C. General K_1

Now we push through the same type of analysis for general $K_1 > 1$.

LEMMA 17.6. *Let Γ be a countable metrically homogeneous graph of generic type with $K_1 > 1$. Suppose in addition that Γ_{K_1} has diameter at least K_1. Then in Γ_{K_1} there is a triangle of type $(K_1, K_1, 1)$.*

PROOF. We use a direct amalgamation argument. Consider the following 2-point amalgamation problem.

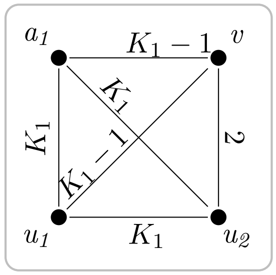


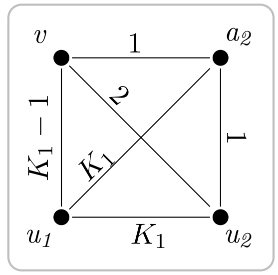
Here the vertices v and u_2 force $d(a_1, a_2) = K_1$. So it will suffice to show that the two factors

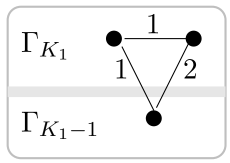
$$(a_1 v u_1 u_2) \text{ and } (a_2 v u_1 u_2)$$

embed into Γ.

Over the basepoint u_1, the factor $(a_2 v u_1 u_2)$ represents a vertex v in $\Gamma_{K_1 - 1}$ at distances $1, 2$ respectively from a pair of adjacent points in Γ_{K_1}.





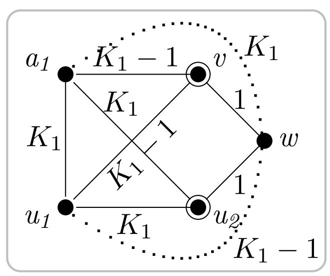


This is available since Γ_{K_1} is connected and the distances $1, 2$ occur between Γ_{K_1-1} and Γ_{K_1}.

So we turn to the configuration

$$(a_1 v u_1 u_2).$$

We amalgamate as follows.

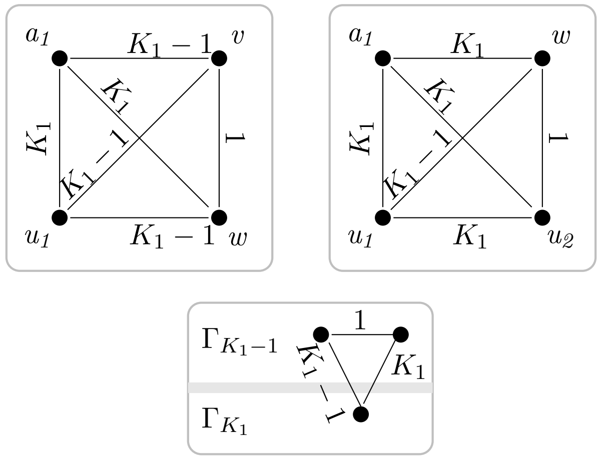


As $K_1 > 1$ this forces $d(u_2, v) = 2$. So it suffices to show that the factors

$$(a_1 v u_1 w) \text{ and } (a_1 u_1 u_2 w)$$

embed into Γ.

Relative to the basepoint u_1, the factor $(a_1 v u_1 w)$ represents a vertex in Γ_{K_1} at distances $K_1 - 1$ and K_1 respectively from a pair of adjacent vertices in Γ_{K_1-1}.



As Γ_{K_1-1} is connected it suffices to check that the distances $K_1 - 1$ and K_1 occur between Γ_{K_1-1} and Γ_{K_1}. And as Γ_{K_1-1} is connected it also follows that the set of distances occurring is an interval.

The distance 1 certainly occurs between Γ_{K_1-1} and Γ_{K_1}, so it suffices to check that the distance K_1 occurs. This corresponds to a triangle of type $(K_1, K_1, K_1 - 1)$. As Γ_{K_1} is connected it suffices at this point to have the diameter of Γ_{K_1} at least $K_1 - 1$.

So the factor $(a_1 v u_1 w)$ occurs in Γ.

Relative to the basepoint a_1, the second factor $(a_1 u_1 u_2 w)$ represents a geodesic of type $(1, K_1 - 1, K_1)$ in Γ_{K_1}.

So if Γ_{K_1} has diameter at least K_1, this factor is also available.

This completes the analysis. $\square$

Now we prove Theorem 1.26.

PROOF OF THEOREM 1.26. We suppose that every countable metrically homogeneous graph of finite diameter is of known type, and we consider a countable metrically homogeneous graph Γ of infinite diameter.

As the countable metrically homogeneous graphs not of generic type are known, we may suppose that Γ is of generic type, and consider the associated parameters, notably K_1. By Remark 17.1 we may suppose that $K_1 < \infty$.

By Lemma 17.5 we may suppose $K_1 > 1$ (or even $K_1 > 2$). By Lemma 17.2 it suffices to show that Γ_{K_1} contains a triangle of type $(K_1, K_1, 1)$, and by Lemma 17.6 it suffices to show that Γ_{K_1} has diameter at least K_1. But as Γ has infinite diameter, Γ_{K_1} has diameter $2K_1$. $\square$

A similar argument applies whenever the diameter δ is at least $2K_1$, and this may be useful even in the case of finite diameter.

CHAPTER 18

APPENDIX
SOME RECENT ADVANCES

At the end of Volume II we will survey a range of problems in the theory of homogeneous structures, most of which arise after one has a classification in hand—or, at least, some particular stock of interesting examples to work with.

Here we discuss progress made on four particular points more closely connected with the material of the present volume, namely the following.

1. The classification of homogeneous multi-orders.
2. The classification conjecture for metrically homogeneous graphs.
3. *Canonical completions* of partial metrically homogeneous graphs.
4. *Twisted automorphisms* of metrically homogenous graphs

As we shall see, the first of these topics has some connection with Part I. And the second simply restates the motivation for much of the work in Part II, as well as that in Amato, Cherlin, and Macpherson [2021]. The last two points concern the further study of the automorphism groups of the known metrically homogeneous graphs, considered as permutation groups, as topological groups, or as abstract groups.

Namely, the third topic turns out to be the main ingredient in the solution of a wide variety of problems concerning the automorphism group of a known metrically homogeneous graph of generic type, whether as a topological group or as an abstract group. It leads in particular to a key finiteness condition which implies all of the following: a structural Ramsey theoretic result (after adding a generic linear order, in most cases), the so-called EPPA property giving descriptive set theoretic information, and a notion of stationary amalgamation in the sense of Tent and Ziegler [2013] which has implications for the structure of the automorphism group as an abstract group.

As the connections between the completion methods for partial structures discussed here and the various applications is part of the general theory, we leave the detailed discussion of these connections to the appendix to Volume II. Here we concentrate on the particular completion procedures relevant to metrically homogeneous graphs. This leads to an interesting incursion into an evolving theory of generalized metric spaces, one which also makes a limited appearance in connection with the first point (§18A). We will however mention a bridge between the completion procedures on the combinatorial side and the applications, namely a finiteness condition for partial substructures.

Our fourth and last topic is not very well known, or, for that matter, very well understood. It concerns the structure of the normalizer $\mathrm{Aut}^*(\Gamma) = N_{\mathrm{Sym}(\Gamma)}(\mathrm{Aut}(\Gamma))$ and the question as to whether the group $\mathrm{Aut}^*(\Gamma)$ splits over $\mathrm{Aut}(\Gamma)$, a point first raised in Cameron and Tarzi [2007]. Though there is still no general theory, there are anecdotal hints of one, and one can say something concrete about this problem in the particular case of metrically homogeneous graphs—beginning with a close analog of a well known result on finite association schemes proved by radically different methods in Bannai and Bannai [1980]; here we consider the more general notion of *twisted isomorphism* before specialzing to the twisted automorphism group.

One should note that the first and third topics involve work published or in course of publication by a variety of authors, while the discussions under the second and fourth headings include assertions taken from articles not in final form, and give my views of these matters as of summer 2021.

18A. Homogeneous multi-orders

We will refer to a structure in a relational language with finitely many linear orders (and no further relations) as a *multi-order.* these are also called *finite-dimensional permutation structures.* In the particular case of two linear orders, they are called *permutation structures,* and are essentially the same thing as permutations, at least in the finite case; in particular the isomorphism type of a permutation structure is a *permutation pattern* in the usual sense. If a permutation is identified with its graph as a function, the two orders correspond to the two axes.

Cameron classified the homogeneous permutation structures, and asked for the generalization to multi-orders (Cameron [2002/03)]).

Problem 4. Classify the (countable) homogeneous multi-orders.

In earlier drafts of the present monograph we discussed Problem 4 as an outstanding problem concerning the classification of homogeneous structures. This problem has now been solved, by striking and powerful new methods from the side of geometrical neostability theory, as we will discuss.

First we point out a connection to the material of Part I.

In Part I we found that when we expand the language of graphs by a symbol for a linear order, the homogeneous structures that arise are not very "new," in the sense that they are derived from homogeneous structures of simpler sorts by a natural expansion process: either adding a generic order, or generically extending a prior partial order to a linear order. Some complications arise as the language of ordered graphs is equivalent in a straightforward sense to the language of ordered tournaments, as well as the language of partial orders with a linear extension, and for the most part homogeneous structures of different kinds—graphs, tournaments, partial orders—give rise to different homogeneous ordered structures when suitably expanded by a linear order.

With this in mind, one might wonder more generally about the relationship of the classification problem for homogeneous structures in a given language with a linear order to the corresponding problem for simpler languages. The base case for this question would take the language of equality as the point of departure, and then add a finite number of linear orders, taking us back to Cameron's problem.

Braunfeld's thesis (Braunfeld [2018]) contained some general results and very general conjectures concerning Problem 4. It was shown there that any finite distributive lattice could be the lattice of definable equivalence relations in a homogeneous multi-order, and that if the reduct of a homogeneous multi-order to the language of its definable equivalence relations is again homogeneous, then the lattice of definable equivalence relations must in fact be distributive (see also Braunfeld [2016]).

The existence statement (realizing an arbitrary finite distributive lattice) is shown by viewing structures with a specified lattice of equivalence relations as *generalized metric spaces* with values in a lattice, and then generically adding orders to make the equivalence relations definable. More explicitly, if E, F are definable equivalence relations with E covering F, there is a quotient structure E/F which is the disjoint union of C/F for C an E-class, and one generically "orders" E/F by a disjoint union of orders on the quotients C/F.[14] At the end, one changes the resulting language back to an equivalent language using global linear orders.

The corresponding classification conjecture was that all homogeneous multi-orders are obtained essentially in this way, with some trivial variants (one order may coincide with another, or the reversal of another, on some of the quotients involved). Part of the difficulty of the problem as originally stated is that the language of multiple linear orders is not the most natural language for the analysis of the structures. (In particular, for a *fixed* number of linear orders, it is hard to say explicitly which of these structures can actually be given in the specified language, though this becomes a finite problem, and may possibly have a neat answer.)

This conjecture has the following consequences. First, the *primitive* case one should have just independent orders, with variants in which some orders are repeated or reversed. Second, in all cases the reduct to the language of definable equivalence relations should be homogeneous, and, less obviously, the lattice of definable equivalence relations should be distributive.

Using amalgamation methods and a very close analysis, the conjectured classification was first verified for the particular case of three linear orders. Then a new insight came from work of Pierre Simon, studying a notion of rank compatible with the presence of orderings, with a classification in the rank 1, $\aleph_0$-categorical case (Simon [2020]). Here the key amalgamation argument comes in just once, to show that the homogeneous multi-orders are in fact finite rank in his sense; in particular the primitive ones are rank 1. Then his general theory gives the conjectured classification of primitive homogenous multi-orders—as well as a classification of their reducts, another subject for which the usual (Ramsey theoretic) methods tend to blow up very considerably with the size of the language.

[14]This construction is inspired by some methods of structural Ramsey theory, connected with the topological dynamics of the automorphism groups of homogeneous generalized metric spaces, a topic we return to below.

Then the imprimitive case of Problem 4 was fully treated in Braunfeld and Simon [2020], showing along the way that the reduct to the language of equivalence relations is indeed homogeneous, and then taking advantage of the resulting distributivity of the lattice.

Once one has the classification it then follows that a general structural Ramsey theorem holds as well, for the whole class of structures.

A byproduct of Pierre Simon's work is an example of a homogeneous binary structure in a finite relational language containing a symbol for a linear order which is not obtainable from a homogeneous structure in a simpler language by adding a suitable linear order. This leaves the results of Part I wholly unexplained on theoretical grounds. One would like to see multi-orders and ordered graphs taken into some "tame" neostability context with applicability to classification problems and related matters which would obviate the need for the very explicit type of analysis in Part I. The existing theory is placed firmly in the context of NIP theories, so it is not at all clear—particularly in view of the limitations coming from Pierre Simon's example—that such a theory should really exist.

It is also natural to ask whether we can do something similar in the line of classification with partial orders in place of linear orders. Notably:

Problem 5.

5.1 Classify the homogeneous structures equipped with two (or, finitely many) partial orderings.
5.2 Classify the homogeneous partially ordered graphs.

It's possible a priori that Problem 5.2 can be handled by the methods used in Part I, but this question has not been explored.

One can also look to combine Parts I and II of the present monograph as follows.

Problem 6. Classify the linearly ordered metrically homogeneous graphs of diameter 3 (or of any diameter, eventually!).

Before getting too carried away witih this particular problem one should recall that some of the homogeneous ordered graphs are in fact ordered homogeneous tournaments (but not ordered homogeneous graphs) and in general if one adds on an ordering to a language one has to be prepared to deal also with all of the proper reducts of the expanded language. On the other hand, given the specific constraints imposed by the triangle inequality, we do not know how concerned to be about that point in the case of this particular problem. And as the prospects for a complete proof of the classification of the metrically homogeneous graphs are looking bright at present (as discussed below), the full version of the question may come into play.

18B. Metrically homogeneous graphs

18B.1. Toward the classification conjecture. Part II of the present monograph left us without any detailed sense of how the general classification conjecture for metrically homogeneous graphs might be proved, though the local analysis of Chapter 15 was intended to supply some useful tools in that direction, and their utility has been illustrated by the proofs of Theorems 1.26 and 1.30.

We did not immediately see anything very general in our treatment of the case of diameter 3 in Amato, Cherlin, and Macpherson [2021], though we did see some suggestive parallels with the treatment of the bipartite case.

On reconsideration, we now see the method of Amato, Cherlin, and Macpherson [2021] as very promising for the treatment of the general case, and the parallels mentioned as central to the proposed strategy. (There may still be traces of a less optimistic, or more agnostic, stance lingering elsewhere in this volume.) As this point dawned on us before (or really during) submission of the final version of that paper, we have included a discussion of this line of thought at the end there, in §7.5 . Here we will flesh this out more. The work remains tentative, and is in progress with Amato. Still we give the plan here in considerable detail. As it depends very heavily at points on work in this volume, this is a good place for this discussion.

The aim is to prove the conjectured classification of metrically homogeneous graphs of generic type and finite diameter in the form given as an explicit catalog on page 219.

18B.1.1. *The framework.* We have explained in Definition 13.6 how to associate a canonical parameter sequence

$$\delta, K_1, K_2, C_0, C_1, \mathcal{S}$$

with any metrically homogeneous graph Γ of generic type.

We will write $\mathcal{A}$ for the class of finite structures embedding in Γ, and $\mathcal{A}^*$ for the class $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$. The classification conjecture may be phrased as follows.

$$\mathcal{A} = \mathcal{A}^*.$$

Note that this notation makes sense whether or not the parameter sequence is admissible; of course, since Γ is homogeneous, it cannot actually hold unless the parameter sequence is admissible, in which case in addition to the class $\mathcal{A}^*$ we have the Fraïssé limit Γ^* and the more recognizable formulation of the conjecture as

$$\Gamma \cong \Gamma^*.$$

The two-step process for proving this is trite, but meaningful.

(1) $\mathcal{A} \subseteq \mathcal{A}^*$.
(2) $\mathcal{A}^* \subseteq \mathcal{A}$.

Actually there will be some overlap between the two steps in practice, and the real dividing line is somewhat past the end of Step 1. So our description of these two steps below will deviate from the simple scheme proposed.

18B.1.2. *The first step.* A more precise statement of the goals in the first phase is as follows.

(1a) $\mathcal{A} \subseteq \mathcal{A}^*$.
(1b) Every triangle in $\mathcal{A}^*$ embeds into Γ.
(1c) The parameters of Γ are admissible.

The content of (1a,1b) together is, mainly, that $\mathcal{A}$ and $\mathcal{A}^*$ contain the same metric triangles. The three points ($1a$–$1c$) taken together correspond to the results of Chapters 13 and 14 of Part II (but there, we dealt only with the 4-trivial case). The idea in general is to follow the same line of argument very closely, after making some improvements in the local analysis, notably around Lemma 15.16.

Thus Step 1 has a preparatory character. Once it is achieved, we can work with Γ^*, the Fraïssé limit of $\mathcal{A}^*$. But going forward the focus in fact remains on the class $\mathcal{A}^*$, and on work with finite configurations.

18B.1.3. *The second step.* Things now become more elaborate and more challenging.

The second step may be phrased as a general embedding theorem: any finite structure compatible with the parameters of Γ embeds isometrically into Γ. The approach suggested by an analysis of Amato, Cherlin, and Macpherson [2021] is to proceed by induction on the diameter of the finite configuration in question.

The following terminology and notation will be convenient.

DEFINITION 18.1. Let Γ be a metrically homogeneous graph of generic type. A finite metric space is Γ-*constrained* if it belongs to $\mathcal{A}^*$.

DEFINITION 18.2. For K_1 finite, and $d \geq 1$, the embedding property $\mathcal{E}_d(K_1)$ is the following.

> *Let* Γ *be a metrically homogeneous graph of generic type and diameter at least* 3 *whose associated parameter* K_1^Γ *equals* K_1.
>
> *Then any finite* Γ*-constrained metric space of diameter at most* d *embeds isometrically into* Γ.

With this notation, the plan is to prove that $\mathcal{E}_d(K_1)$ holds for all d and K_1, by induction on d. The inclusion $\mathcal{A}^* \subseteq \mathcal{A}$ will then be immediate.

The base of this induction is not obvious: it is $d = \max(K_1, 2)$. Accordingly our second step breaks down as follows.

(2a) $\mathcal{E}_2(K_1)$ holds.
(2b) For $K_1 > 1$ finite, $\mathcal{E}_{K_1}(K_1)$ holds.[15]
(2c) For $d > \max(K_1, 2)$, $\mathcal{E}_{d-1}(K_1)$ implies $\mathcal{E}_d(K_1)$.

The first point is a technicality. The treatment of the case $d = K_1$ and the treatment of the inductive step both raise substantive issues.

As far as point (2*c*) is concerned, the plan is to follow and extend the method of Amato, Cherlin, and Macpherson [2021] (this provides a considerably more useful model than our treatment of the bipartite case in Part II), We elaborate.

In retrospect, the main technical result in our treatment of the diameter three case can be viewed as a proof[16] of

$$\mathcal{E}_2(K_1) \Longrightarrow \mathcal{E}_3(K_1)$$

for $K_1 < 3$—at a casual glance the use of the induction hypothesis is a little hard to spot, very brief, and not very explicitly noticed. One can also detect in that analysis the (also brief) moment when the base case $\mathcal{E}_2(K_1)$ is actually proved, for $K_1 \leq 2$. The case $K_1 = 3$ was avoided by a technical device having to do with twisted automorphisms (see §18B.3) which allows for a change of language (and, in particular, a change in the numerical parameters).

[15] One can either set aside the case $K_1 = \infty$ using Theorem [1.30 or adjust the notation so that the case $K_1 = \infty$ is also covered.

[16] Actually, two (similar) proofs, after a division into two cases.

On the other hand, there is really nothing in the prior work that serves as a guide to a proof of point (2b). When $K_1 > \delta/2$ there is an easy inductive argument to handle this, but that does not help us toward the general case. So as far as mapping out a proof strategy is concerned, this is the most troublesome element to be addressed.

We propose to prove the embedding property $\mathcal{E}_{K_1}(K_1)$ by a further inductive argument of an even more specialized character. But this does not seem like the place to elaborate further on this last element. The main point to be made here is that the structure of the proof strategy leads directly to this particular point, and once one focuses on it, one finds there is a reasonably natural approach, where the definition of K_1 comes directly into play.

But we should stiill say something more about how the treatment of (2c) proceeds in general, as there is an important point which was not seen so clearly when working under the assumption that $\delta = 3$. In particular, this will explain why the inductive step (2c) takes place above $\max(K_1, 2)$.

18B.1.4. *Second step: The proof of* (2c). Now we give a reading of the contents of Amato, Cherlin, and Macpherson [2021] from the point of view of the general problem of carrying out the induction step (2c).

Roughly speaking, our line of argument is to show that if $\mathcal{A}^*$ contains any configuration A of diameter at most d which is not in $\mathcal{A}$, then after a suitable minimization this configuration is actually a $(1, d)$-space (or a triangle, but this alternative was already ruled out in Step 1). When $d = \delta$ this suffices as by definition $\mathcal{A}$ and $\mathcal{A}^*$ contain the same $(1, \delta)$-spaces. So to complete the argument we need to eliminate this extremal configuration in the case $d < \delta$. So the first question is how to reduce the general problem to the case of $(1, d)$-spaces—or something very similar (one does not quite reach the desired extremal configuration by very broad arguments).

For this we need useful measures of structural complexity which are minimized by $(1, d)$-spaces. The complexity measures given in Amato, Cherlin, and Macpherson [2021] are suitable, with some further elaboration and adjustment to refer to d rather than δ, (In Amato, Cherlin, and Macpherson [2021], $d = \delta = 3$ throughout the main argument, and one does not notice d as a distinct parameter.)

At this point one needs explicit amalgamation arguments in the vein of Amato, Cherlin, and Macpherson [2021], on a broader scale.

These turn out to depend on the existence of certain auxiliary configurations in $\mathcal{A}^*$ which can be obtained from amalgamation arguments inside the class $\mathcal{A}^*{\restriction}d$ of configurations in $\mathcal{A}^*$ of diameter at most d. At this point, the admissibility of the parameter sequence is of first importance, as our arguments for $\mathcal{A}$ depend on the possibility of solving amalgamation problems in $\mathcal{A}^*$, but somewhat more is needed since we must also control the diameter of the resulting solution. As we will see, the condition $d > \max(K_1, 2)$ comes in precisely at this point.

In the first place, it turns out that $\mathcal{A}^*{\restriction}d$ is itself an amalgamation class for $d \geq \max(K_1, 2)$. This is already very suggestive. In other words, when amalgamating configurations of diameter at most d, we are not forced to introduce any "new" distances greater than d.

Furthermore, when d is strictly greater than $\max(K_1, 2)$ we can also avoid adding new distances equal to either 1 or d in the amalgam. This gives us the necessary control over the structural complexity of the resulting configurations—our measurses of complexity are defined in terms of the associated graph whose edges corresponding to the extreme values for the distance, namely 1 and d.

The necessary amalgamation properties of the class $\mathcal{A}{\restriction}d$ can be checked in various ways—one way, not the most agreeable, would be to make a careful reading of the proofs of amalgamation in this volume. A less cumbersome approach would be to check the admissibility conditions for the parameters of the restricted class in order to replace δ by d, and then to check the earlier discussion of the conditions under which the extremal values $1, \delta$ can be avoided in an amalgam.

A more conceptual line of argument uses the completion process discussed in 18B.2—one of the canonical completion procedures discussed there will do. Here one has a choice of procedures depending on an auxiliary parameter which needs to be at least K_1 and less than d, and this accounts for our restriction $d > K_1$. Evidently if one wishes to avoid the values 1 and d in completing an amalgamation in diameter d, one should also have $d \geq 3$. (A complication arises as the relevant parameter needs also to be at least $\delta/2$, so one may need to iterate the argument to first reduce δ if K_1 is small.)

This explains why the base of the induction is $\max(K_1, 2)$.

Some further difficulties will arise at the end when one has a $(1, d)$-configuration A, or some very similar configuration. But it seems one can also handle the more concrete issues that arise at this point by similar constructions with a more ad hoc quality.

18B.2. Canonical completion of partial substructures. A number of problems relating to automorphism groups of homogeneous structures, both as topological groups and as abstract groups, can be reduced to combinatorial problems for the structures in the associated amalgamation class $\mathcal{A}$. At this point we are going to turn our attention to the key combinatorial problem which arises in the context of metrically homogeneous graphs. We leave the broader discussion of the motivating problems concerning the automorphism group to the appendix to Volume II. The reader familiar with those connections will be thinking of structural Ramsey theory, extensions of partial automorphisms (EPPA), stationary amalgamation, and their implications for the topological dynamics, descriptive set theory, and algebraic structure of the corresponding automorphism groups. Without that motivation, the problem we are about to discuss looks rather more like a topic in the theory of combinatorial algorithms. But however it is viewed, it takes us in very interesting and new directions.

We deal here with the known metrically homogeneous graphs Γ of generic type and their associated amalgamation classes $\mathcal{A}$. But we now shift our attention to the extended class of *partial* (or *weak*) *substructures* of structures in $\mathcal{A}$, which we denote by $\hat{\mathcal{A}}$, and we concern ourselves with the description of the class $\hat{\mathcal{A}}$ and its precise relationship to $\mathcal{A}$.

18B.2.1. *Some examples.* We give a couple of examples of the class $\hat{\mathcal{A}}$ for some specific homogeneous structures (not limited to metrically homogeneous graphs).

(a) If Γ is the rational order, and $\mathcal{A}$ is the class of all finite linear orders, then $\hat{\mathcal{A}}$ is the class of finite acyclic digraphs.
(b) If Γ is the generic metrically homogeneous graph of finite diameter δ, then $\hat{\mathcal{A}}$ is the collection of finite $[\delta]$-edge labeled graph with no *non-metric cycles* (cycles with some edge longer than the length of the complementary path).

We observe that the stated criteria for membership in the class $\hat{\mathcal{A}}$ o are more clearly necessary than sufficient. The way one verifies the correctness of a proposed description of the class $\hat{\mathcal{A}}$ is by giving a *completion procedure* which take a a partial structure A and completes it to a structure $\bar{A}$ in $\mathcal{A}$ whenever this is possible. Typically the criterion for membership in $\hat{\mathcal{A}}$ amounts to the absence of any obstructions to the successful application of the completion procedure.

Ideally the completion procedure should also be *canonical* in the sense that any isomorphism between finite structures gives an isomorphism between their completions; in particular it will follow that automorphisms of A are automorphisms of $\bar{A}$.

Returning to the two examples above, to complete an acyclic digraph to a linear order one first takes its transitive closure to get a partial order, and then one somehow completes the partial order to a linear order. In this case the completion cannot be carried out canonically since the completed structure will be rigid, while the initial structure could be a trivial digraph with any number of vertices and no arcs.

On the other hand, in the case of the generic metrically homogeneous graph of diameter δ, there is a canonical completion, given by the shortest path metric with truncation to diameter δ. In other words, the distance between two points is defined as either the minimum length of a path between them in the original edge-labeled graph, or δ, whichever is least.

There is a connection between canonical completion and canonical amalgamation. If $\mathcal{A}$ is a strong amalgamation class then an amalgamation problem asks for the completion of a particular partial structure, the free join of two structures over a common base. Thus when one has a strong amalgamation class, any canonical amalgamation procedure gives a canonical amalgamation procedure as well. However one should notice that one can also have a canonical completion procedure for $\hat{\mathcal{A}}$ even if $\mathcal{A}$ is not an amalgamation class. We give a class of examples that illustrates this further.

Let $S \subseteq \mathbb{R}_{>0}$ be a finite set of positive real numbers. We consider the class $\mathcal{A}_S$ of finite S-metric spaces. There is a general notion of truncation to S, best expressed in terms of the following operation.

$$a +_S b = \max(s \in S \mid s \leq a + b)$$

When $S = [\delta]$ this is the usual truncation to diameter δ.

Now it turns out (Sauer [2013]) that the class $\mathcal{A}_S$ of finite S metric spaces is an amalgamation class if and only if the operation of $+_S$ is associative, in which case paths have well-defined lengths in the sense of $+_S$. In this case the S-shortest path completion is a completion procedure, and the minimal obstructions to this procedure are the non-metric cycles in the sense of $+_S$.

But one has a similar canonical completion procedure even if the class $\mathcal{A}_S$ is not an amalgamation class. When the operation $+_S$ is non-associative; paths will have multiple possible lengths, but the S-shortest path metric still makes sense, as one minimizes over all possible values. The difference is that free amalgamation of S-metric structures (even of geodesic triangles) can create non-metric cycles in this case; for example, with $S = \{1, 2, 3, 6\}$, amalgamate the geodesic of type $(1, 2, 3)$ with the geodesic of type $(3, 3, 6)$.

18B.2.2. *A finiteness condition.* The following finiteness condition is very useful when it holds, and having a completion procedure usually gives a good handle on whether or not it holds.

DEFINITION 18.3. The class $\mathcal{A}$ (and the structure Γ which is its Fraïssé limit, in the case of an amalgamation class) has *finitely many forbidden partial substructures* if

($\star$) Up to isomorphism there are finitely many minimal structures in the language of $\mathcal{A}$ which do not belong to $\hat{\mathcal{A}}$.

One might say, more simply, that $\mathcal{A}$ is "finitely constrained," but this term would more often be taken to refer to forbidden *induced* substructures. In fact, in finite diameter all of the metrically homogeneous graphs of generic type are finitely constrained in this latter sense, more or less by definition (after paying some attention to the possible Henson constraints).

Now transitive relations usually obstruct this finiteness condition. For example, the minimal obstructions to extending to a linear order are the oriented cycles, and there are infinitely many possibilities (again, these correspond quite directly to the completion procedure mentioned earlier).

Similarly, if $S \subseteq \mathbb{R}$ is finite and $a +_S a = a$ (a "gap"), then the relation $d(x, y) \leq a$ is an equivalence relation, so for $a < \max S$ this provides an infinite set of non-metric cycles. Otherwise, in the absence of gaps, it is easy to see that there are only finitely many non-metric cycles, since S itself is finite. In particular the finiteness condition holds when $S = [\delta]$, bringing us back to the case of the generic metrically homogeneous graph of diameter δ.

Now we return to the known metrically homogeneous graphs of generic type in the general case.

18B.2.3. *The case of metrically homogeneous graphs.* Work on completion procedures for partial metrically homogeneous graphs was motivated initially by the question of the finiteness of the forbidden partial substructures as the most direct route toward a structural Ramsey theorem. A suitable procedure was given for the cases satisfying the following numerical conditions on the parameters in Coulson [2019], with many of the resulting applications to the automorphism groups.

$$C > 2\delta + \max(K_1, \delta/2); \qquad C' = C + 1; \qquad K_2 \geq \delta - 1.$$

A very general completion procedure was found by the "Prague consortium," given in full generality in the preprint Aranda et al. [2017] (57 pp.) and published in a condensed and simplified form (whose generality is less clear) as Aranda et al. [2021]. One extreme case escapes the full version as well, but only because there is no completion procedure with the desired canonicity property in that case. We will refer to the results as given in full in the preprint, and go into more detail in the appendix to Volume II when we discuss the general theory. We note that the imprimitive cases are subtle but are understood as well.

Subsequently it was seen that the completion algorithm founds fits very neatly into a theory of shortest path completions in generalized (and fairly exotic) metric spaces. The material on S-valued metric spaces fits very naturally in the same framework, along with Braunfeld's lattice-valued generalized ultrametric spaces. It is certainly surprising that the integral metric spaces associated with metrically homogeneous graphs are best viewed as *generalized* metric spaces with respect to a highly non-standard addition. On the other hand the generic metrically homogeneous graph of diameter δ is an instance of the S-metric theory and the truncated path completion is the one obtained from that point of view. In this case the "exotic" addition operation is just ordinary addition truncated to δ—so in this case it is not in fact particularly exotic.

A useful detailed account of the completion process in general is also found in Konečný's bachelor's thesis (Konečný [2019a]). As far as the particular case of metrically homogeneous graphs is concerned, this treatment is more complete than the discussion found in his more ambitious and more widely ranging master's thesis (Konečný [2019b]).

It follows from all of this that with few exceptions the amalgamation procedure we gave could be replaced by shortest path completion with respect to a suitable generalized metric space structure. At the moment, oddly enough, the proof of that fact relies on the proof of amalgamation given here, as in order to know that the "completion" process applies to amalgams one needs to know first that the class has strong amalgamation. However, since this is the case, it should be possible to give a direct proof.

In particular, in the primitive case, the numerical constraints defining admissibility must be equivalent to the fact that shortest path completion is both available (via a suitable neutral parameter) and adequate to the task. How exactly that equivalence is manifested working directly from first principles is not entirely clear. One needs a good deal of the numerical information, though not all of it, to show that the relevant interpretation as a generalized metric space exists in the first place, and then the rest of the information will come in when checking the validity of the completion procedure for the classes of interest (which carry additional constraints). At some level this is what Konečný [2019b] aims at, but in a broader setting.

Leaving this digression aside, our immediate task is to explain what the relevant semigroup operations are in our case, what the "shortest path completion" can possibly mean when lengths are not comparable and infima may not even exist, how the metrically homogeneous graphs fit into this framework, and, also, what are the consequences of this procedure for finiteness of forbidden partial structures and related questions.

So we begin.

18B.2.4. *Disjoint sums and neutral distances.* We first describe the most rudimentary instance of the completion procedure: disjoint sum. That is, A and B are finite metric spaces embedding isometrically into a give metrically homogeneous graph Γ, and we aim to embed their disjoint union $A \oplus B$ canonically into Γ.

Definition 18.4. Let A, B be metric spaces and $k \geq 1$. The k-direct sum

$$A \oplus_k B$$

is the metric space on the disjoint union $A \sqcup B$ with $d(a, b) = k$ between A and B.

If Γ is a metrically homogeneous graph of finite diameter δ, then a *neutral distance* M for Γ is a distance such that $\mathcal{A}(\Gamma)$ is closed under the operation $\oplus_M$.

For a known metrically homogeneous graph of generic type and finite diameter $\delta \geq 3$ which has no Henson constraints, the neutral distances are characterized by the conditions

$$\max(K_1, \delta/2) \leq M \leq K_2, (C - \delta - 1)/2$$

When there are Henson constraints it suffices to add the conditions

$$1 < M < \delta$$

Using the numerical characterization of admissibility it can be shown the neutral distances usually exist. The exceptions come in the bipartite case and antipodal case; in the latter case, with $C = 2\delta + 1$, the conditions reduce to $M = \delta/2$ and thus require δ to be even.

18B.2.5. *The distance semigroups $D^{\delta}_{M,C}$.* Given a known metrically homogeneous graph of generic type and finite diameter with parameters $\delta, K_1, K_2, C, C', \mathcal{S}$ and a corresponding neutral parameter M, one defines an associated partially ordered semigroup $D^{\delta}_{M,C}$ as follows.

DEFINITION 18.5. Let δ, M, C be acceptable. Then $D^{\delta}_{M,C}$ is the set $[\delta]$ equipped with the operation $+_{M,C}$ and the relation $\leq_{M,C}$ defined as follows. 1. The operation $+_{M,C}$:

$i +_{M,C} j$ is the number in the interval (d^-, d^+) with

$$d^- = |i - j|;$$
$$d^+ = \min(i + j, C - (i + j) - 1)$$

which is closest to M.

2. The partial order $\leq_{M,C}$:

First define the *natural* partial order $\leq_{M,C,\mathrm{nat}}$.

$$i \leq_{M,C,\mathrm{nat}} j \text{ iff } j = i +_{M,C} x \text{ for some } x \in [\delta].$$

We let $\leq_{M,C}$ be the natural partial order associated with $+_{M,C}$ unless $C = 2\delta + K_1$ and $M = K_1$, in which case we extend it to make $M - 1$ the second largest element:

$$M - 1 > j \text{ for } j > M$$

Remark 18.6. Notice that the value of $i +_{M,C} j$ is M if possible (that is, if M is in the indicated interval) and in any case it is the best approximation to M within that interval (so, either M or an endpoint of the interval).

The order structure on $D^{\delta}_{M,C}$ is as follows (and illustrated below), assuming

$$M < \delta \text{ and } C > 2\delta + 2.$$

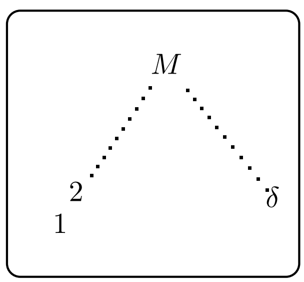
The minimal elements of $D^{\delta}_{M,C}$ are 1 and δ, and M is the unique maximal element. The ordering induced on $[1, M]$ is the usual ordering, and $+_{M,C}1$ agrees with $+1$ there; the ordering on $[M, \delta]$ is the reverse of the usual ordering, and $+_{M,C}1$ agrees with -1 there.

There are order relations between the two sides but they play less of a role, and are not shown in the diagram.

One clarifying and useful point is the relation

$$M +_{M,C} x = M$$

which shows that M is indeed maximum in the ordering on $D_{M,C}$.



$D_{M,C}$

18B.2.6. *Shortest path completions.* It is clear what is meant by the length of a path in a generalized metric space with values in a commutative semigroup. It is considerably less clear what is meant by the shortest path. In fact what we need to define is the shortest path *length*.

If the semigroup is a lower semi-lattice this is reasonably clear: one takes the infimum of all lengths. This does not mean that there actually is a shortest path—a path of that length—though if one completes a partial generalized metric space using the shortest path metric, then there will be such a path, namely the edge between the two vertices.

The semigroups that interest us now are not semi-lattices, for the most part. With very few exceptions they have two minimal elements, 1 and δ. Still the shortest path completion is defined by taking infima—when they exist. This will provide a completion procedure for a class $\hat{\mathcal{A}}$ only if all required infima actually do exist, for any structure in $\hat{\mathcal{A}}$.

This is an obscurely phrased condition and it is not clear how to develop a satisfying theory of partially ordered semigroups with sufficiently many infima—even less clear when one realizes that one will require a distributive law of the form

$$a +_S \inf S = \inf(a + s \mid s \in S)$$

for all "geometrically realizable" instances of the law (where S is a possible set of path lengths).

One case where this works out satisfactorily is the case of the partially ordered semigroups $D^{\delta}_{M,C}$ associated to metrically homogeneous graphs. We have noticed that the two sides of the domain, namely $[0, M]$ and $[M, \delta]$ are linearly ordered. It turns out that for the most part sets of path lengths (computed in the sense of $+_S$) lie just on one side, so all necessary infima not only exist, but are actually realized. There is an exceptional case but it corresponds to the case already mentioned where the partial order is not the natural one, but is adjusted.

For technical reasons, to make this work out, we also need to impose some additional conditions on the neutral parameter M.

Definition 18.7. If Γ is a metrically homogeneous graph of finite diameter δ and M is a neutral distance for Γ then M is *proper* for Γ if, in addition the following conditions hold whenever Case (III) applies to Γ (i.e., in the case in which $C > 2\delta + K_1$).

(P_1) $M + 2K_2 \geq 2\delta$.

(P_2) If $C = 2\delta + K_2$ and $C' > C + 1$ then $M < K_2$.

Again, proper neutral values of the distance exist outside the exceptional cases of antipodal graphs of odd diameter and the bipartite case.

We are now going to attempt to paraphrase in this language a fundamental result given in Konečný [2019a] in a slightly older language.

Similar results are given in the newer form in Konečný [2019b] for a range of examples, but with less detail as far as the case of metrically homogeneous graphs is concerned, so they are a little less suitable for our immediate purpose.

Fact 18.8 (Konečný [2019a, Theorem 5.1, paraphrased]). Let

$$(\delta, K_1, K_2, C_0, C_1)$$

be an admissible sequence of parameters with δ finite, and let A be a partial substructure of

$$\Gamma = \Gamma^{\delta}_{K_1,K_2,C_0,C_1}.$$

Suppose that M is a proper neutral distance for Γ (Definitions 18.4, 18.7).

Then the shortest path completion $(A, d_{M,C})$ of A embeds isometrically into Γ and is M-optimal in the following sense. For any other completion (A, d) of A which embeds isometrically into Γ, and for any vertices $u, v \in A$, one of the following holds.

1. $d(u,v) \leq d_{M,C}(u,v) \leq M$.
2. $d(u,v) \geq d_{M,C}(u,v) \geq M$.
3. $C = 2\delta + K_1$, $C' > C + 1$, $M = K_1$, $d_{M,C}(u,v) = M - 1$, and $d(u,v) > M$.

Furthermore, any automorphism of A is an automorphism of the completion $A_{M,C}$.[17]

One virtue of this statement is that it sheds some light on the two "sides" of the semigroup, the intervals $[1, M]$ and $[M, \delta]$; somewhat more light is shed on this by the proof of the result by induction on path length. And an exceptional case appears explicitly.

This all deserves a fuller account with more attention to various foundational issues and the algebraic background to the theory. The developments in Konečný [2019b] are certainly relevant as well but are focused on the immediate applications more than the foundations of the theory.

One case which may seem troubling—certainly we have found it disconcerting—is the case in which the parameters satisfy $C' > C+1$ and C is not literally a bound on perimeter, which somewhat undercuts the motivation behind the definition of $+_{M,C}$. In particular one

[17] The last point is clear, but it is important to take note of it.

may well wonder why the relevant structures are $D^{\delta}_{M,C}$-metric in this case. So we give an example, in fact one we have seen before.

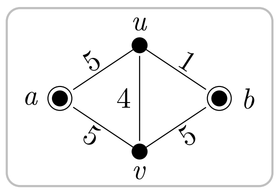
Type	δ	K_1	K_2	C	C'	M
(IIB)	5	3	3	13	16	3

The question is why the triangle type $(5,5,4)$ is metric given that 4 exceeds the "perimeter bound." In fact this is not obvious. What this means is that

$$4 \leq_{M,C} 5 +_{M,C} 5 = 2$$

(here the value of C enters).

However the correctness of this inequality is unclear. It is clarified by the following amalgamation problem.



According to the shortest path completion, the solution must be

$$\inf(5 +_{M,C} 5, 5 +_{M,C} 1) = \inf(4,2)$$

This makes it clear that we must have $4 \leq_{M,C} 2$, and this becomes part of the definition of $\leq_{M,C}$ (in general, in this setting, $M-1$ becomes the unique maximum below M).

So the relevant partially ordered semigroup has a very particular partial order in this case.

This theory casts a great deal of light on the interpretation of the numerical conditions for admissibility we gave in Part II, but this point has not been completely elucidated. We know in particular that our amalgamation method is essentially a shortest path completion (but in our presentation, rather than fixing a single neutral parameter M, we gave a range of possible values in various cases).

One central object of study would be the class of partially ordered semigroups D for which the shortest path completion exists for any D-metric space.[18] However we have seen an example (in the contest

[18] In its most general form one should actually refer to "shortest walks," but the sort of pathology that would require making this distinction has not been encountered to date.

of S-metric spaces, for $S \subseteq \mathbb{R}$) where the natural additive structure is not even associative—associativity being equivalent in that context to the amalgamation property. From this point of view, it remains unclear whether associativity should be a hypothesis or a conclusion. Similar remarks apply to the more obscure question of distributivity in an algebraic setting in which one of the operations (meet) is only partially defined, and needs to be applicable to certain finite sets (whether this reduces to a binary operation in general is unclear).

We insert a few tentative remarks about foundations, as we have no clear prospects of addressing this more systematically in the future.

Call a structure $(S, +, \leq)$ equipped with a binary operation $+$ and a partial order $\leq$ a *distance magma*.[19] We impose no further axioms at the outset. Already one can define the triangle inequality and the notion of S-metric space, and even the length of a walk either as a set of possible values or as their infimum, if it exists. In particular there is a "shortest walk" metric which may not be everywhere defined.

We are interested in the case in which the shortest walk metric provides a canonical amalgamation procedure; call such distance magmas *coherent*, and more particularly call them k-coherent if the shortest walk metric provides 2-point amalgamation over a base of order k. The condition of 1-coherence is simply the existence of all geodesic triangles $(a, b, a+b)$ and seems like a reasonable minimal assumption on the algebraic side. The condition is equivalent to the following pair of conditions.

1. $+$ is commutative.
2. $a \leq (a + b) + b$ for $a, b \in S$.

The second condition is a weak form of *positivity*: $a \leq a + b$.

As an extreme example, if one requires the partial order to be trivial (equality) then the structure $(S, +)$ must be an elementary abelian 2-group. In this case path lengths are unique and the shortest path metric gives an amalgamation procedure for S-metric spaces. The Fraïssé limit is the composition of a trivial structure (with all distances the 0 element of S) and the regular action of S on itself. The limitation to elementary abelian 2-groups corresponds to the symmetry of 2-types.

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[19] Differing sharply from Conant's use of the term in Conant [2017]; for foundational studies some shift in terminology is appropriate.

In general associativity follows from 1-coherence and translation invariance; since the relation of equality is certainly translation invariant this covers the previous examples. One can construct non-associative examples but whether they bring anything of value is unclear.

2-coherence is a sharp form of the well-known 4-*values* condition (amalgamation of triangles over a common edge) considered in Delhommé, Laflamme, Pouzet, and Sauer [2007], and more broadly in Conant [2017], which also characterizes both amalgamation and canonical amalgamation in the settings considered there.

The distributive law $a+\inf(b,c) = \inf(a+b, a+c)$ is equivalent to translation invariance in the linearly ordered case, equivalent to coherence in the case of lattices viewed as distaince magmas $(\Lambda, \sup, \leq)$, and holds in $D^{\delta}_{M,C}$ in the form $a + \inf S = \inf a + S$ (for cases where S is a set of path lengths, at least).

Konečný suggest working in a framework (called "partially ordered semigroups") in which the distance magma $(S, +, \leq)$ satisfies the following conditions: S is a commutative semigroup, $\leq$ a partial order, and positivity and translation invariance hold. He produces sufficient conditions for coherence in terms of forbidden cycles which cover the known examples of interest. From the point of view of homogeneity the question is the following.

Problem 7. Is every infinite primitive homogeneous structure in a finite symmetric binary language the expansion of class of S-metric spaces by forbidden triangles and Henson constraints, where S is a partially ordered semigroup and amalgamation is given by the shortest path metric?

This mixes together two or three problems. One problem is really quite separate: whether the structures in question are given by triangle and Henson constraints. The other part is essentially whether the 3-constrained primitive cases fall under the generalized metric space theory with partially ordered semigroups. On the one hand this asks whether this is the right definition of generalized metric spaces, and on the other hand it asks whether, given the right definition of generalized metric spaces, the 3-constrained case becomes more or less tractable.

Another question embedded in the foregoing is whether primitive infinite structures are associated with strong amalgamation classes.

That is perhaps a separate question and in this problem one may prefer to replace the hypothesis that the structure is infinite by the hypothesis that the associated amalgamation class has strong amalgamation.

Remark 18.9. Formally, one can make a non-commutative theory in the following way. One has a structure $(S, \cdot, *, \leq)$ where $(S, \cdot)$ is a semigroup, $*$ is an involution, that is

(a) $a^{**} = a$.
(b) $(ab)^* = b^* a^*$,
(c) $\inf(a, b)^* = \inf(a^*, b^*)$ when the left side is defined.

and $\leq$ is a partial order. Metrics satisfy

1. $d(y, x) = d(x, y)^*$.
2. $d(x, z) \leq d(x, y)d(y, z)$

This includes the symmetric case with $a^* = a$, and a commutative operation.

It turns out that these axioms are more or less known.[20] In a survey article Kabil and Pouzet [2020], the authors give almost same axioms, in a monoid setting with

$$d(x, y) \leq 1 \iff x = y$$

(a slightly curious formulation); and they give examples and applications. While the survey is recent (in fact, unpublished as of this writing) the work traces its origins to Quilliot [1983]. Since we now have what seems like an urgent need to extend the usual metric theory in this general direction, this line may cast some light on our current concerns as well. We will not attempt anything of that sort here.

But to give a concrete example of the axioms, let $(S, \cdot)$ a group, $a^* = a^{-1}$, and let $\leq$ be equality. Then all metric triangles are geodesics and have (oriented) perimeter 1. Thus path lengths are unique and the shortest path metric is well-defined on partial metric spaces. The Fraïssé limit is dull: it is the composition of a finite structure (the regular action of the group) with a trivial structure ($d(x, y) = 1$ for all x, y).

The symmetric case of this is the case of elementary abelian 2-groups.

[20]Thanks to Matěj Konečný for the pointer.

A more interesting but more problematic example is the structure $\mathbb{S}(n)$ encountered in Volume II, which has types $\xrightarrow{i}$ for $i \in \mathbb{Z}/n\mathbb{Z}$ and the involution $i^* = -i-1$. Addition is problematic: we can choose $x \oplus y = x+y$ or $x+y+1$. If we choose $x \oplus y = x+y$ then the triangle inequality corresponds to the rule

$$z \approx x + y,$$

meaning $z = x+y$ or (sacrificing associativity) $x+y+1$. In other words, if $x \oplus y = x+y$ then we take

$$i \leq i, i-1,$$

which is certainly not a partial order.

To make matters worse, the involution transforms one of the possible addition operations into the other.

Thus one may wonder whether a structure like $\mathbb{S}(n)$ fits into this framework at all. The notion of a shortest path algorithm becomes very problematic as the involution controlling type reversal does not respect addition but rather interchanges two possible operations.

This makes it doubly interesting, from our point of view, that there actually is a known stock of useful non-commutative examples—though whether they are useful for us is a nice open question.

Modulo the question of what the axioms ought to be, whether $\mathbb{S}(n)$ is the sort of thing that should be included, and whether—in that case—one can speak of anything like a shortest path metric, we would ask the following.

Problem 8. Are there further relations between non-commutative metric spaces and 3-constrained homogeneous binary structures in not necessarily symmetric languages, and a notion of *shortest path completion*?'

18B.3. The twisted automorphism group. If Γ is a homogeneous structure in a finite relational language (or more generally an $\aleph_0$-categorical structure) let Γ^* be the structure with the same underlying set whose relations are the equivalence relations E_n on Γ^n given by equality of types: $E_n(\mathbf{a}, \mathbf{b})$ holds iff $\mathbf{a}, \mathbf{b}$ lie in the same orbit under $\mathrm{Aut}(\Gamma)$. Set

$$\mathrm{Aut}^*(\Gamma) = \mathrm{Aut}(\Gamma^*).$$

We call this the group of *twisted automorphisms* of Γ; these may be thought of as automorphisms up to a permutation of the symbols in the language (assuming Γ carries its canonical language). In particular there is a canonical map from $\mathrm{Aut}^*(\Gamma)$ to a group of permutations of the language, and as we suppose Γ is homogeneous for a finite relational language, the image is a finite permutation group. The kernel is $\mathrm{Aut}(\Gamma)$. Defining the *twist group* of Γ as

$$\mathrm{Twist}(\Gamma) = \mathrm{Aut}^*(\Gamma)/\mathrm{Aut}(\Gamma),$$

we have a finite group of *twists of the language* associated with twisted automorphisms.

Classically, one takes $\Gamma = V$ a vector space over a field (a finite field, if one wants to remain in our setting). The field is incorporated into the language as a set of multiplication operators and the twists can be identified with the automorphism group of the field.

Natural questions raised in Cameron and Tarzi [2007] include the following.

(a) What is the group $\mathrm{Twist}(\Gamma)$?
(b) When does the extension $\mathrm{Aut}^*(\Gamma)$ split over $\mathrm{Aut}(\Gamma)$?

For example, the random graph has twisted automorphisms switching edges and non-edges, while the Henson graphs allow no non-trivial twists of the language. In the case of the random graph the extension does not split, since splitting would mean that there is an automorphism α of order two switching edges and non-edges. But then α would leave some pair (v, v^α) invariant, with v, v^α distinct.

Essentially the same argument applies to the random c-edge colored graph Γ_c for c even, so the extension does not split in this case. Cameron and Tarzi show conversely that if c is odd then the extension does split. A reasonable and suggestive way to formulate this result is that the following conditions are equivalent.

1. $\mathrm{Aut}(\Gamma_c^*)$ splits over $\mathrm{Aut}(\Gamma_c)$.
2. Every involution in $\mathrm{Sym}(c)$ lifts to an involution of $\mathrm{Aut}^*(\Gamma_c)$.
3. Every involution in $\mathrm{Sym}(c)$ has a fixed point.

The implications are immediate in the forward directions.

It is unclear whether, for example, the equivalence of the first two conditions is valid in binary homogeneous structures generally. It would seem unlikely.

It is of interest to consider, more generally, the generic c-colored k-hypergraph. Then the method of Cameron and Tarzi gives the equivalence of the following conditions.

1. $\mathrm{Aut}(\Gamma_c^*)$ splits over $\mathrm{Aut}(\Gamma)$.
2. Any subgroup of $\mathrm{Sym}(c)$ whose order divides k has a fixed point.

If k is a prime power then as before this may be expressed numerically as $\gcd(k, m) = 1$. In the general case the correct formulation in numerical terms is the following.

3. c is not a sum of non-trivial divisors of k.

Thus, for example, with $c = 5$ and $k = 6$, while $(c, k) = 1$, this condition is not satisfied,

Another equivalent condition is the following.

4. Each twist of the language lifts to a twisted automorphism of the same order.

While this condition may look more natural than our condition (2), in the context of the proof of equivalence this formulation is less natural.

Non-trivial twisted automorphisms occur in metrically homogeneous graphs as well. In fact, in this context it is also useful to classify *twisted isomorphisms* between metrically homogeneous graphs. Any homogeneous structure will remain homogeneous if the language is permuted, so from that point of view twisted automorphisms are the rule rather than thee exception. However, a metrically homogeneous graph will usually not remain metrically homogeneous under a twist (that is, after relabeling the distances, it is unlikely that the triangle inequality will continue to hold). So twisted isomorphisms between metrically homogeneous graphs should be rare.

Indeed, Rebecca Coulson identified all possible twisted isomorphisms between metrically homogeneous graphs in her thesis (Coulson [2019]), arriving at a very special and also familiar list of possibilities: namely, a list found in connection with an analogous problem for finite association schemes in Bannai and Bannai [1980], by very different methods.[21] In Coulson's work it was not necessary to restrict to the *known* metrically homogeneous graphs, as the existing theory gives sufficient information, at the cost of some additional work and with some additional refinements.

Returning to twisted automorphisms, Coulson's result states that a non-trivial twist associated with a twisted automorphism will have

[21]One could look for a common generalization in the distance transitive case, without assuming finiteness. Neither of the existing approaches seems adequate in that setting.

order 2, so the distinction between the lifting problem for involutions and the splitting problem vanishes.

All of the twists occurring in this case have many fixed points. A typical case would be the involution τ_δ which is defined as the natural *reflection* on $[1,3,\ldots,\ldots,\delta-2,\delta]$, and the *identity* on $[2,4,\ldots,\ldots,\delta-3,\delta-1]$. Thus fixed and non-fixed points alternate, except possibly near the midpoint of the interval. Indeed, with $\delta \equiv 2 \pmod 4$, there is a gap of length two between successive fixed points at the midpoint. In this particular case the twist τ_δ does not lift to an involutory automorphism of the associated metrically homogeneous graph Γ; but in all other cases it does. (There is another involution which is simply the extension of $\tau_{\delta-1}$ to $[\delta]$ fixing δ, and a very similar statement holds in that case.)

In short, once more the criterion for splitting the group $\mathrm{Aut}^*(\Gamma)$ is given in terms of the fixed points of involutory twists (specifically, those near the midpoint of the set of distances). The meaning of this is obscure and one woud like to know more.

At present, I believe the only source for the classification of the twists is the cited thesis by Coulson, and the results on splitting remain in unpublished notes.

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