TWO CARDINAL PROPERTIES OF HOMOGENEOUS GRAPHS

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Abstract. We analyze the two cardinal properties of definable sets in homogeneous graphs.

1. Introduction

A graph is *homogeneous* in Fraïssé's sense if any isomorphism between finite induced subgraphs extends to an automorphism [Fr, Ho]. The countable homogeneous graphs have been classified [LW], and the typical examples are the classical Rado graph, which is the graph on a countable set of vertices which is obtained up to isomorphism with probability one by choosing edges randomly and independently with probability 1/2, and the analogous "generic" K_n-free graph, the unique homogeneous countable graph containing no n-clique, and embedding every finite K_n-free graph as an induced subgraph. There are also some finite examples and some whose connected components are complete; furthermore, the complement of a homogeneous graph is also homogeneous.

J. Burdges and S. Warner raised the question of the *2-cardinal* properties of the Rado graph. Given a first order formula $\phi(x, \mathbf{y})$ and a structure $\mathcal{M}$, the *2-cardinal spectrum* of ϕ relative to $\mathcal{M}$, denoted $\mathrm{Spec}\,(\phi, \mathcal{M})$, is defined as the set of all pairs (κ, λ) of infinite cardinals such that:

> There is a structure $\mathcal{M}^* \cong \mathcal{M}$ of cardinality λ, and a choice of parameters $\mathbf{b}$ in $\mathcal{M}^*$, so that the set defined by $\phi(x, \mathbf{b})$ in $\mathcal{M}^*$ has cardinality κ.

Taking $\mathcal{M}$ to be the Rado graph or the generic K_n-free graphs, and the formula $\phi(x, \mathbf{y})$ to be the edge relation $E(x, y)$, or its complement $\neg E(x, y)$, we determine the 2-cardinal spectra explicitly:

Theorem 1. *If G is the Rado graph or the generic K_n-free graph, and if $\kappa \leq \lambda$ are infinite cardinals, then the following are equivalent:*
1. $\lambda \leq 2^\kappa$;

2. There is a graph G^* elementarily equivalent to G of cardinality λ, and a vertex $v \in V(G^*)$ for which $|\Delta(v)| = \kappa$;
3. There is a graph G^* elementarily equivalent to G of cardinality λ, and a vertex $v \in V(G^*)$ for which $|\Delta'(v)| = \kappa$.

Here $\Delta(v)$ denotes the set of neighbors of v in the graph G^*, and $\Delta'(v)$ is its complement.

The notion of elementary equivalence can be decoded into explicit graph theoretic language:

Fact 1.1. *1. A graph G is elementarily equivalent to the Rado graph if and only if it satisfies the following extension property P_k for each k:*
(P_k)For $C \subseteq V(G)$ with $|C| = k$ and for $C' \subseteq C$, there is $v \in V(G) \setminus C$ with $\Delta(v) \cap C = C'$

2. A K_n-free graph G is elementarily equivalent to the generic K_n-free graph if and only if it satisfies the following extension property P^n_k for each k:
(P^n_k)For $C \subseteq V(G)$ with $|C| = k$ and for $C' \subseteq C$, if the induced graph on C' is K_{n-1}-free, then there is $v \in V(G) \setminus C$ with $\Delta(v) \cap C = C'$

This is easily seen on the basis of the general theory [Ho]. On the basis of this theory, our theorem above, and the classification in [LW], one can determine the 2-cardinal spectra of arbitrary formulas in arbitrary homogeneous graphs.[1] On the other hand the 2-cardinal properties of homogeneous structures in general remain open, even in the case of binary relational languages.

The theorem will be proved by a construction which is based on a standard construction of 2^κ *independent subsets* of a set of cardinality κ [Ku, p. 288].

2. The construction

Proposition 2.1. *Let G be the Rado graph or the generic K_n-free graph with $n \geq 3$, and κ an infinite cardinal. Then there is a graph G^* elementarily equivalent to G with the following properties*
1. $V(G^*) = V_0 \cup V_1$, a disjoint union, with $|V_0| = \kappa$ and $|V_1| = 2^\kappa$;
2. V_1 is an independent set of vertices;
3. There is a vertex $v_* \in V_0$ with $|\Delta(v_*)| = 2^\kappa$.
4. For each vertex $v \in V(G^*)$ we have $|\Delta(v) \cap V_0| = |\Delta'(v) \cap V_0| = \kappa$.
5. For any set of vertices $V \subseteq V(G^*)$ containing V_0, the restriction of G^* to V is elementarily equivalent to G.

Before entering into the construction, let us check that this implies the main theorem.

[1] *Erraturm.* This is overstated. See [Ack].

Corollary 2.2. *Let G be the Rado graph or the generic K_n-free graph with $n \geq 3$, and κ an infinite cardinal. Then there are graphs G', G'' elementarily equivalent to G of cardinality 2^κ containing vertices v' and v'' respectively such that:*

$$|\Delta(v')| = \kappa; \quad |V(G'') \setminus \Delta(v'')| = \kappa$$

Proof. : We may take $G' = G^*$ as in the Proposition, with v' any vertex in V_1. For the graph G'' we take $v'' = v_* \in V(G^*)$ as in the Proposition, and let G'' be the subgraph of G^* induced on $V_0 \cup \Delta(v_*)$. □

Thus in the cases of interest to us we find that Spec(ϕ, G) contains $(\kappa, 2^\kappa)$ for all κ, and hence on general principles contains (κ, λ) for all $\kappa \leq \lambda \leq 2^\kappa$. The only other point that needs to be made is the following: if G is the Rado graph or the generic K_n-free graph, and if v is a vertex of G for which either $\Delta(v)$ or its complement has cardinality κ, then $|V(G)| \leq 2^\kappa$. Consider for example the case in which $|\Delta(v)| = \kappa$. Then for $w_1, w_2 \notin \Delta(v)$ we will have $\Delta(v) \cap \Delta(w_1) \neq \Delta(v) \cap \Delta(w_2)$, using the appropriate 3-extension property P_3 or P_3^n. Thus $|V(G) \setminus \Delta(v)| \leq 2^\kappa$.

Thus our main Theorem follows directly from the Proposition. For the proof of the Proposition we now make an explicit construction modeled on the method described in [Ku, p. 288]. We shall suppose that G is the generic K_n-free graph with $n \geq 3$; the case of the Rado graph is simpler.

Let I be the set of all functions with domain a finite subset of κ and with range contained in $\{0, 1\}$. Let V_0 be the set of finite subsets of I. Thus $|V_0| = \kappa$. Let $V_1 = 2^\kappa$, the set of all functions from κ to $\{0, 1\}$. Let $\mathcal{U}$ be the set of all quadruples of the form $U = (A, A', X, F)$ with $A \subseteq V_0$ finite, $A' \subseteq A$, $X \subseteq \kappa$ finite, and $F \subseteq 2^X$. Order V_0 with order type κ, and choose distinct vertices $v_U \in V_0$ with the following properties for $U \in \mathcal{U}$:

(i) There is a finite set $Y \supseteq X$ such that $v_U = \{f \in 2^Y : f \restriction X \in F\}$.

(ii) $v_U >$ sup A.

Now we impose an edge relation on $V_0 \cup V_1$ in which every edge involves at least one of the vertices v_U for some $U \in \mathcal{U}$. The definition is by induction on the well ordered set $\{v_U : U \in \mathcal{U}\}$, beginning with an empty edge relation. Let $v_U^< = \{v \in V_0 : v < v_U\}$. At each stage we will specify $\Delta(v_U) \cap (v_U^< \cup V_1)$.

For a fixed $U = (A, A', X, F)$, as sup $A < v_U$ the induced graph on $A \cup V_1$ is determined. If A' contains a clique of order $n - 1$ then take $\Delta(v_U) \cap (v_U^< \cup V_1) = \emptyset$. If A' is K_{n-1}-free then we take $\Delta(v_U) \cap v_U^< = A'$, and for $f \in V_1$, we take $f \in \Delta(v_U)$ if and only if

(a) $\exists Y\, f \restriction Y \in v_U$, or equivalently, $f \restriction X \in F$; and

(b) $\{f\} \cup A'$ contains no clique of order $n - 1$.

This completes the construction of the graph G^*. Let v_* be the first vertex in V_0 of the form v_U for some $U = (A, A', X, F) \in \mathcal{U}$. We may assume that the set F is nonempty. Then $v_*^< \cup V_1$ is an independent set of vertices, so $\Delta(v_*) \cap V_1$ is $\{f \in V_1 : f \restriction X \in F\}$, a set of cardinality 2^κ.

The rest of (1–4) is clear. It remains to check that the restriction of G^* to any set V with $V_0 \subseteq V \subseteq V(G^*)$ is elementarily equivalent to the specified graph G, or in other words that G^* is K_n-free and satisfies the extension properties P_k^n for all k, with witnesses to the extension properties taken in V_0.

If $C \subseteq V(G^*)$ is an n-clique and $v = \max(C \cap V_0)$, then $v = v_U$ for some $U \in \mathcal{U}$, say $U = (A, A', X, F)$, and then $(C \cap V_0) \setminus \{v_U\} \subseteq A'$, while $|C \cap V_1| \leq 1$. Thus the precautions taken in the construction will ensure that G^* is K_n-free.

Now suppose $C \subseteq V(G^*)$ is finite, $C' \subseteq C$, and C' contains no $(n-1)$-clique. Let $A = C \cap V_0$, $A' = C' \cap V_0$, $B = C \cap V_1$, and $B' = C' \cap V_1$, and choose $X \subseteq \kappa$ finite so that the functions $f \restriction X$ for $f \in B$ are distinct. Let $F = \{f \restriction X : f \in B'\}$ and $U = (A, A', X, F)$. Then $\Delta(v_U) \cap C = C'$, and $v_U \in V_0$. □

References

[Ack] N. L. Ackerman, "On n-cardinal spectra of ultrahomogeneous theories," ca. 2013, Prprint, `https://people.math.harvard.edu/~nate/papers/index.html`.

[Fr] R. Fraïssé, "Sur l'extension aux relations de quelques propriétés des ordres," *Ann. Ecole Norm. Sup.* **7** (1954), 361-388.

[Ho] W. Hodges, **Model Theory**, Cambridge University Press, Encylopedia of Mathematics and its Applications **42**, 1993.

[Ku] K. Kunen, **Set Theory: An Introduction to Independence Proofs**, Studies in Logic **102**, North Holland, Amsterdam, 1988.

[LW] A. Lachlan and R. Woodrow, "Countable ultrahomogeneous undirected graphs," *Trans. Amer. Math. Soc.* **262** (1980), 51-94.

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