Ramsey Expansions of 3-Hypertournaments

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Abstract. We study Ramsey expansions of certain homogeneous 3-hypertournaments. We show that they exhibit an interesting behaviour and, in one case, they seem not to submit to current gold-standard methods for obtaining Ramsey expansions. This makes these examples very interesting from the point of view of structural Ramsey theory as there is a large demand for novel examples.

Keywords: Homogeneous hypertournaments · Ramsey property

Structural Ramsey theory studies which homogeneous structures have the so-called Ramsey property, or at least are not far from it (can be *expanded* by some relations to obtain a structure with the Ramsey property). Recently, the area has stabilised with general methods and conditions from which almost all known Ramsey structures follow. In particular, the homogeneous structures offered by the classification programme are well-understood in most cases. Hence, there is a demand for new structures with interesting properties.

In this abstract we investigate Ramsey expansions of four homogeneous 4-constrained 3-hypertournaments identified by the first author [3] and show that they exhibit an interesting range of behaviours. In particular, for one of them the current techniques and methods cannot be directly applied. There is a big demand for such examples in the area, in part because they show the limitations of present techniques, in part because they might lead to a negative answer to the question whether every structure homogeneous in a finite relational language has a Ramsey expansion in a finite relational language, one of the central questions of the area asked in 2011 by Bodirsky, Pinsker and Tsankov [2].

1 Preliminaries

We adopt the standard notions of languages (in this abstract they will be relational only), structures and embeddings. A structure is *homogeneous* if every

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isomorphism between finite substructures extends to an automorphism. There is a correspondence between homogeneous structures and so-called *(strong) amalgamation classes* of finite structures, see e.g. [5]. A structure $\mathbf{A}$ is *irreducible* if every pair of vertices is part of a tuple in some relation of $\mathbf{A}$.

In this abstract, an *n-hypertournament* is a structure $\mathbf{A}$ in a language with a single n-ary relation R such that for every set $S \subseteq A$ with $|S| = n$ it holds that the automorphism group of the substructure induced on S by $\mathbf{A}$ is precisely $\mathrm{Alt}(S)$, the alternating group on S. This in particular means that exactly half of n-tuples of elements of S with no repeated occurrences are in $R^{\mathbf{A}}$. For $n = 2$ we get standard tournaments, for $n = 3$ this correspond to picking one of the two possible cyclic orientations on every triple of vertices. It should be noted however, that another widespread usage, going back at least to Assous [1], requires a unique instance of the relation to hold on each n-set. A *holey n-hypertournament* is a structure $\mathbf{A}$ with a single n-ary relation R such that all irreducible substructures of $\mathbf{A}$ are n-hypertournaments. A *hole* in $\mathbf{A}$ is a set of 3 vertices on which there are no relations at all.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be structures. We write $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ to denote the statement that for every 2-colouring of embeddings of $\mathbf{A}$ to $\mathbf{C}$, there is an embedding of $\mathbf{B}$ to $\mathbf{C}$ on which all embeddings of $\mathbf{A}$ have the same colour. A class $\mathcal{C}$ of finite structures has the *Ramsey property* (*is Ramsey*) if for every $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ there is $\mathbf{C} \in \mathcal{C}$ with $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ and $\mathcal{C}^+$ is a *Ramsey expansion* of $\mathcal{C}$ if it is Ramsey and can be obtained from $\mathcal{C}$ by adding some relations. By an observation of Nešetřil [8], every Ramsey class is an amalgamation class under some mild assumptions.

1.1 Homogeneous 4-Constrained 3-Hypertournaments

Suppose that $\mathbf{T} = (T, R)$ is a 3-hypertournament and pick an arbitrary linear order $\leq$ on T. One can define a 3-uniform hypergraph $\hat{\mathbf{T}}$ on the set T such that $\{a, b, c\}$ with $a \leq b \leq c$ is a hyperedge of $\hat{\mathbf{T}}$ if and only if $(a, b, c) \in R$. (Note that by the definition of a 3-hypertournament, it always holds that exactly one of (a, b, c) and (a, c, b) is in R.) This operation has an inverse and hence, after fixing a linear order, we can work with 3-uniform hypergraphs instead of 3-hypertournaments. There are three isomorphism types of 3-hypertournaments on 4 vertices:

- $\mathbf{H}_4$ The homogeneous 3-hypertournament on 4 vertices. For an arbitrary linear order $\leq$ on H_4, $\hat{\mathbf{H}}_4$ contains exactly two hyperedges. Moreover, they intersect in vertices $a < b$ such that there is exactly one $c \in H_4$ with $a < c < b$.
- $\mathbf{O}_4$ The odd 3-hypertournament on 4 vertices. For an arbitrary linear order $\leq$, $\hat{\mathbf{O}}_4$ will contain an odd number of hyperedges. Conversely, any ordered 3-uniform hypergraph on 4 vertices with an odd number of hyperedges will give rise to $\mathbf{O}_4$.
- $\mathbf{C}_4$ The cyclic 3-hypertournament on 4 vertices. There is a linear order $\leq$ on C_4 such that $\hat{\mathbf{C}}_4$ has all four hyperedges. In other linear orders, $\hat{\mathbf{C}}_4$ might have no hyperedges or exactly two which do not intersect as in $\mathbf{H}_4$.

We say that a class $\mathcal{C}$ of finite 3-hypertournaments is *4-constrained* if there is a non-empty subset $S \subseteq \{\mathbf{H}_4, \mathbf{O}_4, \mathbf{C}_4\}$ such that $\mathcal{C}$ contains precisely those finite 3-hypertournaments whose every substructure on four distinct vertices is isomorphic to a member of S. There are four 4-constrained classes of finite 3-hypertournaments which form a strong amalgamation class [3]. They correspond to the following sets S:

$S = \{\mathbf{C}_4\}$ The *cyclic* ones. These can be obtained by taking a finite cyclic order and orienting all triples according to it. Equivalently, they admit a linear order such that the corresponding hypergraph is complete.

$S = \{\mathbf{C}_4, \mathbf{H}_4\}$ The *even* ones. The corresponding hypergraphs satisfy the property that on every four vertices there are an even number of hyperedges.

$S = \{\mathbf{C}_4, \mathbf{O}_4\}$ The $\mathbf{H}_4$-*free* ones. Note that in some sense, this generalizes the class of finite linear orders: As $\mathrm{Aut}(\mathbf{H}_4) = \mathrm{Alt}(4)$, one can define $\mathbf{H}_n$ to be the $(n-1)$-hypertournament on n points such that $\mathrm{Aut}(\mathbf{H}_n) = \mathrm{Alt}(n)$. For $n = 3$, we get that $\mathbf{H}_3$ is the oriented cycle on 3 vertices and the class of all finite linear orders contains precisely those tournaments which omit $\mathbf{H}_3$.

$S = \{\mathbf{C}_4, \mathbf{O}_4, \mathbf{H}_4\}$ The class of all finite 3-hypertournaments.

2 Positive Ramsey Results

In this section we give Ramsey expansions for all above classes with the exception of the $\mathbf{H}_4$-free ones. Let $\mathcal{C}_c$ be the class of all finite cyclic 3-hypertournaments. Let $\overrightarrow{\mathcal{C}_c}$ be a class of finite linearly ordered 3-hypertournaments such that $(A, R, \leq) \in \mathcal{C}_c$ if and only if for every $x < y < z \in A$ we have $(x, y, z) \in R$. Notice that for every $(A, R) \in \mathcal{C}$ there are precisely $|A|$ orders $\leq$ such that $(A, R, \leq) \in \overrightarrow{\mathcal{C}_c}$ (after fixing a smallest point, the rest of the order is determined by R), and conversely, for every $(A, R, \leq) \in \overrightarrow{\mathcal{C}_c}$ we have that $(A, R) \in \mathcal{C}_c$.

It is a well-known fact that every Ramsey class consists of linearly ordered structures [7]. We have seen that after adding linear orders freely, the class of all finite ordered even 3-hypertournaments corresponds to the class of all finite ordered 3-uniform hypergraphs which induce an even number of hyperedges on every quadruple of vertices. These structures are called *two-graphs* and they are one of the reducts of the random graph (one can obtain a two-graph from a graph by putting hyperedges on triples of vertices which induce an even number of edges). Ramsey expansions of two-graphs have been discussed in [4] and the same ideas can be applied here.

Let $\overrightarrow{\mathcal{C}_e}$ consist of all finite structures $(A, \leq, E, R)$ such that $(A, \leq)$ is a linear order, (A, E) is a graph, (A, R) is a 3-hypertournament and for every $a, b, c \in A$ with $a < b < c$ we have that $(a, b, c) \in R$ if and only if there are an even number of edges (relation E) on $\{a, b, c\}$. Otherwise $(a, c, b) \in R$.

Theorem 1. *The 4-constrained classes of finite 3-hypertournaments with* $S \in \{\{\mathbf{C}_4\}, \{\mathbf{C}_4, \mathbf{H}_4\}, \{\mathbf{C}_4, \mathbf{O}_4, \mathbf{H}_4\}\}$ *all have a Ramsey expansion in a finite language. More concretely:*

1. *$\overrightarrow{\mathcal{C}_c}$ is Ramsey.*
2. *$\overrightarrow{\mathcal{C}_e}$ is Ramsey.*
3. *The class of all finite linearly ordered 3-hypertournaments is Ramsey.*

We remark that these expansions can be shown to have the so-called *expansion property* with respect to their base classes, which means that they are the optimal Ramsey expansions (see e.g. Definition 3.4 of [6]).

Proof. In $\overrightarrow{\mathcal{C}_c}$, R is definable from $\leq$ and we can simply use Ramsey's theorem. Similarly, in $\overrightarrow{\mathcal{C}_e}$, R is definable from $\leq$ and E, hence part 2 follows from the Ramsey property of the class of all ordered graphs [9].

To prove part 3, fix a pair of finite ordered 3-hypertournaments $\mathbf{A}$ and $\mathbf{B}$ and use the Nešetřil–Rödl theorem [9] to obtain a finite ordered holey 3-hypertournament $\mathbf{C}'$ such that $\mathbf{C}' \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. The holes in $\mathbf{C}'$ can then be filled in arbitrarily to obtain a linearly ordered 3-hypertournament $\mathbf{C}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

3 The $\mathbf{H_4}$-Free Case

Let $\mathbf{A} = (A, R)$ be a holey 3-hypertournament. We say that $\overline{\mathbf{A}} = (A, R')$ is a *completion* of $\mathbf{A}$ if $R \subseteq R'$ and $\overline{\mathbf{A}}$ is an $\mathbf{H}_4$-free 3-hypertournament. Most of the known Ramsey classes can be proved to be Ramsey by a result of Hubička and Nešetřil [6]. In order to apply the result for $\mathbf{H}_4$-free 3-hypertournaments, one needs a finite bound c such that whenever a holey 3-hypertournament has no completion, then it contains a substructure on at most c vertices with no completion. (Completions defined in [6] do not directly correspond to completions defined here. However, the definitions are equivalent for structures considered in this paper.) We prove the following.

Theorem 2. *There are arbitrarily large holey 3-hypertournaments* $\mathbf{B}$ *such that* $\mathbf{B}$ *has no completion but every proper substructure of* $\mathbf{B}$ *has a completion.*

This theorem implies that one cannot use [6] directly for $\mathbf{H}_4$-free hypertournaments. However, a situation like in Theorem 2 is not that uncommon. There are two common culprits for this, either the class contains orders (for example, failures of transitivity can be arbitrarily large in a holey version of posets) or it contains equivalences (again, failures of transitivity can be arbitrarily large). In the first case, there is a condition in [6] which promises the existence of a linear extension, and thus resolves the issue. For equivalences, one has to introduce explicit representatives for equivalence classes (this is called *elimination of imaginaries*) and unbounded obstacles to completion again disappear.

For $\mathbf{H}_4$-free hypertournaments neither of the two solutions seems to work. This means that something else is happening which needs to be understood in order to obtain a Ramsey expansion of $\mathbf{H}_4$-free tournaments. Hopefully, this would lead to new, even stronger, general techniques.

In the rest of the abstract we sketch a proof of Theorem 2.

Lemma 1.

1. *Let* $\mathbf{G} = (G, R)$ *be a holey 3-hypertournament with* $G = \{1, 2, 3, 4\}$ *such that* $(1, 3, 4) \in R$*,* $(1, 4, 2) \in R$ *and* $\{1, 2, 3\}$ *and* $\{2, 3, 4\}$ *are holes. Let* (G, R') *be a completion of* $\mathbf{G}$*. If* $(1, 2, 3) \in R'$*, then* $(2, 3, 4) \in R'$*.*
2. *Let* $\mathbf{G}^\neg = (G, R)$ *be a holey 3-hypertournament with* $G = \{1, 2, 3, 4\}$ *such that* $(2, 4, 3) \in R$*,* $(1, 4, 2) \in R$ *and* $\{1, 2, 3\}$ *and* $\{1, 3, 4\}$ *are holes. Let* (G, R') *be a completion of* $\mathbf{G}^\neg$*. If* $(1, 2, 3) \in R'$*, then* $(1, 3, 4) \notin R'$*.*

Proof. In the first case, suppose that $(1, 2, 3) \in R'$. If $(2, 4, 3) \in R'$, then (G, R') is isomorphic to $\mathbf{H}_4$. Hence $(2, 3, 4) \in R'$. The second case is proved similarly.

Suppose that $\mathbf{A} = (A, R)$ is a holey 3-hypertournament. For $x, y, z, w \in A$, we will write $xyz \Rightarrow yzw$ if the map $(1, 2, 3, 4) \mapsto (x, y, z, w)$ is an embedding $\mathbf{G} \to \mathbf{A}$ and we will write $xyz \Rightarrow \neg xzw$ if the map $(1, 2, 3, 4) \mapsto (x, y, z, w)$ is an embedding $\mathbf{G}^\neg \to \mathbf{A}$. Using the complement of $\mathbf{G}$, we can define $\neg xyz \Rightarrow \neg yzw$, and using the complement of $\mathbf{G}^\neg$ we can define $\neg xyz \Rightarrow xzw$. This notation can be chained as well, e.g. $xyz \Rightarrow yzw \Rightarrow zwu \Rightarrow \neg zuv$ means that all of $xyz \Rightarrow yzw$, $yzw \Rightarrow zwu$, $zwu \Rightarrow \neg zuv$ are satisfied.

Let $n \geq 6$. We denote by $\mathbf{O}_n = (O_n, R)$ the holey 3-hypertournament with vertex set $O_n = \{1, \ldots, n\}$ such that

$$123 \Rightarrow 234 \Rightarrow 345 \Rightarrow \cdots \Rightarrow (n-2)(n-1)n \Rightarrow \neg(n-2)n1 \Rightarrow \neg n12 \Rightarrow \neg 123.$$

All triples not covered by these conditions are holes.

Lemma 2.

1. *There is a completion* (O_n, R') *of* $\mathbf{O}_n$*.*
2. *If* (O_n, R') *is a completion of* $\mathbf{O}_n$*, then* $(1, 2, 3) \notin R'$*.*
3. *For every* $v \in O_n \backslash \{1, 2, 3\}$ *there is a completion* $(O_n \backslash \{v\}, R')$ *of the structure induced by* $\mathbf{O}_n$ *on* $O_n \backslash \{v\}$ *such that* $(1, 2, 3) \in R'$*.*

Proof. For part 1, observe that every set of four vertices of $\mathbf{O}_n$ with at least two different subsets of three vertices covered by a relation is isomorphic to $\mathbf{G}$, $\mathbf{G}^\neg$ or the complement of $\mathbf{G}$. It follows that whenever $x, y, z \in O_n$ is a hole such that $x < y < z$, we can put (x, z, y) and its cyclic rotations in R' to get a completion. Part 2 follows by induction on the conditions.

For part 3, we put $(1, 2, 3), (2, 3, 4), \ldots (v-3, v-2, v-1) \in R'$, $(v+1, v+3, v+2), \ldots, (n-2, n, n-1) \in R'$ and $(n-2, n, 1), (n, 1, 2) \in R'$. It can be verified that this does not create any copies of $\mathbf{H}_4$. A completion of (O_n, R') exists as the class of all finite $\mathbf{H}_4$-free tournaments has strong amalgamation.

Similarly, for $n \geq 6$, we define $\mathbf{O}_n^\neg = (O_n^\neg, R)$ the holey 3-hypertournament with vertex set $O_n^\neg = \{1, \ldots, n\}$ such that

$$\neg 123 \Rightarrow \neg 234 \Rightarrow \neg 345 \Rightarrow \cdots \Rightarrow \neg(n-2)(n-1)n \Rightarrow (n-2)n1 \Rightarrow n12 \Rightarrow 123$$

and there are no other relations in R. In any completion $(O_n^\neg, R')$ of $\mathbf{O}_n^\neg$ it holds that $(1, 2, 3) \in R'$, in fact, an analogue of Lemma 2 holds for $\mathbf{O}_n^\neg$.

Let $\mathbf{B}_n$ be the holey 3-hypertournament obtained by gluing a copy of $\mathbf{O}_n$ with a copy of $\mathbf{O}_n^-$, identifying vertices 1, 2 and 3. (This means that $\mathbf{B}_n$ has $2n-3$ vertices). We now use $\{\mathbf{B}_n : n \geq 6\}$ to prove Theorem 2.

Proof (of Theorem 2). Assume that (B_n, R') is a completion of $\mathbf{B}_n$. So in particular, it is a completion of the copies of $\mathbf{O}_n$ and $\mathbf{O}_n^-$. By Lemma 2 and its analogue for $\mathbf{O}_n^-$, we have that $(1,2,3) \notin R'$ and $(1,2,3) \in R'$, a contradiction.

Pick $v \in B_n$ and consider the structure $\mathbf{B}_n^v$ induced by $\mathbf{B}_n$ on $B_n \setminus \{v\}$. We prove that $\mathbf{B}_n^v$ has a completion. If $v \notin \{1,2,3\}$, one can use part 3 of Lemma 2 and its analogue for $\mathbf{O}_n^-$ to complete the copy of $\mathbf{O}_n$ and $\mathbf{O}_n^-$ (one of them missing a vertex) so that they agree on $\{1,2,3\}$. Using strong amalgamation, we get a completion of $\mathbf{B}_n^v$. If $v \in \{1,2,3\}$, we pick an arbitrary completion of $\mathbf{O}_n$ and $\mathbf{O}_n^-$, remove v from both of them, and let the completion of $\mathbf{B}_n$ to be the strong amalgamation of the completions over $\{1,2,3\} \setminus v$.

The following question remains open.

Question 1. What is the optimal Ramsey expansion for the class of all finite $\mathbf{H}_4$-free hypertournaments? Does it have a Ramsey expansion in a finite language?

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