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SEMISIMPLE TORSION IN GROUPS OF FINITE MORLEY RANK

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We prove several results about groups of finite Morley rank without unipotent p-torsion: p-torsion always occurs inside tori, Sylow p-subgroups are conjugate, and p is not the minimal prime divisor of our approximation to the "Weyl group". These results are quickly finding extensive applications within the classification project.

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0. Introduction

A group of finite Morley rank is a group equipped with a notion of dimension satisfying various natural axioms [8, p. 57]; these groups arise naturally in model theory, expecially geometrical stability theory. The main examples are algebraic groups over algebraically closed fields, where the notion of dimension is the usual one, as well as certain groups arising in applications of model theory to diophantine problems, where the notion of dimension comes from differential algebra rather than algebraic geometry. The dominant conjecture is that all such simple groups are algebraic.

Algebraicity Conjecture (Cherlin/Zilber). *A simple group of finite Morley rank is an algebraic group over an algebraically closed field.*

Much work towards this conjecture involves local analysis in an inductive setting reminiscent of the classification of the finite simple groups, but without transfer arguments or character theory.

Other methods have emerged recently in the study of groups of finite Morley rank, and have led to a number of advances. Among the characteristic features of this recent work are the systematic use of *generic covering* arguments, which will be met with below, as well as the study of divisible abelian p-subgroups (commonly known as *p-tori*), with which we will also be occupied here.

Such p-tori may always be viewed as *semisimple*. However, there are difficulties when one wishes to view individual p-elements as either semisimple or unipotent. For example, even a connected solvable p-group of a group of finite Morley rank is merely a central product, not necessarily a direct product, of a p-torus and a definable connected nilpotent p-subgroup of bounded exponent (commonly known as a *p-unipotent subgroup*). Elements in the intersection have an ambiguous character. Our main objective here is to obtain several results concerning that p-torsion in connected groups of finite Morley rank which is semisimple in a robust sense involving the absence of p-unipotent subgroups.

As our groups have elements of infinite order, we say $p^\perp$-group and $\pi^\perp$-group when the group has no p-torsion, or no p-torsion for any prime $p \in \pi$. Alternatively, a group G of finite Morley rank is said to have $p^\perp$ *type* if it contains no nontrivial unipotent p-subgroup, and similarly $\pi^\perp$ *type* if $p^\perp$ type for any prime $p \in \pi$. For the case $p = 2$, the connected $2^\perp$-groups of finite Morley rank are exactly the connected groups of *degenerate type* by [5], while the groups of finite Morley rank of $2^\perp$ type comprise both *odd and degenerate type*. The classification project now focuses exclusively upon simple groups of $2^\perp$ type because the Algebraicity Conjecture holds in the presence of a 2-unipotent subgroup [2]. The results of the present paper will have numerous applications to classification problems, beginning with the Generation Theorem of [7] — or strictly speaking, beginning with some earlier papers that could have been shortened had the results been available at the time.

The main results are Theorems 1–5 below. We expect each of them to find further use. The last three are intended to be less technical and more readily applicable than the first two, but they do not exhaust the information that can be extracted from the more technical results. Two of the results, stated as Theorems 2* and 3* below, are given in more general forms in the text.

The first section expands upon [10] and clarifies the nature of the generic element of G.

Theorem 1. *Let G be a connected group of finite Morley rank, p a prime, and let a be a generic element of G. Then*

(1) *the element a commutes with a unique maximal p-torus T_a of G,*
(2) *the definable hull $d(a)$ contains T_a, and*
(3) *if G has $p^\perp$ type then $d(a)$ is p-divisible.*

The next section contains a new genericity argument for cosets.

Theorem 2*. *Let G be a group of finite Morley rank, let a be a p-element in G such that $C_G(a)$ has $p^\perp$ type, and let T be a maximal p-torus of $C_G(a)$ (possibly* *trivial). Then*

$$\bigcup (aC_G^\circ(a,T))^{G^\circ} \text{ is generic in } aG^\circ.$$

These two technical results are the main ingredients in the following robust criterion for semi-simplicity.

Theorem 3*. *Let G be a connected group of finite Morley rank, p a prime, and a any p-element of G such that $C_G^\circ(a)$ has $p^\perp$ type. Then a belongs to a p-torus.*

Theorem 3* has an immediate consequence for the structure of Sylow p-subgroups, i.e. maximal solvable p-subgroups.

Corollary 3*. *Let G be a connected group of finite Morley rank of $p^\perp$ type, and T a maximal p-torus of G. Then any p-element of $C_G(T)$ belongs to T.*

Our fourth section further exploits the genericity argument for cosets to prove conjugacy of Sylow p-subgroups.

Theorem 4. *Let G be a group of finite Morley rank of $p^\perp$ type. Then all Sylow p-subgroups are conjugate.*

We note that the Sylow 2-subgroups are known to be conjugate in general groups of finite Morley rank [8, Theorem 10.11], while for general p, conjugacy is known in solvable groups of finite Morley rank [8, Theorem 9.35].

Our last topic concerns the so-called *Weyl group*, which for our present purposes may be defined as follows.

Definition. Let G be a group of finite Morley rank, and T a maximal divisible abelian torsion subgroup of G. The *Weyl group* of G is the group $N_G(T)/C_G^\circ(T)$.

The maximal divisible abelian torsion subgroups of G are conjugate by [10], so this group is well-defined up to conjugacy and in particular up to isomorphism. Furthermore, it is finite since $N_G^\circ(T) = C_G^\circ(T)$ [8, Theorem 6.16].

Theorem 5. *Let G be a connected group of finite Morley rank. Suppose the Weyl group is nontrivial and has odd order, with r the smallest prime divisor of its order. Then G contains a unipotent r-subgroup.*

In particular, a connected group G of finite Morley rank of degenerate type with a nontrivial Weyl group will contain unipotent torsion.

All of these results will be needed in [7]. The Torality Theorem (Theorem 3) should be quite useful subsequently in the analysis of particular configurations associated with classification problems in odd type groups. Indeed, the corollary to Theorem 3 is also given for $p = 2$ in [6], where it is applied to the study of generically multiply transitive permutation groups.

Outside material will be introduced as needed, but much of this occurs already in the first section. Any facts used without explicit mention can be found in [8].

1. Generic *p*-Divisibility

We begin by analyzing the generic element of a connected group of finite Morley rank. We use the notation $d(a)$ for the definable hull of an element a, defined to be the intersection of all definable subgroups containing a. The definable hull of a divisible abelian torsion subgroup of G is called a *decent torus*.

Theorem 1. *Let G be a connected group of finite Morley rank, p a prime, and let a be a generic element of G. Then*

(1) *the element a commutes with a unique maximal decent torus T_a of G,*
(2) *the definable hull $d(a)$ contains T_a, and*
(3) *if G has $p^\perp$ type then $d(a)$ is p-divisible.*

Here we consider only elements a whose *type* over $\emptyset$ is generic. When G is of $p^\perp$ type there need not be any generic definable *set* X such that $d(a)$ is p-divisible for every element of X. Indeed, with G an algebraic torus in characteristic other than p, that stronger claim fails. In this case the generic element has infinite order, but every infinite definable set contains p-elements of finite order, and for such elements $d(a) = \langle a \rangle$.

The idea of the proof is to replace the group G by the centralizer of one of its maximal decent tori. This depends on the main result of [10], which is closely related to point (1) above.

Fact 1.1 [10]. *Let G be a group of finite Morley rank. Then all maximal decent tori of G are conjugate. Furthermore, if T is a maximal decent torus of G, then there is a generic subset X of the group $C_G^\circ(T)$ such that*

(1) *$X \cap C_G^\circ(T)^g = \emptyset$ for $g \notin N_G(T)$, and*
(2) *$\bigcup X^G$ is generic in G.*

In particular Fact 1.1 states that any element of the generic definable set $\bigcup X^G$ commutes with a unique conjugate of T, or in other words with a unique maximal decent torus of G. So we have our first point:

Lemma 1.2. *Let G be a connected group of finite Morley rank, and let $a \in G$ be generic over $\emptyset$. Then $C_G^\circ(a)$ contains a unique maximal decent torus T_a of G.*

Our next point is that T_a is contained in $d(a)$, and at this point we must work not with the generic set X, but with the type of a itself generic. In this situation there are two notions of genericity which are relevant: genericity in the group G, and genericity in the subgroup $C_G^\circ(T_a)$, but again by Fact 1.1 these two notions can be correlated. Indeed, the next result is a direct reformulation of Fact 1.1.

Lemma 1.3. *Let G be a connected group of finite Morley rank, and T_0 a maximal decent torus. Then an element $a \in C_G^\circ(T_0)$ is generic over $\emptyset$ in G if and only if the following hold.*

(1) *T_0 is generic over $\emptyset$, in the set of maximal decent tori;*
(2) *The element a is generic in the group $C_G^\circ(T_0)$ over the canonical parameter for T_0.*

A word on terminology: as a definable set, T_0 can be viewed as an "imaginary element" of G, and the canonical parameter for T_0 is simply this element. As this may be identified with T_0 itself, one may speak of genericity "over T_0". The natural language for discussing the group $C_G^\circ(T_0)$ treats this parameter as a distinguished constant; it is interdefinable with $C_G^\circ(T_0)$.

Proof. Suppose first that a is generic. Then $T_0 = T_a$.

We show that T_0 is generic over $\emptyset$. If T_0 belongs to a $\emptyset$-definable family $\mathcal{T}$ in $G^{\rm eq}$ (a uniformly definable family in G) then a belongs to the set $U := \bigcup_{T \in \mathcal{T}} C_G^\circ(T)$. If the family $\mathcal{T}$ is nongeneric in the set of maximal decent tori, then U is nongeneric in G, a contradiction — the failure of genericity is immediate by a rank calculation.

Now we show a is generic over the canonical parameter t_0 for T_0, in $C_G(T_0)$. If a belongs to some nongeneric set $Y_{t_0} \subseteq C_G^\circ(T_0)$, where Y_{t_0} is defined over t_0, then a belongs to the $\emptyset$-definable nongeneric set $\bigcup_{T_t \in \mathcal{T}} Y_t$, where again the nongenericity follows by a direct rank calculation.

Now suppose T_0 is generic over $\emptyset$ and $a \in C_G^\circ(T_0)$ is generic over the parameter t_0. Suppose that a belongs to the $\emptyset$-definable subset Y of G. Let $Y_0 = Y \cap C_G^\circ(T_0)$. As $a \in Y_0$, the set Y_0 is generic in $C_G^\circ(T_0)$. The set $\mathcal{T}$ of conjugates T of T_0 for which $Y \cap C_G^\circ(T)$ is generic in $C_G^\circ(T)$ is $\emptyset$-definable and contains T_0, and hence $\mathcal{T}$ is generic in the set of conjugates of T_0. It then follows from Fact 1.1(2) that Y is generic in G. □

We will also need some general properties of definable quotients.

Lemma 1.4. *Let G be a group of finite Morley rank, $A \subseteq G$, H a normal A-definable subgroup of G, and $\bar{G} := G/H$.*

(1) *If an element $a \in G$ is generic over A then its image $\bar{a}$ in $\bar{G}$ is generic over A.*
(2) *If T is a maximal decent torus of G, and H is solvable, then the image $\bar{T}$ of T in $\bar{G}$ is a maximal decent torus of $\bar{G}$.*

The first point has already occurred in a special form above, and is also contained in [11, Lemme 6.2].

Proof. *Ad 1.* Suppose $\bar{V}$ is an A-definable subset of $\bar{G}$ containing $\bar{a}$, with preimage V in G. As V contains a it is generic. But $\mathrm{rk}(V) = \mathrm{rk}(\bar{V}) + \mathrm{rk}(H)$ and thus $\bar{V}$ is generic in $\bar{G}$. Thus $\bar{a}$ is generic in $\bar{G}$.

Ad 2. Let T_p be the p-torsion subgroup of T. It suffices to show that $\bar{T}_p$ is a maximal p-torus of $\bar{G}$. Let S_p be the preimage in G of a maximal p-torus $\bar{S}_p$ of $\bar{G}$ containing $\bar{T}_p$. We may suppose that $G = d(S_p)$ and thus G is solvable. Now let P be a Sylow p-subgroup of G containing T_p. Then $\bar{P}$ is a Sylow p-subgroup of $\bar{G}$ by [4]. It now follows from conjugacy of Sylow p-subgroups that $\bar{P}$ contains a maximal p-torus of $\bar{G}$. But as P is a solvable p-group, $P^\circ = T_p * U_p$ with U_p unipotent [8, Corollary 6.20], so $\bar{T}_p$ is the maximal p-torus of $\bar{P}$, and hence is a maximal p-torus of $\bar{G}$. □

Lemma 1.5. *Let G be a connected group of finite Morley rank and $a \in G$ generic. Then $d(a)$ contains T_a.*

Proof. Treating the parameter T_a as a constant, and bearing in mind Lemma 1.3, we may suppose that a is a generic element of $C_G^\circ(T_a)$, and T_a is $\emptyset$-definable. Hence we may replace G by $C_G^\circ(T_a)$, assuming therefore that

G contains a unique maximal decent torus T, which is central in G.

Let T_1 be the definable hull of the torsion subgroup of $d(a) \cap T$. As T is taken to be $\emptyset$-definable, the torsion subgroup of T is contained in $\mathrm{acl}(\emptyset)$ and hence the definable set T_1, treated as another parameter, is also in $\mathrm{acl}(\emptyset)$. Thus $\bar{a}$ is generic in the quotient $\bar{G} := G/T_1$, and in this quotient $\bar{T} := T/T_1$ is a maximal decent torus. So replacing G by $\bar{G}$, we may suppose that $d(a) \cap T$ is torsion free. It suffices to show that $T = 1$.

By [8, Ex. 10, p. 93], $d(a) = A \oplus C$ is the direct sum of a divisible abelian group A and a finite cyclic group C. If $n = |C|$, then for any multiple N of n, we have $d(a^N) = A$. On the other hand, for any torsion element $t \in T$, the element $a' = at$ is also generic over $\emptyset$ and hence a' and a realize the same type. Letting N be a multiple of n and the order of t, it follows that $d((a')^n) = d((a')^N) = d(a^N) = d(a^n)$ and thus $t^n \in d(a^n) \le d(a)$. Now by our reductions $d(a)$ contains no p-torus for any p, and hence the torsion part of $d(a)$ has bounded exponent. Thus t^n has bounded exponent, with t varying and n fixed, and so $T = 1$ as claimed. □

For the final point in Theorem 1 we prepare the following, which is a minor variation on Fact 1.7 below.

Lemma 1.6. *Let G be a connected group of finite Morley rank, p a prime, and T a maximal p-torus of G. Suppose that T is central in G and a is a p-element of G not in T. Then $C_G^\circ(a)$ contains a nontrivial p-unipotent subgroup. In particular, if G is of $p^\perp$ type then all p-elements in G belong to T.*

Here we employ one of the main results of [5].

Fact 1.7 [5, Theorem 4]. *Let G be a connected group of finite Morley rank, and let $a \in G$ be a nontrivial p-element. Then $C_G(a)$ contains an infinite abelian p-subgroup.*

Proof of Lemma 1.6. Observe first that the p-torsion subgroup of $d(T)$ is T, and thus $a \notin d(T)$. Now passing to a quotient as in the previous argument we may suppose that $T = 1$ and G contains no p-torus. So $C_G^\circ(a)$ contains a nontrivial p-unipotent subgroup by Fact 1.7. □

We turn to the last point in Theorem 1.

Lemma 1.8. *Let G be a connected group of finite Morley rank of $p^\perp$ type, and $a \in G$ a generic element. Then $d(a)$ is p-divisible.*

Proof. As we have seen above, we may suppose that T_a is central in G and $\emptyset$-definable. The group $d(a)$ is an abelian group of finite Morley rank, and hence has the form $A \oplus C$ for some p-divisible abelian group A, and some p-group C of bounded exponent by [8, Ex. 10, p. 93]. Since G is of $p^\perp$ type, $C \leq T_a$ by Lemma 1.6. As $T_a \leq d(a)$ by Lemma 1.5, $T_a \leq A$ and $d(a) = A$ is p-divisible. □

Now Theorem 1 is contained in Lemmas 1.2, 1.5, and 1.8.

2. Coset Genericity

In this section, we prove a *generic covering theorem*. Theorems of this type have played an increasing role in the analysis of connected groups of finite Morley rank. Our aim here is to show that for a p-element a of a group G of $p^\perp$ type, the union of the conjugates of $C_G^\circ(a)$ is generic in G. This improves on the analysis carried out in [5] for groups of p-degenerate type. In order to prove this, we need to sharpen it substantially and identify a subgroup of $C_G^\circ(a)$ actually responsible for the genericity. The precise result we aim at is the following, which generalizes the result in several directions, notably by allowing the element a to lie outside the connected component of G.

We formulate this analysis using a more general set of primes π, as opposed to the single prime p used in the introductory statement. A π-torus is a divisible abelian π-group. Similarly $\pi^\perp$ type means $p^\perp$ type for all $p \in \pi$.

Theorem 2. *Let G be a group of finite Morley rank, let a be a π-element in G such that $C_G(a)$ has $\pi^\perp$ type, and let T be a maximal π-torus of $C_G(a)$ (possibly trivial). Then*

$$\bigcup (aC_G^\circ(a,T))^{G^\circ} \quad \textit{is generic in } aG^\circ.$$

In particular

$$\bigcup (aC_G^\circ(a)^{G^\circ}) \quad \textit{is generic in } aG^\circ.$$

Generic covering theorems have involved definable subgroups more often than cosets. The following covering lemma, given in [5], is well adapted to the case of cosets.

Fact 2.1 [5, Lemma 4.1]. *Let G be a group of finite Morley rank, H a definable subgroup of G, and X a definable subset of G. Suppose that*

$$\mathrm{rk}\left(X \Bigg\backslash \bigcup_{g \notin H} X^g\right) \geq \mathrm{rk}(H).$$

Then $\mathrm{rk}(\bigcup X^G) = \mathrm{rk}(G)$.

The following property of generic subsets of cosets is very well known for subgroups, but occurs more rarely in its general form.

Lemma 2.2. *Let G be a group of finite Morley rank, H a connected definable subgroup, and X a definable generic subset of the coset aH. Then $\langle X \rangle = \langle aH \rangle = \langle a, H \rangle$.*

Proof. The second equality is purely algebraic, and clear. For the first, an application of genericity and connectedness shows that $H \leq \langle X \rangle$, and thus $aH \subseteq \langle X \rangle$. □

Proof of Theorem 2. We will use the notation $N_G(X)$ here for arbitrary subsets of G, not just subgroups, with its usual meaning: the setwise stabilizer of X under the action of G by conjugation.

Let $\mathcal{T}$ be the set of maximal π-tori of $C_G^\circ(a)$. We observe first that $\mathcal{T}$ may be identified with a definable set in G^{eq}. Indeed, it follows from the conjugacy of maximal decent tori that maximal π-tori are conjugate under the action of the group

$$G_a = C_G^\circ(a)$$

so $\mathcal{T}$ corresponds naturally to the right coset space $N_{G_a}(T)\backslash G_a$ for any $T \in \mathcal{T}$, and $N_{G_a}(T) = N_{G_a}(d(T))$ is definable. As the elements of $\mathcal{T}$ are not necessarily definable themselves, this identification should be used with circumspection.

As the maximal π-tori of $C_G(a)$ are conjugate, we may suppose that the π-torus $T \in \mathcal{T}$ is chosen generic over a. Set

$$H := C_G^\circ(\langle a, T \rangle)$$

which enters the picture most naturally here as $C^\circ_{C_G^\circ(a)}(T)$. Then T is the unique maximal π-torus in H, and we aim to show that

$$\mathrm{rk}\left(\bigcup (aH)^{G^\circ}\right) = \mathrm{rk}(G^\circ).$$

Let $\hat{H}$ be the generic stabilizer of aH, defined as

$$\{g \in G : \mathrm{rk}((aH) \cap (aH)^g) = \mathrm{rk}(aH)\}.$$

This is a definable subgroup of G. We claim

$$\mathrm{rk}(\hat{H}) = \mathrm{rk}(H). \tag{2.1}$$

Since a is an element of finite order normalizing (even centralizing) H, the group $\langle a, H\rangle$ is definable, with $\langle a, H\rangle^\circ = H$. Applying the preceding lemma,

$$\hat{H} \le N_G(\langle a, H\rangle) \le N_G(\langle a, H\rangle^\circ) = N_G(H) \le N_G(T).$$

Thus $\hat{H}^\circ \le C_G(T)$.

We claim that any π-element u of $\langle a, H\rangle$ lies in the abelian group $\langle a, T\rangle$: indeed, the π-group $U = \langle u, a\rangle$ has the form $U_0\langle a\rangle$ with $U_0 = U \cap H$. We claim that $U_0 \le T$. For this, it suffices to show that any π-element $u' \in U_0$ belongs to T. But this holds by Lemma 1.6.

Therefore $\langle a, H\rangle$ contains only finitely many elements of the same order as a, and as $\hat{H}$ acts by conjugation on these elements, we have $\hat{H}^\circ \le C^\circ_G(a)$ and thus $\hat{H}^\circ \le C^\circ_G(\langle a, T\rangle) = H$. So (2.1) holds.

We would like to apply the generic covering lemma, Fact 2.1, with $X = aH$ and with H (in the lemma) equal to $\hat{H}$ (here). For this, it suffices to verify the condition

$$\mathrm{rk}\left(aH \Big\backslash \bigcup_{g \notin \hat{H}} (aH)^g\right) = \mathrm{rk}(H). \tag{$*$}$$

Now suppose $x \in aH$ is generic over the parameters a and T (really, $d(T)$). We claim that both

$$x \text{ centralizes a unique maximal } \pi\text{-torus of } C^\circ_G(a), \text{ namely } T, \text{ and} \tag{2.2}$$

$$a \in d(x) \tag{2.3}$$

Clearly $x = ah$ with $h \in H$ generic over a and T. Since T is itself generic over a, h realizes the type of a generic element of $C^\circ_G(a)$ over a (Lemma 1.3). By Theorem 1(1) for $p \in \pi$, h centralizes a unique maximal p-torus of $C^\circ_G(a)$ for $p \in \pi$, and hence so does x. So (2.2) follows.

As $C^\circ_G(a)$ has $\pi^\perp$ type, $d(h)$ is π-divisible by Theorem 1(3). So, for any π-number q, the quotient $d(h)/d(h^q)$ is a q-divisible group of exponent at most q, and hence trivial: $d(h) = d(h^q)$. Let q be the order of a. Then $d(x^q) = d(a^q h^q) = d(h^q) = d(h)$. So $h \in d(x)$, and hence also $a \in d(x)$, giving (2.3).

If $(*)$ fails, then $aH \cap \bigcup_{g \notin \hat{H}} (aH)^g$ is generic in aH, so, as $x \in aH$ is generic over the parameters a and T, we have $x^g \in aH$ for some $g \notin \hat{H}$. As $x, x^g \in aH$, $d(x)$ and $d(x^g)$ both commute with T, and therefore $d(x)$ also commutes with $T^{g^{-1}}$. Since $a \in d(x)$, we have $T^{g^{-1}} \le C^\circ_G(a)$. By (2.2) it follows that $T = T^{g^{-1}}$, that is $g \in N_G(T)$.

Again, since $a \in d(x)$, the element $a^g \in d(x^g)$ lies inside $\langle a, H\rangle$. Of course $a^g \in \langle a, T\rangle$ since a has order q. Since $g \in N_G(T)$ this gives $g \in N_G(\langle a, T\rangle)$. So g normalizes $H = C^\circ_G(\langle a, T\rangle)$ as well. Therefore $(aH)^g = (xH)^g = x^g H^g = x^g H = aH$, and $g \in \hat{H}$, a contradiction. So $(*)$ holds, and our result follows by Fact 2.1.

□

3. Torality

We now prove the main result of the paper. Again, we formulate this in a technical form slightly more general than the original statement, using a set of primes π.

Theorem 3. *Let G be a connected group of finite Morley rank, π a set of primes, and a any π-element of G such that $C_G^\circ(a)$ has $\pi^\perp$ type. Then a belongs to a π-torus.*

Theorem 3 has the following direct corollary.

Corollary 3.1. *Let G be a connected group of finite Morley rank with a π-element a such that $C_G(a)$ has $\pi^\perp$ type. Then a belongs to any maximal π-torus of $C_G(a)$.*

Proof. By Theorem 3, there is a maximal π-torus T containing a. By Fact 1.1, any maximal π-torus in $C_G(a)$ is $C_G^\circ(a)$-conjugate to T, and so contains a. □

This imposes very strong restrictions on a simple group G of finite Morley rank. For $\pi = \{2\}$, the outstanding structural problems concern groups of $2^\perp$ type (i.e. odd or degenerate type). In this context, our results impose constraints on the structure of a Sylow 2-subgroup, which will be developed in [7].

For the proof, we use the following variation on Fact 1.7 [5, Theorem 4]. This lemma is due to Tuna Altinel.

Lemma 3.2 (Altinel). *Let G be a connected group of finite Morley rank, and let $a \in G$ be a nontrivial π-element. Then $C_G(a)$ contains an infinite abelian p-subgroup for some $p \in \pi$.*

For the proof, we require the following.

Fact 3.3 ([3]; [9, Fact 3.2]). *Let $G = H \rtimes T$ be a group of finite Morley rank with H and T definable. Suppose T is a solvable π-group of bounded exponent and $Q \lhd H$ is a definable solvable T-invariant $\pi^\perp$-subgroup. Then*

$$C_H(T)Q/Q = C_{H/Q}(T).$$

Proof of Lemma 3.2. We may take G to be a minimal counterexample. So in particular $C_G^\circ(a)$ is a $\pi^\perp$-group by Fact 1.7. Of course, G does contains an infinite abelian p-group for some $p \in \pi$ by Fact 1.7. So clearly $a \notin Z(G)$.

As $Z^\circ(G)$ has no π-torsion, $C_{G/Z^\circ(G)}(a) = C_G(a)/Z^\circ(G)$ by Fact 3.3. So $C_{G/Z^\circ(G)}^\circ(a)$ has no π-torsion by [8, Ex. 11, p. 93 or Ex. 13c, p. 72]. Thus $Z^\circ(G) = 1$ by minimality of G.

We now show that $a \in d(x) \cap aC_G^\circ(a)$ for any $x \in aC_G^\circ(a)$. Let $K := d(x) \cap C_G^\circ(a)$. So x is a π-element in $d(x)/K$. By [8, Ex. 11, p. 93], $xd^\circ(x)$ contains a π-element b. As $C_G^\circ(a)$ is a $\pi^\perp$-group, a is the unique π-element in $aC_G^\circ(a) \supseteq xK$. Thus $a = b \in d(x)$, as desired.

By Theorem 2, $\bigcup(aC_G^\circ(a))^G$ is generic in G. We show that G has no divisible torsion. Otherwise, choose a maximal decent torus T of G. By Fact 1.1, $\bigcup C_G^\circ(T)^G$

is generic in G too, and hence meets $aC_G^\circ(a)$ in an element x. So $a \in d(x)$ lies inside some $C_G^\circ(T)^g$ with $g \in G$. But $C_{C_G^\circ(T)^g}(a)$ is still a $\pi^\perp$-group, contradicting the minimality of G.

As $C_G^\circ(a^{-1}) = C_G^\circ(a)$, $\bigcup(a^{-1}C_G^\circ(a))^G$ is also generic in G, by Theorem 2. So there is some $x \in a^{-1}C_G^\circ(a) \cap (aC_G^\circ(a))^g$ for some $g \in G$. As above a^g and a^{-1} are the only π-elements in $(aC_G^\circ(a))^g$ and $a^{-1}C_G^\circ(a)$, respectively. So $a^g = a^{-1}$. It follows that G has an involution in $d(g)$.

We recall that $B(G)$ is the subgroup of G generated by all its 2-unipotent subgroups. As G has no divisible torsion, G has even type by [5], but has no algebraic simple section. So $B(G)$ is a 2-unipotent subgroup normal in G, by the Even Type Theorem [2, Main Theorem and Proposition II 6.2]. Now $Z^\circ(B(G)) \neq 1$ by [8, Lemma 6.2].

Since $a^g = a^{-1}$, there is a 2-element u in $d(g)$ such that $a^u = a^{-1}$. As $G/B(G)$ is 2-torsion-free by [5], we find $u \in B(G)$ and $a^{-2} = [a, u] \in B(G)$. Then a is a 2-element, so a belongs to $B(G)$ and 2 belongs to π, contradicting that $C_G^\circ(a) \geq Z^\circ(B(G))$ is a $\pi^\perp$-group. □

Proof of Theorem 3. We may suppose that a is nontrivial. By Lemma 3.2, there is a non-trivial π-torus T of $C_G^\circ(a)$, which we take maximal in $C_G^\circ(a)$. Set $H := C_G^\circ(a, T)$. By Theorem 2, the set $\bigcup(aH)^G$ is generic in G. So after conjugating we may suppose that some $x = ah \in aH$ is generic in G. We claim that $a \in d(x)$.

Since x is generic in G, $C_G(x)$ contains a unique maximal π-torus S of G, which lies inside $d(x)$, by Theorem 1. Clearly $T \leq S$ since $T \leq C_G(x)$. The definable hull $d(x)$ contains a π-element x' with $x'H = xH = aH$. So $x'a^{-1} \in H$ is a π-element. Now H is connected, T is a maximal π-torus of H, T is central in H, and H has $\pi^\perp$-type. So decomposing $x'a^{-1}$ into a product of p-elements in H, it follows from Lemma 1.6 that $x'a^{-1} \in T \leq S \leq d(x)$. Hence $a \in d(x)$, as claimed.

Again since x is generic in G, we have $x \in C_G^\circ(S)$ by Fact 1.1, and hence $a \in C_G^\circ(S)$. If $a \notin S$, then the image $\bar{a}$ of a in $\bar{C} = C_G^\circ(S)/d(S)$ is nontrivial. By Lemma 3.2, there is an infinite abelian p-subgroup $\bar{U}$ in $C_{\bar{C}}(\bar{a})$, for some $p \in \pi$, and $\bar{U}$ must be p-unipotent. It follows that there is a p-unipotent subgroup U of $C_G^\circ(S)$ with $[a, U] \leq d(S)$. Now a normalizes U because a normalizes the central product $Ud(S)$. It follows that $[a, U]$ is a connected subgroup of $U \cap d(S)$, which is finite, so $U \leq C(a)$, a contradiction. □

For applications to the structure of Sylow p-subgroups in connected groups of $p^\perp$ type and low Prüfer p-rank, especially Prüfer rank 1, see [7].

4. Conjugacy of Sylow p-Subgroups

We define *Sylow p-subgroups* of a group G of finite Morley rank as maximal solvable p-subgroups. One arrives at the same class of subgroups by imposing local finiteness or local nilpotence in place of solvability [8, Sec. 6.4]. If S is a Sylow p-subgroup of G then S° will be a central product of a p-unipotent subgroup and a p-torus, and in particular S° is nilpotent. So if S is a Sylow p-subgroup and X a proper subgroup of S, then $N_S(X) > X$.

Our goal in the present section is the following.

Theorem 4. *Let G be a group of finite Morley rank of $p^\perp$ type. Then all Sylow p-subgroups are conjugate.*

The conjugacy result is also known for solvable groups, as a special case of the theory of Hall subgroups [8, Theorem 9.35] and for arbitrary groups of finite Morley rank when $p = 2$ [8, Theorem 10.11].

As an immediate consequence we can strengthen [5, Theorem 3].

Corollary 4.1. *Let G be a connected group of finite Morley rank and $p^\perp$ type. If some Sylow p-subgroup of G is finite then G contains no elements of order p.*

The critical case for the proof is the case in which at least one Sylow p-subgroup is finite; which proves the corollary itself. It also shows that Sylow p-subgroups are conjugate if all lie outside G°.

Lemma 4.2. *Suppose G is a group of finite Morley rank and $p^\perp$ type containing a finite Sylow p-subgroup P. Then all Sylow p-subgroups of G are conjugate.*

Proof. Consider a counterexample G of minimal Morley rank and degree. Thus Sylow p-subgroups are conjugate in proper definable subgroups of G.

Let $O_p(G)$ denote the subgroup of G generated by its solvable normal p-subgroups. Such a p-subgroup must be contained in P and thus $O_p(G) \leq P$ is finite, and is the largest finite normal p-subgroup of G. In $\bar{G} = G/O_p(G)$ we have $O_p(\bar{G}) = 1$ and $\bar{P} = P/O_p(G)$ is a finite Sylow p-subgroup of $\bar{G}$, and if we prove the claim for $\bar{G}$ it follows for G. So we may suppose

$$O_p(G) = 1. \tag{4.1}$$

Let D be a subgroup of P of maximal order subject to the condition: D is contained in a solvable p-subgroup of G which has no conjugate contained in P. Let R be such a p-subgroup. Let $D_1 = N_P(D)$, $D_2 = N_R(D)$. By the maximality of D, any p-Sylow subgroup P_1 of $N_G(D)$ containing D_1 is conjugate to a subgroup of P. Let R_1 be a Sylow p-subgroup of $N_G(D)$ containing D_2. If R_1 is conjugate to P_1, then R_1 is conjugate to a subgroup of P. In particular, D_2 is then conjugate to a subgroup of P and R is conjugate to a group meeting P in a subgroup of order greater than $|D|$. But this contradicts the choice of D.

It follows that in $N_G(D)$ we have nonconjugate Sylow p-subgroups, so by the minimality of G, we find $D \lhd G$ and thus $D \leq O_p(G) = 1$. Hence any solvable p-subgroup which meets P nontrivially is conjugate to a subgroup of P.

Fix $a \in P$ nontrivial. We claim

$$C_{G^\circ}(a) \text{ is a } p^\perp\text{-group.} \tag{4.2}$$

If this fails, take $x \in C_{G^\circ}(a)$ a nontrivial p-element. By Fact 1.7, $C_G(x)$ contains an infinite abelian p-subgroup A. As $O_p(G) = 1$, we have $C_G(x) < G$ and hence the Sylow p-subgroups of $C_G(x)$ are conjugate. Taking Sylow p-subgroups Q and R of $C_G(x)$ containing $\langle a, x\rangle$ and A respectively, we find that Q is conjugate to a subgroup of P since Q meets P nontrivially, and hence the infinite group R is conjugate to a subgroup of the finite group P, a contradiction.

Now let b be an arbitrary p-element of the coset aG°, and T_b a maximal p-torus of $C_G^\circ(b)$. Then $\bigcup(bC_G^\circ(b, T_b))^{G^\circ}$ is generic in aG° by Theorem 2. This applies in particular to a, with $T_a = 1$. As we have generic sets associated to a and b in the coset aG°, their intersection is nontrivial, giving

$$aC_G^\circ(a) \cap b'C_G^\circ(b', T_{b'}) \neq \emptyset \quad \text{for some conjugate } b' \text{ of } b. \tag{4.3}$$

Fix $h \in aC_G^\circ(a) \cap b'C_G^\circ(b', T_{b'})$. As $h \in aG^\circ$, there is a p-element $h' \in d(h) \cap aG^\circ$. Of course, $d(h)$ lies inside both $\langle a\rangle C_G^\circ(a)$ and $\langle b'\rangle C_G^\circ(b', T_{b'})$. As $\langle a\rangle \cap G^\circ = 1$ by (4.2), it follows that $d(h) \cap aG^\circ$ is contained in $aC_G^\circ(a)$. So $h' \in aC_G^\circ(a)$. We also deduce from (4.2) that a is the unique p-element in $aC_G^\circ(a)$, and then $h' = a$ since h' is a p-element. As $a \in d(h)$, we know a centralizes $T_{b'}$, and therefore $T_{b'} = 1$ by (4.2). Now $C_G^\circ(b')$ is also a $p^\perp$-group by Lemma 1.6. So a similar argument shows that $d(h) \cap aG^\circ$ is contained inside $b'C_G^\circ(b', T_{b'})$, and therefore $h' \in b'C^\circ(b')$, as $C_G^\circ(b')$ is now a $p^\perp$-group, b' is the unique p-element in $b'C_G^\circ(b')$, and $h' = b'$. We conclude that $a = b'$ and therefore

$$\text{For } a \in P^\#, \text{ any two } p\text{-elements in } aG^\circ \text{ are conjugate.} \tag{4.4}$$

Now fix an arbitrary Sylow p-subgroup Q of G. We will show that P and Q are conjugate.

Let $\bar{G} = G/G^\circ$ and let $\bar{R}$ be a Sylow p-subgroup of $\bar{G}$ containing $\bar{P}$. We may suppose after conjugating Q that $\bar{Q} \leq \bar{R}$. We claim

$$\bar{R} = \bar{P}. \tag{4.5}$$

Assuming the contrary, let R be the preimage in G of $\bar{R}$. We have $N_{\bar{R}}(\bar{P}) > \bar{P}$ and thus $N_R(PG^\circ) > PG^\circ$. As $PG^\circ < R \leq G$, Sylow p-subgroups of PG° are conjugate and therefore $N_R(PG^\circ) = G^\circ N_R(P)$ by a Frattini argument. Thus, $N_R(P)$ covers $N_{\bar{R}}(\bar{P})$ and therefore there is a p-element $x \in N_R(P) \setminus P$. But then P is not a Sylow p-subgroup, a contradiction. So (4.5) holds.

Hence $QG^\circ \leq PG^\circ$. If $QG^\circ < PG^\circ$, then Sylow p-subgroups of QG° are conjugate, and $N_{PG^\circ}(Q) \not\leq QG^\circ$ by a Frattini argument. In this situation, we similarly discover a p-element outside QG° that normalizes Q, which then contradicts Q being a Sylow p-subgroup of G. So therefore $QG^\circ = PG^\circ$. In particular there are $a \in Q^\#$ and $b \in P$ with $aG^\circ = bG^\circ$, and thus a, b are conjugate. Hence some conjugate of Q meets P, and as we have shown this conjugate of Q must itself be conjugate to P. □

Proof of Theorem 4. We have G a group of finite Morley rank of $p^\perp$ type and P_1, P_2 Sylow p-subgroups. We may suppose that Sylow p-subgroups in proper definable subgroups of G are conjugate, and we wish to prove the same for G.

Let T_1, T_2 be the maximal p-tori in P_1, P_2 respectively. We may suppose $T_1 \leq T_2$. If P_1 is finite the preceding lemma applies. So we may suppose T_1 is nontrivial.

If $N_G(T_1) < G$ then conjugacy holds in $N_G(T_1)$ and thus T_2 is conjugate to a subgroup of P_1. In this case $T_1 = T_2$, so $P_1, P_2 \leq N_G(T_1)$ and our claim follows.

So suppose $T_1 \lhd G$. Then passing to $\bar{G} = G/d(T_1)$, the image $\bar{P}_1$ of P_1 is finite. We claim that $\bar{P}_1$ is a Sylow p-subgroup of $\bar{G}$. Let $\bar{Q}_1$ be a solvable p-group containing $\bar{P}_1$, set $\bar{Q} = d(\bar{Q}_1)$, and let Q be the preimage in G of $\bar{Q}$. Then Q is solvable. By [4], $\bar{P}_1$ is a Sylow p-subgroup of $\bar{Q}$, and hence $\bar{P}_1 = \bar{Q}_1$. That is, $\bar{P}_1$ is a Sylow p-subgroup of $\bar{G}$.

By the previous lemma, $\bar{P}_1$ and $\bar{P}_2$ are conjugate, and we may suppose they are equal. Let $\bar{P} = d(\bar{P}_1)$ and let P be the preimage in G of $\bar{P}$. Then P is solvable and $P_1, P_2 \leq P$, so by [4] the groups P_1, P_2 are conjugate, as claimed. □

5. Weyl Groups

A suitable notion of "Weyl group" in the context of groups of finite Morley rank is the following. This definition is well-defined up to conjugacy in G, and finite.

Definition 5.1. Let G be a group of finite Morley rank. Then the *Weyl group* associated to G is the abstract group $W = N_G(T)/C_G^\circ(T)$ where T is a maximal decent torus.

In algebraic groups, Weyl groups are Coxeter groups, and in particular are generated by involutions. We note that, by a Frattini argument using Fact 1.1, the "Weyl group" associated to some non-maximal decent torus is a section of the Weyl group associated to a maximal decent torus.

Theorem 5. *Let G be a connected group of finite Morley rank. Suppose the Weyl group is nontrivial and has odd order, with r the smallest prime divisor of its order. Then G contains a unipotent r-subgroup.*

In fact, we prove that either

(H1) An r-element representing an element of order r in W centralizes a unipotent r-subgroup, or else

(H2) Some toral r-element centralizes a unipotent r-subgroup.

Corollary 5.2. *Let G be a minimal connected simple group of finite Morley rank. Suppose the Weyl group is nontrivial and has odd order, with r the smallest prime divisor of its order. Then* (H1) *holds in G.*

Proof. Otherwise there is an r-element x representing a Weyl group element of order r such that $C_G(x)$ has $r^\perp$ type. Then x is toral by Theorem 3. Of course (H2) holds by Theorem 5. So there is a toral r-element y such that $C_G(y)$ contains a unipotent r-subgroup, and thus $U_r(C_G(y)) \neq 1$ by solvability. We may assume that x and y belongs to the same r-torus by [10]. Then $x \in T \leq C_G^\circ(y)$ and $U_r(C_G(y)) \leq C_G(T) \leq C_G(x)$, a contradiction. □

Proof of Theorem 5. We consider a connected group G with maximal decent torus T, we let $W = N_G(T)/C_G^\circ(T)$ be the associated Weyl group, and we take $r = r_W$ the minimal prime divisor of $|W|$. We suppose toward a contradiction that (H1) and (H2) both fail in G, and subject to this, we first minimize the Morley rank of G, then minimize the order of $Z(G)$ if that group is finite. We will argue below that $Z(G) = 1$ using these reductions.

Let Z denote $Z(G)$ if this group is finite, and $Z^\circ(G)$ otherwise. Let $\bar{G} = G/Z$, and let $\bar{T}$ be the image of T in $\bar{G}$, a maximal decent torus of $\bar{G}$ by Lemma 1.4(2). Let $\overline{W} = N_{\bar{G}}(\bar{T})/C_{\bar{G}}^\circ(\bar{T})$ be the corresponding Weyl group, and $\bar{r} = r_{\overline{W}}$ the least prime dividing its order. We claim first that

$$\bar{r} = r. \tag{5.1}$$

The preimage of $\bar{T}$ in G is TZ, and T is the unique maximal decent torus of TZ, so $N_G(T)$ is the preimage of $N_{\bar{G}}(\bar{T})$. Therefore, $C_G^\circ(T) = N_G^\circ(T)$ maps onto $C_{\bar{G}}^\circ(\bar{T}) = N_{\bar{G}}^\circ(\bar{T})$, and the preimage of $C_{\bar{G}}^\circ(\bar{T})$ in $N_G(T)$ is $ZC_G^\circ(T)$. Thus, there is a natural surjection $W \to \overline{W}$ with kernel isomorphic to $Z/(Z \cap C_G^\circ(T))$. If $Z(G)$ is infinite, then as Z is connected this map is an isomorphism, and in particular $\bar{r} = r$. So suppose that $Z = Z(G)$ is finite. Then $\bar{r} \geq r$ and we must show that r divides $|\overline{W}|$.

Let Z_r denote the r-torsion subgroup of Z. We claim that $Z_r \leq C_G^\circ(T)$. If $z \in Z_r$ is not toral, then $C_G^\circ(z)$ contains unipotent torsion by Theorem 3. As (H1) fails in G, and z is central, we have $z \in C_G^\circ(T)$ in this case. On the other hand, if $z \in Z_r$ is toral, then as z is central we have $z \in T \leq C_G^\circ(T)$. Thus $Z_r \leq C_G^\circ(T)$. Therefore $|W|_r = |\overline{W}|_r$ and in particular $\bar{r} = r$. This proves (5.1).

Now we show

$$Z(G) = 1. \tag{5.2}$$

By the choice of G, if $Z(G) > 1$, then $\bar{G}$ satisfies (H1) or (H2) with respect to the same value of r. So there is an r-element x of G such that $C_{\bar{G}}(\bar{x})$ contains a nontrivial unipotent r-subgroup, with $\bar{x}$ either a representative for a nontrivial element of $\overline{W}$, or toral. Then there is a nontrivial unipotent r-subgroup U of $C_G(x \mod Z)$, in other words $[x, U] \leq Z$. Letting $U_0 = [x, U]$ if this is nontrivial, and $U_0 = U$ otherwise, we have a unipotent r-subgroup U_0 centralizing x.

If $\bar{x}$ represents a nontrivial element of $\overline{W}$, then x represents a nontrivial r-element of W, and (H1) holds in G, a contradiction. So $\bar{x}$ is toral, and we may suppose $\bar{x} \in \bar{T}$, that is $x \in TZ$. Then there is an r-element $x' \in xZ \cap T$, and again $U_0 \leq C_G(x')$, giving $(H2)$ in G, and a contradiction. Thus (5.2) holds.

Now fix an r-element $a \in N_G(T)$ representing a nontrivial element of W. We notice that there is no proper definable connected subgroup H of G containing $\langle a, C_G^\circ(T)\rangle$. Otherwise, the corresponding Weyl group $W_H = N_H(T)/C_H^\circ(T) = N_H(T)/C_G^\circ(T)$ would again have r as the minimal prime divisor of its order, and then this would violate the minimal choice of G.

Let T_r be the maximal r-torus of T, which is nontrivial by Theorem 3. We claim

$$C_{T_r}(a) \text{ is finite.} \tag{5.3}$$

Otherwise, let $T_1 = C_{T_r}^\circ(a) > 1$, and consider $H = C_G^\circ(T_1)$. Then $H < G$ since $Z(G) = 1$, and $C_G^\circ(T) \leq H$. Now $C_G^\circ(a)$ is of $r^\perp$ type since (H1) fails in G. So if T_2 is a maximal torus of $C_G^\circ(a)$ containing T_1, then $a \in T_2$ by Theorem 3. Hence $a \in H$, and we arrive at a contradiction. This proves (5.3).

It will follow that T_r acts transitively on the coset aT_r:

$$a^{T_r} = aT_r. \tag{5.4}$$

We consider the endomorphism of T_r given by commutation with a. By (5.3) this has finite kernel, so it is surjective [8, Ex. 9, p. 93]. Therefore $[a, T_r] = T_r$, and multiplication by a on the left gives (5.4).

As $\Omega_1(T_r)\cdot\langle a\rangle$ is a finite r-group, we have $C_{T_r}(a) > 1$. Fix $b \in C_{T_r}(a)$ an element of order r. Our goal is to show that b and b^2 are conjugate under the action of W, using (5.4) as well as a variant with the roles of a and b reversed. From this it will follow quickly that r is not in fact minimal, to conclude the argument. To begin with, we have a conjugate to ab by (5.4). We next concern ourselves with reversing the roles of a and b.

We claim

$$a \notin C_G^\circ(b) \text{ and } b \notin C_G^\circ(a). \tag{5.5}$$

Consider the group $H = C_G^\circ(b)$. We have $H < G$ since $Z(G) = 1$, and $C_G^\circ(T) \leq H$. Thus $a \notin H = C_G^\circ(b)$.

Suppose $b \in C_G^\circ(a)$. By the failure of (H2) in G, $C_G^\circ(b)$ is a group of $r^\perp$ type. By Theorem 3, b belongs to a torus of $C_G^\circ(a)$. Let T_1 be a maximal torus of $C_G^\circ(a)$ containing b. Then again by Theorem 3, $a \in T_1 \leq C_G^\circ(b)$, a contradiction. Thus (5.5) holds.

Theorem 4 applies to $C_G(a)$, so its Sylow r-subgroups are conjugate. As $b \in C_G(a)$, it follows that b normalizes a maximal r-torus R of $C_G(a)$. Now $a \in R$ by Corollary 3.1, and R is a maximal r-torus of G. At this point, b represents a nontrivial element of $N_G(R)/C_G^\circ(R)$.

Now extend R to a maximal decent torus S in G. By a Frattini argument, $N_G(R) = C_G^\circ(R)N_{N_G(R)}(S)$, so the coset $bC_G^\circ(R)$ meets $N_G(S)$. Let $b' \in bC_G^\circ(R) \cap$

$N_G(S)$. Then $(b')^r \in C_G^\circ(R) \cap N_G(S)$, so there is an r-element $b'' \in b'(C_G^\circ(R) \cap N_G(S))$. Then $b'' \in bC_G^\circ(R)$ is an r-element normalizing S.

Since (H2) fails in G, the group $N_G(R)$ has $r^\perp$ type. In particular, Theorem 4 applies to the subgroup $\langle b, C_G^\circ(R)\rangle$ of $N_G(R)$, and in this group one Sylow r-subgroup has the form $R \cdot \langle b\rangle$. Hence b'' is conjugate under the action of $C_G^\circ(R)$ to an element b^* of $R \cdot \langle b\rangle$. Then b^* normalizes a conjugate S^* of S containing R, and since $b^* \notin C_G^\circ(R)$, $b \in \langle R, b^*\rangle$ also normalizes S^*. Therefore, we can now reverse the roles of a and b, and conclude that b is conjugate to ab under the action of R, and thus a and b are conjugate. As $r > 2$, we find similarly that b^2 is conjugate to a, and thus that b, b^2 are conjugate in G.

Now we return to the action of the Weyl group. The group $N_G(T)$ controls fusion in T. That is, if $X \subseteq T$ and $X^g \subseteq T$, there is an element of $N_G(T)$ carrying X to X^g. Indeed, we have $T, T^g \leq C_G(X^g)$, so T^g is conjugate to T in $C_G(X^g)$, and thus X is conjugate to X^g under $N_G(T)$.

By the control of fusion, b and b^2 are conjugate by an element $w \in W$. Then some power of w has order a prime divisor ℓ of the order of 2 in the multiplicative group modulo r, and in particular ℓ divides $r-1$, so $\ell < r$, a contradiction. This final contradiction completes the proof. □

Corollary 5.3. *Let G be a connected group of finite Morley rank without unipotent torsion. If the Weyl group is nontrivial then it has even order. In particular, the group G is not of degenerate type in this case.*

Theorem 5 could be proved more efficiently by applying [1, Theorem 1], which says that the centralizer of a p-torus in a connected group of finite Morley rank is connected. We note that the proof of [1, Theorem 1] employs our Lemmas 1.3 and 2.2.

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