PERMUTATION GROUPS OF FINITE MORLEY RANK

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Introduction

Groups of finite Morley rank made their first appearance in model theory as *binding groups*, which are the key ingredient in Zilber's ladder theorem and in Poizat's explanation of the Picard-Vessiot theory. These are not just groups, but in fact permutation groups acting on important definable sets. When they are finite, they are connected with the model theoretic notion of algebraic closure. But the more interesting ones tend to be infinite, and connected.

Many problems in finite permutation group theory became tractable only after the classification of the finite simple groups. The theory of permutation groups of finite Morley rank is not very highly developed, and while we do not have anything like a full classification of the simple groups of finite Morley rank in hand, as a result of recent progress we do have some useful classification results as well as some useful structural information that can be obtained without going through an explicit classification. So it seems like a good time to review the situation in the theory of permutation groups of finite Morley rank and to lay out some natural problems and their possible connections with the body of research that has grown up around the classification effort.

The study of transitive permutation groups is equivalent to the study of pairs of groups (G, H) with H a subgroup of G, and accordingly one can read much of general group theory as permutation group theory, and vice versa, and, indeed, a lot of what goes on in work on classification makes a good deal of sense as permutation group theory—including even the final identification of a group as a Chevalley group, which can go via Tits' theory of buildings, or in other words by recognition of the natural permutation representations of such groups. Many special topics in permutation groups tied up with structural issues were discussed in [7, Chapter 11], with an eye toward applications. See also Part III of [15].

The most important class of permutation groups consists of the *definably primitive* permutation groups, and in finite group theory one has the O'Nan-Scott-Aschbacher classification of these groups into various families, determined mainly by the structure of the socle and the way it meets a point stabilizer. This theorem has been adapted to the context of finite Morley rank by Macpherson and Pillay [14], and is the one really general piece of work in the area to date. We will refer to this fundamental result (or set of results) as MPOSA. Also noteworthy is the classification by Hrushovski of groups acting faithfully and definably on strongly minimal sets [17, Th. 3.27], found here as Proposition 2.4, and the study by Gropp [11] of the rank two case.

It turns out that basic notions of permutation group theory such as primitivity and multiple transitivity have more than one useful analog in the context of groups of finite Morley rank, for two reasons: (a) we are interested particularly in connected groups (and, by implication, sets of Morley degree 1); (b) we are interested in generic behavior. Of course we also impose definability constraints. So we have definable primitivity and some analogs involving connectivity, and we have generic n-transitivity, which is far more common than ordinary n-transitivity. Indeed, sharp 4-transitivity cannot occur on an infinite set [12], while $\mathrm{AGL}(V)$ acts generically sharply $(n+1)$-transitively on V if V has dimension n, with $\mathrm{PGL}(V)$ generically sharply $(n+1)$-transitive on projective space, with similar, though less extreme, statements for other classical groups acting naturally.

In our first section we will explore some of the fundamental definitions and their natural variations. After that we will focus on the following problem, which can be taken up from a number of points of view:

Problem 1. *Bound the rank of a definably primitive permutation group of finite Morley rank in terms of the rank of the set on which it acts.*

We will show, using soft methods, that there is some such bound. Here we combine MPOSA with some ideas that have come recently out of the classification project. There are two points to the analysis. One aims to drive the stabilizer of a sufficiently long sequence of generic and independent elements to the identity, bounding the length of the sequence. This divides into two parts: (1) getting started: first bound the possible degree of generic multiple transitivity, which is the length of time one waits before anything happens; (2) moving along: once the chain of point stabilizers begins to decrease, argue that the process runs out in bounded time. In the first stage we are not very precise in our estimates, but one may expect that very good bounds should hold in this part of the process, as generically highly transitive groups should be rare outside of known examples. This is a problem which makes sense and is interesting for groups of finite Morley rank in general, for simple algebraic groups acting definably, and even for simple algebraic groups acting algebraically. In the latter case it has been solved in characteristic 0 by Popov [19], using some results of Kimura et al. on rational representations with an open orbit.

The following result, controlling what one might reasonably call "Lie rank", plays an important role in our "soft" analysis and could also be of use in more concrete approaches.

Lemma 3.8. *Let (G, Ω) be a definably primitive permutation group of finite Morley rank, T a definable divisible abelian subgroup of G, T_0 its torsion subgroup, and $O(T)$ the largest definable torsion free subgroup of T. Then $\mathrm{rk}(T/O(T)) \leq \mathrm{rk}(\Omega)$.*

The present paper has an improvisational character. In the light of recent progress (including some directly attributable to the month devoted to this topic at the Newton Institute) it occurred to us that this could be a good time to take up the topic of permutation groups afresh. We thought that it would in particular provide an interesting setting for a review of some of the neglected but interesting areas (representation theory, cohomology) where the theory in the algebraic case offers considerable food for thought. No doubt this is the case, but that is not the paper that has emerged in this round. Rather, the rank-bounding problem described above wound up giving us a sharper but somewhat narrower focus, in which the existing theory plays a substantial role, and problems of linearization are particularly highlighted. While as a result some of our favorite problems are not represented here, we found some others, some entirely new, along with some new reasons for repeating old questions. We hope some of our readers will answer some of these questions quickly, and find better versions of some of the others. Also, we hope that the optimistic tone with which various approaches are described will not give the impression that they do not require proper proofs. The line between the proven and the unproven is certainly fuzzy here and in the long run caution is the best policy (in spite of Keynes).

We also hope that we have dealt with the foundations in a satisfactory way, or, if not, that someone will take up the matter further.

We would like to take this opportunity to thank the staff at the Newton Institute (especially the gentleman who repaired the espresso machine) and the organizers, with extra thanks to Zoé Chatzidakis.

Contents

(1) Foundations
(2) Bounds on rank: examples and a reduction
(3) The definable socle
(4) Actions of finite groups on connected groups
(5) Simple permutation groups
(6) Generic multiple transitivity: extremes
(7) Problem list

1. FOUNDATIONS

We review some basic model theoretic and permutation group theoretic notions in the context of definable group actions, for the most part under the assumption of finite Morley rank, though it may well be worth taking the extra care necessary to work in the stable category systematically. The main notions are transitivity, primitivity, and multiple transitivity and a number of related variants, along with definability, genericity, and connectivity. Most of this is already present in one form or another in [14] but we think it is still worth while to consider the foundations at leisure, and separately from more technical matters. Indeed, we have the feeling that there is still something to be done here, at least at the level of collecting illuminating examples.

The term "permutation group" is generally taken here to refer to a group equipped with a faithful action. But in dealing with intransitive actions, we may consider the restriction of the group to an individual orbit, and there may be a kernel in this case. Sometimes this actually matters. We will be casual about this below; strictly speaking one should insert the term "faithful" almost everywhere below, taking pains to omit it occasionally.

1.1. **Transitivity, genericity, connectivity.** We work in the definable category, that is with groups and definable actions. We write actions on the right ("α^g"). The following is completely elementary but fundamental.

Lemma 1.1. *Let (G, Ω) be a transitive permutation group, and G_α a point stabilizer (with $\alpha \in \Omega$). Then the action of G on Ω is equivalent to the action of G on the coset space $G_\alpha \backslash G$. In particular the action is interpretable in G if and only if Ω and G_α are interpretable in G, in which case the two actions are definably equivalent.*

One often treats permutation groups (G, Ω) as structures with underlying set Ω and some inherited relations. The most satisfying choice of relations for our present purposes, where the group G is of interest in its own right, are those definable without parameters in the structure (G, Ω).

Lemma 1.2. *Let (G, Ω) be a transitive permutation group and view Ω as a structure equipped with all relations 0-definable in the two-sorted structure (G, Ω), equipped with the action. View G as a structure equipped with its group operation and a distinguished subgroup G_α for some $\alpha \in \Omega$. If the structure on G is stable then G is interpretable in Ω. In particular, if G is stable and Ω has finite Morley rank, then G has finite Morley rank.*

Proof. The point stabilizers form a uniformly definable family of subgroups. If G is stable it follows that arbitrary intersections of point stabilizers are in fact point stabilizers of finite sets. In particular the identity subgroup is the stabilizer of some finite subset $A \subseteq \Omega$. Hence the map $g \mapsto A^g$ is $1-1$. Therefore the elements of G and the group structure on G are encoded in Ω. $\square$

Problem 2. *If (G, Ω) is a permutation group and Ω is stable in the induced language, does it follow that G is stable? Does this hold at least when Ω has finite Morley rank?*

Note that in the present paper we will treat the structure (G, Ω) as given, and not just the group G. So in that context the action of interest is always definable, and whether it is definable in some previously given language on G is a question that rarely arises for us here, though when G is, for example, an algebraic group, it may be an important issue.

Genericity is a fundamental notion in groups, and passes to transitive permutation groups.

Lemma 1.3. *Let (G, Ω) be a stable transitive permutation group, $\alpha \in \Omega$, and X a definable subset of Ω. Then the following conditions are equivalent.*

    (1) *$\{g \in G : \alpha^g \in X\}$ is generic in G*
    (2) *Finitely many G-translates of X cover Ω.*

If Ω has finite Morley rank then an equivalent condition is

    (3) *$\mathrm{rk}(X) = \mathrm{rk}(\Omega)$*

Such a set is called *generic* in Ω.

We note that after identifying Ω with $G_\alpha \backslash G$, the set defined in point (1) is $\bigcup X$. We now look at connected components.

Lemma 1.4. *Let (G, Ω) be a transitive permutation group of finite Morley rank. Let Ω_0 be an orbit for the connected component G°, Then $\mathrm{rk}(\Omega_0) = \mathrm{rk}(\Omega)$, $\deg(\Omega_0) = 1$, and the orbits of G_0 are conjugate under the action of G.*

Proof. Since $G^\circ \triangleleft G$, the orbits of G° are conjugated by G, and as the action of G is transitive it follows that these orbits are conjugate. As there are finitely many such orbits they have the same rank as Ω. Finally, there is a definable bijection between Ω_0 and a coset space for G°, so the Morley degree of Ω_0 is 1. $\qquad\square$

We note that the setwise stabilizer $G_0 = G_{\{\Omega_0\}}$ of Ω_0 may be larger than G° and that the former is really the group induced "by G" on Ω_0.

It is useful to extend the notion of genericity to permutation groups which are not necessarily transitive, at least in the context of groups of finite Morley rank. This we do using the rank directly.

Definition 1.5. *Let X be a definable set in a structure of finite Morley rank. A definable subset Y of X is* strongly generic *in X if $\mathrm{rk}(Y \setminus X) < \mathrm{rk}(X)$, and* weakly generic *in X if $\mathrm{rk}(Y) = \mathrm{rk}(X)$. When X has Morley degree 1, the two notions are equivalent, and are referred to as* genericity. *Otherwise, it is prudent to specify which version is meant.*

Lemma 1.6. *Let (G, Ω) be a permutation group of finite Morley rank and g an element of G which fixes a strongly generic subset of Ω pointwise. Then $g = 1$.*

Proof. Let $H = \{g \in G : g$ fixes a strongly generic subset of Ω pointwise$\}$. Then H is a definable normal subgroup of G.

Take $h \in H^\circ$ generic, and $\alpha \in \Omega$ generic over h. Then h and α are independent, so α is fixed by a generic subset of H°, and hence by H°. As H° is normal in G and G acts transitively, the group H° acts trivially on Ω. That is, $H^\circ = 1$ and H is finite.

Let X be the fixed point set for H in Ω. As H is finite, the set X is generic, hence nonempty. Since X is also G-invariant, we have $X = \Omega$ and thus $H = 1$. $\quad\square$

Definition 1.7. *Let (G, Ω) be a permutation group.*
  (1) *G is* generically transitive *on Ω if G has a strongly generic orbit.*
  (2) *G is* generically n-transitive *on Ω if the induced action on Ω^n is generically transitive.*
  (3) *G is* generically sharply n-transitive *on Ω if the induced action on Ω^n has a strongly generic orbit on which G acts regularly.*

We will not actually take up the generically sharply transitive case per se, but it is worth mentioning as the subject of [11], particularly since such groups certainly exist.

Example 1. *The natural representation of $\mathrm{AGL}(n)$ affords a generically sharply $(n+1)$-transitive action, and the natural projective representation of $\mathrm{PGL}(n)$ affords a generically sharply $(n+1)$-transitive action.*

Problem 3. *Find all the generically sharply n-transitive actions of algebraic groups over algebraically closed fields, for $n \geq 2$.*

Note that by a theorem of Hall [12] there cannot be a sharply n-transitive permutation group on an infinite set for $n \geq 4$, which stands in sharp contrast to the above.

We would suggest that it is reasonable and interesting, though certainly challenging, to aim eventually at an identification of all the generically highly transitive groups of finite Morley rank.

While there is no close connection in general between generic n-transitivity and n-transitivity, it is reasonable to work with transitive actions throughout (unless the set Ω carries some useful structure supported on several orbits, as is the case in most natural representations). For $n = 1$ one just restricts to the strongly generic orbit (possibly picking up a kernel). For $n > 1$ one makes use of the following.

Lemma 1.8. *Let (G, Ω) be a generically n-transitive permutation group of finite Morley rank.*

(1) *(G, Ω) is generically m-transitive for $m \leq n$.*
(2) *If X is the strongly generic orbit for G, then G is generically n-transitive on X*
(3) *If $n > 1$ then Ω has Morley degree 1.*

Proof. The first two points are clear. For the last point, we may suppose that G is transitive on Ω. If Ω has Morley degree greater than 1, then Ω contains distinct G°-orbits Ω_0 and Ω_1. But then no element of $\Omega_0 \times \Omega_0$ is conjugate to any element of $\Omega_0 \times \Omega_1$ under the action of G, and we have a contradiction. $\qquad\square$

The last argument is a very weak analog, but the best we have, for the statement that a doubly transitive group is primitive.

Lemma 1.9. *Let (G, Ω) be a generically n-transitive permutation group, and let Ω_0 be a G°-orbit. Then (G°, Ω_0) is generically n-transitive.*

Proof. If $n = 1$ then this is clear. If $n > 1$ then Ω has Morley degree 1, so Ω^n also has Morley degree 1. If O is the strongly generic orbit for G in Ω^n, and O_0 is a G°-orbit inside O, then $\mathrm{rk}(O_0) = \mathrm{rk}(O) = \mathrm{rk}(\Omega^n)$ and as the Morley degree is 1 it follows that O_0 is strongly generic in Ω^n. $\qquad\square$

1.2. **Notions of Primitivity.** Let us propose a number of notions of primitivity.

Definition 1.10. *Let (G, Ω) be a permutation group.*

(1) *The action is* primitive *if there is no nontrivial G-invariant equivalence relation.*
(2) *The action is* definably primitive *if there is no nontrivial definable G-invariant equivalence relation.*
(3) *The action is* virtually definably primitive *if any G-invariant definable equivalence relation has either finite classes or finitely many classes.*

We are not much interested in primitivity per se, as this is too not natural in the definable category. On the other hand it happens to be the case that almost all definably primitive permutation groups are primitive ([14], see Lemma 1.17).

It is less clear which of the definable versions of primitivity is to be preferred, and we have a third possibility to offer in a moment (this in turn raises the question as to what the O'Nan-Scott-Aschbacher Theorem should be about in our category). Certainly one desirable criterion is that any permutation representation that may

interest us should have a nontrivial primitive quotient. These will be definable, and frequently of Morley degree 1 (or, indeed, with the acting group G connected).

We intend to lay out the relationships among these notions in detail. First we give their translations into "internal" group theoretic terms. For this we will need to restrict attention to transitive group actions, so we begin with this point, which is completely straightforward.

Lemma 1.11. *Let (G, Ω) be a permutation group.*

(1) *If the action is definably primitive, then it is transitive.*
(2) *If the action is virtually definably primitive then either it has finitely many orbits, and is virtually definably primitive on each orbit, or else it has finite orbits, and the set of orbits carries no definable equivalence relation with infinitely many infinite classes.*

Proof. Possibly the context for the second point needs to be elucidated. The notion of definability is relative to the structure (G, Ω) containing the action and whatever additional structure it may carry (on G, on Ω, or on both together). We consider the equivalence relation given by the orbits themselves, which is definable in this context. If the orbits are finite then this is not very interesting, but it may meet our definition. $\square$

Remark 1.12. *Let (G, Ω) be a stable permutation group with all orbits finite. Then G is finite, since the kernel (which is assumed trivial) is the stabilizer of a finite set, and therefore G acts faithfully on a finite union of orbits.*

This remark is actually of some use, when one comes across permutation groups carrying a definable invariant equivalence relation with finite classes. The lemma states that the kernel of the action on the set of classes is finite.

It should now be reasonably clear that nothing of any significance would be lost by including transitivity in the definition of virtual definable primitivity.

Lemma 1.13. *Let (G, Ω) be a transitive permutation group, $\alpha \in \Omega$ fixed, and G_α the point stabilizer.*

(1) *The action is primitive if and only if G_α is a maximal proper subgroup of G.*
(2) *The action is definably primitive if and only if G_α is a maximal definable subgroup of G.*
(3) *The action is virtually definably primitive if and only if for any definable subgroup H of G containing G_α, either $[H : G_\alpha]$ or $[G : H]$ is finite.*

Proof. The G-invariant equivalence relations on Ω are classified by the subgroups of G containing G_α. More explicitly, if C is a set containing α, and $G_{\alpha:C} = \{g \in G : \alpha^g \in C\}$, then the following are equivalent

(1) $G_{\alpha:C}$ is a subgroup of G;
(2) $G_{\alpha:C}$ is the setwise stabilizer $G_{\{C\}}$ of C in G;
(3) $\{C^g : g \in G\}$ is a partition of Ω.

Furthermore, when this holds we have $|C| = [G_{\{C\}} : G_\alpha]$ and $|\{C^g : g \in G\}| = [G : G_{\{C\}}]$. $\square$

There is a rule of thumb that says one is always interested in the "connected" versions of classical notions. So we propose a notion of *c-primitivity* intended to be the connected version of primitivity, in two variants.

Definition 1.14. *Let* (G, Ω) *be a transitive permutation group of finite Morley rank,* $\alpha \in \Omega$, *and* G_α *the point stabilizer.*

(1) *The action is* c-*primitive if* G_α *is a maximal proper definable connected subgroup of* G°.
(2) *The action is virtually* c-*primitive if* $G_\alpha{}^\circ$ *is a maximal proper definable connected subgroup of* G°.

One may wish to translate this back into permutation group theoretic terms. Here one needs the notion of a *finite cover* $(G, \hat{\Omega})$ of a permutation group (G, Ω), which is given by a surjective G-invariant map $\pi : \hat{\Omega} \to \Omega$ with finite fibers (really the map $(1, \pi) : (G, \hat{\Omega}) \to (G, \hat{\Omega})$ is the morphism).

Lemma 1.15. *Let* (G, Ω) *be a transitive permutation group of finite Morley rank.*

(1) *The action is virtually* c-*primitive if and only if every definable finite cover is virtually definably primitive.*
(2) *The action is* c-*primitive if and only if every finite cover is an isomorphism, and the action is virtually definably primitive.*

Proof. Only the first point requires unwinding, and this is simply a matter of working out the content of the cumbersome criterion given, which can be read as follows: for every definable subgroup G_0 of G_α of finite index, and for every definable subgroup H of G containing G_0, either $[H : G_0] < \infty$ or $[G : H] < \infty$. This can be decoded further to: either $H^\circ = G_\alpha{}^\circ$ or $H^\circ = G^\circ$—at which point we have virtual c-primitivity. $\qquad\square$

One of the goals of permutation group theory is to provide a convenient language for saying useful things about maximal subgroups, and in a finite Morley rank context one thing that one could reasonably ask of the parallel theory is a useful way of looking at maximal connected subgroups. But this is not really the point of view that has been taken to date.

We now take note of reductions to Morley degree 1.

Lemma 1.16. *Let* (G, Ω) *be a transitive permutation group of finite Morley rank, and* Ω_0 *an orbit for* G° *in* Ω. *Let* $G_0 = G_{\{\Omega_0\}}$.

(1) *If* (G, Ω) *is definably primitive then* Ω *has Morley degree 1 and* $\Omega_0 = \Omega$, $G_0 = G$.
(2) *If* (G, Ω) *is virtually definably primitive then any* G_0-*invariant definable relation on* Ω_0 *has finite classes.*
(3) *If* (G, Ω) *is* c-*primitive or virtually* c-*primitive, then* (G_0, Ω_0) *has the same property.*

On the other hand, this does not mean that we can reduce the group itself to its connected component.

Example 2. *Let* (H, X) *be a definably primitive permutation group of finite Morley rank with* H *connected. Let* K *be a finite permutation group acting transitively on a set* I. *Then the wreath product* $H \wr K$ *acting on* X^I *is a definably primitive permutation group of finite Morley rank whose connected component* H^I *leaves invariant the equivalence relations* E_i $(i \in I)$ *defined by*

$$E_i(a, b) \iff a_i = b_i$$

Thus (H^I, X^I) *is not even virtually definably primitive.*

One may still find it profitable in practice to pay particular attention to connected permutation groups, but there is no general reduction of the full theory to that case, and this leads to practical difficulties in situations where one would like to make inductive arguments.

The foregoing example is typical in the sense that the class of definably primitive groups is much richer than the set of connected definably primitive groups precisely because the finite sections can play such an important role. The general MPOSA classification includes several cases that cannot arise as actions of connected groups.

1.3. **Primitivity vs. Definable Primitivity.**

Example 3. *A definably primitive action which is not primitive.*
Take a large torsion free divisible abelian group in its natural language (or as a $\mathbb{Q}$-vector space) and consider the regular action. This is definably primitive and c-primitive but not primitive.

One may be a bit suspicious of this example since the point stabilizer is trivial, and with good reason. If a regular action is definably primitive then the group G contains no nontrivial proper definable subgroups and hence is abelian and (if infinite) torsion free. On the other hand according to [14, Prop. 2.7] a definably primitive group is primitive unless the point stabilizer is finite. We elaborate slightly on this point.

Lemma 1.17 ([14]). *Let (G, Ω) be a permutation group of finite Morley rank which is definably primitive but not primitive, and let G_α be a point stabilizer. Then G_α is finite, and if $G_\alpha > 1$ then G° is either abelian or quasisimple.*

Proof. Suppose that $G_\alpha < H < G$. Let H_0 be the normal closure in H of $G_\alpha{}^\circ$. Then by a lemma of Zilber H_0 is definable and connected. Since $G_\alpha \leq H_0 G_\alpha \leq H$ it follows by definable primitivity that $H_0 = G_\alpha{}^\circ$, that is $G_\alpha{}^\circ \triangleleft H$. As $G_\alpha < N(G_\alpha{}^\circ) \leq G$ it follows that $G_\alpha{}^\circ \triangleleft G$. Now G acts transitively on Ω so it follows that $G_\alpha{}^\circ$ acts trivially on Ω, hence $G_\alpha{}^\circ = 1$ and G_α is finite.

Now if K is a definable connected normal subgroup of G then $KG_\alpha = G$ and thus $K = G^\circ$. So if G° is nonabelian then G° is quasisimple, that is perfect with finite center and simple factor group. $\square$

This result has a converse: if the point stabilizer is finite it cannot be maximal, at least if we pass to an uncountable model.

Examples of such groups with G° abelian coming from irreducible representations of finite groups are mentioned in [14]. Another type of example is the following (responding to a question raised in the last paragraph of §2 of [14]).

Example 4. *Let $G = \mathrm{PSL}_2(\mathbb{C})$ and consider the action on cosets of $H = \mathrm{Alt}(5)$. Then H is maximal among Zariski-closed proper subgroups, but not among (e.g., countable) proper subgroups. So in the algebraic category this provides a definably primitive but not primitive action.*

It can be shown with some additional effort that this action will remain definably primitive in any enrichment of the language for which the group has finite Morley rank.

1.4. **Definably Primitive Quotients.** One would like to think that the general permutation group of finite Morley rank can be analyzed in terms of transitive and even primitive constituents. The reduction to the transitive case already involves very substantial complications. Even if the action is generically transitive, the action on a generic orbit may have nontrivial kernel, and there may be infinitely many nongeneric orbits.

As far as primitivity is concerned, one would like to find a nontrivial definably primitive quotient of any transitive permutation group of finite Morley rank, but even this is too much to ask.

Example 5. *Let G be a Chevalley group over an algebraically closed field of positive characteristic, and let $G(q)$ be the subgroup of $\mathbb{F}_q$-rational points. Typically the proper Zariski closed subgroups of G containing $G(q)$ are the $G(q')$ for q' a power of q. Thus taking G with its structure as an algebraic group, any proper definable subgroup containing $G(q)$ is finite, and thus if we consider the action of G on $G(q)\backslash G$, this has no definable and definably primitive quotient.*

On the other hand, this example is again fairly typical.

Lemma 1.18. *Let (G, Ω) be a transitive permutation group of finite Morley rank. Then the following hold.*

* (1) *There is a nontrivial definable quotient of (G, Ω) which is virtually definably primitive.*
* (2) *If (G, Ω) is virtually definably primitive then either (G, Ω) is a finite cover of a definably primitive permutation group, or the point stabilizer is finite.*

Proof. Let G_α be a point stabilizer.

The first point is immediate: extend G_α to a proper definable subgroup of maximal rank.

For the second point, if $N(G_\alpha{}^\circ) < G$ then $[N(G_\alpha{}^\circ) : G_\alpha] < \infty$ by virtual definable primitivity, and we look at $N(G_\alpha{}^\circ)\backslash G$. On the other hand, if $G_\alpha{}^\circ \lhd G$ then by transitivity $G_\alpha{}^\circ$ acts trivially, $G_\alpha{}^\circ = 1$. □

The most useful instance of this is the following.

Corollary 1.19. *Let (G, Ω) be a transitive permutation group of finite Morley rank with G connected simple. Then either (G, Ω) has a nontrivial definably primitive quotient, or the point stabilizer is finite.*

Proof. We apply both parts of the previous corollary, getting a nontrivial quotient of (G, Ω) satisfying one of our two conclusions. Note however that if the point stabilizer in the quotient is finite then it was finite in (G, Ω): as G is simple, there is no kernel in this action. □

Problem 4. *Suppose that (G, Ω) is a virtually definably primitive permutation group of finite Morley rank with which is not a finite cover of a definably primitive permutation group. Show that G is a Chevalley group of positive characteristic, and the point stabilizer is contained in $G(\mathbb{F}_q)$ for some finite field $\mathbb{F}_q$.*

As we have already noticed, *c*-primitivity is just virtual definable primitivity together with the condition that the point stabilizer be connected. On the other hand, in general virtual definable primitivity has little to do with *c*-primitivity, with examples again afforded by actions with finite point stabilizers.

1.5. **Generic n-Transitivity Revisited.** We have the natural inductive principle: if (G, Ω) is a generically n-transitive permutation group of finite Morley rank, and G_α a point stabilizer with α in the generic orbit, then G_α acts generically $(n-1)$-transitively on Ω.

However, one might also expect generically highly transitive actions to be definably primitive, and this fails badly.

Example 6. *Let (H_i, X_i) $(i = 1, 2)$ be generically n-transitive permutation groups of finite Morley rank. Then $(H_1 \times H_2, X_1 \times X_2)$ is generically n-transitive.*

One may ask whether there is, nonetheless, a general theory of generically n-transitive groups. There is an initial reduction to the primitive case. Recall that generic n-transitivity passes to connected components.

Lemma 1.20. *Let (G, Ω) be a transitive and generically n-transitive group of finite Morley rank with G connected. Then the following hold.*

(1) $\mathrm{rk}(G_\alpha) \geq (n-1)\,\mathrm{rk}(\Omega)$ *for $\alpha \in \Omega$.*
(2) *Any definable quotient of (G, Ω) is generically n-transitive.*
(3) *If $n > 1$ then there is an infinite, definable, definably primitive quotient $(\bar{G}, \bar{\Omega})$.*

Proof. The first two points are immediate. For the last point, since G is connected, "infinite" is the same as "nontrivial".

Now Lemma 1.18 tells us that we have an infinite definable and virtually definably primitive quotient $(\bar{G}, \bar{\Omega})$ and that this is either a finite cover of a definably primitive permutation group, or has a finite point stabilizer. But $\mathrm{rk}(\bar{G}_{\bar{\alpha}}) > 0$ so the latter possibility is excluded. $\qquad\square$

We now have two good reasons for restricting our attention to the primitive case: (1) in the present article we need to deal with generically n-transitive groups in order to analyze primitive ones; (2) it seems that results on the primitive case may bear strongly on the general case, in view of the foregoing.

Problem 5. *Is there an O'Nan-Scott-Aschbacher analysis of generically 2-transitive groups which are not necessarily definably primitive? Are all such groups essentially products of generically n-transitive primitive groups (or generically n'-transitive groups, with n' not much smaller than n)?*

This concludes our review of the fundamental definitions. The central notion of [14] was definable primitivity, and this review suggests that one might on occasion prefer to broaden the notion a little, but that the impact of this would be marginal.

2. Bounds on rank

We will organize our discussion around the following result and problem.

Theorem 1. *There is a function $\rho : \mathbb{N} \to \mathbb{N}$ such that the following holds. For any virtually definably primitive permutation group (G, Ω) of finite Morley rank we have*

$$\mathrm{rk}(G) \leq \rho(\mathrm{rk}(\Omega))$$

Problem 6. *Find good bounds on ρ, where $\rho(r)$ is the maximum rank of a virtually definably primitive permutation group (G, Ω) of finite Morley rank, with $\mathrm{rk}(\Omega) = r$.*

Notice the following corollary.

Corollary 2.1. *There is a function $\tau : \mathbb{N} \to \mathbb{N}$ such that the following holds. For any virtually definably primitive permutation group (G, Ω) of finite Morley rank which is generically t-transitive,*

$$t \leq \tau(\mathrm{rk}(\Omega))$$

Problem 7. *Find good bounds on τ, where $\tau(r)$ is the maximum degree of generic transitivity associated to a virtually definably primitive permutation group (G, Ω) of finite Morley rank, with $\mathrm{rk}(\Omega) = r$.*

We particularly like this last problem, because we have as yet no decent bounds on τ and we imagine there should be very good ones. The gap between τ and ρ is not so large, as we shall show.

Proposition 2.2. *For any r we have*

$$r\tau(r) \leq \rho(r) \leq r\tau(r) + \binom{r}{2}$$

The proof of this Proposition will be given shortly.

Lemma 2.3. *We may take Ω to have Morley degree 1, and we may replace "virtually definably primitive" by "definably primitive," in the definitions of ρ and τ, without altering the values.*

Proof. We have seen in Lemma 1.16 that we may restrict attention to Morley degree 1 actions, with all G-invariant relations finite. Furthermore we know that each relevant permutation group (G, Ω) is either a finite cover of a definably primitive one, or has finite point stabilizer. It suffices to observe now that neither the rank nor the degree of generic transitivity can be maximized in the presence of a finite point stabilizer. $\square$

2.1. **Examples.** Our first order of business is to justify our primitivity hypotheses. After all, any permutation group on a set of rank 0 has rank 0, which is a very good bound. There is also an excellent bound for the rank of a stable group acting transitively on a rank 1 set, due to Hrushovski.

Proposition 2.4. *Let (G, A) be a stable permutation group with A strongly minimal. Then the Morley rank of G is at most 3, and either G° is abelian and regular on A, or G is isomorphic with a Zariski closed subgroup of $\mathrm{PSL}_2(K)$ for some algebraically closed and strongly minimal field K interpretable in G.*

In particular, if the rank of G is $n \leq 3$ then the action of G° is sharply n-transitive on A.

But above rank 1 things becomes more complicated. Examples of the following type are given in [11], and show that nontransitive permutation groups are not so easily reduced to transitive ones.

Example 7. *Let V be a vector space over an algebraically closed field K, L a 1-dimensional vector space over K, $E_V = \mathrm{End}(V)$, $\lambda : V \to L$ a nonzero linear map, and $f : L \to V$ a map from the line L into a space curve in V not contained in a proper subspace of V. We associate with these data the permutation group (E_V, L^2) with E_V viewed as an additive group and with action*

$$A.(x, y) = (x, y + \lambda(A.f(x)))$$

(using left-handed notation). Evidently this is faithful, and L^2 has rank two.

The orbits are the rank one sets $L_x = \{x\} \times L$. On L_x, if $v = f(x)$ then the kernel is

$$\{A : A.v \in \ker \lambda\}$$

giving a parametrized family of subspaces of $\mathrm{End}(V)$.

Even in the transitive case there are somewhat similar examples.

Example 8. *Let G be an algebraic group with a rational linear representation on the vector space V and let W be a subspace of V which does not contain any nontrivial G-invariant subspace of V. Let $\hat{G} = V \rtimes G$ and consider the transitive permutation group*

$$(\hat{G}, W \backslash \hat{G})$$

This has rank $\mathrm{rk}(G) + \mathrm{rk}(V/W)$, and is not definably primitive since $W < V$. On the other hand it is faithful by the choice of W.

We want $\mathrm{rk}(\hat{G})$ unbounded and $\mathrm{rk}(W \backslash \hat{G})$ bounded, for which we take G fixed and $\dim(V/W)$ bounded. For example G could be simple, V irreducible, and W a hyperplane. Or G could be a torus acting so V so that all weight spaces are 1-dimensional, and W could be a hyperplane avoiding all weight spaces.

In particular G could be a 1-dimensional torus and then $W \backslash \hat{G}$ would be a Morley rank two representation of a group of arbitrarily large rank.

Here a nontrivial invariant equivalence relation is given by the orbits of the normal subgroup V, the quotient has kernel V, and the induced action is the regular action of G on itself. On the other hand the stabilizer of a single equivalence class will be the group V, with kernel a conjugate of W, and this portion of the action resembles the previous example.

These examples are not particularly outlandish, but they do suggest that it may be difficult to get even the coarsest degree of control over imprimitive representations.

On the other hand, we will eventually get a bound on the rank of a *simple* group of finite Morley rank acting definably on a set of specified rank, and from this we will get our general bound in the primitive case.

2.2. **Reduction to generic multiple transitivity.** We take up the proof of Proposition 2.2. The idea is that if a point stabilizer does not act generically transitively, then the ranks of the orbits of successive point stabilizers (taken along a sequence of independent generic points of Ω) decrease steadily. One difficulty that immediately comes to mind, in view of the last example, is that we have no assurance that we can usefully pass to simple constituents of the induced permutation groups along the way. So we need to apply the initial primitivity hypothesis for the full group at each step.

On the other hand we will see later that by making use of MPOSA one can *sometimes* usefully recover primitivity for such constituents. So in all probability the very soft argument given here can be usefully refined, and the bounds sharpened, using both MPOSA and the structure theory of groups of finite Morley rank. In the present article we will only explore this approach in the opposite case, under the hypothesis of generic multiple transitivity, where our soft bounds are very loose, and one should expect very good bounds by more concrete methods.

Definition 2.5. *Let (G, Ω) be a permutation group of finite Morley rank with Ω of Morley degree 1. Let o_k denote the* generic rank *of an orbit for the group $G_\alpha{}^\circ$ where α is an independent k-tuple of generic elements of Ω. That is, the following set should be generic in Ω:*

$$\{\omega \in \Omega : \mathrm{rk}(\omega^{G_\alpha{}^\circ}) = o_k\}$$

We would get the same values without taking connected components, but in fact it is the connected components of the stabilizers which interest us.

Lemma 2.6. *Let (G, Ω) be a primitive permutation group, $k \geq 1$, and suppose that $0 < o_k < \mathrm{rk}(\Omega)$. Then $o_{k+1} < o_k$.*

Proof. Assume the contrary, $o_{k+1} = o_k$. Our idea is then that the orbits of a stabilizer of a sequence of length $k + 1$ of independent generic elements do not really depend on the last of the elements, or for that matter any of them, and hence there should be a corresponding G-invariant equivalence relation after eliminating spurious dependencies. This is reminiscent of various standard lines of argument in stability theory and may well be a special case of one of them.

We use the symbol $\approx$ to denote *generic equality* for definable sets: so $A \approx B$ means that $\mathrm{rk}(A \Delta B) < \mathrm{rk}(A), \mathrm{rk}(B)$, and in particular $\mathrm{rk}(A) = \mathrm{rk}(B)$. We also use the *generic quantifier* $\forall^* x$ with the meaning "for a strongly generic set of x". Let $r_0 = \mathrm{rk}(\Omega)$. Our first claim is the following.

(1)   For all sequences α, β of either k or $k + 1$ independent generic elements of Ω we have

$$\forall^* x \in \Omega \quad [G_\alpha{}^\circ \cdot x \approx G_\beta{}^\circ \cdot x]$$

Consider the bipartite graph Γ whose vertices are sequences of length k or $k+1$ of independent generic elements of Ω, with edges corresponding to inclusion or reverse inclusion. As this graph is connected, and the displayed condition in (1) defines an equivalence relation on arbitrary sequences α, β of elements from Ω, it suffices to verify that condition (1) holds for a pair of vertices α, β in Γ joined by an edge. But this is immediate as the orbits in question have Morley degree one and (generically) equal rank. So (1) holds.

Now we define an equivalence relation $E(x, y)$ on Ω as follows:

$$\forall^* \alpha \in \Omega^k \, (G_\alpha{}^\circ \cdot x = G_\alpha{}^\circ \cdot y)$$

This is a definable G-invariant equivalence relation on Ω. An equivalent condition is the following, writing $\Omega^{(k)}$ for the set of sequences of k independent generic elements of Ω.

$$\exists \alpha \in \Omega^{(k)} \quad (\alpha \downarrow x, y) \, \& \, (G_\alpha{}^\circ \cdot x = G_\alpha{}^\circ \cdot y)$$

Indeed, as the type over x, y of a sequence of generic elements of Ω independent over x, y is unique, if $G_\alpha{}^\circ \cdot x = G_\alpha{}^\circ \cdot y$ for one such sequence α, then the same holds for all such sequences.

We will write E_x for the E-equivalence class of x. The next point is really part of a larger topic (finiteness of weight) but can be dealt with in an *ad hoc* way here. Cf. [8, §5.6]

(2) $\forall^* x \, \mathrm{rk}(E_x) \geq o_k$

Let $\alpha^{(i)}$ be a sequence of $2r_0 + 1$ independent generic elements of $\Omega^{(k)}$. Then for any pair $x, y \in \Omega$ there is an $i \leq 2r_0 + 1$ such that $xy \downarrow \alpha^{(i)}$.

Take $x \in \Omega$ generic over $\alpha^{(1)} \ldots \alpha^{2r_0+1}$, and let $C = \bigcap_{i \leq 2r_0+1} G_{\alpha^{(i)}}{}^{\circ} \cdot x$. Then by (1) $\mathrm{rk}(C) = o_k$. Choose $y \in C$ with $\mathrm{rk}(y/x) \geq o_k$. Choose i so that $x, y \downarrow \alpha^{(i)}$. Then $E(x, y)$ holds by our second criterion. As E_x is x-definable and contains a point y of rank at least o_k over x, $\mathrm{rk}(E_x) \geq o_k$. This proves (2).

(3) $\forall^* x \; \mathrm{rk}(E_x) \leq o_k$

Take $x \in \Omega$ generic and let $C = E_x$. Choose $y \in C$ generic and independent from x over the parameter C. Then $\mathrm{rk}(C) = \mathrm{rk}(y/\text{``}C\text{''})$ (there is a definable set C on the left side, and a type over the "element" C on the right). Take $\alpha \in \Omega^{(k)}$ with $\alpha \downarrow x, y$. Then $y \in G_\alpha \cdot x$.

As the element "C" is definable from x, and $x, y \downarrow \alpha$, we have $y \downarrow_{x, \text{``}C\text{''}} \alpha$. But $y \downarrow_{\text{``}C\text{''}} x$, so

$$y \underset{\text{``}C\text{''}}{\downarrow} \alpha, x$$

In other words, $\mathrm{rk}(y/\alpha x) = \mathrm{rk}(y/\text{``}C\text{''})$. Therefore $\mathrm{rk}(C) = \mathrm{rk}(y/\text{``}C\text{''}) = \mathrm{rk}(y/\alpha x) \leq \mathrm{rk}(G_\alpha{}^\circ \cdot x) = o_k$. $\qquad \square$

Proof of Proposition 2.2. The claim is

$$r\tau(r) \leq \rho(r) \leq r\tau(r) + \binom{r}{2}$$

Let $t = \tau(r)$ and let (G, Ω) be primitive and generically t-transitive with Ω of rank r. Our claim is that

$$rt \leq \mathrm{rk}(G) \leq rt + \binom{r}{2}$$

The first inequality is immediate as Ω^t has rank rt.

Now by the definition of τ, (G, Ω) is not generically $(t+1)$-transitive and therefore $o_t < \mathrm{rk}(G)$. Hence by repeated application of the foregoing lemma, $o_{t-1+i} \leq \mathrm{rk}(G) - i$ until o_{t-1+i} becomes 0. In particular $o_{t-1+r} = 0$.

Now let α be a sequence of independent generic elements of Ω, of length $t + r - 1$. Then $G_\alpha{}^\circ$ has finite orbits, generically, hence fixes a generic set of points in Ω, and by Lemma 1.6 we find $G_\alpha = 1$, and $\mathrm{rk}(G_\alpha) = 0$.

Now writing G_i for the connected component of the pointwise stabilizer of $(\alpha_1, \ldots, \alpha_i)$ for $i \leq t + r - 1$ (beginning with $G_0 = G^\circ$), we find

$$\mathrm{rk}(G) = \sum_{i < t+r-1} \mathrm{rk}(G_i / G_{i+1})$$

Furthermore $\mathrm{rk}(G_i / G_{i+1}) = \mathrm{rk}(\alpha_{i+1}^{G_i}) = o_i$, so

$$\mathrm{rk}(G) \leq \sum_{i \geq 0} o_i \leq rt + \sum_{0 \leq i \leq r-1} (r - 1 - i) = rt + \binom{r}{2}$$

$\qquad \square$

3. The definable socle

Using some very general results on the definable socle of a definably primitive permutation group [14], we will eventually reduce the proof of Theorem 1 to the following Theorem 2. The material in [14] is really the beginning of the subject, and if we have addressed other more specialized matters first, it was only to get them out of the way. We will not give a full account of [14] here, but certainly everything there needs to be taken into account and it would also be desirable to continue that analysis further, if possible, at some more substantial level of generality than we envision here. But the MPOSA tool in its present form is more than adequate for applications of the sort we consider here, and here we focus more on the other problems which arise in such cases.

Theorem 2. *There is a function σ which bounds the rank of a simple permutation group of finite Morley rank in terms of the rank of any set on which it acts faithfully.*

Again, this reduces via Proposition 2.2 to the following special case.

Proposition 3.1. *There is a function τ' which bounds the degree of generic multiple transitivity of a simple permutation group of finite Morley rank in terms of the rank of the set on which it acts.*

We will review the necessary information from [14] and then make the reduction of Theorem 1 to Theorem 2 (and hence to Proposition 3.1).

3.1. **The main case division.** We now exploit the structure of the socle in a definably primitive permutation group of finite Morley rank. A good deal is known about the possibilities here, and for full details we refer to [14]. Here we emphasize those points that give immediate information relevant to bounds on rank, using comparatively soft methods, though certainly not as soft as those of the previous section. This involves the *definable socle*, which is the subgroup generated by its minimal definable normal subgroups. In any group G of finite Morley rank, it is easy to see that the definable socle is itself definable, and is a finite direct product of finitely many simple groups and an abelian group. If there is no abelian normal subgroup then the socle contains its own centralizer. In particular, if G is infinite, then the definable socle is infinite, and is a direct product of a connected normal subgroup of G (the connected socle) with a finite normal subgroup of G, the latter commuting with G°.

In definably primitive groups the situation is considerably tighter. To begin with we have the following, which is largely the preamble to Theorem 1.1 of [14].

Theorem 3 ([14, 1.1]). *Let (G, Ω) be a definably primitive permutation group of finite Morley rank with Ω infinite. Let B be the definable socle of G. Then one of the following occurs.*

(1) *Affine type: B is abelian, is the unique minimal normal definable subgroup of G, and acts regularly on Ω. G splits as $B \rtimes G_\alpha$ for any $\alpha \in \Omega$ and the action of G on Ω is equivalent to the action of G on B with B acting by translation and G_α by conjugation. Furthermore, B is either torsion free divisible, or an elementary abelian p-group, and there is no nontrivial G_α-invariant*

(2) *B is a finite direct product $T_1 \times \cdots \times T_k$ of isomorphic connected simple groups.*

What is missing in the above summary is the further subdivision of the case 2 into three further subcases, each obtained from wreath products of a corresponding basic type. These types may be referred to suggestively as *regular simple type*, *nonregular almost simple type*, and *simple diagonal action*, and are listed as types 2, 3, and 4(a) in [14]; here type 1 is the affine type, and type 4(b) consists of wreath products of types 2, 3, or 4(a) as well as certain intermediate permutation groups lying between the specified socle and the full wreath product. The further results (1.2-1.4) of [14] then cast additional light on the individual types.

We will in fact need a little more from [14] but we will first see how far Theorem 3 takes us.

We make a few comments on the proof of the statement as we have given it. Note first that by definable primitivity, any definable normal subgroup acts transitively on Ω. It follows easily that if G has an abelian normal subgroup then any minimal abelian normal subgroup A acts regularly and the group splits as described in case (1); furthermore the definable socle centralizes A and hence also acts regularly, forcing the socle to reduce to A. So one may suppose that all minimal normal subgroups are nonabelian. Furthermore, on general principles each nonabelian minimal definable normal subgroup of G has the structure indicated in (2), so we need only consider the case in which there is more than one minimal definable normal subgroup of G, and they are all nonabelian. In particular, as they commute and act transitively, it follows that each acts regularly. Then for any two minimal normal subgroups H_1, H_2 of G it is easy to see that the point stabilizer $(H_1 H_2)_\alpha$ is the graph of an isomorphism of H_1 with H_2 (and this is the starting point for a more careful analysis of B_α).

3.2. **The torsion-free divisible case.** We give a first indication that this case division is helpful for our present concerns.

Lemma 3.2. *Let (G, Ω) be a definably primitive permutation group whose socle A is torsion free divisible, and let $r = \mathrm{rk}(\Omega)$. Then*

$$\mathrm{rk}(G) \leq r^2 + r$$

Proof. This is very much like saying $\dim(\mathrm{AGL}(V)) \leq \dim(V)^2 + \dim(V)$ and really has the same proof.

By Theorem 3 we can identify Ω with A and G with $A \rtimes G_0$ (note that G_0 is actually the point stabilizer of 0 under this identification).

Choose a sequence $(a_1, a_2, \dots)$ as long as possible with $a_{i+1} \notin d(a_1, \dots, a_i)$ (in particular $a_1 \neq 0$). Let $A_i = d(a_1, \dots, a_{i-1})$. As the A_i are also torsion free and divisible, $\mathrm{rk}(A_i) \geq i$ and therefore the sequence $(a_1, \dots)$ is finite of length $n \leq r$.

Now as $A_n = A$, the point stabilizer $G_{a_1, \dots, a_n}$ is trivial. Hence writing G_i for the point stabilizer $G_{a_1, \dots, a_i}$ we have

$$\mathrm{rk}(G_0) = \sum_1^n \mathrm{rk}(G_i / G_{i-1}) \leq nr \leq r^2$$

and our claim follows. $\square$

More to the point is the question as to why this doesn't work in other cases. In the case of elementary abelian socles what is missing is a linear structure to bound the "dimension" (in the sense implicit above) in terms of the rank. In the case of simple socles this argument actually works quite well, as we will see next, but only

bounds the rank of G in terms of the rank of the socle, which must still be related to the rank of Ω, since the socle usually does not act regularly.

3.3. **Simple socles.** Let us extract what we can from the last argument for the case of simple socles, and more generally for nonabelian socles.

Lemma 3.3. *Let S be an $\aleph_0$-saturated simple group of finite Morley rank and rank n. Then there are elements $a_0, \ldots, a_n$ which definably generate S in the sense that*

$$S = d(a_0, \ldots, a_n)$$

Notice that we need the saturation hypothesis even in the algebraic case: in positive characteristic there are many locally finite simple algebraic groups.

Proof. We claim that we can choose a sequence $(a_0, a_1, \ldots)$ so that setting $S_i = d(a_0, a_1, \ldots, a_i)$ we have $\mathrm{rk}(S_i) \geq i$ until $S_i = S$. This will certainly prove the lemma, padding the sequence if necessary.

We begin with the case $i = 1$, which requires us to choose the two elements a_0, a_1. If G does not have bounded exponent then by the saturation hypothesis we can choose $a_0 = a_1$ to be an element of infinite order. If G does have bounded exponent than for the purposes of this step we may work in a minimal connected simple definable section $\bar{S}$ and lift back to S afterward.

Now as $\bar{S}$ is minimal simple and of bounded exponent it follows easily that its Borel subgroups are nilpotent, and thus $\bar{S}$ is a so-called "bad group". For these there is some useful structural information and in particular any nontrivial finite subgroup lies in a unique Borel subgroup [7]. So if we take a_0, a_1 to be nontrivial elements in two distinct Borel subgroups then $d(a_0, a_1)$ is infinite, as required.

With the base of the induction out of the way, we continue as follows. We suppose that S_i has rank at least i, with $i \geq 1$, and we wish to choose $a_{i+1} = a$ in S so that

$$\mathrm{rk}(d(S_i, a)) > \mathrm{rk}(S_i)$$

If we cannot do this, then we find

$$d(S_i, a)^\circ = S_i{}^\circ$$

for all $a \in S$, and hence $S_i{}^\circ \lhd S$, forcing $S_i{}^\circ = S$, and we are already done. $\square$

And now as before one has the following conclusion.

Lemma 3.4. *Let G be a group of finite Morley rank and suppose that G has a simple definable socle S of rank n. Then $\mathrm{rk}(G) \leq n^2 + n$.*

Proof. Consider the action of G on S by conjugation, which is faithful. Let the sequence $(a_0, a_1, \ldots, a_n)$ be chosen in accordance with the previous lemma. Then the point stabilizer of the sequence fixes S pointwise and is therefore trivial, so we may repeat the computation of Lemma 3.2. $\square$

We have not yet considered the case of a general nonabelian socle, and we will return to this below.

3.4. **Elementary abelian socles.** Now we take up the case of an elementary abelian socle. For the present subsection, let us fix the notation as follows.

Notation 3.5. (H, A) *is a pair consisting of an infinite group H acting definably, definably irreducibly, and faithfully on an elementary abelian p-group A, and the pair has finite Morley rank.*

We now apply the version of Zilber's field theorem given in [7, Theorem 9.5].

Theorem 4. *Let $V \rtimes G$ be a connected group of finite Morley rank with V, G definable, V abelian and G-minimal, and $C_G(V) = 1$. Suppose that G has an infinite definable abelian normal subgroup K. Then $C_V(G) = 1$, K is central in G, and there is an interpretable algebraically closed field over which V becomes a finite dimensional vector space, K becomes a group of scalars generating the field, and G acts linearly.*

Applying this in the present instance we get the following.

Lemma 3.6. *Suppose that H° has a nontrivial definable connected abelian normal subgroup and let K be a minimal such subgroup. Then there is a definable H°-invariant subgroup A_0 of A, and a finite subgroup K_0 of K, such that K/K_0 acts freely on A_0 and A is a direct sum of H-conjugates of A_0. Furthermore K is central in H°.*

Proof. Let $A_0 \leq A$ be a minimal definable connected H°-invariant subgroup. Then A is the sum of the conjugates of A_0 under H. In particular at least one of the conjugates of A_0 is not centralized by K, and we may take A_0 to have this property. Then $K_0 = C_K(A_0)$ is finite by the minimality of K.

Let $\bar{H}^\circ = H^\circ/C_{H^\circ}(A_0)$. Apply the previous theorem to $A_0 \rtimes H^\circ/C_{H^\circ}(A_0)$, with the subgroup $\bar{K} \cong K/C_K(A_0)$. Then $[K, H^\circ] \leq K_0$ and by connectedness K is central in H°.

Furthermore there is a vector space structure on A_0 with respect to which $\bar{K}$ acts by scalars and $H^\circ/C_{H^\circ}(A_0)$ acts linearly. $\square$

This setup produces the natural bound on the rank of H.

Lemma 3.7. *Suppose H° has a nontrivial definable connected abelian normal subgroup. Then $\mathrm{rk}(H^\circ) \leq \mathrm{rk}(A)^2$.*

Proof. Let K be a minimal definable connected abelian normal subgroup of H° and let A_0 be a minimal definable connected H°-invariant subgroup of A.

Let $r = \mathrm{rk}(A)$, $r_0 = \mathrm{rk}(A_0)$, and let A be the sum of k H-conjugates of A_0 and no fewer. By the minimality of A_0, A modulo a finite subgroup is a direct sum of k subgroups of rank r_0 and thus $r = kr_0$.

The group $\bar{H}^\circ = H^\circ/C_{H^\circ}(A_0)$ acts as a linear group on A_0. If A_0 has dimension d then $\mathrm{rk}(\bar{H}^\circ) \leq d^2(r_0/d) = r_0 d$. Now H° embeds into the direct product of k groups definably isomorphic to $\bar{H}^\circ$ and hence $\mathrm{rk}(H) = \mathrm{rk}(H^\circ) \leq kr_0 d = rd \leq r^2$. $\square$

We still have to deal with the very reasonable possibility that the definable socle of H° is semisimple. In this case what is needed, initially, is a bound on the number of simple factors. For this we need the following preparation, which says, roughly speaking, that tori tend to act generically freely, thereby bounding their ranks. Here the groups playing the role of tori are the definable hulls of divisible abelian torsion subgroups, taken modulo any definable torsion free subgroups.

Lemma 3.8. *Let (G, Ω) be a definably primitive permutation group of finite Morley rank, T a definable divisible abelian subgroup of G, T_0 its torsion subgroup, and $O(T)$ the largest definable torsion free subgroup of T. Then $\mathrm{rk}(T/O(T)) \leq \mathrm{rk}(\Omega)$.*

Proof. Note that T_0 is a countable subgroup of T. Therefore in a saturated model we may take a point $\alpha \in \Omega$ generic over T_0. Suppose that a torsion element $t \in T_0$ fixes α. Then t fixes a generic subset of Ω pointwise. Since Ω has Morley degree 1, by Lemma 1.6, we find $t = 1$. In other words, the point stabilizer T_α is torsion free and thus contained in $O(T)$. Hence $\mathrm{rk}(T/O(T)) \leq \mathrm{rk}(T/T_\alpha) \leq \mathrm{rk}(\Omega)$. $\square$

For the next step, we need a result of Wagner on fields of finite Morley rank in positive characteristic.

Theorem 5 ([21], cf. [2]). *Let F be a field of finite Morley rank of even type and positive characteristic. Then every definable subgroup of $F^\times$ is the definable hull of its torsion subgroup.*

Accordingly we make the following definition.

Definition 3.9. *A divisible abelian group of finite Morley rank is called a* good torus *if every definable subgroup is the definable hull of its torsion subgroup.*

The preceding theorem has the following consequence. Recall that a *unipotent* subgroup in a group of finite Morley rank is a solvable definable connected subgroup of bounded exponent. Such groups are in fact nilpotent. There are also "characteristic 0" versions of unipotence but we do not use them here; our unipotence is π-unipotence for some finite set of primes π.

Lemma 3.10. *Let H be a connected solvable group of finite Morley rank acting faithfully on a unipotent group V of bounded exponent. Let $U(H)$ be the maximal unipotent subgroup of H. Then $H/U(H)$ is a good torus.*

Proof. Let $1 = V_0 < V_1 < \cdots < V_n = V$ be a chain of definable H-invariant subgroups of V such that successive quotients $A_i = V_i/V_{i-1}$ are H-minimal, that is infinite and without proper definable H-invariant subgroups. It is easy to see that the joint kernel of all the actions of H on the A_i is $U(H)$. Thus we reduce easily to the case in which V is H-minimal and $U(H) = 1$, replacing V by the A_i and H by its various quotients $H/C_H(A_i)$.

Now V is contained in the Fitting subgroup of VH and is self-centralizing in VH, so the connected component $F^\circ(VH)$ of the Fitting subgroup of VH is unipotent, and hence $F^\circ(VH) = V$ since $U(H) = 1$. But by the structure theory for connected solvable groups, $VH/F^\circ(VH)$ is divisible abelian and thus H is divisible abelian. Now Zilber's Field Theorem applies and H acts on V like a subgroup of the multiplicative group of a field. By Wagner's theorem, it is a good torus. $\square$

Lemma 3.11. *Let $K = L_1 \times \cdots \times L_n$ be a product of infinite simple groups acting faithfully on an elementary abelian p-group V, with the pair (V, K) having finite Morley rank. Then $n \leq \mathrm{rk}(V)$.*

Proof. We argue by induction on the rank of V. Easily K is faithful on V° so we may suppose V is connected. We may also suppose that $C_V(K) = 1$. If V is not definably K-irreducible and V_0 is a minimal nontrivial K-invariant subgroup, then V_0 is infinite and connected and induction easily yields our claim. So we suppose V is definably K-irreducible. Then $(V \rtimes K, V)$ is a primitive permutation group.

Suppose first that

(1) $L = L_n$ contains a nontrivial unipotent subgroup.

Then $W = C_V{}^\circ(U) < V$ is nontrivial.

Let $K_1 = L_1 \times \cdots \times L_{n-1}$. Then K_1 acts on W and if the action is faithful we find $n - 1 \leq \mathrm{rk}(W) < \mathrm{rk}(V)$ and our claim follows.

So we may suppose that L_1 acts trivially on W. Let $V_1 = C_V(L_1) < V$. This contradicts the definable K-irreducibility of V.

So now we consider the alternative, which we may take to be the following.

(2)   No factor L_i contains a nontrivial unipotent subgroup

Now if B is a Borel subgroup of any factor L_i, then $U(B) = 1$ and by the preceding lemma B is a good torus. Therefore we can find a good torus T in K of rank at least n. Then $O(T) = 1$ and thus by Lemma 3.8 $\mathrm{rk}(T) \leq \mathrm{rk}(V)$. So $n \leq \mathrm{rk}(V)$ as claimed. $\square$

Now we use this information to give a bound on the rank of H in terms of the ranks of A and the ranks of the definable simple nonabelian subgroups of H°.

Lemma 3.12. *Suppose that every definable simple nonabelian subgroup of H° has rank at most s, and $r = \mathrm{rk}(A)$. Then the rank of H is at most*

$$\max(r^2, r(s^2 + s))$$

Proof. In view of Lemma 3.7 we will suppose that all minimal normal definable subgroups of H° are simple. Let there be n such. For L a definable normal simple subgroup of H°, let $H_L = H^\circ / C_{H^\circ}(L)$. Then there is an embedding $H^\circ \to \prod_L H_L$ and thus $\mathrm{rk}(H^\circ) \leq n \max_L(\mathrm{rk}(H_L))$

Now identify L with its image in H_L. Then L is a definable simple normal subgroup of H_L, and we claim that L is the definable socle of H_L. Let K be the preimage in H° of $C_{H_L}(L)$. Then $[K, L] \leq C_L(L) = 1$, so $K = C_{H^\circ}(L)$. Thus L is self-centralizing in H_L and is the definable socle.

So by Lemma 3.4 we have $\mathrm{rk}(H_L) \leq s^2 + s$ and thus $\mathrm{rk}(H) \leq n(s^2 + s) \leq r(s^2 + s)$, taking into account Lemma 3.11. $\square$

Now we go back to the case of nonabelian socles and make a similar reduction.

3.5. **Nonabelian socles.** Throughout the present subsection (G, Ω) denotes a permutation group of finite Morley rank with nonabelian definable socle. At this point one can usefully examine the subdivision into cases afforded by the full statement of MPOSA, but we continue on with less precise methods. Set $r = \mathrm{rk}(\Omega)$.

Lemma 3.13. *If G has more than one minimal definable normal subgroup, then*

$$\mathrm{rk}(G) \leq r^2 + 2r$$

Proof. Let L be a minimal normal subgroup of G. By our hypothesis $C_G(L)$ is nontrivial and hence by definable primitivity L and $C_G(L)$ both act transitively on Ω. It follows that each acts regularly and in particular $\mathrm{rk}(C_G(L)) \leq r$, so $\mathrm{rk}(G) \leq r + \mathrm{rk}(G/C_G(L))$.

Now write $L = L_1 \times \cdots \times L_n$ with the L_i isomorphic definable simple normal subgroups of G°. As L acts regularly on Ω we have $\mathrm{rk}(L_i) = r/n$. Then Lemma 3.4 yields

$$\mathrm{rk}(G/C_G(L)) \leq n((r/n)^2 + (r/n)) = r^2/n + r$$

and our claim follows.   □

Now suppose that G contains a unique minimal definable normal subgroup L, which is nonabelian. We should note that if G were connected then that subgroup would in fact be simple, as G acts transitively on the simple factors of a nonabelian minimal normal subgroup, and then the situation would be quite transparent. But what this suggests, correctly, is that primitive groups will have a strong tendency to be disconnected, in general.

In this case we finally need to use the more precise information from [14], describing the point stabilizer L_α, which is a characteristic feature of the OSA point of view. There are the following two possibilities:

(a) L is a product of simple factors L_i and $L_\alpha = \prod_i (L_i)_\alpha$ (possibly $L_\alpha = 1$ here);

(b) L is a product of ℓ groups L_i, each of which is a product of k simple factors L_{ij}, with $k \geq 2$, and $(L_i)_\alpha$ is a diagonal subgroup of L_i, and in particular $\mathrm{rk}((L_i)_\alpha)$ is definably isomorphic to L_{ij} for each j.

Lemma 3.14. *If G has a unique minimal normal subgroup L, and its simple factors have rank s, then corresponding to cases (a, b) above we have the following estimates.*

(a) $\mathrm{rk}(G) \leq r(s + s^2)$;

(b) $\mathrm{rk}(G) \leq 2(r^2 + r)$

Proof. Let L be the product of n simple factors, each of rank s. Then our estimate for $\mathrm{rk}(G)$ is

$$n(s^2 + s)$$

In case (a) we have $n \leq r$. So we consider case (b).

If L acts regularly on Ω then $s = r/n \leq r$.

If the point stabilizer L_α is a product of ℓ diagonal subgroups then $n = k\ell$ for some k and $r = \mathrm{rk}(L) - \mathrm{rk}(L_\alpha) = ns - \ell s = (k-1)\ell s$, $s = r/(k-1)\ell = r/(n-\ell) \leq (2/n)r$. So $n(s^2 + s) \leq (4/n)r^2 + 2r \leq 2(r^2 + r)$ since $n \geq 2$.   □

The upshot of all of this is that we need an estimate for the rank of a simple group acting on Ω in terms of the rank of Ω, or equivalently, a bound on the degree of multiple transitivity for such an action.

Of course, this reduction of a general problem on primitive permutation groups to the simple case is a typical application of the OSA point of view.

4. ACTIONS OF FINITE GROUPS ON CONNECTED SOLVABLE GROUPS

The present section has a preparatory character. In the next section we use that fact that generically n-transitive groups have the symmetric group $\mathrm{Sym}(n)$ as a section. Here we examine definable actions of $\mathrm{Sym}(n)$, and related groups, on connected solvable groups of finite Morley rank, looking for lower bounds on the rank. This is an issue which has not arisen in the past and has some affinities with linear representation theory, though an action on a connected abelian group is not necessarily very closely connected with a linear representation, as far as we know. Getting sharp bounds for this sort of problem seems challenging.

There are three natural variations, all of which come into play.

Problem 8. *Let Σ be a finite group. Find lower bounds for each of the following.*

(1) *The minimal rank of a connected solvable group of finite Morley rank which affords a faithful representation of Σ.*
(2) *The minimal rank of a connected solvable group of finite Morley rank which affords a faithful representation of a central extension of Σ.*
(3) *The minimal rank of a connected solvable group of finite Morley rank which affords a faithful representation of a group $\hat{\Sigma}$ which covers Σ, i.e. maps homomorphically onto Σ.*

We will work with abelian groups rather than solvable groups, but in most cases we can reduce to Σ-minimal groups, in which case there is no difference. But the question makes sense also more generally, without even the hypothesis of solvability.

Actually, as we will see we are interested particularly in actions of finite groups on divisible abelian groups. This brings us a little closer to the characteristic zero linear theory.

The estimate we will need is the following. This is off the main line of our discussion, and any estimate of the kind will suffice for our subsequent purposes.

Proposition 4.1. *Let A be a connected abelian group of finite Morley rank, and Σ a finite group acting definably and faithfully on A. Suppose that Σ maps surjectively onto $\mathrm{Sym}(n)$. Then $\mathrm{rk}(A) \geq \lfloor n/4 \rfloor$*

We return to the main line in the next section.

4.1. **Generalities.** We dispose of some formal points before taking up anything concrete. Recall that the *Frattini* subgroup $\Phi(\Sigma)$ of a finite group Σ is the intersection of its maximal subgroups, and is nilpotent.

Definition 4.2. *Let $\Sigma \to \bar{\Sigma}$ be a surjection between finite groups. The map, or by abuse Σ itself, is called a* Frattini cover *of $\bar{\Sigma}$ if the kernel of the map is contained in the Frattini subgroup of Σ.*

An equivalent condition is this: no proper subgroup of Σ covers $\bar{\Sigma}$. So we have a trivial but useful starting point.

Lemma 4.3. *Let Σ_0 be a finite group, Σ a finite group mapping homomorphically onto Σ_0. Then Σ contains a subgroup Σ_1 which is a Frattini cover of Σ_0.*

For the proof just take Σ_1 minimal. Notice that if we also have a faithful representation of Σ on a connected abelian group of finite Morley rank, then we have in particular a faithful representation of a Frattini cover. This has the advantage that the kernel of a Frattini cover is nilpotent, so we are getting much more control over the group. What we would prefer is to reduce even further, to central extensions.

Now we take a preliminary look at the minimal modules for our finite groups.

Lemma 4.4. *Let Σ be an almost simple group (i.e., the socle is simple). Let A be a connected abelian group of minimal Morley rank such that Σ acts faithfully and definably on A. Then Σ acts faithfully on some Σ-minimal definable section of A.*

Proof. We argue by induction on $\mathrm{rk}(A)$. Let $A_0 \leq A$ be Σ-minimal. If Σ acts faithfully on A_0 or A/A_0 we conclude directly or by induction, respectively. In the alternative case the socle Σ_0 of Σ acts trivially on both factors and therefore each element $a \in A$ gives rise via commutation to a homomorphism $\alpha : \Sigma_0 \to A_0$, from a simple group to an abelian group. So in this case Σ_0 acts trivially on A, a contradiction. $\qquad\square$

Next we prepare some general estimates for actions of elementary abelian groups.

Lemma 4.5. *Let E be an elementary abelian p-group and $m = m_p(A)$. If E acts faithfully on the connected p-divisible abelian group A of finite Morley rank, then $\mathrm{rk}(A) \geq m$.*

Proof. Let $V \leq A$ be E-minimal. Let E_0 be the kernel of the action of E on V, a subspace of E of codimension 1. If E_0 acts faithfully on A/V then we conclude by induction.

So let E_1 be the kernel of the action of E_0 on A/V. Then E_1 acts trivially on the factors of the chain $1 < V < A$. So for $e \in E_1^{\#}$ commutation with e gives a homomorphism $\epsilon : A/V \to A$ with image a connected elementary abelian p-group, contradicting our hypothesis. $\square$

4.2. **Actions of symmetric groups.** We aim here at lower bounds for the rank of a connected abelian group on which $\mathrm{Sym}(n)$ acts faithfully, particularly when the abelian group in question is the definable closure of a p-torus for some p. Of course, we will really need to deal with groups covering $\mathrm{Sym}(n)$ and these do not necessarily behave in the same way, so the reader may prefer to pass on to the next subsection which returns to the more general problem, with worse estimates.

Here it would be very useful to have a result of the following kind: if T is the definable closure of a p-torus and is F-minimal under the action of a finite group F, then the corresponding Tate module is F-irreducible. Unfortunately this is nonsensical, as one sees already by taking $F = 1$, but perhaps something can be rescued in this direction.

Before stating our next lemma we note that the symmetric group $\mathrm{Sym}(6)$ is isomorphic to the symplectic group $\mathrm{Sp}(4, 2)$ and therefore has a faithful representation on a connected abelian group of Morley rank 4, namely the corresponding vector space over the algebraic closure of $\mathbb{F}_2$.

Lemma 4.6. *Let A be an abelian group on which $\mathrm{Sym}(n)$ acts definably and faithfully, with A either a finite elementary abelian p-group or a connected abelian group. Let $\delta(A)$ be $m_p(A)$ in the first case and $\mathrm{rk}(A)$ in the second case. Then $\delta(A) \geq n-1$ unless $n = 6$, $d = 4$, and A is an elementary abelian 2-group (finite or connected).*

Proof. We proceed by induction on n and then by induction on $\delta(A)$. If $n = 2$ our claim is vacuous. If $n \geq 3$ and $d = 1$ then the transpositions must act by inversion on G, and the action cannot be faithful. So we may suppose throughout that $d \geq 2$ and $n \geq 4$. Write Σ for $\mathrm{Sym}(n)$.

We treat separately the case $n = 4$. Let V be the Klein 4-group in Σ. Let $A_0 < A$ be V-minimal. Then V has a kernel on A_0 and this kernel has three conjugates in Σ. It follows easily that $d \geq 3$. So from now on we suppose

$$n \geq 5$$

In particular Σ is almost simple and A may be taken to be Σ-minimal in the connected case, irreducible in the finite case. In particular A is either an elementary abelian 2-group, or is 2-divisible.

We deal first with the case of an elementary abelian 2-group. We will use the connected component notation below with an eye on the case in which A is connected. This operator should be interpreted as vacuous ($X^{\circ} = X$, not $X^{\circ} = 1$) when A itself is finite.

Fix a transposition $\tau \in \Sigma$. Then $1 \leq [\tau, A] \leq C_A{}^{\circ}(\tau) \leq A$. If the three sections $A_1 = [\tau, A]$, $A_2 = C_A{}^{\circ}(\tau)/[\tau, A]$, and $A_3 = A/C_A{}^{\circ}(\tau)$ are all nontrivial, consider the action of the group $\Sigma_0 = C_\Sigma(\tau) \cong \mathrm{Sym}(n-2)$. If $n \geq 7$ so that $n - 2 \geq 5$, then Σ_0 acts faithfully on at least one of the sections A_i and thus we find $\delta(A_i) \geq n - 3$, $\delta(A) \geq (n - 3) + 2 = n - 1$ as required. If $n = 6$ then the socle of $\mathrm{Sym}(4)$ acts faithfully on at least one of these sections, which therefore has rank at least 2. So we get our lower bound $\delta(A) \geq 4$ in this case. If $n = 5$ we have only to deal with the case in which $\delta(A_i) = 1$ for $i = 1, 2, 3$. Then $\mathrm{Sym}(3)$ is generated by involutions and each of them acts trivially on each section, so $\mathrm{Sym}(3)$ acts trivially on each section and therefore an element of order 3 acts trivially on A, a contradiction.

This disposes of all cases in which the elementary abelian 2-group A has the three sections A_1, A_2, A_3 all nontrivial. So at least one of these sections is trivial. But τ acts faithfully on A, so we find $C_A{}^{\circ}(\tau) = [\tau, A]$ or in other words $[\ker(1 - \tau)]^{\circ} = \mathrm{im}(1 - \tau)$. So $\delta(A_1) = \delta(A_3)$. If $n \geq 7$ then again Σ_0 acts faithfully on one of these sections and thus $\delta(A) \geq 2(n - 3) \geq n - 1$.

We now have to treat the small cases with $n = 5, 6$.

Suppose first $\delta(A_1) = \delta(A_3) = 1$. Let V be a 4-group in Σ. Then $C_A{}^{\circ}(V)$ is nontrivial and hence $C_A{}^{\circ}(V) = C_A{}^{\circ}(i)$ for $i \in V^\#$ or in other words $C_A{}^{\circ}(i) = C_A{}^{\circ}(j)$ for commuting involutions i, j, and as $n \geq 5$ we conclude that $C_A{}^{\circ}(i) = C_A{}^{\circ}(\Sigma)$ which may be supposed trivial. Thus we have a contradiction.

If $\delta(A_1) = \delta(A_3) = 2$ and $\delta(A) = 4$ then the only remaining case is $n = 6$, which we allow.

So now we pass to the main case.

A is 2-divisible

Recall that A is either connected, or is a finite elementary abelian p-group with p odd, now.

Consider an involution τ and let $\Sigma_0 = C_\Sigma(\tau) \cong \mathrm{Sym}(n-2)$. We have

$$A = A_\tau^+ + A_\tau^-$$

with A_τ^+, A_τ^- the subgroups centralized by or inverted by τ respectively. If A is infinite (connected) then the intersection $A_\tau^+ \cap A_\tau^-$ is finite and these groups are connected.

We have actions of Σ_0 on A_τ^+ and A_τ^-, and at least one of these actions is faithful. So $\delta(A) \geq (n - 3) + 1 = n - 2$ with equality only if one of the factors has rank 1 and the other has rank $n - 3$.

Suppose $\delta(A_\tau^+) = 1$. The same holds for any transposition as they are conjugate. So if τ, τ' are transpositions whose product σ has order 3, then τ and τ' invert subgroups of corank 1 in A, and hence σ inverts a subgroup of corank at most 2. So in this case $\delta(A) = 2$ and thus by the above $n - 3 = 1$, $n = 4$. Then consider a pair of commuting transpositions τ_1, τ_2. We have $A_{\tau_1}^\pm = A_{\tau_2}^\pm$ in some order and hence $\tau_1 \tau_2$ acts either trivially or by inversion on A. Then the action of $\tau_1 \tau_2$ commutes with the action of Σ on A and therefore this action is not faithful. We conclude

$$\delta(A_\tau^-) = 1, \delta(A_\tau^+) = n - 3 > 1$$

for any transposition $\tau \in \Sigma$. In particular $n \geq 5$.

For any pair of commuting transpositions τ, τ', the element τ' centralizes A_τ^-, as otherwise we would have $A_\tau^- = A_{\tau'}^-$ for all such commuting pairs and then A_τ^- is independent of τ, since $n \geq 5$.

For any transposition τ, the action of τ on A/A_τ^- is trivial and in particular any subgroup containing A_τ^- is τ-invariant.

Now we need to work out the action in more detail. Fix two noncommuting transpositions τ_1, τ_2 and let $A_i = A_{\tau_i}^-$ for $i = 1, 2$. Take $a_1 \in A_1$ and choose $a_2 \in A_2$ so that

$$a_1^{\tau_2} = a_1 + a_2$$

Let $\sigma = \tau_2 \tau_1$. Then $a_1^\sigma = -a_1 + a_2^{\tau_1}$ and σ carries A_1 to A_2, so $a_2^{\tau_1} \in a_1 + A_2$ and therefore $a_2^{\tau_1} = a_1 + a_2$. Thus $a_2 = a_1^\sigma$. Similarly $a_2^{\tau_1} = a_2^{\sigma^{-1}} + a_2 = a_1 + a_2$.

Now let τ_i be the elementary transposition $(i, i+1)$ and let $A_i = A_{\tau_i}^-$. Assuming $\delta(A) < n - 1$, let k be minimal so that $\delta(A_1 + \cdots + A_k) < k$. Then $A_k \leq A_1 + \cdots + A_{k-1}$ and therefore $A_1 + \cdots + A_k$ is Σ-invariant, so $k = n - 1$ and $A = A_1 + \cdots + A_{n-2}$ is an almost direct sum.

Consider the cycle $\sigma = (12 \ldots n)$. For any $i < n - 1$ if $\sigma_i = \tau_{i+1} \tau_i$ then $\sigma \sigma_i^{-1}$ commutes with τ_i and thus σ acts on A_i as σ_i does. This holds also for A_{n-1} with a notational variation. So for $a \in A_i$ we have $a^{\tau_{i+1}} = a + a^\sigma$, and similarly $a^{\tau_{i-1}} = a + a^{\sigma^{-1}}$.

Now consider any relation

$$a_1 + \cdots + a_{n-1} = 0$$

with $a_i \in A_i$. Applying τ_i we get

$$a_{i-1}^{\tau_i} - a_i + a_{i+1}^{\tau_i} = a_{i-1} + a_i + a_{i+1}$$

or

$(*)$ $$a_{i-1}^\sigma + a_{i+1}^{\sigma^{-1}} = 2a_i$$

From this we easily derive

$$i a_{i+1} = (i+1) a_i^\sigma$$

for all $i < n - 1$. For $i = 1$ the equation $(*)$ reduces to the desired form and after that induction applies. But there is also a final equation for $i = n - 1$ which reduces to

$$0 = n a_{n-1}^\sigma$$

or $n a_{n-1} = 0$. As A is divisible it follows that there are only finitely many possible values for a_{n-1}, which contradicts our assumption. $\square$

4.3. Groups covering $\mathrm{Sym}(n)$**.** Now we look at actions on connected abelian groups of groups Σ mapping homomorphically onto $\mathrm{Sym}(n)$.

We first mention some comparatively low dimensional representations of double covers of $\mathrm{Sym}(n)$, and the like. The Schur multiplier of $\mathrm{Sym}(n)$ is $\mathbb{Z}/2\mathbb{Z}$, but their are two nonisomorphic double covers.

(1) $2.\mathrm{Sym}(4) = \mathrm{GL}(2, 3)$ has a 2 dimensional representation.
(2) Both of the double covers of $\mathrm{Sym}(7)$ have 4-dimensional representations in characteristic 7.
(3) $\mathrm{Alt}(8) = \mathrm{SL}(4, 2)$ has 4-dimensional representations in characteristic 2.

Now we give our main estimate for groups covering $\mathrm{Sym}(n)$.

Proof of Proposition 4.1. We may suppose that Σ is a Frattini cover of $\mathrm{Sym}(n)$.

Suppose first that $\mathrm{rk}(A) = 1$. In this case any abelian subgroup of Σ is cyclic and in particular the Sylow 2-subgroup T of Σ contains a unique involution. So this involution is in the kernel of the map $\Sigma \to \mathrm{Sym}(n)$, and T itself is either cyclic or a generalized quaternion subgroup. So the image of T in $\mathrm{Sym}(n)$ is abelian, contradicting the structure of $\mathrm{Sym}(n)$.

So we may suppose $\mathrm{rk}(A) \geq 2$. That being the case, we may also suppose $n \geq 8$, but as one might reasonably aim at a sharper lower bound, e.g. $(n-2)/2$, we will only assume here that

$$n \geq 5$$

Let Γ be the kernel of the given homomorphism $\Sigma \to \mathrm{Sym}(n)$. Since $\Sigma \to \mathrm{Sym}(n)$ is a Frattini cover, Γ is nilpotent. Let Γ_2 be the Sylow 2-subgroup. If $\Gamma_2 = 1$ then one expects the same estimates as in the preceding subsection, but in any case Σ contains an elementary abelian 2-subgroup E with $m(E) = \lfloor n/2 \rfloor$ and this suffices for our estimate. So we suppose

$$\Gamma_2 > 1$$

Let $A_0 \leq A$ be Σ-minimal. Let Γ_0 be the kernel of the action of Σ on A_0. If Γ_0 is contained in Γ then we can look at the action of Σ/Γ_0 on A_0 and conclude by induction unless $A = A_0$. On the other hand, if Γ_0 is not contained in Γ then Γ_0 covers $\mathrm{Alt}(n)$ and hence Σ is generated by Γ_0 and a 2-element, and $\Sigma' \leq \Gamma_0$. If $A_0 < A$ we can consider the action of Σ on A/A_0 similarly and conclude by induction unless Σ' acts trivially on A/A_0. So finally Σ' acts trivially on A_0 and A/A_0 and hence the subgroup of Σ' generated by its $2'$-elements acts trivially on A, a contradiction as this covers $\mathrm{Alt}(n)$. So we may suppose that A is Σ-minimal.

Now let Σ_1 be any subgroup of Σ covering $\mathrm{Alt}(n)$. We claim that A is Σ_1-minimal. We may replace Σ_1 by a minimal subgroup covering $\mathrm{Alt}(n)$ and then $\Sigma_1 = \Sigma_1'$ and thus $\Sigma_1 = \Sigma^{(\infty)}$ is normal in Σ. Now suppose A is not Σ_1-minimal. Let $A_0 < A$ be Σ_1-minimal. Then A is the almost direct sum of two conjugates of A_0. Thus $\mathrm{rk}(A) = 2\,\mathrm{rk}(A_0)$. Now there is a subgroup of Σ_1 covering a copy of $\mathrm{Sym}(n-2)$ inside $\mathrm{Alt}(n)$, so we may suppose by induction that $\mathrm{rk}(A_0) \geq \lfloor (n-2)/4 \rfloor$ and thus $\mathrm{rk}(A) \geq \lfloor n/4 \rfloor$ and we conclude. So we may suppose that A is Σ_1-minimal for all such Σ_1.

Now we return to the structure of the nilpotent group Γ and its Sylow 2-subgroup Γ_2. Suppose that Γ_2 is noncyclic and take k minimal so that $Z_k(\Gamma_2)$ is noncyclic. Let $C = Z_{k+1}(\Gamma_2)$ and let Q be the preimage in Γ_2 of $\Omega_1(Z_k(\Gamma_2)/C)$. Let $Q_0 = C_Q(C)$.

Now Γ_2 acts trivially on Q/C and thus there is an action of $\mathrm{Sym}(n)$ on Q/C. If this action is faithful then $m_2(Q/C) \geq 4$ and hence $Q_0 > C$. In this case we can consider the action of $\mathrm{Sym}(n)$ on Q_0/C, and if this is faithful then $m_2(Q_0/C) \geq n-1$ unless $n = 6$ and $m_2(Q_0/C) = 4$.

Suppose $m_2(Q_0/C) \geq n - 1$. If Q_0 is elementary abelian then we get a strong estimate. Otherwise, $Q_0' \leq C$ is the subgroup of order 2 and the commutator map $Q_0/C \times Q_0/C \to Q_0'$ gives a symplectic structure on Q_0/C, possibly degenerate. We may take a totally isotropic subspace of Q_0/C of dimension $(n-1)/2$, which lifts back to an abelian subgroup $V \leq Q_0$ for which $m_2(V/C) = (n-1)/2$. Thus V contains an elementary abelian subgroup of rank $(n-1)/2$ and we get the desired estimate.

Now suppose $m_2(Q_0/C) = 4$ and $n = 6$, with $\mathrm{Sym}(n)$ acting faithfully and with $Q_0' > 1$. Let Q_1 be the preimage in Q_0 of $\Omega_1(Q_0/Q_0')$. Then Q_1/Q_0' is elementary abelian of rank 4. As $\mathrm{Sym}(n)$ respects the symplectic structure it coincides with $\mathrm{Sp}(4,2)$ and in particular acts transitively on Q_1/Q_0'. Then as $Q_1 \setminus C$ contains involutions, Q_1 has exponent two and thus is elementary abelian of rank 5. So $\mathrm{rk}(A) \geq 5$.

So we are left with the case in which $\mathrm{Alt}(n)$ centralizes Q/C and hence a subgroup of Σ covering $\mathrm{Alt}(n)$ centralizes Q. If Q contains a noncyclic elementary abelian subgroup E then let $A_0 \leq A$ be E-minimal and let E_0 be the kernel of the action of E on A_0. Then $C_A{}^\circ(E_0)$ is $C_\Sigma(Q)$-invariant and thus $C_A{}^\circ(E_0) = A$, implying $E_0 = 1$ and E is cyclic after all.

So Q contains a unique involution and is therefore either cyclic or generalized quaternion. But by our current assumption Q is not cyclic. On the other hand by construction $Q/Z(Q)$ is elementary abelian, and so Q is a quaternion group of order 8, and is centralized by a subgroup of Σ covering $\mathrm{Alt}(n)$.

Taking a 2-element $t \in \Sigma$ which represents a transposition in $\mathrm{Sym}(n)$, we may suppose after adjustment that t centralizes an element a of order 4 in Q_0. Then $C(a)$ covers $\mathrm{Sym}(n)$ so by minimality $a \in Z(\Sigma)$ and in particular $a \in Z(Q)$, a contradiction.

All of this shows that Γ_2 is cyclic. Now one can show in the same way that Γ is cyclic and centralized by $\mathrm{Alt}(n)$ and pursue the matter further, but we will stop here with a simple estimate. We have Γ_2 centralized by a subgroup of Σ covering $\mathrm{Alt}(n)$. Taking an elementary abelian subgroup of $\mathrm{Alt}(n)$ of rank at least $2\lfloor n/4 \rfloor$, and lifting to a 2-subgroup of Σ, we get an elementary abelian subgroup of rank at least $\lfloor n/4 \rfloor$ in Σ, and conclude. $\qquad\square$

5. Simple permutation groups

In the present section we will bring to bear a good deal of recent work on simple groups of finite Morley rank to the problem of bounding their rank or degree of multiple transitivity in terms of the rank of a set acted upon faithfully. We begin with by recalling some of the relevant information.

The material in subsection 5.4 is in a very provisional state and may not even be internally consistent.

5.1. **The four types.** The 2-Sylow theory for groups of finite Morley rank gives a broad division of all such groups into four types, according to the structure of their 2-Sylow° subgroups (maximal connected 2-subgroups). In general these have the form

$$U * T$$

a central product with finite intersection of a 2-unipotent group U (a connected, definable 2-group of bounded exponent) and a 2-torus T (a divisible abelian 2-group, not definable). The group G is said to be of even, odd, mixed, or degenerate type according as $S = U$, $S = T$, $U, T > 1$, or $S = 1$. A delicate analysis depending on a body of technology developed in the finite case, and some other ingredients, shows the following.

Theorem 6. *A simple group of finite Morley rank of even type is algebraic.*

Theorem 7. *Let G be a group of finite Morley rank. Then G contains normal subgroups $T_2(G)$, $U_2(G)$ of odd and even types respectively which commute, with $G/[T_2(G) * U_2(G)]$ of degenerate type.*

Here $T_2(G)$ is the subgroup generated by all definable hulls of 2-tori in G, and $U_2(G)$ is the subgroup generated by all unipotent 2-subgroups. Theorem 7 depends on Theorem 6. In fact what one actually shows is that Theorem 7 holds for groups whose definable simple sections of even type are all algebraic.

These theorems have the following consequence, which can however be proved much more directly [6].

Theorem 8. *Let G be a simple group of finite Morley rank containing no nontrivial divisible torsion subgroup. Then G is of degenerate type.*

Another result of a general character with a comparatively short proof is the following [5].

Theorem 9. *Let G be a connected group of finite Morley rank of degenerate type. Then G contains no involutions.*

Actually this can be extended to a similar result holding for any prime p. In that form it reads as follows.

Theorem 10 ([5])**.** *Let G be a connected group of finite Morley rank containing no infinite abelian p-subgroup. Then G contains no elements of order p.*

The classification of simple groups of even type has various structural consequences which are probably not accessible without that classification. For example we have the following [1].

Proposition 5.1. *Let G be a connected group of finite Morley rank of even type and suppose that G contains no nontrivial normal 2-unipotent subgroup. Then G is a central product of definable subgroups*

$$E(G) * \hat{O}(G)$$

where $E(G)$ is a product of quasisimple algebraic groups in characteristic two, and $\hat{O}(G)$ has degenerate type.

These notions apply in our context as follows.

Lemma 5.2. *Let G be a definably primitive permutation group of finite Morley rank. Then G° is not of mixed type.*

Proof. Supposing the contrary, $U_2(G)$ is nontrivial. If G contains a nontrivial normal 2-unipotent subgroup then the definable socle of G is an elementary abelian 2-group. But then the definable socle of G is disjoint from $T_2(G)$ and thus $T_2(G)$ is trivial.

Similarly, if G contains no nontrivial normal 2-unipotent subgroup, then the definable socle of $U_2(G)$ is a product of simple groups of even type and thus the definable socle of G is also a product of simple groups of even type, and in fact all of the factors are isomorphic. Again, this forces the socle to miss $T_2(G)$ and thus $T_2(G) = 1$. □

We can refine this a little when we have generic 2-transitivity as well.

Lemma 5.3. *Let (G, Ω) be a definably primitive and generically 2-transitive permutation group of finite Morley rank. Then G° is either of odd or of even type.*

Proof. We have already eliminated mixed type. On the other hand, as G is generically 2-transitive, also G° is generically 2-transitive, and hence there is an involution in G° swapping two generic and independent elements of Ω. Since G° is connected and contains an involution, it is not of degenerate type. $\square$

This really does require primitivity in addition to generic 2-transitivity.

Example 9. *Let $G = G_0 \times G_1$, where $G_i = \mathrm{GL}(V_i)$ and V_i are vector spaces over fields K_i, with K_0 of characteristic two and K_1 not of characteristic two. Then G has mixed type and acts generically n-transitively on $V_1 \times V_2$, where n is the minimal dimension involved.*

5.2. **Even type groups.** In the simple case, we can use the classification in even type to reduce the analysis to the specific case of Chevalley groups. The theory in odd type is not sufficiently advanced to allow this kind of sweeping reduction to known groups, though there is a large body of work also in this direction.

Lemma 5.4. *Let (G, Ω) be a definably primitive group of finite Morley rank, where the group G is isomorphic as an abstract group to a Chevalley group over an algebraically closed field. Then the algebraic dimension of a maximal torus T of G is at most $\mathrm{rk}(\Omega)$, and if the characteristic of the base field is positive then $\mathrm{rk}(T) \leq \mathrm{rk}(\Omega)$.*

Proof. By Lemma 3.8, $\mathrm{rk}(T/O(T)) \leq \mathrm{rk}(\Omega)$. Now T is definably a direct product of 1-dimensional tori T_i and $O(T) = \prod O(T_i)$, so $T/O(T)$ is also a product $\prod T_i/O(T_i)$ and therefore $\mathrm{rk}(T/O(T))$ is at least the number of factors.

Now in the case of positive characteristic we also know that T is a good torus by Wagner's results, and thus $O(T) = 1$. $\square$

Proposition 5.5. *Let (G, Ω) be a simple permutation group of finite Morley rank and even type. Then the rank of G is bounded by a function of the rank of Ω (e.g., $4\,\mathrm{rk}(\Omega)^2$).*

Proof. If d is the algebraic dimension of G and ℓ the Lie rank, then for the classical groups of types $A_\ell, B_\ell, C_\ell, D_\ell$ we have $d = \ell(\ell + 2)$ or $\ell(2\ell \pm 1)$, so at most $2\ell^2 + \ell$. This leaves aside only the exceptional groups of types $E_6 - E_8, F_4, G_2$, for which the Lie ranks and dimensions are: $(6, 78); (7, 133); (8, 248); (4, 52); (2, 14)$ respectively.

Now if f is the rank of the base field (visible in various ways in the group, notably via the root groups), then $\mathrm{rk}(G) = df$ and $\mathrm{rk}(T) = \ell f$. So it suffices to bound d/ℓ in terms of $\mathrm{rk}(\Omega)$, and since $\ell \leq \mathrm{rk}(T) \leq \mathrm{rk}(\Omega)$, this is certainly possible. $\square$

So the problem now is to get some control over actions of simple groups of odd type without an explicit classification (and even for Chevalley groups in characteristic zero there is something still to analyze).

5.3. **Odd type groups.** We now need some general structural properties of odd type groups, and we have to elaborate a little on the existing theory. Our starting point is the following.

Definition 5.6. *A decent torus is a divisible abelian group of finite Morley rank which is the definable hull of its torsion subgroup.*

Theorem 11 ([9])**.** *Let T be maximal definable decent torus in a connected group G of finite Morley rank. Then the generic element of G belongs to a unique conjugate of $C^\circ(T)$. Furthermore, any two maximal decent tori of G are conjugate.*

The following inessential variant can be proved the same way, or simply deduced from the foregoing.

Proposition 5.7. *Let G be a connected group of finite Morley rank and T a maximal p-torus of G. Then the generic element of G lies in one and only one conjugate of $C^\circ(T)$. Furthermore, any two maximal p-tori of G are conjugate in G.*

Proof. Since any maximal p-tori extend to maximal decent tori, the conjugacy statement here follows from the preceding.

Now let $\hat{T}$ be a maximal decent torus containing the given p-torus T. Then for a generic set of elements $a \in G$, a lies in one and only one conjugate of $C^\circ(\hat{T})$, and in particular a centralizes a conjugate of T. We may suppose a centralizes $\hat{T}$.

If a centralizes the maximal p-torus T_1 of G, and $\hat{T}_1$ is a maximal decent torus in $C^\circ(a)$ containing T_1, then T and T_1 are conjugate in $C^\circ(a)$ and hence $\hat{T}_1$ is also a maximal decent torus of G. Thus $\hat{T}_1 = \hat{T}$ (by our choice of a) and $T_1 \le \hat{T}$. So by maximality of T and T_1, we have $T = T_1$. $\square$

In a similar vein the following will be useful.

Lemma 5.8. *Let G be a connected group of finite Morley rank containing no nontrivial unipotent p-subgroup. Then for a generic element a of G, the group $d(a)$ is p-divisible.*

Proof. Let T be a maximal p-torus of G. Let $X \subseteq G$ be the set of elements belonging to a unique conjugate of $C^\circ(T)$. This set is generic in G and disjointly covered by conjugates of $X \cap C^\circ(T)$. So it suffices to look at generic elements of $C^\circ(T)$. In other words, we may suppose that the maximal p-torus T is central in G.

We claim then that all elements of order p in G belong to T. Let $\hat{T} = d(T)$ be the definable hull of T, a divisible abelian group whose p-torsion lies in T. Our claim is that $G/\hat{T}$ contains no p-torsion. Otherwise, $\bar{G} = G/\hat{T}$ contains a nontrivial connected definable abelian p-subgroup $\bar{K}_0$, whose definable hull is the image of a connected nilpotent subgroup K of G. Now K is a connected nilpotent group containing p-torsion not in T, and as T is a maximal p-torus in K there must be a nontrivial p-unipotent subgroup in K, contradicting our hypothesis.

Now consider $a \in G$ generic over T (which is a countable set). Then $d(a)$ has the form $A \times T_0$ with A p-divisible and T_0 a finite cyclic p-group. In particular $T_0 \le T$.

Now $a = bt$ with $b \in A$ and $t \in T_0$. So $AT_0 = d(a) \le d(b) \times \langle t \rangle$ and thus $A = d(b)$, $T_0 = \langle t \rangle$. But $b = at^{-1}$ is also generic over T, and $d(b)$ is p-divisible. As there is a unique generic type in G, we must have $d(g)$ p-divisible for any realization g of this type. $\square$

Unlike most results of this general character, the lemma does not yield a generic *definable set* of elements g for which the definable hull $d(g)$ is p-divisible. For example, working in an ordinary algebraic torus over an algebraically closed field, any generic set will contain p-elements for all primes p different from the characteristic, and for such elements the definable hull is the finite cyclic group generated by them.

The following specialized result will be quite useful for the analysis of groups of odd type as permutation groups.

Proposition 5.9. *Let G be a connected group of finite Morley rank and odd type, and let T be a maximal 2-torus of G. Then T contains all the involutions in $C(T)$.*

Proof. It is convenient to generalize this a little, allowing G also to be of degenerate type (i.e. $T = 1$), in which case we arrive at the nontrivial but known result that degenerate type groups contain no involutions.

We suppose toward a contradiction that G is a minimal counterexample, and that $i \in G$ is an involution centralizing a maximal torus T of G, with $i \notin T$. We make a number of reductions. Note that i cannot belong to any proper definable connected subgroup containing T. Furthermore G is nonabelian.

$$(1) \qquad\qquad\qquad\qquad Z^\circ(G) = 1$$

The connected abelian subgroup $Z^\circ(G)d(T)$ does not contain i. Therefore in the quotient $\bar{G} = G/Z^\circ(G)$, the subgroup $\bar{T}$ is a maximal 2-torus centralizes by an involution $\bar{i}$ which lies outside $\bar{T}$. If $Z^\circ(G) > 1$ then we contradict the minimality of G (or $T \le Z^\circ(G)$ and then we reduce to the degenerate case). So $Z^\circ(G) = 1$.

$$(2) \qquad\qquad \text{Without loss of generality, } Z(G) = 1$$

What we actually have is $Z(G)$ finite, and we wish to factor out the center. For this it suffices to check first that $i \notin Z(G)T$. Assuming the contrary, we may even take $i \in Z(G)$.

Now a generic element of G lies in a conjugate of $C^\circ(T)$. So for a generic set of elements $g \in G$, both g and gi lie in subgroups of the form $C^\circ(T_1)$, $C^\circ(T_2)$ for some conjugates T_1, T_2 of T. In particular $T_1, T_2 \le C^\circ(g)$. So T_1 and T_2 are conjugate in $C^\circ(g)$. Since $g \in C^\circ(T_1)$, we find $g \in C^\circ(T_2)$ as well. Since $gi \in C^\circ(T_2)$ we find $i \in C^\circ(T_2)$. So conjugating in G, we find $i \in C^\circ(T)$ as well. Now $C^\circ(T)/d(T)$ has degenerate type and hence contains no involutions. That is, $i \in d(T)$ and thus $i \in T$, a contradiction. Replacing G by $\bar{G} = G/Z(G)$, we may suppose $Z(G) = 1$.

As $C^\circ(i) < G$, minimality of G yields

$$(3) \qquad\qquad\qquad\qquad i \notin C^\circ(i)$$

This is the sort of pathological situation in which a genericity argument becomes available.

We consider the coset $iC^\circ(i)$. A generic element of this coset has the form ia with a a generic element of $C^\circ(i)$ (treating i as a fixed parameter), and in particular there is one and only one conjugate T_a of T in $C^\circ(i)$ centralizing a.

Now $d(a)$ is 2-divisible by Lemma 5.8, and $[d(a) : d(a^2)] \le 2$, so $d(a) = d(a^2)$ and thus $a \in d(ai)$ and $d(ai) = d(a, i)$. Hence $C(ai) = C(a) \cap C(i)$ and thus T_a is the is also the unique conjugate of T in G commuting with ai.

Let $X = \{aC^\circ(i) : a \text{ is generic in } C^\circ(i) \text{ over } i\}$. Suppose that $x \in X \cap X^g$ for some $g \in G$, so that

$$x = ai = (bi)^g$$

with a and b generic over i in $C^\circ(i)$. Then $T_a = T_{ai}$ and $T_b = T_{bi}$. For notational simplicity suppose that $T_a = T$. Then $T_a = T_b^g$. There is also some $h \in C^\circ(i)$ with $T^h = T_b$. Then $T^{hg} = T$ and $hg \in N(T)$.

Furthermore we have $i^g = i^{hg} \in C(T) \cap iC^\circ(i)$, since $i^g \in d(bi)^g = d(ai)$. Writing $i^g = ij$ with $j \in C^\circ(i)$, we see that $j \in C^\circ(i)$ commutes with T, so by induction $j \in T$. Thus $i \in i\Omega_1(T)$. This is a finite set.

Now suppose the element x lies in $X^{g_1} \cap X^{g_2}$ and the conjugates i^{g_1}, i^{g_2} are equal. Then $g_1 g_2^{-1} \in C(i)$, and therefore $X^{g_1} = X^{g_2}$. So all of this shows that x belongs to only finitely many distinct conjugates of X. Let the number of such conjugates be k.

The type X (over i) does not have $k+1$ distinct conjugates which intersect. So the same applies, by compactness, to some i-definable subset $\hat{X}$ of $iC^\circ(i)$ containing X. We claim that $\bigcup \hat{X}$ is generic in G.

We have $N(\hat{X}) \leq N(C^\circ(i)\langle i \rangle)$ since $\hat{X}$ generates $C^\circ(i)\langle i \rangle$. So by a Frattini argument we have $N(\hat{X}) \leq C(i)N(T\langle i \rangle)$. Now $N^\circ(T\langle i \rangle) = C^\circ(T\langle i \rangle) \leq C(i)$, so $N^\circ(\hat{X}) = C^\circ(i)$. Now a standard rank computation (first used in the theory of bad groups, and much used recently, cf. [13]), shows that the union $\bigcup \hat{X}^G$ is generic in G.

However for all $x \in \hat{X}$, the group $d(x)$ is not 2-divisible, while for generic $g \in G$, the group $d(g)$ is 2-divisible, and we arrive at a contradiction. $\square$

We tidy this up a little more.

Corollary 5.10. *Let G be a connected group of finite Morley rank and odd type, and let T be a maximal 2-torus of G. Then T contains all the 2-elements in $C(G)$.*

Proof. Supposing on the contrary that t is a 2-element in $C(T)$ outside T, we may take t to have order 2 modulo T. Then taking $t_0 \in T$ with $t_0^2 = t^2$ we have $tt_0^{-1} \in T$ by the previous proposition. $\square$

5.4. **Primitive simple groups of odd type.** Now we come back to permutation groups. We are interested in working with the action of a maximal 2-torus in a definably primitive simple permutation group of odd type.

Lemma 5.11. *Let (G, Ω) be a definably primitive permutation group of finite Morley rank, and let $r = \mathrm{rk}(\Omega)$. Then for any k there is an $N \leq kr + 1$ such that for any sequence*

$$x_1, \ldots, x_N, \ldots, x_{N+k}$$

of independent generic elements of Ω, the point stabilizer $G_{N+k} = G_{x_1, \ldots, x_{N+k}}$ contains a maximal 2-torus of $G_N = G_{x_1, \ldots, x_N}$.

Proof. Take an infinite sequence $x_1, x_2, \ldots$ of independent generic elements of Ω. Let G_i be the point stabilizer $G_{x_1, \ldots, x_i}$. Let T_i be the definable hull of a maximal 2-torus of G_i, and let $r_i = \mathrm{rk}(T_i/O(T_i))$.

The sequence r_i is monotonically decreasing and takes on at most $r + 1$ values in view of Lemma 3.8. Thus over the interval $[1, \ldots, k(r+1) + 1]$ there must be an interval $[N, N+k]$ of length k over which the function is constant. Here $N \leq kr + 1$.

Suppose now that T_1 is the definable hull of a maximal 2-torus of G_{N+k} and that T_2 is the definable hull of a maximal 2-torus of G_N with $T_1 \leq T_2$. As $r_N = r_{N+k}$ we have

$$\mathrm{rk}(T_1/O(T_1)) = \mathrm{rk}(T_2/O(T_2))$$

Thus $T_2 = T_1 O(T_2)$ and hence the maximal 2-torus of T_2 is contained in T_1, as claimed. $\square$

Proposition 5.12. *Let (G, Ω) be a definably primitive permutation group of finite Morley rank with G connected and of odd type. Then the degree of generic transitivity of G is at most $4r^2 + 5r + 1$*

Proof. Take an infinite sequence $x_1, x_2, \ldots$ of independent generic elements of Ω. Let $k = 4r + 4$. Suppose that G is generically $[k(r + 1) + 1]$-transitive. Apply the preceding lemma to get a corresponding value of N. Let H be the point stabilizer $G_{x_1, \ldots, x_N}$ and Let F be the set

$$\{x_{N+1}, \ldots, x_{N+k}\}$$

Then H_F is the point stabilizer $H_{x_{N+1}, \ldots, x_{N+k}}$. As G is generically $(N+k)$-transitive, the setwise stabilizer $H_{\{F\}}$ induces the full symmetric group $\mathrm{Sym}(k)$ on F.

Fix a maximal 2-torus T_0 of H_F and let T be its definable hull. By the Frattini argument there is a subgroup of $H_{\{F\}}$ normalizing T and covering $\mathrm{Sym}(k)$. Modulo the kernel of the action on $T/O(T)$, the group still covers $\mathrm{Sym}(k)$, as its 2-elements are forced to act nontrivially on T_0 by Corollary 5.10.

This then yields $\mathrm{rk}(T/O(T)) \geq \lfloor k/4 \rfloor > r$ violating our estimate in Lemma 3.8.   □

Note that for $r = 2$ this gives a bound on generic transitivity of at most 27 for a definably primitive action. Everything depends on the bound in Lemma 4.1 and there is considerable room for improvement here, particularly when r is low.

The bound in [11] for *sharp* generic transitivity without assuming definable primitivity is 6. In the actual example of $\mathrm{SL}(n)$, with generic sharp $(n+1)$-transitivity, a maximal torus of rank $n - 1$ appears in the stabilizer of n independent points with a group $\mathrm{Sym}(n)$ acting on it, so the various estimates do not give up too much up to this point. But after fixing another generic independent point one loses the whole torus in this case. It would also be helpful to understand whether double covers of $\mathrm{Sym}(n)$ can occur in the normalizer of a maximal 2-torus.

6. Generic multiple transitivity

6.1. **An extremal case.** The following conjecture has the potential to involve a good deal of the existing theory of groups of finite Morley rank.

Problem 9. *Let G be a connected group of finite Morley rank acting faithfully, definably, transitively and generically $(n+2)$-transitively on a set Ω of Morley rank n. Then the pair (G, Ω) is equivalent to the projective linear group $\mathrm{PGL}_{n+1}(F)$ acting on the projective space $\mathbb{P}^n(F)$ for some algebraically closed field F.*

Of course the transitivity hypothesis is superfluous as one may restrict to the relevant orbit in any case. It is here just to keep the notation clean. Note that we aim to show in particular that the action of G is primitive.

This conjecture would give natural bounds for generic multiple transitivity of permutation groups of finite Morley rank. But the main point is that it provides a convenient sandbox for trying out recently developed methods, in particular those used in the ongoing analysis of groups of odd type. A major part of this conjecture is the "affine group" case which might come in to the analysis of the point stabilizer of the group from Problem 9; see Problem 13 below.

One reason that induction may be useful is the following.

Lemma 6.1. *Let (G, Ω) be a transitive and generically n-transitive permutation group of finite Morley rank. Let E be a definable G-invariant equivalence relation on Ω. Then $(G, \Omega/E)$ is generically n-transitive (though not necessarily faithful).*

Proof. As G acts transitively on Ω/E, the equivalence classes have constant rank. So generic sets in Ω and Ω/E correspond. □

Taking an inductive approach to Problem 9, then under the stated hypotheses it follows that any proper G-invariant definable equivalence relation E will have finite classes. One can then show that there is a maximal such relation E, definable by $E(\alpha, \beta) \iff G_\alpha{}^\circ = G_\beta{}^\circ$. There is a potential issue with the uniform definability of $G_\alpha{}^\circ$ but that is not a problem for transitive actions. Modulo this maximal E, we have a definably primitive action. So if we have the desired result for definably primitive actions, then the pair $(G/K, \Omega/E)$ is identified, with K the kernel of the action on the quotient. Now K leaves each class of E invariant, so K° acts trivially on Ω, hence K is finite and G is a central extension of G/K. In particular K is contained in a maximal torus T of G. Now T leaves some class C of E invariant, and being connected fixes C pointwise. Hence K has fixed points, and being central acts trivially on Ω. So $K = 1$. So without loss of generality one may suppose (G, Ω) is definably primitive in Problem 9.

Thus we may bring [14] to bear again, and now the group G is connected. This is a very special case, and the classification then reduces to the following possibilities.

(1) The affine case: $G = A \rtimes H$ with H a point stabilizer acting generically $(n + 1)$-transitively on an elementary abelian group of rank n.

(2) The socle is a simple group L acting regularly on Ω and $G = L \rtimes H$ with H a point stabilizer acting generically $(n+1)$-transitively on the simple group L of rank n as a group of automorphisms.

(3) The socle is a simple group L acting nonregularly and G lies between L and $\mathrm{Aut}(L)$.

(4) Diagonal action: the socle is the product of two isomorphic simple groups $L_1 \times L_2$. $L_1 \times L_2 \leq G \leq \mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2)$ and furthermore $G/(L_1 \times L_2) \leq \Delta(\mathrm{Out}(L))$ where $L \cong L_1 \cong L_2$ and $\Delta(\mathrm{Out}(L))$ is the subgroup of $\mathrm{Out}(L_1) \times \mathrm{Out}(L_2)$ corresponding to the diagonal subgroup via an isomorphism of L_1 with L_2. In other words, if $\hat{G}$ is the subgroup of $\mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2)$ consisting of pairs (α, β) with $\alpha \equiv \beta \mod \mathrm{Inn}(L)$, we want

$$L_1 \times L_2 \leq G \leq \hat{G}$$

The set Ω is defined by specifying the point stabilizer, which is the diagonal subgroup $\Delta(\mathrm{Aut}(L))$ (or its intersection with G).

In the third and fourth cases the condition of finiteness of Morley rank is extremely restrictive. If L is a Chevalley group then any connected subgroup of $\mathrm{Aut}(L)$ containing L will coincide with L itself, while $\mathrm{Aut}(L)$ includes everything induced by a field automorphism.

In Problem 9 our target is type (3) and therefore one of the things one wants to do, presumably, is to remove the other possibilities. However another idea is to use induction and focus on the point stabilizer. Probably one should do both, but the second seems like the main avenue of attack.

Let us examine the basic case of type (4), where $G = L \times L$ and $L_\alpha = \Delta(L)$. This is equivalent to the natural action of $L^{\mathrm{op}} \times L$ on L on the left and right, which

as an action of $L \times L$ becomes $(g, h).a = g^{-1}ah$. Thus $G_1 = \Delta(L)$ and the action of G_1 on L is equivalent to the action of L on L by conjugation.

The general case of (4) is similar. The set Ω can be identified with $\Delta(L)\backslash L \times L$ which has as representatives either copy of L, and after this identification G_1 becomes a subgroup of $\mathrm{Aut}(L)$ containing L and acting on Ω as it acts on L.

Now in both cases (2) and (4) there is a definable simple normal subgroup L acting regularly on Ω, on which the point stabilizer G_α acts as a group of automorphisms. Thus in these cases $\mathrm{rk}(L) = n$ and G_α acts generically $(n + 1)$-transitively on L. By the proof of Lemma 3.3 if $a_1, \ldots, a_n$ are n independent generic elements of L, then $d(a_1, \ldots, a_n) = L$ unless $\mathrm{rk}(d(a_1, \ldots, a_\ell)) \leq \ell - 1$ for all $\ell \leq n$. In particular the generic element of L has finite order.

Problem 10. *Let L be a connected $\aleph_0$-saturated group of finite Morley rank such that for two independent generic elements a, b we have $\mathrm{rk}(d(a, b)) \leq 1$. Show that L is nilpotent of bounded exponent.*

This, or something similar, would get rid of these two cases for our present problem.

Alternatively, as we will see later, in many such cases one can force the socle to act generically doubly transitively and in particular get an involution into L, in which case it is definitely of unbounded exponent, as it is not of degenerate type.

One may also use Lemma 3.8 to study the point stabilizer $M = G_\alpha$ on an appropriate definable quotient Ω/E. Here one hopes to arrive at a configuration where modulo the kernel K of the action, M/K contains a subgroup $\mathrm{PGL}_n(F)$, while K has Prüfer rank at most 1. This configuration deserves close study. In particular, one should try to deal with the following.

Problem 11. *If a simple algebraic group L acts definably on a group K of degenerate type, then $[K, L]$ is nilpotent.*

There is a similar conjecture for actions of the definable hull of a 2-torus on a degenerate type group, which would give the solution to this problem as a special case. But perhaps this version is more accessible.

6.2. **The affine case.** For the affine case, we want the following.

Problem 12. *Let G be a connected group of finite Morley rank acting faithfully, definably, and generically t-transitively on an abelian group V of Morley rank n. Then $t \leq n$.*

Later, with an eye on the point stabilizer, we may also want the extremal affine case treated fully. Whether or not it enters directly into the treatment of Problem 9, it is of independent interest.

Problem 13. *Let G be a connected group of finite Morley rank acting faithfully, definably, and generically n-transitively on a connected abelian group V of Morley rank n. Then V has a structure of a n-dimensional vector space over an algebraically closed field F of Morley rank 1, and G is $\mathrm{GL}_n(F)$ in its natural action on F^n.*

This problem connects closely to the structure theory for groups of finite Morley rank. Note that Problem 12 is a special case of Problem 13.

In either of these problems A should be G-minimal, if we are in an inductive setting. We have only to consider the possibility of a nontrivial G-invariant finite

subgroup A_0, in which case G fixes A_0 pointwise. Passing to A/A_0 we either have a contradiction or an identification of G/K acting on A/A_0 with $\mathrm{GL}_n(F)$ in its natural action, where $K = C_G(A_0)$. As usual K° acts trivially A and K is finite. As G is connected we find $K = 1$. So now we have

$$1 \to A_0 \to A \to A/A_0 \to 1$$

a sequence of G-modules with A_0 finite and trivial, and A/A_0 natural. The associated 2-cocycle with values in A_0 is "generically constant" in each variable, the extension so should split definably, another point to be checked. As A is connected we have finally $A_0 = 1$.

In particular A is either torsion free or an elementary abelian p-group for some p. In the context of Problem 12 only the elementary abelian case is possible, and in the context of Problem 13 one may have the torsion free case, with each subgroup of the form $d(a)$ of rank 1. But in the latter case one has also that the definable closure of k independent generic elements has rank k and that its setwise stabilizer acts generically k-transitively. This seems like enough to complete the analysis of the torsion free case in the full generality of Problem 13, by direct induction (perhaps by characterizing the underlying projective geometry, or by more group theoretic methods). Some particular arguments of a more concrete character may also be needed in low rank cases.

Now return to Problem 12 in the case in which A is an elementary abelian p-group.

Consider any nontrivial definable G-invariant equivalence relation E such that a generic element of A belongs to an infinite class. Then G acts generically t-transitively on the quotient of the generic orbit modulo E and if $t > n$ this forces the ranks of the generic classes to be 1, by induction on n, and then the action on the quotient and the induced group G/K can be identified with $(\mathrm{PGL}(n-1, F), \mathbb{P}^{n-1}(F))$. In particular the setwise stabilizer of a single generic E-class C is its pointwise stabilizer in this representation, and acts on the class C, which however has rank 1. It follows that the whole class C has a nontrivial pointwise stabilizer G_C in G. Let $\hat{C}$ be the fixed point set of G_C. This is a proper subgroup of A. One can recover $\hat{C}^\circ$ from its generic element. So this gives rise to another G-invariant equivalence relation on the generic orbit in A, again with classes of rank 1, but now each class is generic in the group it generates. So we cover A by a collection of G-invariant subgroups with finite intersections. Actually the intersection of each subgroup with the union of its distinct conjugates is finite and G-invariant so by G-minimality these intersections are trivial.

Now it seems the kernel K of the quotient action is acting as an abelian subgroup on each class C (group $C \cup \{0\}$) and hence is itself abelian. And also a good torus. So central in G. Hence semi-regular. So if this is nontrivial then we have just put a linear structure on A and that is the end of generic $(n+1)$-transitivity. So K is finite and G is a covering group of $\mathrm{PGL}(n-1, F)$. But this acts generically sharply n-transitively on the quotient so after fixing n generic independent points in A we are out of group elements.

All of which says that as one expects any G-invariant equivalence relation has generically finite classes. And for such a class C and $a \in C$, C is contained in the fixed point set for $G_a{}^\circ$, which is a finite subgroup of A ($a \neq 0$). So we have again a primitive quotient of (G, A) or rather (G, O) with O the generic orbit, and there is at worst a finite kernel. So G is itself subject to MPOSA. An infinite abelian

normal subgroup in G would linearize A again and lead to a contradiction, so the socle of G is again simple or a product of two simple groups; actually quasisimple allowing for the finite kernel.

Two simple groups will lead again to linearization and a contradiction so the socle is a single simple subgroup L. As L acts transitively on the primitive quotient of the generic orbit it also acts transitively on that orbit.

Suppose A contains a nontrivial infinite definable proper subgroup. Then fixing a minimal one A_0, A is more or less a direct sum of conjugates of A_0 by finitely many elements of L. Then the generic element of A is a sum of independent generic elements in these conjugates (which can all be identified with A_0). So the generic $(n+1)$-transitivity of G on A gives the same thing on A_0, in lower rank, and a contradiction. So A is minimal in this strong sense.

Now G contains involutions since it is generically doubly transitive. An involution has finite centralizer on A so inverts A. But we have a 2-element interchanging generic independent elements so an involution swaps them or fixes them, a contradiction.

So it seems that in Problem 9 our original group G must have a simple socle L acting nonregularly.

If this socle is of even type then it is a Chevalley group, and $G = L$, and one is in the algebraic case, more or less. Actually as the language is enriched there are potentially more permutation representations than in the algebraic category. For the truly algebraic case generically multiply transitive actions of simple groups (also reductive groups) have been studied by Popov [19] by a method which is limited to characteristic zero for technical reasons. One does not get beyond generic 4-transitivity except in the case of the series A_ℓ, as anticipated.

Problem 14. *Extend Popov's work to all characteristics and to the finite Morley rank permutation group category in which the groups are Chevalley groups, or products of Chevalley groups and tori.*

In our category the base field can vary from group to group in a direct product, and it is not clear what a "torus" actually is, but good tori would be a natural candidate for the role. In any case it is the base case of Chevalley groups which is critical here.

This approach also leaves low values of n to deal with. Passing through this kind of classification is cumbersome for low values of n and one should look for more intrinsic methods.

The basis of the induction is discussed again, more generally, in Problems 15 and 16 formulated below.

If the even type case is eliminated then while G is of odd type, the socle of G may, in principle, be of either degenerate or odd type. It is not clear how to limit things further on an a priori basis, so it would seem to be high time to take up the point stabilizer, which also has a generically highly transitive action.

Note incidentally that we may assume

<p style="text-align:center">The point stabilizer is connected.</p>

This amounts to replacing Ω by a finite cover where the point stabilizer is replaced by its connected component.

If one succeeds in identifying the point stabilizer (and the ambient group) after this change, then all that remains is to check that it is self-normalizing in the larger group.

6.3. **The point stabilizer.** In the context of Problem 9 it would be nice to show that the point stabilizer has a faithful primitive permutation representation, and to argue that it falls in the affine class, then identify the affine action explicitly as the action of $\mathrm{GL}(n)$ on its natural module. After which one would still have to identify the original group, and the whole of the point stabilizer. However it is not so clear how to extract an appropriate representation from the data.

Let us consider the action of the point stabilizer G_α on Ω. This may be definably imprimitive. In that case one gets equivalence classes of rank 1 and a quotient of G_α to which induction applies. The connected component of the kernel operates on a collection of rank 1 sets and is therefore either solvable or has factors of type $\mathrm{PSL}(2)$. So one has, in this case, a great deal of information about G_α and one may hope to disentangle the situation from this point.

On the other hand the action of G_α on its generic orbit may be definably primitive, and while this again restricts the structure of G_α, it is not so strong, and needs to be driven toward an eventual contradiction. In particular G_α may have simple socle, in which case we are forced again to deal with simple groups having an unclear structure.

However, if L is the simple socle of G, then L_α contains the socle of G_α and by primitivity acts transitively on the generic orbit of G_α. So in this case at least L acts generically doubly transitively and in particular contains involutions, so is of odd type.

6.4. **Small affine groups.** Returning now to the affine case, we comment on the case of low rank, which would seem to require separate treatment.

Problem 15. *Let G be a connected group of finite Morley rank acting faithfully and definably on an abelian group V of Morley rank 2. Then either G is solvable, or V has a structure of a 2-dimensional vector space over an algebraically closed field F and G is one of the groups $\mathrm{SL}_2(F)$ and $\mathrm{GL}_2(F)$ in their natural action on $V = F^2$.*

Problem 15 is believed to be solved, though all such claims must be checked.

Problem 16. *Let G be a connected group of finite Morley rank acting faithfully, definably and irreducibly on an abelian group V of Morley rank 3. Then one of the following holds:*

- *V is of finite exponent $p > 2$ and G is a simple p-group.*
- *V has a structure of a 3-dimensional vector space over an algebraically closed field F and $G = ZL$ where Z is a subgroup of the group of scalar matrices and L is one of the groups $\mathrm{PSL}_2(F)$ (in its irreducible representation as a 3-dimensional orthogonal group) or $\mathrm{SL}_3(F)$ (in its natural action on F^3).*

For the treatment of Problem 16, one would have to collate all known information on "small" groups of odd type, including Altseimer's work on characterization of PSL_3 by centralizers of involutions. When G happens to be $\mathrm{PSL}_2(F)$, turning V into a vector space over F could be a problem (unless this case is already covered by Meierfrankenfeld [16]—but we have not checked the details).

The $\mathrm{PSL}_2(F)$ case of Problem 16 is likely to come out of the configuration which appears to be covered by the recent theorem by Adrian Delore [10] on minimal simple groups of Prüfer rank 1.

Fact 6.2. *Let G be a minimal simple group of odd type and Prüfer rank 1. Assume that the centralizer of a toric involution is not a Borel subgroup in G. Then $G \simeq \mathrm{PSL}_2(F)$ for an algebraically closed field F.*

Eric Jaligot has informed us that the proof of Delore's theorem as such uses the solvability assumption only for "local" subgroups in G: that, subgroups of the form $N_G^\circ(A)$ for an abelian subgroup $A < G$. Exactly this condition, solvability of local subgroups, will naturally appear in the treatment of Problem 4. It would be nice if someone would undertake the task of checking that all papers on minimal simple groups actually use only this weaker assumption; this would make Fact 6.2 immediately applicable in the analysis of Problem 16.

The following fact could be useful in proofs of Problems 15 and 16. (Again, the proof of this "fact" is being written up.)

Fact 6.3. *Let G be a simple group of finite Morley rank acting faithfully and definably on an elementary abelian p-group V. If G contains no infinite elementary abelian p-subgroups then the following statements hold.*

- *Every connected solvable subgroup of G is abelian.*
- *If C is a Carter subgroup of G then $C_G(C)$ is generous in G.*
- *Carter subgroups of G are conjugate.*
- *G contains no involutions.*

In the course of the proof of Fact 6.3 the following observation arises; one would expect it to be very useful in the proof of Problem 16.

Fact 6.4. *Let H be a connected group of finite Morley rank acting faithfully and definably on an elementary abelian p-group V. Assume that $H = LT$ where $L \lhd H$ and T is an divisible abelian group equal to the definable closure of its 2-torsion. If L contains no infinite elementary abelian p-subgroups then $[L, T] = 1$.*

6.5. **An inductive step for Problem 13.** The inductive step in the treatment of Problem 13 also appears to be natural: we take a generic point $v \in V$ and consider

$$U = C_V(C_G(v));$$

the aim is to show that $\mathrm{rk}(U) = 1$ and that therefore $H = N_G^\circ(U)/U$ acts generically $(n-1)$-transitively on V/U, which would allow us to apply the inductive assumption and conclude that $\bar{H} = H/C_H(V/U)$ is $\mathrm{GL}_{n-1}(F)$.

The next step is to glue together a Curtis-Tits system in a big subgroup G^* of G from a Curtis-Tits system in $[\bar{H}, \bar{H}] \simeq \mathrm{SL}_{n-1}(F)$ and from a Curtis-Tits system in a similar section of an appropriate conjugate of H. We discuss Curtis-Tits systems in the next subsection.

Finally, if a Curtis-Tits system in G^* is constructed and G^* is identified with $\mathrm{GL}_n(F)$ for an algebraically closed field F, the structure of an F-vector space on V is introduced by Meierfrankenfeld's characterization of natural modules for classical groups [16].

6.6. **The Curtis-Tits Theorem.** We start by quoting a very general form of the Curtis-Tits Theorem in the formulation due to Timmesfeld [20].

Fact 6.5 ([20])**.** *Let Φ be an irreducible spherical root system of Tits rank at least 3, with fundamental system Π and Dynkin diagram Δ. Let G be any group generated by rank one groups $X_r = \langle A_r, A_{-r}\rangle$ for $r \in \Pi$, with unipotent subgroups A_r, A_{-r} satisfying the condition*

$$N_{X_r}(A_r) \cap N_{X_r}(A_{-r}) \le N(X_s)$$

for all $r, s \in \Pi$. Set $X_{rs} = \langle X_r, X_s\rangle$ for $r, s \in \Pi$ distinct, and assume the following all hold.

(1) *X_r, X_s commute for r, s not connected in Δ.*
(2) *If r, s are connected in Δ, then there is a group $\bar{X} = \bar{X}_{rs}$ of Lie type with root system Φ_{rs} (the span of r, s in Φ), which is generated by subgroups $\bar{A}_\alpha$ for $\alpha \in \Phi_{r,s}$, and there is a surjective homomorphism $\phi_{rs} : X_{rs} \to \bar{X}_{rs}$, such that:*
   (a) *$\phi_{rs}[A_\alpha] = \bar{A}_\alpha$ for $\alpha \in \Phi_{rs}$;*
   (b) *$\ker \phi_{rs} \le Z(X_{rs})$;*
   (c) *If $\bar{X}_{rs}$ is defined over a field of order 2 or 3, or is of the form $\mathrm{PSL}_3(4)$, then $\ker \phi_{rs}$ is a $2'$-group or a $3'$-group respectively.*

Then there is a group $\bar{G}$ of Lie type $\mathcal{B}$, with root system Φ and with fundamental system Π, and there is a surjective homomorphism $\sigma : G \to \bar{G}$ mapping the groups $A_{\pm r}$ for $r \in \Pi$ onto the corresponding fundamental root groups and their opposites in $\bar{G}$. Furthermore, $\ker \sigma \le Z(G) \cap H$, where H is the subgroup generated by the groups $H_r = N_{X_r}(A_r) \cap N_{X_r}(A_{-r})$ for $r \in \Pi$.

The following case is the one which concerns us here.

Proposition 6.6. *Let Φ be an irreducible root system (of spherical type) and rank at least 3, and let Π be a system of fundamental roots for Φ. Let X a group generated by subgroups X_r for $r \in \Pi$, Set $X_{rs} = \langle X_r, X_s\rangle$. Suppose that X_{rs} is a group of Lie type Φ_{rs} over an infinite field, with X_r and X_s corresponding root SL_2-subgroups with respect to some maximal torus of X_{rs}. Then $X/Z(X)$ is isomorphic to a group of Lie type via a map carrying the subgroups X_r to root SL_2-subgroups.*

Since the only simple algebraic groups which we expect to appear in Problems 9 and 13 are groups PSL_n and SL_n over an algebraically closed field, we can specialize the Curtis-Tits Theorem even further.

The following fact is essentially Fact 5.2 of [4].

Proposition 6.7. *Let G be a group of finite Morley rank generated by a family of subgroups K_i, $i = 1, \ldots, n - 1$, $n \geqslant 3$. Assume that the following conditions hold:*

- *All K_i are isomorphic to $\mathrm{SL}_2(F)$ for some algebraically closed field F of characteristic $\neq 2$.*
- *$[Z(K_i), Z(K_j)] = 1$ for all $i, j = 1, \ldots, n - 1$.*
- *$[K_i, K_j] = 1$ if $|i - j| > 1$.*
- *$\langle K_i, K_j\rangle \simeq \mathrm{SL}_3(F)$ if $|i - j| = 1$.*

Then G is isomorphic to a factor group of the group $\mathrm{SL}_n(F)$ by a (finite) subgroup from the center.

Problem 16, which describes "3-dimensional" groups, could be very useful in the control of groups generated by two "root" SL_2-subgroups.

6.7. **Pseudoreflection groups.** Finally, Problem 13 may reduce to yet another Problem, also concerned with very familiar objects.

Problem 17. *Let G be a connected group acting definably, faithfully and irreducible on a abelian group V (written additively). Assume that G contains a* pseudoreflection subgroup, *that is, an abelian subgroup R such that*

(1) $V = [V, R] \oplus C_V(R)$
(2) R *acts transitively on the set of non-trivial elements in $[V, R]$.*

Then V has a structure of a vector space over an algebraically closed field F such that $[V, R]$ is one-dimensional subspace and R acts on $[V, R]$ as the multiplicative group of F, and $G = \mathrm{GL}(V)$.

It would be interesting to see whether the theory of pseudoreflection groups can be transferred from the case of groups of even type, where it features prominently in the classification theory of simple groups of finite Morley rank and even type, to the context of groups of odd type.

Notice that in the special case when we have $\mathrm{rk}([V, R]) = 1$, which is the only case needed for treating Problem 13, Problem 17 should follow easily from the results of Problems 15 and 16. However, at this point another theme comes into the plot: *classical involutions.*

Indeed, for any conjugate R^g of R, the group $L = L_g = \langle R, R^g \rangle$ centralizes $U = C_V(R) \cap C_V(R^g)$ and acts on V/U. But $\mathrm{rk}(V/U) \leqslant 2$, and it will follow from Problem 15 that $\bar{L} = L/C_L(V/U)$ is isomorphic either to an abelian group $R * R^g$ (central product), or to a soluble group $F^+ \rtimes F^*$, or to $\mathrm{GL}_2(F)$ for an algebraically closed field F, with R and R^g being one-dimensional algebraic tori.

The case in which all of the subgroups $\bar{L}_g$ are solvable, $g \in G$, is very peculiar and should lead to a contradiction with the assumption that G acts on V irreducibly.

Otherwise we choose L such that $\bar{L} \simeq \mathrm{GL}_2(F)$. It should be easy to show that $C_L(V/U) = 1$; after that, if we denote by J the derived subgroup of $L \simeq \mathrm{GL}_2(F)$, we get a remarkable subgroup:

- $J \simeq \mathrm{SL}_2(F)$;
- if z is the involution from $Z(J)$ then $J \lhd C_G(z)$;
- Let J' be a conjugate of J and z' be the involution in $Z(J')$. If z and z' commute then

$$\langle J, J' \rangle \simeq J \times J', J * J' \text{ or } \underset{3}{\mathrm{SL}}(F)$$

(this is where Problem 16 would be useful).

The classical involution analysis in the spirit of Aschbacher and Berkman [4] should hopefully lead to the configuration of the Curtis-Tits Theorem (in the version of Proposition 6.7), and ultimately yield a subgroup $G^* \simeq \mathrm{SL}_n(F)$.

We can repeat a comment made in §6.5: if a Curtis-Tits system in G^* is constructed and G^* is identified with $\mathrm{SL}_n(F)$ for an algebraically closed field F, the structure of a F-vector space on V is introduced by Meierfrankenfeld's characterization of natural modules for classical groups [16]. After that the identification of G with $\mathrm{GL}(V)$ is likely to be very straightforward.

Notice also that Problem 17 could possibly provide a very efficient way to develop the Inductive Step in Problem 13: if, in the notation of §6.5, we can indeed conclude that $H/C_H(V/U)$ is $\mathrm{GL}_{n-1}(F)$, then a pseudoreflection subgroup from $\mathrm{GL}_{n-1}(F)$ in its action on V/U is likely to induce a pseudoreflection subgroup action on V.

6.8. **One more problem.** The following arises in conjunction with attempts to analyze intersections of Carter subgroups in groups of degenerate type, which is a major focus of interest in the general theory, and may not be entirely out of place here.

Problem 18. *Let A be a connected abelian group of Morley rank n, and let $(A_i : i \in I)$ be a uniformly definable family of pairwise distinct connected definable subgroups of rank k in A. Show that $\mathrm{rk}(I) \leq \binom{n}{k}$.*

The bound is achieved when A is a vector space over a field of rank 1. The case $n = 2$ and $k = 1$ is of particular interest.

7. Problem list

We list here the problems that turned up along the way.

Problem 1 (Introduction). *Bound (tightly, if possible) the rank of a definably primitive permutation group of finite Morley rank in terms of the rank of the set on which it acts.*

Problem 2 (§1). *If (G, Ω) is a permutation group and Ω is stable in the induced language, does it follow that G is stable? Does this hold at least when Ω has finite Morley rank?*

Problem 3 (§1). *Find all the generically sharply n-transitive actions of algebraic groups over algebraically closed fields, for $n \geq 2$.*

Problem 4 (§1). *Suppose that (G, Ω) is a virtually definably primitive permutation group of finite Morley rank with which is not a finite cover of a definably primitive permutation group. Show that G is a Chevalley group of positive characteristic, and the point stabilizer is contained in $G(\mathbb{F}_q)$ for some finite field $\mathbb{F}_q$.*

Problem 5 (§1). *Is there an O'Nan-Scott-Aschbacher analysis of generically 2-transitive groups which are not necessarily definably primitive? Are all such groups essentially products of generically n-transitive primitive groups (or generically n'-transitive groups, with n' not much smaller than n)?*

Problem 6 (§2). *Find good bounds on ρ, where $\rho(r)$ is the maximum rank of a virtually definably primitive permutation group (G, Ω) of finite Morley rank, with $\mathrm{rk}(\Omega) = r$.*

Problem 7 (§2). *Find good bounds on τ, where $\tau(r)$ is the maximum degree of generic transitivity associated to a virtually definably primitive permutation group (G, Ω) of finite Morley rank, with $\mathrm{rk}(\Omega) = r$.*

Problem 8 (§5). *Let Σ be a finite group. Find lower bounds for each of the following.*

   (1) *The minimal rank of a connected solvable group of finite Morley rank which affords a faithful representation of Σ.*
   (2) *The minimal rank of a connected solvable group of finite Morley rank which affords a faithful representation of a central extension of Σ.*
   (3) *The minimal rank of a connected solvable group of finite Morley rank which affords a faithful representation of a group $\hat{\Sigma}$ which covers Σ, i.e. maps homomorphically onto Σ.*

Problem 9 (§6). *Let G be a connected group of finite Morley rank acting faithfully, definably, transitively and generically $(n+2)$-transitively on a set Ω of Morley rank n. Then the pair (G, Ω) is equivalent to the projective linear group $\mathrm{PGL}_{n+1}(F)$ acting on the projective space $\mathbb{P}^n(F)$ for some algebraically closed field F.*

Problem 10 (§6). *Let L be a connected $\aleph_0$-saturated group of finite Morley rank such that for two independent generic elements a, b we have $\mathrm{rk}(d(a,b)) \leq 1$. Show that L is nilpotent of bounded exponent.*

Problem 11 (§6). *If a simple algebraic group L acts definably on a group K of degenerate type, then $[K, L]$ is nilpotent.*

Problem 12 (§6). *Let G be a connected group of finite Morley rank acting faithfully, definably, and generically t-transitively on an abelian group V of Morley rank n. Then $t \leq n$.*

Problem 13 (§6). *Let G be a connected group of finite Morley rank acting faithfully, definably, and generically n-transitively on a connected abelian group V of Morley rank n. Then V has a structure of a n-dimensional vector space over an algebraically closed field F of Morley rank 1, and G is $\mathrm{GL}_n(F)$ in its natural action on F^n.*

Problem 14 (§6). *Extend Popov's work to all characteristics and to the finite Morley rank permutation group category in which the groups are Chevalley groups, or products of Chevalley groups and tori.*

Problem 15 (§6). *Let G be a connected group of finite Morley rank acting faithfully and definably on an abelian group V of Morley rank 2. Then either G is solvable, or V has a structure of a 2-dimensional vector space over an algebraically closed field F and G is one of the groups $\mathrm{SL}_2(F)$ and $\mathrm{GL}_2(F)$ in their natural representations.*

Problem 16 (§6). *Let G be a connected group of finite Morley rank acting faithfully, definably and irreducibly on an abelian group V of Morley rank 3. Then one of the following holds:*

- *V is of finite exponent $p > 2$ and G is a simple p-group.*
- *V has a structure of a 3-dimensional vector space over an algebraically closed field F and $G = ZL$ where Z is a subgroup of the group of scalar matrices and L is one of the groups $\mathrm{PSL}_2(F)$ (in its irreducible representation as a 3-dimensional orthogonal group) or $\mathrm{SL}_3(F)$ (in its natural action on F^3).*

Problem 17 (§6). *Let G be a connected group acting definably, faithfully and irreducible on a abelian group V (written additively). Assume that G contains a pseudoreflection subgroup, that is, an abelian subgroup R such that $V = [V, R] \oplus C_V(R)$ and R acts transitively on the set of non-trivial elements in $[V, R]$. Then V has a structure of a vector space over an algebraically closed field F such that $[V, R]$ is one-dimensional subspace and R acts on $[V, R]$ as the multiplicative group of F, and $G = \mathrm{GL}(V)$.*

Problem 18 (§6). *Let A be a connected abelian group of Morley rank n, and let $(A_i : i \in I)$ be a uniformly definable family of pairwise distinct connected definable subgroups of rank k in A. Show that $\mathrm{rk}(I) \leq \binom{n}{k}$.*

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