GENERICITY, GENEROSITY, AND TORI

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1. Introduction

Bruno Poizat is associated, among other things, with broadening the scope of the theory of groups of finite Morley rank to the level of stable groups [P87], in keeping with the general trend of the development of model theory from Morley to Shelah. But this has not prevented him from taking an active interest in such aspects of the theory as are associated more particularly with the case of finite Morley rank, and in particular the algebraicity conjecture of Zilber and the present author: *A simple group of finite Morley rank is algebraic.* In particular he joined with Borovik in one of the early manifestations of the project of bringing the techniques of finite group theory to bear on this problem, in an article on Sylow theory entitled *Tores et p-groups* [BP90], two topics which, for $p = 2$, will be central to our discussion on this occasion. The 2-Sylow theory became the basis of a vigorous programme of close analysis, reminiscent of substantial portions of the much larger and more elaborate developments leading to the classification of the finite simple groups, and over time has achieved results considerably exceeding our initial hopes for these methods, in part because of advances on the model theoretic side, notably Wagner's work on fields of finite Morley rank [W01].

As a result, enough is known about simple groups of finite Morley rank to give useful information about connected groups of finite Morley rank generally, particularly those containing involutions. We will illustrate this point by discussing two problems of a very general character where the structure theory can be usefully applied even though the Algebraicity Conjecture remains unresolved. The first of these was proposed by Poizat, as an example of a problem which, while trivial in an algebraic group, seemed out of reach of the theory of groups of finite Morley rank. Namely, suppose a connected group of finite Morley rank is generically of finite exponent, in other words the equation

$$x^n = 1$$

holds generically (on a set of full Morley rank). Does the same equation then hold identically? In an algebraic group, or whenever there is a reasonable topology in sight, this is trivial by density and continuity. This problem of Poizat is a contender for the title of the simplest thing we do not know about groups of finite Morley rank,

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The peculiar efficacity of the known structure theory with respect to the prime 2 allows us to solve this problem completely when n is a power of 2, and more generally to reduce to the case of n odd. But this analysis actually requires some consideration of p-groups for all p, and fortunately there are parts of the theory of torsion in connected groups of finite Morley rank which hold quite generally, notably those involving p-tori (divisible abelian p-groups).

Accordingly, after presenting the relevant part of the structure theory for connected groups of finite Morley rank (§2), in a considerably condensed form, we will discuss its application to this problem of Poizat (§3).

Going in a quite different direction, Borovik proposed a problem in the theory of permutation groups of finite Morley rank which appeared to call for structural analysis of a similar kind. The problem is the following: *bound the rank of a definably primitive permutation group of finite Morley rank in terms of the rank of the set on which it acts.* Recall that a permutation group is *primitive* if it preserves no nontrivial equivalence relation; it is natural to work with the definable version of this notion in our category. Using structure theory, one can indeed prove the existence of such a bound. The bound obtained is quite weak, but to achieve any bound at all appears to require some structural analysis, just as many results in finite permutation group theory appear out of reach except via the classification of the finite simple groups. This parallel goes fairly far: in finite group theory, the starting point for the analysis of primitive permutation groups is an analysis due to O'Nan, Scott and Aschbacher of the structure of the socle, and this has been carried over to our context by Macpherson and Pillay [MP85].

In that analysis, quite general arguments reduce one to what may be called the *generically multiply transitive* case. A permutation group is t-transitive if it acts transitively on t-tuples of distinct elements from the underlying set. The "generic" version of this notion requires that there be at least one generic orbit in this space. The key to the solution of Borovik's problem lies in bounding the degree of generic t-transitivity in terms of the rank of the underlying set. This can be done by first bounding the Morley rank of a maximal 2-torus and then using this to control the parameter t [BC07] (§4).

Coming back to the structure theory itself, as opposed to its applications, I would like to examine two ingredients of that theory which have been quite active in recent years, namely the theory of *Carter subgroups*, and the use of *genericity arguments* (§§5,6). The use of genericity arguments goes back to the earliest work on "bad groups" but they have played an increasingly central role of late. The role of Carter subgroups is a little less clear, and partly conjectural at this point. What these topics have in common is an obsession with conjugacy theorems or their more model theoretic counterparts, generic covering theorems.

It may seem odd that *partial* results on the Algebraicity Conjecture can yield any sort of general results at all. The first explicit, and practical, suggestion as to how this might actually be achieved is found in Altınel's habilitation [A01]. This largely determines the form in which we present the structure theory, as we shall see next.

2. Connected groups of finite Morley rank

In the present section we set out a structure theory for connected groups of finite Morley rank. This includes a user-friendly repackaging of a portion of the known results on the classification of simple groups of finite Morley rank. Our formulation captures somewhat more than half of what is currently known about the Algebraicity Conjecture; the balance does not fit into the framework we have chosen here.

2.1. **Fundamental notions.** We consider a group G of finite Morley rank. The Morley rank of a definable set X is denoted $\mathrm{rk}(X)$. A definable subset X of G is *generic* if $\mathrm{rk}(X) = \mathrm{rk}(G)$.

The notion of connectivity may be defined in two distinct, but fortunately equivalent, ways. Namely a group G is said to be connected if:

- There is no proper definable subgroup of G of finite index.
- The intersection of two generic subsets is generic.

With the first definition, it is easy to show that G contains a unique definable connected subgroup of finite index, denoted G°. When one already has a connected group in hand, it is the second definition which is useful.

If X is an arbitrary subset of G (by no means required to be definable) we let $d(X)$ be the subgroup of G *definably generated* by X, that is the smallest definable subgroup of G containing X. If we write $\langle X\rangle$ for the subgroup generated by X in the usual sense, we have $d(X) = d(\langle X\rangle)$, and in the context of algebraic groups this would be the Zariski closure of $d(\langle X\rangle)$.

One can extend the notion of connectivity to groups which are not necessarily definable as follows: $H^\circ = H \cap d(H)^\circ$.

2.2. **p-tori and p-unipotent groups.** For p a prime, and P a p-group, we say that P is a *p-torus* if it is divisible abelian, and P is *p-unipotent* if it is definable, connected, of bounded exponent, and nilpotent.

These notions are of value for all primes p, but of particular value for $p = 2$ in view of the following.

Theorem 1. *Let S be a maximal connected 2-subgroup of a group G of finite Morley rank, not necessarily definable. Then S has the following structure:*

$$U * T$$

where U is 2-unipotent and T is a 2-torus.

Maximal connected 2-subgroups of G are referred to as *Sylow*$^\circ$ *2-subgroups*; they are all conjugate. In this structure theorem, the product $U * T$ is a central product: the factors commute, and in addition have finite intersection.

In algebraic groups, the structure of Sylow$^\circ$ 2-subgroups depends on the characteristic of the base field: in characteristic two they will be unipotent ($S = U$, $T = 1$) and in other characteristics they will be 2-tori ($S = T$, $U = 2$). It is therefore natural to adopt the following classification of groups of finite Morley rank according to the structure of the Sylow$^\circ$ 2-subgroup:

- Even type: $S = U$;
- Odd type: $S = T$;
- Mixed type: $U, T > 1$;
- Degenerate type: $S = 1$

The study of *simple* groups of finite Morley rank goes very much according to this classification, with different methods being applied in each case. We will not dwell on this here, as for our purposes a much coarser classification is appropriate.

Namely, we will ask only whether the group G does, or does not, contain a nontrivial 2-torus. In the absence of nontrivial 2-tori we have very good structural information, and in their presence matters are less clear. One can however make some good use of the 2-torus itself, as we shall see.

2.3. **Main results.** Suppose that G is a connected group of finite Morley rank.

Definition 2.1.

- $O_2(G)$ *is the largest normal unipotent* 2*-subgroup of* G.
- $U_2(G)$ *is the subgroup of* G *generated by its unipotent* 2*-subgroups.*
- $\hat{O}(G)$ *is the largest connected normal definable subgroup of* G *of degenerate type.*

The main structural result is the following.

Theorem 2 (Groups without 2-tori)**.** *Let* G *be a connected group of finite Morley rank containing no nontrivial* 2*-torus, and suppose* $O_2(G) = 1$. *Then*

$$G = U_2(G) * \hat{O}(G)$$

Furthermore $U_2(G)$ *is a product of simple algebraic groups over algebraically closed fields of characteristic two, and* $\hat{O}(G)$ *contains no involutions.*

This has the following useful consequence. We say that a group G is of *unipotent type* if it contains no nontrivial p-torus, for any p.

Corollary 2.2. *Let* G *be a connected group of finite Morley rank of unipotent type. Then* $G/O_2(G)$ *contains no involutions.*

When the group in question contains a nontrivial 2-torus we lack such definite structural information. However the following result holds quite generally.

Theorem 3 ([Ch05])**.** *Let* G *be a connected group of finite Morley rank. Then the generic element of* G *belongs to* $C^\circ(T)$ *for some unique maximal* 2*-torus* T.

Of course, in the absence of 2-tori this says nothing; but then our structural result applies.

The foregoing is more or less everything needed for the applications considered later, but we will go into more detail below. And we will refer to some of these details in the sequel.

2.4. **Groups without** 2**-tori.** The structure theorem for groups without nontrivial 2-tori (Theorem 2) incorporates a great deal of information about simple groups of finite Morley rank in even and degenerate types. The two main ingredients are as follows.

Theorem 4.

(1) *A simple group of finite Morley rank of even type is algebraic.*
(2) *A connected group of degenerate type contains no involutions.*

The first result, on groups of even type, is proved by a close structural analysis heavily inspired by parts of the classification of the finite simple groups as well as the amalgam method. The published articles in this direction assume the group in question has no simple definable sections of degenerate type. As we have mentioned, Altınel's habilitation pointed the way toward the "absolute" result given here. Implementing that required making major changes in some of the earlier analysis, in some cases invoking [W01], and making at least minor changes in the remainder. One point that emerged in this, most clearly in [1], was the importance of "good tori," to which we return below. Only portions of the analysis can be found in the journal literature. A full account will be given in [ABC07].

The result on groups of degenerate type has a short and self-contained proof inspired in part by techniques from "black box" group theory [BBC07a]. Another way of stating this result is as follows: if a connected group of finite Morley rank contains an involution, then it contains an infinite 2-subgroup.

2.5. **Good Tori.** The proof that a simple group of even type is algebraic came in three waves. In the first instance, two extra hypotheses were imposed: that the group in question has no nontrivial definable simple sections of degenerate type, and that it interprets no "bad fields;" this second hypothesis came to be eliminated fairly quickly and so the first wave was rapidly engulfed by the second, retaining the hypothesis on degenerate type sections. As noted above, the practical possibility of eliminating the latter hypothesis was opened up in [A01], whose implementation was found to depend on the results of [W01], most conveniently expressed in terms of the notion of a *good torus.*

Definition 2.3. *A definable divisible abelian subgroup T of a group of finite Morley rank is a* good torus *if every definable subgroup of T is the definable hull of its torsion subgroup.*

These tori have the following excellent rigidity properties, which follow fairly directly from the definitions.

R-I $N^\circ(T) = C^\circ(T)$;
R-II Any uniformly definable family of subgroups of T is finite.
R-III If H is any definable section of the ambient group G, then any uniformly definable family of homomorphisms from H to T is finite.

Furthermore, the result of [W01] can be expressed in the following terms: the multiplicative group of a field of finite Morley rank and positive characteristic is a good torus.

In [Ch05] the following conjugacy theorem is proved, making heavy use of all three rigidity properties.

Theorem 5. *Let G be a group of finite Morley rank. Then any two maximal good tori of G are conjugate.*

This theory was applied in the analysis of even type groups in [1] in the following form.

Corollary 2.4. *Let $\mathcal{F}$ be a uniformly definable family of good tori in a group of finite Morley rank G. Then under the action of G, $\mathcal{F}$ breaks up into finitely many conjugacy classes.*

Proof. As mentioned, maximal good tori are conjugate in G and hence we may fix one such, T, and assume that the tori in the family $\mathcal{F}$ are all subtori of T. At this point $\mathcal{F}$ is finite, by property R-II. □

The finiteness result needed in [1] was actually proved first, and the more general formulation given as Corollary 2.4 was subsequently disengaged from the particular context in which it first arose.

The good tori which actually come into consideration are the tori in copies of $\mathrm{SL}_2(K)$ sitting inside the ambient group G, where K has characteristic two (so that Wagner's result applies). We will say more on about this in §6.2.

2.6. **Groups of unipotent type.** As we have seen, if G is a connected group of finite Morley rank of unipotent type, then $G/O_2(G)$ has degenerate type, and this follows from the general structure theory for groups without 2-tori, which depends on a very elaborate analysis of all simple groups of even type. But this particular result also has a direct proof, given in detail in [BBC07b]. We will sketch that proof here.

Let U be the connected component of a Sylow° 2-subgroup of G. As G has unipotent type, U is unipotent and in particular definable. We claim $U = O_2(G)$, or in other words we must show that $G = N(U)$.

One may suppose G is a minimal counterexample. Let $M = N_G(U)$. One argues that M has the property of *strong embedding:* for $g \in G$, if $M \cap M^g$ contains an involution, then $g \in M$. By a well known and elementary group theoretic argument, if $M < G$ then this implies that all of the involutions of U are conjugate under the action of M.

But on the other hand, if one studies the action of the connected group M° on U by conjugation it follows that $M^\circ = UC^\circ(U)$. In the contrary case one could extract a section of U on which a connected section of M acts like a multiplicative subgroup of a field, and then in consequence of [W01] this group must be a good torus, and thus contain a nontrivial p-torus for some p. But this contradicts the assumption that G has unipotent type.

Now it suffices to put these two facts together: M° acts trivially on the involutions of U, while if $M < G$, then M acts transitively on the same set, forcing the unipotent group U to be trivial. But then $M = G$ in any case.

2.7. **Maximal p-tori.** The result given earlier for $p = 2$ actually holds more generally.

Theorem 6. *Let G be a connected group of finite Morley rank, and p a prime. Then the generic element of G belongs to $C^\circ(T)$ for some unique maximal p-torus T.*

Let us focus on the following variant.

Theorem 7 (T_p)**.** *Let G be a group of finite Morley rank, T a p-torus, and $H = C^\circ(T)$. Then the union of the conjugates of H is generic in G.*

This quite properly puts the emphasis on the group H, and the bulk of the argument is aimed at establishing the following points.

Lemma 2.5. *With the notation of Theorem T_p, the group H has the following properties.*

- *H is almost self-normalizing (i.e., $H = N^\circ(H)$);*
- *H is generically disjoint from its conjugates (i.e., $H \setminus (\bigcup H^{[G\setminus N(H)]})$ is generic in H.*

The work comes in the derivation of the second property, which we pass over. Given that, the rest of the argument is entirely soft:

Lemma 2.6 (Genericity Lemma)**.** *If G is a group of finite Morley rank and H a definable subgroup which is almost self-normalizing and generically disjoint from its conjugates, then*

(1) $\bigcup H^G$ *is generic in G;*
(2) *For $X \subseteq H$, the set $\bigcup X^G$ is generic in G if and only if $\bigcup X^H$ is generic in H.*

After a time, one gets tired of repeating the phrase "the union of the conjugates is generic in G." It was Jaligot who first got sufficiently tired of this form of words to introduce a shorter term.

Definition 2.7. *A definable subset X of G is* generous *in G if the union of its conjugates is generic in G.*

We may rephrase our genericity lemma as a generosity lemma.

Lemma 2.8 (Generosity Lemma)**.** *If G is a group of finite Morley rank and H a definable subgroup which is almost self-normalizing and generically disjoint from its conjugates, then*

- *H is generous in G;*
- *For $X \subseteq H$, the set X is generous in G if and only if it is generous in H.*

That's better!

Now it is time to look at Poizat's problem and see if we have learned anything.

3. Generic equations

3.1. The result.

Conjecture 1. *Let G be a connected group of finite Morley rank which satisfies the equation*

$$x^n = 1$$

generically. Then the equation holds identically.

We have the following partial result, and somewhat more.

Theorem 8. *Let G be a connected group of finite Morley rank which satisfies the equation*

$$x^n = 1$$

generically, with n a power of 2. Then the equation holds identically.

The more general form reads as follows.

Theorem 9. *Let G be a connected group of finite Morley rank which satisfies the equation*

$$x^n = 1$$

generically, and write $n = 2^k n_O$ with n_O odd. Then we may decompose G into a central product as follows:

$$G = U * G_1$$

with U 2-unipotent and so that G/U satisfies the equation

$$x^{n_O} = 1$$

generically.

In other words, one may suppose n to be odd.

3.2. **The analysis.** Under the hypotheses of Theorem 9, the proof takes place in two stages, as follows.

- G contains no nontrivial p-torus
- $G = U * G_1$ where U is a 2-group of bounded exponent and G/U contains no involutions.

Most of the work goes into the first point, after which the second is a modest refinement of Corollary 2.2. So let us see how the first point is argued.

Suppose on the contrary that G contains a nontrivial p-torus, and let T_p be a maximal one, and $T = d(T_p)$, $H = C^\circ(T_p)$. We then obtain, successively:

- $x^n = 1$ generically in G;
- $x^n = 1$ generically in H;
- $x^n = 1$ generically in some coset Ta with $a \in H$;
- $x^n = 1$ generically in T;
- and a contradiction.

The transition from the first point (our assumption) to the second point is most clearly expressed by the Generosity Lemma 2.8. The passage from the second to the third is a sort of Fubini principle (or additivity of rank). From the third to the fourth we use the fact that a and T commute.

And that is all there is to it.

4. Permutation Groups

We consider permutation groups as structures (G, X) consisting of a group G, a set X, and a faithful action of G on X. And we will always suppose that the permutation group under consideration has finite Morley rank.

4.1. The main result, and a special case.

Definition 4.1. *A permutation group (G, X) is* definably primitive *if there is no nontrivial G-invariant definable equivalence relation on X.*

The result we aim at is the following.

Theorem 10 ([BC07])**.** *Let (G, X) be a definably primitive permutation group of finite Morley rank. Then the rank of G is bounded by a function of the rank of X.*

This leads us directly to the notion of generic multiple transitivity.

Definition 4.2. *Let (G, X) be a permutation group of finite Morley rank. The action is* generically t-transitive *if there is an orbit of G in X^t which is generic in X^t.*

A generically t-transitive group has rank at least $t\,\mathrm{rk}(X)$, so bounding this parameter t in terms of $\mathrm{rk}(X)$ is certainly an essential part of the problem. In fact one can show that the whole problem reduces to this. So we will focus on this special case.

Theorem 11. *There is a function τ such that for any definably primitive and generically t-transitive permutation group (G, X) with $\mathrm{rk}(X) = r$, we have*

$$t \leq \tau(r)$$

4.2. **A bound on t.** We discuss the proof of Theorem 11, So let (G, X) be a definably primitive permutation group of finite Morley rank.

The first point is to get some control over divisible torsion subgroups of G, and this is expressed as follows.

Lemma 4.3. *If T is a definable divisible abelian subgroup of G and T_∞ is its maximal definable torsion free subgroup, then*

$$\mathrm{rk}(T/T_\infty) \leq \mathrm{rk}(X)$$

The point here is that if a point of X is generic over the torsion subgroup of T, then the stabilizer of that point is torsion free and hence contained in T_∞.

Now using general results on definably primitive permutation groups due to O'Nan, Scott, and Aschbacher in the finite case and Macpherson and Pillay in our context [MP85], one comes down eventually to the case in which G is simple. If G contains no nontrivial 2-torus then it is algebraic in characteristic two, and in particular its maximal torus T is a good torus, so $T_\infty = 1$ and we have a bound on the rank of T, giving us a bound on both the rank of the underlying field, and the Lie rank of the group, from which one may bound the Morley rank of the group.

So we focus on simple groups with nontrivial 2-tori. In that case if we assume a high degree of generic multiple transitivity, then after some maneuvering we may consider the definable hull of a maximal 2-torus, not in G, but in the point stabilizer of a suitably chosen and not too large (compared to t) set of points in X, and the generic multiple transitivity will give us something like an action of the symmetric group Sym_n on this torus with a little effort. We say "something like" for two reasons: really the group acting is some finite group which has the symmetric group as a quotient, and secondly it is not clear that the action is faithful. This question of faithfulness is an important one, and the analysis bifurcates at this point.

- If there is a faithful action of a group like Sym_n (with n large) then this can be used to blow up the rank of T/T_∞; this takes some argument but is intrinsically very plausible.
- On the other hand, if there is no such action then the situation becomes delicate again. We will take this up separately.

In dealing with the second point we need to be more explicit about what our setup is and how we actually choose the point stabilizer of interest to us. Let $x_1, x_2, \ldots$ be a sequence of t independent generic points in X. Rather than working in the original group G, we want to work in the connected component of the stabilizer of some initial segment of these points, and then consider the subgroup stabilizing a further n points (where n is proportional to t). Call these two groups H and H_0. The main point is that one can choose these groups so that a maximal 2-torus of H_0 is maximal in H; in other words, the maximal 2-torus in successive point stabilizers cannot decrease steadily. Now by generic t-transitivity the group H will induce an action of Sym_n on the n points whose stabilizer in H is H_0, and thus on H_0, and with a little adjustment also on a maximal 2-torus of H_0.

This then sets us up to apply the following specialized result on torsion in connected groups of finite Morley rank, which says that this induced action is faithful.

Lemma 4.4. *Let G be a connected group of finite Morley rank with no unipotent 2-subgroup, and let T be a maximal 2-torus of G. Then T contains all the 2-elements in $C(G)$.*

This is a sharp statement, implying for example our earlier claim that connected degenerate type groups have no involutions, and its proof develops that line of analysis further.

5. Carter subgroups

5.1. Carter subgroups and their kin.

Definition 5.1. *Let G be a group of finite Morley rank. A* Carter subgroup *of G is a connected definable nilpotent subgroup which is almost self-normalizing in G.*

Theorem 12 ([FJ05])**.** *A group of finite Morley rank has a Carter subgroup.*

The existence proof uses Burdges' "graduated" unipotence theory (viewed upside down, and equally legitimately, as a graduated semisimplicity theory). The general idea is to take the most semisimple (and, simultaneously, largest) group one can find among all connected nilpotent groups, building it up by degrees.

The role that should perhaps be played by Carter subgroups in our theory is often fulfilled in an approximate way by the subgroups $C^\circ(T)$ with T some sort of torus (e.g., maximal good). That there is some underlying logic to this is suggested by a result of Frécon:

Theorem 13 ([F05])**.** *Suppose the group G involves no bad groups and no bad fields, and T_0 is a maximal divisible torsion subgroup of G. Then $C^\circ(T_0)$ is a Carter subgroup.*

5.2. **Generosity and conjugacy.** One would like to know that Carter subgroups are also generous and conjugate. This is rather more than is really needed however. What one really would like to know is that some Carter subgroups are generous, and that those are conjugate; one would then be entitled to build in the generosity condition as part of the "right" definition of Carter subgroup.

A step in this direction has been taken by Jaligot.

Theorem 14 ([J06])**.** *Let G be a group of finite Morley rank, and Q_1, Q_2 Carter subgroups. If both are generous in G, then they are conjugate.*

More recently Frécon has also announced the conjugacy of Carter subgroups in minimal simple groups. These issues take us deeply into the theory of degenerate type groups, in particular, a subject which for a very long time seemed entirely shrouded in mystery.

This takes us a little outside the part of the theory that we can really apply at this stage, but as Theorem 13 suggests, this is the setting we would like to be in.

6. Generosity arguments

As we have noticed, generosity arguments have been used in our subject from the earliest work on bad groups to some of the most recent work on Carter subgroups. As the level of generality at which we work has increased, we have been led to rely more and more on such arguments, and correspondingly a little less on special features of the theory of solvable groups or algebraic groups.

We have already seen one of the basic methods used, in the Generosity Lemma 2.8. The method is a little more versatile than that particular formulation would suggest, but the essence of the method is there. On the other hand, actually getting to the point of being able to apply that method usefully generally involves plunging into some quite tightly prescribed configurations that arise here and there in various classification problems. I would like to open this up a bit and actually examine three such specific situations a little more closely; they have all been touched on above. I have the following three cases in view:

(1) Degenerate type groups;
(2) Weak embedding;
(3) Toricity arguments

6.1. **Degenerate type groups.** We will discuss the role played by generosity arguments in the proof that a connected degenerate type group G of finite Morley rank contains no involutions. Proceeding inductively, one may suppose that any proper definable connected subgroup contains no involutions.

There are three points to be established:

(1) We may suppose G is simple;
(2) A Sylow subgroup of G is elementary abelian;
(3) And then a final contradiction

For the first point, the claim is not that our minimal counterexample G is necessarily simple, but that $Z(G)$ is finite and $\bar{G} = G/Z(G)$ is again a minimal counterexample, and simple. The delicate point here is the proof that $G/Z(G)$ will still contain an involution, if G does. We will not dwell on this point now; but observe that it should not be passed over as obvious!

The second point is proved by a generosity argument, and this is the one on which we will elaborate.

The third point is the core of the argument, and exploits an idea from black box group theory. By good fortune, it leads to a precisely opposite conclusion to the second point. One may interchange these two steps, as the arguments involved are entirely independent of one another.

Let us then take up the task of showing that in our minimal counterexample the Sylow 2-subgroup, which is in any case finite (that is what degenerate type means), is elementary abelian. Or in other words, we must show that there are no elements of order 4.

The key points are these.

Lemma 6.1.

(1) *For any two distinct nontrivial* 2*-elements* $t, t' \in G$*, the cosets* $tC^\circ(t)$ *and* $t'C^\circ(t')$ *are disjoint.*
(2) *For any nontrivial* 2*-element* $t \in G$*, the coset* $tC^\circ(t)$ *is generous in* G*.*

These two points together show that all 2-elements of G have the same order, as otherwise we have two disjoint generic subsets of G. For the first point, if $a \in tC^\circ(t)$, then easily t is the unique 2-element in $d(a) \cap tC^\circ(t)$, and the claim follows. So we come down to the generosity argument needed for the second point, which is a variation on the basic Generosity Lemma 2.8. Namely, the coset $tC^\circ(t)$ has the following properties:

- $N^\circ(tC^\circ(t)) = C^\circ(t)$
- Distinct conjugates of $tC^\circ(t)$ are pairwise disjoint.

The notation $N^\circ(tC^\circ(t))$ is unusual, but we interpret $N(tC^\circ(t))$ as the stabilizer of $tC^\circ(t)$ under conjugation. Since t is the unique 2-element in $tC^\circ(t)$, the first property is immediate. And the second property is a special case of the one just discussed above. Now an easy rank computation suffices:

$$\begin{aligned}\mathrm{rk}(\bigcup[tC^\circ(t)]^G) &= \mathrm{rk}(G/N(tC^\circ(t))) + \mathrm{rk}(tC^\circ(t)) \\ &= \mathrm{rk}(G) - \mathrm{rk}(C^\circ(t)) + \mathrm{rk}(C^\circ(t)) = \mathrm{rk}(G)\end{aligned}$$

and the union is generic.

6.2. **Weak embedding.** In a series of four articles Altınel and I took up the line proposed in his habilitation and reworked the first chapter of the classification of K^*-groups of even type, which had already been carried out twice under successively weaker side conditions. The more or less descriptive titles of the first three papers in this series gave way to a more evocative title in the fourth: *Limoncello.* If one takes these articles in order then the situation should be clear enough by the time one hits the fourth. But I propose to jump plunge into the fourth, take a look around, and see what sort of generosity argument comes into play. At this point one finds good tori on the scene.

The setting for the whole series which ends with “Limoncello” is a so-called *uniqueness case*, specifically the case of weak embedding. One has a simple group G of finite Morley rank of even type, all of whose proper definable simple sections of even type are algebraic, and one has in addition a proper definable subgroup M which is *weakly embedded* in the sense that for $g \in G$:

$$M \cap M^g \text{ has an infinite Sylow 2-subgroup if and only if } g \in M$$

The aim of the four articles is to show that $G \cong \mathrm{SL}_2$ (with M a Borel subgroup).

Now a weakly embedded subgroup has the following property: for any nontrivial 2-unipotent subgroup U of M, the normalizer $N(U)$ is contained in M. This sort of condition has been referred to as a “black hole” property. It tends to nerf standard approaches to analyzing group theoretic configurations, and one needs to prepare special tools to deal with such extreme situations.

At the stage of analysis that concerns us, we already have a normal definable elementary abelian subgroup $A \leq M$ with $M/C^\circ(A)$ containing no involutions. This is compatible with what is expected in the target group but leaves the structure of $C^\circ(A)$ completely up in the air.

One expects in a general way to build a copy of SL_2 inside the group G, after which one would show this group is in fact all of G. This suggests that we would be spending some time on the particular configuration in which SL_2 occurs as a proper subgroup of G. As so often happens, it is convenient to choose the case division a little subtly. The case at which one arrives, at the end, is actually the following.

(∗) There are two distinct conjugates A_1, A_2 of A in G for which

$$H = C^\circ(A_1, A_2) > 1$$

In this situation we set $L = \langle A_1, A_2 \rangle \leq C^\circ(H) < G$ and then we find easily that $L \cong \mathrm{SL}_2$ (essentially, by induction). It is this more tightly constrained configuration that needs to be handled separately—and at this point we are no longer looking for an identification of G, but for an outright contradiction.

Now in looking at the treatment of this in [J01], one notices that a key point is to consider a maximal torus T of L, its conjugates in G, and more particularly those of its conjugates lying in M. The latter turn out to be conjugate also under the action of M. This part of the argument originally made use of the fact that M was known to be solvable, a point which is no longer available at our current level of generality. However the tori we are looking at are good tori, and so we have a problem relating to conjugacy of good tori. In particular we know on completely general grounds, namely Corollary 2.4, that this family breaks up into *finitely many* conjugacy classes under the action of M. As mentioned earlier, the arguments given for this in the first draft, in fact the first $n-1$ drafts, of "Limoncello" were more ad hoc, but pointed in the direction of the more abstract result.

The various rank computations that take place afterward, with this finiteness result in hand, will not be rehearsed here. For a while one can use the finiteness result to get slightly weakened forms of the estimates in [J01]. These are not actually strong enough to take us to the end of the analysis, but they are strong enough to allow us to show ultimately that the number of conjugacy classes of the relevant tori under the action of M is in fact just 1, after which one more or less returns to the main line of the older analysis, mutatis mutandis.

This is of course a highly technical business, but we have the advantage at the outset of knowing the role played by similar results in finite group theory, as well as the experience gained in two previous waves of analysis in the context of groups of finite Morley rank.

6.3. **Toricity.** The fact that connected degenerate type groups have no involutions turns out to be the leading edge of the study of p-torsion in groups of "$p^\perp$ type": by this, we mean groups containing no nontrivial p-unipotent subgroups. For $p = 2$, these are the odd type and degenerate groups, and that is the case which most concerns us.

A p-element is called "toral" if it belongs to a p-torus. In a number of concrete configurations in simple groups of finite Morley rank, one prefers to work with toral involutions, and the others play a ghostly role in that they never actually appear inside proper connected subgroups of the ambient group (where most of the analysis actually takes place). Needless to say, this leads to technical issues, and as it turns out they can all be avoided in the configurations that actually arise.

Theorem 15 ([BC0x])**.** *Let G be a connected group of finite Morley rank and $p^\perp$ type. Then every p-element is toral.*

One gets as a corollary the useful Lemma 4.4 which came up in our discussion of definably primitive groups of finite Morley rank, in its general form.

Corollary 6.2. *Let G be a connected group of finite Morley rank and $p^\perp$ type, and let T be a maximal p-torus of G. Then T contains all the p-elements in $C(T)$.*

Proof. Let $a \in C(T)$ be a p-element, and T_0 a maximal p-torus of G containing a. Then T and T_0 are maximal p-tori of $C(a)$, and and are therefore conjugate under the action of $C(a)$. As $a \in T_0$ it follows that $a \in T$. □

The point of this discussion is that once more generosity arguments play a large role in the proof of the toricity theorem 15. This time we proceed as follows. We have a p-element $a \in G$, and we consider a generic maximal p-torus of $C^\circ(a)$, and the group $H = C^\circ(a, T) = C^\circ(\langle a, T\rangle)$.

If $a \in H$ then one looks at the image of a in $\bar{H} = H/T$, where $\bar{H}$ is "p-degenerate" (which for $p = 2$ is the degenerate case). As in the case $p = 2$, something quite special has to be done in that base case, but at the moment we are more interested in the induction that moves us onward to the general case, so we will let that go.

So suppose $a \notin H$. The claim then is:

$$(*) \qquad \text{The coset } Ha \text{ is generous in } G.$$

Now for any element $g \in Ha$, the group $d(g)$ is not p-divisible. On the other hand, one can show using our previous generosity results for centralizers of p-tori that exactly the opposite holds in any group of $p^{\perp}$ type. So the generosity statement (*) suffices to conclude the proof of the toricity theorem.

This generosity statement should have a familiar look to it by now, and the proof runs much as usual: the coset Ha is almost self-normalizing and generically disjoint from its conjugates, in the appropriate sense, as in the last subsection. We will not run through the details here, and accordingly the choices we have made with some precision in our set-up will not be justified here. Needless to say, such choices need to be managed with care, and tend to reflect specific features of the configuration under consideration.

7. Conclusion

I would like to stress once more what this article is, and is not, about. It is certainly not about the Algebraicity Conjecture as such, though most of our discussion is firmly anchored in work originally undertaken in the light of that conjecture. Along the way, and mainly in the last few years, results of considerable generality have been brought to the table, sometimes by judicious improvement on a known recipe, as in the case of Altınel's habilitation, with a dash of new model theoretic ingredients (Wagner), and sometimes by elaborating on improvisations which appeared first as responses to the needs of the moment.

There is a large body of work on groups on odd type which has been undergoing transformations of a very similar kind. While the issue of "bad fields" was more or less eliminated from the scene in groups of even type at an early stage, it is only with Burdges' thesis [B04], that this issue began to come under control in the odd type setting, and it still simmers on as something of a challenge at each further step in this direction. And a strategy to provide "absolute" results on groups of odd type without first clearing up the situation in degenerate type is not presently in sight (at least, not during one's waking moments).

As I have said, I chose to focus here on what is for me the least expected side of the theory: the possibility of cooking up some very general conclusions, taking a very incomplete structure theory as the source of all ingredients. In this casserole, it may be observed, there lingers the distinct aroma of three generations of Lyonnaise cuisine.

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