SIMPLE GROUPS OF FINITE MORLEY RANK OF UNIPOTENT TYPE

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ALEXANDRE BOROVIK
SCHOOL OF MATHEMATICS,
THE UNIVERSITY OF MANCHESTER, U.K.,
JEFFREY BURDGES
SCHOOL OF MATHEMATICS,
THE UNIVERSITY OF MANCHESTER, U.K.,
AND
GREGORY CHERLIN
DEPARTMENT OF MATHEMATICS,
RUTGERS UNIVERSITY, U.S.A.

Abstract. A simple group of finite Morley rank of unipotent type contains no involutions. We explain why this is true, and why we think it is interesting. The story involves model theory, finite group theory, and some aspects of the theory of algebraic groups.

1. Introduction

About thirty years ago it was conjectured by Cherlin and Zilber that every simple group of finite Morley rank is algebraic, more precisely a Chevalley group over an algebraically closed field. In the latter case the Morley rank agrees with the algebraic dimension of the Zariski closure.

But when we assume that a group has finite Morley rank, we are not assuming that the rank function has any topological content. Similar issues arise in the classification of the finite simple groups. In most cases the problem is to identify a given finite simple group with a group of Lie type. In the absence of more geometric tools, one aims to reconstruct the underlying combinatorial geometry (the associated building) via identification of a suitable (B,N)-pair.

There is now a considerable body of work on simple groups of finite Morley rank which incorporates many techniques originating in the classification of the finite simple groups. These techniques become quite effective as soon as one has some involutions in the group (the more, the better). Of course in finite simple group theory one begins with the Feit-Thompson Odd Order Theorem, after which involutions are guaranteed (and with the assistance of some character theory the supply of involutions increases rapidly). We do not have an analog of the Odd Order Theorem, and indeed there is a very real possibility that torsion-free simple

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groups of finite Morley rank exist. In fact, this remains the most plausible scenario for a counterexample to the Algebraicity Conjecture (rather than, say, some minor variation on simple algebraic groups, or an analog of some sporadic finite group).

Work on the Algebraicity Conjecture borrows heavily from the theory of finite simple groups, while incorporating features (like connectivity) belonging to the domain of algebraic groups. Indeed, the notion of Morley rank has no useful analog in the finite theory. There are deep theorems suggesting that the dimensions of definable sets should be reflected by their cardinalities in sufficiently large finite groups (more precisely, by $\log_q$ of the cardinality, with q the order of the base field), but there is no sensible way to bring this idea to bear on the finite case.

However, there is another branch of finite group theory which does make use of something analogous to dimension: black box group theory, in which a large finite group is hidden in a box and elements may be picked randomly and independently, and various computations made with them. In this context "high probability" corresponds well with "full dimension" (or, as one says, "genericity"). We have a little more in the finite Morley rank context: we can speak of relative genericity with respect to definable subgroups, something which is difficult in the black box case, where in addition to identifying the subgroup one must also somehow equip it with a suitable probability measure; this is sometimes possible, and in one specific case there has been a flow of useful technical ideas back and forth between the black box case and the finite Morley rank case, beginning with work of Altseimer in both areas, cf. [AlBo01]. Using these techniques, one can prove the following [BBC05].

Theorem 1. *Let G be a connected group of finite Morley rank containing an involution. Then G contains an infinite 2-subgroup.*

One may say this more pungently. Borrowing the term "Sylow 2-subgroup" from finite and locally finite group theory, the theorem states that if the Sylow 2-subgroup of a connected group of finite Morley rank is nontrivial, then it is infinite. Thus, while we cast no light on the Odd Order Theorem per se, we do get a dichotomy: once one has an involution, one has something substantial to work with. In particular, we prefer to work with "Sylow° 2-subgroups", defined as maximal *connected* 2-groups. In that language, our theorem says that when the Sylow 2-subgroups are nontrivial, the Sylow° 2-subgroups are also nontrivial.

A group in which the Sylow° 2-subgroups are trivial is said to be of *degenerate type*. We now know in view of the above that these groups contain no involutions. One reason to prefer the 2-Sylow°s to the 2-Sylows is the following structure theorem [BP90, BN94].

Theorem 2. *Let S be a Sylow° 2-subgroup in a group G of finite Morley rank. Then*

$$S = U * T$$

with U definable, connected, nilpotent, of bounded exponent and T 2-divisible abelian.

The "$*$" here represents a central product, and the intersection $U \cap T$ is also finite in this case.

Accordingly, once one moves beyond the degenerate type case, one has either U or T (and possibly both) nontrivial, giving a substantial foothold for further analysis.

Now U and T are in some abstract sense the *unipotent* and *semisimple* parts of S. We make this formal as follows.

Definition 1.1. *Let p be a prime.*

(1) *A p-*unipotent *group is a definable connected p-group of bounded exponent.*
(2) *A p-*torus *is a divisible abelian p-group.*

Note that while we do not incorporate nilpotence into the definition of p-unipotence here, it would be reasonable to do so. For $p = 2$, the nilpotence can be proved in any case. For p odd it remains open.

In the present paper we will deal with another notion of unipotence, the broadest one we can imagine.

Definition 1.2.
A group G of finite Morley rank is of unipotent type *if it contains no p-torus, for any prime p.*

Our discussion here will be centered on the following result, which was used in [BBC05].

Theorem 3. *If G is a simple group of finite Morley rank of unipotent type, then G contains no involutions.*

There are two ways to prove this. The long way round is first to invoke the absence of 2-tori to prove that G is either algebraic in characteristic two or of degenerate type, and then to conclude by invoking the structure of simple algebraic groups. This proof takes something like 200 pages and involves material which is not yet fully published [ABC0x]. Another proof can be extracted from the long one by paying a little attention to what the relevant portions of the proof are, and this saves approximately 199.5 pages. We will give that proof here (in [BBC05] we saved the full 200 by referring to [ABC0x]).

There are two reasons to take the foregoing theorem seriously. In [BBC05] our concern was the analysis of connected groups satisfying a *generic equation* of the form

$$x^n = 1 \text{ (generically)}$$

In particular we showed there that for n a power of 2, such an equation must hold everywhere if it holds generically, and more generally we reduced the analysis of such equations to the case in which n is odd. Thus we cast some light on a long-standing test problem put forward by Poizat, in a way which has only become possible recently as the work on the Algebraicity Conjecture has been brought to bear to reveal structural features of groups of finite Morley rank which are not tied to a purely inductive context.

But apart from this specific application, there is also a methodological point here, and the latter is really our motivation for taking the matter up again. The reason that the long version of the proof is so very long, is that it is based on a full classification result in much the spirit of finite simple group theory. That result runs as follows.

Theorem 4. *Let G be a simple group of finite Morley rank containing a nontrivial 2-unipotent subgroup. Then G is an algebraic group over an algebraically closed field of characteristic two.*

One may distinguish three phases in the development of the proof of this classification theorem, in which auxiliary hypotheses were gradually stripped out. In the first phase one supposed that the group G involved no "bad field" and no degenerate type groups (the latter is reasonable if one takes an inductive approach to the Algebraicity Conjecture; but then one is committed to dealing with the Odd Order Theorem at some point). We will not detail the "bad fields" hypothesis; it was excised relatively early on (though it has not entirely vanished from the landscape, nor should it). Bypassing the Odd Order Theorem comes as more of a surprise, and the experience of finite group theory would not be encouraging in this respect. But as Altınel's habilitation showed [Alt01], there is one very special feature in our situation:

Theorem 5. *A unipotent 2-group acting on a degenerate type group must act trivially.*

This tends to "uncouple" the odd order problems from the rest of the theory when there are unipotent 2-groups available.

Still, problems arise as one implements this idea. Some of them are addressed in [AC03, AC04, AC05a, AC05b], and a full account is in preparation as [ABC0x] T. Altınel, A. V. Borovik, and G. Cherlin, **Simple Groups of Finite Morley Rank**. One of the features which has emerged over time is the importance of p-tori for the analysis. This came into view only gradually and was eventually extracted in an explicit way in [Ch05], just in time to be incorporated into the final draft of [AC05b].

In retrospect, we find it very striking that if one assumes p-tori out of existence, everything collapses quickly back to the degenerate case. At the other extreme, the paper [AC05b] is largely concerned with the analysis, or exploitation, of the p-tori which arise in a configuration where a group which ought to be SL_2 turns out in fact to have copies of SL_2 (and, in particular, their tori) inside it. In earlier treatments under more restrictive hypotheses, the existing theory of Carter subgroups in solvable groups gave sufficient control, and there was no need to focus one's attention on p-tori as such.

When this broad classification project got fairly under way, under the initiative of the first author, the type of analysis that took place was very much bound up with various special assumptions. Lately there has been a discernible flow back toward structural results which apply quite broadly to groups of finite Morley rank. Several such results have been detected at first within very concrete configurations associated with specific classification problems.

At the same time, some of the early results already fit into this framework, notably the 2-Sylow theory itself (which is a necessary prerequisite for this approach to the Algebraicity Conjecture), as well as the lemma on which Altınel's habilitation is based. Both of these results are proved inductively, but in a self-contained way. Subsequently there have been very general results on "characteristic 0 unipotence" theory feeding back into the theory of Carter subgroups and other areas [Bu06?, FJ05]. The theory of "good tori" emerged in the series [AC03, AC04, AC05a, AC05b], notably in the last of these, and was finally detached from specific contexts in [Ch05]. The Carter theory has also developed further at a very general level [FJ05, Ja06?]. There are strong parallels between the Carter theory and the theory of p-tori, and an ideal theory would combine the best features of both. We will come back to this at the end.

2. Theorem 3

While the hypothesis of simplicity in Theorem 3 simplifies the statement, it makes more sense to prove the result in a more global form, as follows.

Theorem 6. *Let G be a connected group of finite Morley rank of unipotent type. Then $G/O_2{}^\circ(G)$ contains no involutions.*

Here we use the notation "$O_2{}^\circ(G)$" to denote the largest *connected* normal definable 2-subgroup of G, without attempting to define $O_2(G)$ separately, which can be problematic, in general. Of course, our theorem implies that $O_2{}^\circ(G) = O_2(G)$ for any reasonable definition of the latter!

In the proof of this result, we make free use of Theorem 1, a substantial result in its own right, and so we aim only at showing that $G/O_2{}^\circ(G)$ is of degenerate type. We may pass to a quotient and suppose that

$$O_2{}^\circ(G) = 1$$

and then our claim is that G is itself of degenerate type.

Before entering into the proof of Theorem 6, which will be given in the following section, there is more to be said both about both its hypothesis and its conclusion.

2.1. **Tori.** Our assumption on G is that there is no nontrivial p-torus for any prime p. In practice one prefers to work with definable subgroups as far as possible, and this leads to two variations on the notion of p-torus which are of considerable utility.

Definition 2.1.

(1) *A* decent torus *is a definable divisible abelian group T which is the definable hull of its torsion subgroup T_{tor} (that is, the smallest definable subgroup containing T_{tor}).*
(2) *A* good torus *is a definable divisible abelian group T such that every definable subgroup T_0 of T is the definable hull of its torsion subgroup $(T_0)_{\text{tor}}$.*

Good tori have remarkable rigidity properties. For example, any uniformly definable family of subgroups of a good torus is finite [AC04]. Furthermore, the multiplicative group of a field of finite Morley rank and nonzero characteristic is a good torus. This follows from the main result of [Wa01], and is made explicit in [AC04]. So the following clears the air considerably.

Lemma 2.2. *Let G be a group of finite Morley rank. Then the following are equivalent.*

(1) *G contains no nontrivial decent torus.*
(2) *There is no prime p for which G contains a nontrivial p-torus.*
(3) *No definable section of G is a good torus.*

These conditions are inherited under passage to definable sections, or to elementary extensions.

Proof. We show first that condition (2) passes to elementary extensions. Suppose that in an elementary extension G^* of G we have some nontrivial p-torus T_0, and consider $A = d(T_0)$. Then A is definable and p-divisible, and contains p-torsion; the existence of such a group passes to the elementary substructure G and contradicts (2). So (2) is preserved by passage to elementary extensions.

As the third condition is inherited by definable sections, it will be sufficient to check the stated equivalences.

The equivalence of the first two is clear.

To see that the third condition implies the first, it suffices to show that any nontrivial decent torus T has a nontrivial good torus as a definable quotient. Indeed, let T_0 be any maximal proper connected definable subgroup of T, and pass to $\bar{T} = T/T_0$. Then $\bar{T}$ is again a decent torus, and now any proper definable subgroup of $\bar{T}$ is finite. So $\bar{T}$ is in fact a good torus.

Finally let us check the implication ($2 \implies 3$). Suppose that (2) holds and (3) fails, and that H is a definable section of G (that is $H = K/N$ with K, N definable and $N \lhd K$), which is a good torus. Then H contains a nontrivial p-torus $\bar{T}$ for some p. Each element $\bar{t}$ of $\bar{T}$ lifts to a p-element t of G (as H is a *definable* section) and hence the abelian groups $Z(C(t))$ as t varies over G contain p-subgroups of unbounded order. It follows easily that some elementary extension of G contains a nontrivial p-torus, and thus as we have seen G also contains a nontrivial p-torus, contradicting (2). So ($2 \implies 3$). $\square$

2.2. **The structure of G.** The conclusion of Theorem 6 also deserves further elucidation. That theorem says that the 2-elements in a group of unipotent type really must behave in a unipotent way. We can say a little more: these elements cannot interact in a serious way with the rest of the group.

Proposition 2.3. *Let G be a connected group of finite Morley rank containing no decent torus and let $U = {O_2}^\circ(G)$. Suppose that G/U is of degenerate type. Then $G = U \cdot C_G(U)$.*

As we will see, this is a combination of Zilber's Field Theorem with Wagner's results in the case of characteristic two, which takes on the following form.

Proposition 2.4. *Let H be a connected solvable $p^\perp$-group of finite Morley rank acting faithfully on a nilpotent p-group V of bounded exponent. Then H is a good torus.*

Here a $p^\perp$-group is a group which contains no element of order p. Elements of infinite order are permitted.

Proof. We work in the group $G = V \rtimes H$, which is again a solvable group, and we use the theory of the Fitting subgroup.

Observe that $F(G) = V(F(G) \cap H)$ and that $F(G) \cap H$ centralizes V since it is a $p^\perp$-subgroup of $F(G)$. As the action of H is faithful, the intersection $F(G) \cap H$ is trivial. Thus $F(G) = V$, and $H \cong G/F(G)$ is abelian divisible by the structure theory for connected solvable groups of finite Morley rank (an analog of Lie-Kolchin).

Take a G-invariant normal series $V = V_0 > V_1 > \cdots > V_n = (0)$ with successive quotients finite or G-minimal (that is, having no proper definable G-invariant subgroups). The stabilizer of this chain in H (that is, the subgroup acting trivially on each factor) is trivial since the chain consists of definable p-groups, and H is a $p^\perp$-group; this is again an analog of a standard fact from finite group theory which goes over to our context.

Now consider the combined action of H on all of the quotients $A_i = V_i/V_{i+1}$. In other words, if $\bar{H}_i$ is the image of H in $\mathrm{Aut}(A_i)$, we have a definable injection of H into $\prod_i \bar{H}_i$.

Now by Zilber's Field Theorem, the groups $\bar{H}_i$ are subgroups of multiplicative groups of fields of finite characteristic (two). As H is connected, these groups are good tori by Wagner's theorem, and a connected subgroup of a product of good tori is again a good torus, as is easily checked [AC04]. So H is a good torus. □

Proof of Proposition 2.3. Let G be a counterexample of minimal rank. Let $A = Z^\circ(U)$. Then $\bar{G} = G/C_G(A)$ is a group of degenerate type acting faithfully on A. Since $\bar{G}$ has degenerate type, its Borel subgroups are $2^\perp$-groups, either by Theorem 1 or by the considerably more classical solvable case of the same result. So by Proposition 2.4, the Borel subgroups of $\bar{G}$ are good tori, hence trivial. So the connected group $\bar{G}$ is trivial, or in other words

$$G \text{ centralizes } A.$$

On the other hand, by the minimality of G, the group $G/A = (U/A)\cdot C_{G/A}(U/A)$. Let $H/A = C_{G/A}(U/A)$. Then $G = UH$ and $[H,U] \le A$.

For any definable $2^\perp$-subgroup X of H, we have $[X,U] = 1$ since X acts trivially on both factors U/A and A of the chain $1 \le A \le U$: see [ABC99, Cor. 2.45] (or consider $[X,u]$ for $u \in U$).

It follows that $H/C_H(U)$ is a 2-group, since for each $a \in H$ the definable hull of a is the sum of a 2-group and a $2^\perp$-group, and thus $H \le UC_H(U)$, and $G = UC_G(U)$. □

3. Proof of Theorem 6

Our claim is the following: for G connected of unipotent type, $O_2{}^\circ(G)$ is a Sylow° 2-subgroup. Let us introduce the notation $U_2(G)$ for the subgroup of G generated by all its unipotent 2-subgroups. By an early result of Zilber, analogous to a well-known lemma in the theory of algebraic groups, the group $U_2(G)$ is definable. Another way of phrasing our claim is that $O_2{}^\circ(G) = U_2(G)$. Notice that in a simple algebraic group G in characteristic two, $U_2(G)$ will be the whole group.

We now consider the structure of a minimal counterexample to the claim.

Lemma 3.1. *Let G be a group of finite Morley rank and unipotent type, and suppose that $O_2{}^\circ(G)$ is not a Sylow° 2-subgroup of G. Suppose further that G is of minimal rank among all such groups. Then the following hold.*

1. *$Z(G)$ is finite and $G/Z(G)$ is simple.*
2. *For $U \le G$ a nontrivial definable 2-subgroup, not contained in $Z(G)$, setting $H = N^\circ(U)$ and $V = O_2{}^\circ(H)$, we have $H = VC_H{}^\circ(V)$ and H/V is of degenerate type.*

Proof.

Ad (1). It follows from the minimality hypothesis that $O_2{}^\circ(G) = 1$ and that $U_2(G) = G$. As G is connected, any finite normal subgroup is central. We must show that G contains no nontrivial proper definable connected normal subgroup.

Supposing the contrary, then for H a nontrivial proper definable connected subgroup of G, setting $\bar{G} = G/H$, our minimality hypothesis implies that $\bar{G}/O_2{}^\circ(\bar{G})$ is of degenerate type. Since we also have $\bar{G} = U_2(\bar{G})$, we find that $\bar{G} = O_2{}^\circ(\bar{G})$ is a 2-group. Let S be a Sylow° 2-subgroup of G. Then $G = HS$ ([PW00]), and by Theorem 5 we have $S \le C(H)$. Thus $S \le O_2{}^\circ(G)$, so $S = 1$ and $G = H$, a contradiction.

This proves the first point.

Ad (2). With U, H, and V as specified, observe that $H = N^\circ(U) < G$. Thus by minimality H/V is of degenerate type, and by Proposition 2.3 we have $H = VC_H(V)$. □

Now let us fix our notation in accordance with the preceding lemma. We may take G to be a group of finite Morley rank and unipotent type, with $O_2{}^\circ(G) < U_2(G)$, and of minimal rank among such groups, and we may factor out $Z(G)$. We then have the following conditions.

(∗) $G = U_2(G)$ is simple, and for $U \leq G$ a nontrivial definable 2-subgroup, setting $H = N^\circ(U)$ and $V = O_2{}^\circ(H)$, we have $H = VC_H{}^\circ(V)$, with H/V of degenerate type.

We observe that this condition also implies that the Sylow° 2-subgroups of G are unipotent.

Now, using the notion of strong embedding, borrowed directly from finite group theory, we can arrive quickly at a proof of Theorem 6.

Definition 3.2. *Let G be any group, M a subgroup. Then M is* strongly embedded *in G if the following conditions are satisfied:*

(1) *For $g \in G$ $M^g \cap M$ contains an involution if and only if $g \in M$.*
(2) *$M < G$.*

In groups of finite Morley rank a weaker variant of this notion is at least as useful, because it is more easily verified and has similar consequences.

Definition 3.3. *Let G be any group, and M a subgroup. Then M is* weakly embedded *in G if the following conditions are satisfied:*

(1) *For $g \in G$ $M^g \cap M$ contains an infinite 2-subgroup if and only if $g \in M$.*
(2) *$M < G$.*

There are more convenient criteria for strong and weak embedding in groups of finite Morley rank, valid more generally whenever there is a 2-Sylow theory available. For strong embedding, once G is known to contain an involution it suffices to check that M contains a Sylow 2-subgroup S of G, and the centralizer of each involution in S. For weak embedding, once it is known that G has a nontrivial and unipotent Sylow° 2-subgroup, it suffices to check that M contains a Sylow° 2-subgroup S of G, and the normalizer of every nontrivial definable connected subgroup of S.

One of the useful consequences of strong embedding is the following, whose proof in our context is just as in the finite case.

Lemma 3.4. *Let M be a strongly embedded subgroup in the group G of finite Morley rank. Then the involutions of M are conjugate in M.*

Lemma 3.5. *Let G be a group of finite Morley rank satisfying the conditions (∗) above. Let S be a Sylow° 2-subgroup of G. Then $N(S)$ is strongly embedded in G.*

Proof. We show first that

$$N(S) \text{ is weakly embedded in } G$$

For this, it suffices to show that for $U \leq S$ nontrivial, connected, and definable, we have $N(U) \leq N(S)$.

Supposing the contrary, take $U \le S$ maximal connected definable such that $H = N(U)$ is not contained in $N(S)$. Evidently $U < S$. Let $V = O_2{}^{\circ}(N(U))$. As V is a Sylow° 2-subgroup of H, it follows that $U < V$, and by maximality of U we have $N(V) \le N(S)$. Now $N(U) \le N(V) \le N(S)$, a contradiction. This proves that $N(S)$ is weakly embedded in G.

Now it suffices to prove that $C(i) \le N(S)$ for any involution $i \in N(S)$ (note that $N(S)$ contains a Sylow 2-subgroup of G). Let $U = C_S{}^{\circ}(i)$, a nontrivial connected definable 2-group (this uses strongly the fact that U is 2-unipotent). Then $N(U) \le N(S)$. Let $V = O_2{}^{\circ}(C(i))$. Then V is a Sylow° 2-subgroup of $C(i)$ and thus contains U. Furthermore $N_V(U) \le N(S)$ so $N_V{}^{\circ}(U) \le C_S{}^{\circ}(i) = U$. It follows that $V = U$ and $N(V) \le N(S)$. In particular $C(i) \le N(S)$, as claimed. □

Now we can prove Theorem 6 and in particular Theorem 3.

Proof of Theorem 6. Assuming the theorem fails, we find a group G of finite Morley rank satisfying the hypothesis $(*)$ above, and in particular a Syow° 2-subgroup S of G is unipotent. Then $N(S)$ is strongly embedded in G. Setting $H = N^{\circ}(S)$, we have $H = S \cdot C_H(S)$.

Now all involutions of $N(S)$ are conjugate under the action of $N(S)$ by Lemma 3.4. In particular they all lie in $Z(S)$. But $H = S \cdot C_H(S)$ acts trivially on $Z(S)$, and $N(S)/H$ is finite, so S contains only finitely many involutions. But S is 2-unipotent and hence contains an infinite elementary abelian subgroup. This is a contradiction. □

4. Poizat's problem

As we have said, the original point of Theorem 3 was that it was needed to clarify a longstanding problem put forward by Poizat, concerning groups which are generically of finite exponent. Our best result in that direction is the following [BBC05].

Proposition 4.1. *Let G be a connected group of finite Morley rank containing a definable generic subset whose elements are of order n for some fixed n. Then a Sylow 2-subgroup U of G is unipotent, $G = U * C_G(U)$ is a central product, and G/U is a group without involutions whose elements are generically of order n_0, where n_0 is the odd part of n.*

For the proof, one begins by defining U not as a Sylow 2-subgroup of G but as a Sylow° 2-subgroup, and one proves the corresponding result; of course, one first gets G/U to be a group of degenerate type, and then one invokes Theorem 1, which is indeed the main result of [BBC05], to tie the final knots.

The connection between Poizat's problem and our Theorem 3 or 6 can be seen in the following result.

Proposition 4.2. *Let G be a connected group of finite Morley rank containing a definable generic subset whose elements are of order n for some fixed n. Then G contains no nontrivial p-torus for any p.*

Now this comes from a rather different, and more central, place in the theory of groups of finite Morley rank: it is really a statement about what happens when one *does* have nontrivial p-tori. Up to this point in the present article, p-tori have been conspicuous primarily by their absence. But of course the moral of our story is that we need good techniques for exploiting p-tori when we have them, and this is fortunately the case. What follows is a variant of the main result of [Ch05], which can easily be reduced to the form given there.

Theorem 7. *Let T be a maximal p-torus in a connected group G of finite Morley rank, and let $H = C^\circ(T)$, the connected centralizer. Then there is a definable subset $X \subseteq H$, generic in H, such that any two conjugates of X are either disjoint or equal, and such that the union of the conjugates of X is a generic subset of G.*

This means that as far as rank computations are concerned, G can be replaced by a cartesian product $X \times Y$ with Y parametrizing the set of conjugates of X (in fact Y can be identified with the coset space $N(T)\backslash G$ in this situation).

The proof of this theorem involves all the rigidity properties of good tori and the connection between p-tori and good tori via decent tori. It was of course directly inspired by the situation in algebraic groups, and more particularly the theory of reductive algebraic groups, where $C^\circ(T)$ reduces to a maximal torus of G.

There is a competing theory which is very useful in classification results, namely the theory of Carter subgroups. This theory provides another analog of conjugacy of maximal tori, valid in arbitrary solvable finite groups, and, suitably adjusted, also in arbitrary groups of finite Morley rank with no solvability hypothesis. The Carter theory and the theory of maximal decent tori have much in common. We will return to this point in the next section.

Now we can sketch the proof of Theorem 4.1. The main step from our present point of view is the proof of Proposition 4.2, that is the elimination of p-tori.

Proof of Proposition 4.2. Invoke Theorem 7. So the notation is as follows: T is a maximal p-torus, $H = C^\circ(T)$, $X \subseteq H$ is the generic subset of H afforded by that result. Set $\hat{T} = d(T)$, the definable hull of T. One can then easily move the initial hypothesis around as follows:

Generically $x^n = 1$ in G
Generically $x^n = 1$ in X
Generically $x^n = 1$ in H
Generically $x^n = 1$ in some coset of $\hat{T}$ in H
Generically $x^n = 1$ in $\hat{T}$
Identically, $x^n = 1$ in $\hat{T}$
Identically, $x^n = 1$ in T

And the last condition forces $T = 1$. An important point here is that the elements of any coset of $\hat{T}$ in H commute; this commutativity is what we were aiming at, as it allows the transition from a generic condition to an identical condition. □

The hypothesis that $x^n = 1$ generically passes to definable quotients but not to subgroups; but the conclusion on the absence of p-tori is fully inductive, and this gave us the necessary flexibility for our analysis.

5. Carter subgroups

In finite group theory, a Carter subgroup is a selfnormalizing nilpotent subgroup, and it is a remarkable fact that in solvable finite groups there is always a unique conjugacy class of Carter subgroups; neither existence nor uniqueness is evident. In groups of finite Morley rank it is more convenient to work with the connected analog of this notion: so now a Carter subgroup will be a connected definable nilpotent subgroup which is almost selfnormalizing, that is it coincides with the connected component of its normalizer. These behave as well in solvable groups of finite Morley rank as they do in the finite theory; but in fact *every* group of finite Morley rank has Carter subgroups in this sense [FJ05]. On the other hand, the issue of conjugacy is still open. There are other properties which are desirable in the context of finite Morley rank, notably the following.

Definition 5.1. *A subgroup H of a connected group G of finite Morley rank is* genererous *if the union of the conjugates of H is generic in G.*

For example, centralizers of decent tori are generous; this is a variant of Theorem 7. If one considers centralizers of maximal decent tori one gets a sharper version as in that theorem, but even this looser version is useful.

In [Ja06?] one finds the following.

Theorem 8. *Let G be a connected group of finite Morley rank. Then there is at most one conjugacy class of generous Carter subgroups.*

Here existence remains an open problem. Thus we have the existence of Carter subgroups and the uniqueness of generous Carter subgroups, and if the gap between these two could be closed we would have a theory which in important ways is sharper than the parallel theory for centralizers of maximal decent tori, which are not known to be nilpotent.

Of course, for T a maximal decent torus and $H = C^\circ(T)$, the group H/T contains no nontrivial decent torus, that is it is of unipotent type, and thus H has a unipotent 2-subgroup U for which H/TU is of degenerate type as well as unipotent type. It is in fact unlikely that all such groups are nilpotent. Apart from the possible existence of simple groups of this type, it is quite likely that there are solvable nonnilpotent torsion-free groups of this type, a point tied up with the possible existence of "bad fields".

Frécon [Fr05] has shown that the two theories, based alternatively on Carter subgroups or centralizers of p-tori, essentially coincide in the *tame* case (with no "bad groups" or "bad fields" involved), so that one gets all the features of both in this case. As things now stand, in order to bring the theory based on tori to a comparable degree of completeness one would at a minimum need to eliminate simple groups of degenerate type. On the other hand, there is no such obvious obstruction to improving the Carter theory by proving the existence of generous Carter subgroups in full generality. If this could be done, it would then become appropriate to redefine "Carter subgroups" so as to include generosity.

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E-mail address: `borovik@manchester.ac.uk`

E-mail address: `burdges@math.rutgers.edu`