Minimal connected simple groups of finite Morley rank with strongly embedded subgroups [☆]

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Abstract

Let G be a simple K^*-group of finite Morley rank of odd type which is not algebraic. Then G has Prüfer 2-rank at most two. This improves on an earlier result of the second and third authors for the tame case. In the critical case G is minimal connected simple of odd type with a proper definable strongly embedded subgroup. The bulk of the analysis relies on the first author's theory of 0-unipotence and the related 0-Sylow subgroup theory, as well as the so-called Bender method adapted to this context.
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1. Introduction

The Algebraicity Conjecture states that a simple group of finite Morley rank should be isomorphic with an algebraic group. A program initiated by Borovik aims at controlling the 2-local structure in a hypothetical minimal counterexample to the Algebraicity Conjecture. There is now a large body of work on this program. A fundamental division arises at the outset, according to the structure of a Sylow 2-subgroup. In algebraic groups this structure depends primarily on the characteristic of the base field. In groups of finite Morley rank in general, in addition to the *even* and *odd* type groups, which correspond naturally to the cases of characteristic two or not two, respectively, we have two more cases, called *mixed* and *degenerate* type. In the degenerate case the Sylow 2-subgroup is finite.

The cases of even and mixed type groups are well in hand, and work in course of publication will show that the simple groups of finite Morley rank of these two types are algebraic. Work on degenerate type has hardly begun, though recently some interesting approaches have emerged. We deal here with odd type groups exclusively. In this context, the "generic" case is generally considered to be that of groups of Prüfer 2-rank three or more. The following result enables us to complete the analysis of the generic case.

Theorem 1. *Let G be a minimal connected simple group of finite Morley rank and of odd type. Suppose that G contains a proper definable strongly embedded subgroup M. Then G has Prüfer 2-rank one.*

Combining this with known results, we will derive the following.

Theorem 2. *Let G be a simple K*-group of finite Morley rank of odd type, which is not algebraic.*

(1) *Then G has Prüfer 2-rank at most two.*
(2) *If G is tame and minimal connected simple, and all the involutions in a standard Borel subgroup of G are central, then G has Prüfer 2-rank one.*

The proof of Theorem 1 will be self-contained, while for Theorem 2 we will need to invoke an extensive body of material, some in course of publication (or, alternatively, available in [Bur04]). We will now enlarge on the terminology used above.

The Prüfer 2-rank of a group of finite Morley rank is the Prüfer rank of a maximal 2-divisible abelian subgroup, and is always finite.

A group of finite Morley rank is *tame* if it involves no bad field, and is a K^*-group if every proper definable infinite simple section is algebraic. The Borovik program was initially directed primarily at tame K^*-groups. It is known that a *tame* nonalgebraic simple K^*-group of finite Morley rank and odd type has Prüfer 2-rank at most two, which gives a qualitative version of the Borovik program in the tame case; most of the analysis does not depend on tameness, up to the point of a reduction to the minimal connected simple case, treated in [CJ04] by methods specific to the tame case. Each clause of Theorem 2 improves on this result; the first clause eliminates the tameness hypothesis, while the second clause eliminates one of the configurations in Prüfer rank two which was left open in [CJ04].

The first clause of Theorem 2 completes the analysis of the "generic" K^*-group of finite Morley rank of odd type. An outline of the various special cases which require further analysis, and some ideas as to how they may be approached, is given in Chapter 11 of [Bur04].

In the next section we will give the derivation of Theorem 2 from Theorem 1, and we will also make some preliminary remarks concerning the proof of Theorem 1. Subsequent sections are de-

voted to the proof of Theorem 1, which is divided into two cases. These two cases will be treated by quite distinct methods. The first is handled relatively rapidly in Section 4, while the other occupies us for another three sections. Our method of analysis is heavily influenced by [CJ04], which obtained similar results under the hypothesis of tameness, and also gave an analysis of the problematic configurations in Prüfer 2-ranks one and two (with two such configurations in each rank). As noted, our theorem also eliminates one of these four problematic configurations.

A case division exploited throughout [CJ04], and which we continue to make use of, is the following. Let a *standard* Borel subgroup be a Borel subgroup which contains a Sylow° 2-subgroup of G. Then our case division is as follows: either (I) all the involutions of a standard Borel subgroup are central, or (II) not. In the former case, the method used in [CJ04] in high Prüfer 2-rank is inappropriate outside the tame case, and is replaced here by another method which, as noted, yields a sharper result in Prüfer 2-rank two. In the latter case, one becomes involved in a close analysis of intersections of Borel subgroups. The basic lemma concerning such intersections in the tame case is the following.

Fact 1.1. *(See [CJ04, §3.4].) Let G be a tame minimal connected simple group of odd type and finite Morley rank. Assume that B_1 and B_2 are distinct Borel subgroups of G with $O(B_1), O(B_2) \neq 1$. Then $F(B_1) \cap F(B_2) = 1$.*

An important consequence of this lemma is that $B_1 \cap B_2$ is abelian under the stated hypotheses (as $B_1/F(B_1)$ and $B_2/F(B_2)$ are abelian). It would be equally useful to have $(B_1 \cap B_2)°$ abelian. We see no direct way of proving anything as strong as this in general, but we continue to work in this general direction, at the price of a much more elaborate analysis. This was initiated in [Bur04], whose Theorem 9.2 gives a very elaborate analog of Fact 1.1, describing the configuration that arises from the situation in which $(B_1 \cap B_2)°$ is nonabelian.

In the tame case, in high Prüfer 2-rank, the case in which a standard Borel subgroup B contains an involution not in its center was disposed of in short order via the powerful Fact 1.1, whereas the analysis of the other case was quite long. Here it appears the situation is entirely reversed, with the case of central involutions being dispatched relatively expeditiously via the construction of two disjoint generic subsets. However, in [CJ04] a portion of the preliminary analysis of the case of central involutions actually went toward showing that the hypotheses of our main theorem are satisfied in that case.

2. Preliminaries

2.1. Theorem 2

We will now discuss the prior results which link Theorems 1 and 2. The two facts which require explanation are as follows.

Fact 2.1. *Let G be a simple nonalgebraic K^*-group of finite Morley rank and odd type, and Prüfer 2-rank at least three. Then G is minimal connected simple, and has a proper definable strongly embedded subgroup.*

Fact 2.2. *Let G be a tame and minimal connected simple group of finite Morley rank and odd type, and Prüfer 2-rank at least two, and suppose that there is a standard Borel subgroup B of G such that every involution of B lies in $Z(B)$. Then G has a proper definable strongly embedded subgroup.*

We deal first with Fact 2.1.

A general reference for this fact, and for a great deal of prior material, is the thesis [Bur04], notably Theorem 10.15 in that reference, combined with Theorems 8.14 and 8.18. The latter two theorems may be found in [BBN], while Theorem 10.15 will appear in [Bur0x]. These results depend in turn on a very substantial body of material due to Borovik and others. For the reader's convenience we give some additional details.

Theorem 10.15. *Let G be a simple K^*-group of finite Morley rank and odd type with Prüfer 2-rank at least three. Then either G has a proper 2-generated core, or G is an algebraic group over an algebraically closed field of characteristic not two.*

The 2-*generated core* is defined as follows. Let S be a Sylow 2-subgroup of G. Let $\Gamma_{2,S}^e(G)$ be the definable closure of the subgroup of G generated by all subgroups of the form

$$N_G(V) \quad \text{(for } V \leqslant S, \ V \text{ an elementary abelian 2-group of rank two).}$$

Then $\Gamma_{2,S}^e(G)$ is the 2-*generated core* of G with respect to S, and as S varies the groups $\Gamma_{2,S}^e$ are conjugate. So the 2-generated core of G is well defined up to conjugacy.

Having a proper 2-generated core is weaker than having a strongly embedded subgroup, but is the first step in this direction. The next step is given by the following, found in [BBN]; see also [Bur04, Chapter 8],

Strong Embedding Theorem 1. *Let G be a simple K^*-group of finite Morley rank and odd type, and suppose that G has normal 2-rank at least three and Prüfer 2-rank at least two. Let S be a Sylow 2-subgroup of G, and $M = \Gamma_{2,S}^e(G)$ the associated 2-generated core. If M is a proper subgroup of G, then G is a minimal connected simple group and M is strongly embedded in G. Furthermore M° is a Borel subgroup of G and S is connected.*

Note that Fact 2.1 now follows.

We turn next to Fact 2.2, where we deal with the tame case. This fact concerns one of the four configurations which were considered explicitly in [CJ04], in the context of tame minimal connected simple groups of finite Morley rank and odd type, and which were not eliminated there. It shows that the strong embedding hypothesis with which we will be working is valid in that case. The groups under consideration in [CJ04] are tame groups of finite Morley rank, minimal among connected simple groups. The configuration in which all involutions in a standard Borel subgroup B are central, and with the additional hypothesis of Prüfer rank at least two, is taken up in the first part of Section 7 of [CJ04].

There it is shown, first, that a standard Borel subgroup B is nilpotent, and then, after some further analysis, that the normalizer of B is strongly embedded. It is also clear in this case that each Sylow° 2-subgroup is contained in a unique standard Borel subgroup, and in particular the standard Borel subgroups are conjugate, so they all have the same properties.

Note that the case division considered in [CJ04, §§6,7] is equivalent to the following: a standard Borel subgroup is, or is not, strongly embedded. In all cases that survive analysis the centralizer of an involution contains a standard Borel subgroup. Here we will begin with a strongly embedded subgroup, and then see quickly that the normalizer of a standard Borel subgroup is strongly embedded; for us, the interesting case division lies farther on, within this configuration, and the harder of the two cases is one that vanished after a few lines of analysis in the tame context.

2.2. The context of Theorem 1

We now turn to Theorem 1. Suppose that G is a minimal connected simple group of finite Morley rank and odd type, and that M is a proper definable strongly embedded subgroup of G. We also assume that

(†) G has Prüfer 2-rank at least two.

We set $B = M^\circ$, and to justify this notation we prove the following.

Lemma 2.3. *B is a standard Borel subgroup of G, and $M = N(B)$.*

Proof. Since M is a strongly embedded subgroup of G, it contains a Sylow 2-subgroup of G, and hence B contains a Sylow$^\circ$ 2-subgroup S of G. By the minimality of G, B is solvable.

Now suppose $B \leqslant H$ with H a Borel subgroup of G. We claim that $B = H$. Let $V \leqslant \Omega_1(S)$ have order four. As $V \leqslant H$, in particular V acts on H, and as H is connected solvable of odd type, a fundamental generation property given in [Bor95, 5.14] implies that

$$H = \langle C_H^\circ(v) \colon v \in V^\# \rangle.$$

On the other hand, by strong embedding of M, we have $C_H^\circ(v) \leqslant B$ for all involutions $v \in V^\#$, and thus $H = B$. So B is a standard Borel subgroup of G.

As $B = M^\circ$, we have $M \leqslant N(B)$. Conversely, as $B \leqslant M$ contains involutions and M is strongly embedded, we conclude $N(B) \leqslant M$. Thus $M = N(B)$. □

Part of the foregoing argument can be strengthened as follows to a "black hole" principle (the term goes back to Harada). We record this for future reference.

Lemma 2.4. *Let H be a connected definable proper subgroup of G and V an elementary abelian 2-subgroup of B of rank 2. If V normalizes H then $H \leqslant B$.*

Now we intend to make a case division based on whether or not all the involutions of B are central in B. As M is strongly embedded in G, all involutions of M are conjugate in M. Hence all involutions of M lie in B, and if one involution is central in B then all involutions of B lie in its center. So our case division is actually the following: either all involutions of B are central in B, or none are.

Before entering into the analysis of individual cases, we give a genericity result that holds in both cases. This was given already in [Bur04] and somewhat more explicitly in [BBN], under stronger hypotheses which are not actually used at this point in the argument. For the reader's convenience we indicate the gist of the argument, which depends on the following generic covering lemma, a result for which we will have further use in Section 6.

Fact 2.5. *(See [CJ04, 3.3].) Let G be a connected group of finite Morley rank, and B a definable subgroup of finite index in its normalizer. Suppose that there is a nongeneric subset X of B such that $B \cap B^g \subseteq X$ for $g \in G \setminus N(B)$. Then $\bigcup_{g \in G} B^g$ is generic in G.*

Lemma 2.6. *The union $\bigcup_{g \in G} B^g$ is a generic subset of G.*

Proof. By strong embedding, $M \cap M^g$ contains no involutions for $g \notin M$, and in particular $B \cap B^g$ is a $2^\perp$-group for $g \notin M$. By a very general lemma given as [Bur04, 8.17] or [BBN, 5.10], the union $\bigcup_{g \in G \setminus N(B)} (B \cap B^g)$ is contained in a proper definable subgroup of B, in fact a $2^\perp$-group, and hence is not generic in B. Then by the generic covering lemma the claim follows. □

In the remainder of the article, we will prove Theorem 1, dividing the analysis into two very different cases, which rely on very different methods. We first summarize some very general background material, used throughout, and sufficient for the treatment of the first of our two cases. The second case will involve a further body of material which is both more extensive, and of more recent vintage.

3. Background material

The material of the present section is for the most part well known. The most subtle item comes from the theory of Carter subgroups and is due to Frécon: Fact 3.10, and its consequence Lemma 3.11, below.

3.1. Unipotence

Definition 3.1. Let p be a prime, G a group of finite Morley rank, and P a subgroup of G.

(1) P is said to be *p-unipotent* if P is a solvable connected definable p-subgroup of G of finite exponent.
(2) $U_p(G)$ is the largest normal p-unipotent subgroup of G.

The group $U_p(G)$ is well defined, by an elementary argument.

Fact 3.2. *(See [Nes90].) Let H be a solvable group of finite Morley rank and P a p-unipotent subgroup of H. Then $P \leqslant U_p(H)$.*

This is phrased somewhat differently in [Nes90]. The essential point is that $H^\circ / F^\circ(H^\circ)$ is divisible abelian. From this it follows that $P \leqslant F^\circ(H)$, and then the structure of nilpotent groups applies, as in [BN94, §6.4].

The next lemma is a weak form of Fact 1.1, and has a similar proof, given also in [Bur05a]. The virtue of this lemma is that it holds without any assumption of tameness. While this will suffice for the purposes of the next section, we will need much more subtle variations subsequently.

Lemma 3.3. *Let G be a minimal connected simple group of finite Morley rank, p a prime, and P a nontrivial p-unipotent subgroup of G. Then P is contained in a unique Borel subgroup of G.*

Proof. Suppose on the contrary that B_1 and B_2 are two Borel subgroups containing P, and chosen so that $Q = U_p(B_1 \cap B_2)$ is maximal. Then $Q \leqslant U_p(B_1)$.

Suppose $Q < U_p(B_1)$. Then $N^\circ_{U_p(B_1)}(Q) > Q$ by the normalizer condition [BN94, §6.4]. Put $N^\circ(Q)$ in a Borel subgroup B_3; by the maximality of Q, we find $B_3 = B_1$. Then $B_3 \neq B_2$, so we must have $Q = U_p(B_2)$. Then $B_2 \leqslant N^\circ(Q) \leqslant B_1$, a contradiction.

There remains the possibility that $Q = U_p(B_1) = U_p(B_2)$. But then $B_1, B_2 = N^\circ(Q)$, again a contradiction. □

3.2. Sylow subgroups

Lemma 3.4. *Let P be a Sylow p-subgroup of a connected solvable group of finite Morley rank. Then P decomposes as $U * T$, a central product, with U p-unipotent and T a p-torus (that is, a divisible abelian p-group).*

Proof. This is essentially [BN94, §6.4], where a similar structure theorem is given more generally for p-subgroups of solvable groups of finite Morley rank. When H is in addition connected, then by [BN94, 9.39], its Sylow (or indeed its Hall) subgroups are connected, and this gives the connectivity of the factor U. $\quad\square$

Lemma 3.5. *Let G be a group of finite Morley rank, P a Sylow 2-subgroup, H a normal subgroup. Then*:

(1) $P \cap H$ *is a Sylow 2-subgroup of H.*
(2) *If H is definable and $\overline{G} = G/H$, then $\overline{P}$ is a Sylow 2-subgroup of $\overline{G}$.*

The first follows directly from the conjugacy of Sylow 2-subgroups. The second point is given in [PW00].

3.3. Genericity

Fact 3.6. *(See [CJ04, 3.4].) Let G be a connected group of finite Morley rank. Suppose that B is a definable subgroup of finite index in its normalizer such that $\bigcup_{g \in G} B^g$ is generic in G. Suppose that $x \in N_G(B) \setminus B$. Let X be the set*

$$\left\{ x' \in xB \colon x' \in \left(\langle x \rangle B \right)^g \text{ for some } g \in G \setminus N(B) \right\}.$$

Then X is generic in xB.

Fact 3.7. *(See [CJ04, 3.6].) Let H be a group of finite Morley rank such that H° is abelian, and let xH° be a coset whose elements are generically of fixed order n. Then every element of xH° is of order n.*

Lemma 3.8. *Let H be a group of finite Morley rank with H° solvable, and H/H° of prime order p. Suppose the elements of every coset of H° other than H° are generically of order p. If some element of $H \setminus H^\circ$ has an infinite centralizer in H°, then H° contains a nontrivial p-unipotent subgroup.*

Proof. This was proved under the assumption that H° is nilpotent in [CJ04, 3.7]. However, under the stated hypotheses (or slightly weaker ones; one such coset suffices) the solvability of H° implies its nilpotence, by [JW00, Corollary 16]. $\quad\square$

3.4. Carter subgroups

A Carter subgroup of a solvable group of finite Morley rank is a definable self-normalizing nilpotent subgroup. If H is a connected solvable group of finite Morley rank, then by [Wag94,

5.5.10, 5.5.12] and [Fré00, 1.1], it has a Carter subgroup, and any two such are conjugate; and by [Fré00, 3.2], its Carter subgroups are connected.

Fact 3.9. *(See [Fré00, 3.2].) Let H be a connected solvable group of finite Morley rank, and Q a nilpotent subgroup of H with $[N_H(Q) : Q]$ finite. Then Q is a Carter subgroup of H.*

This includes the more elementary result that the centralizer of any element of finite order in a connected solvable group is infinite (compare [J01]).

Fact 3.10. *(See [Fré00, 7.15].) Let H be a connected solvable group of finite Morley rank, and R a Hall π-subgroup of H for some set π of primes. Then $N_H(R)$ contains a Carter subgroup of H.*

This has the following important consequence.

Lemma 3.11. *Let H be a connected solvable group of finite Morley rank with $U_p(H) = 1$, and let Q be a Carter subgroup of H. Then Q contains a Sylow p-subgroup of H.*

Proof. Let S be a Sylow p-subgroup of H. By Fact 3.10, $N_H(S)$ contains a Carter subgroup Q_0 of H. Q_0 is connected. It follows easily from Lemma 3.4 that S is a p-torus, that is, abelian and p-divisible. So $Q_0 \leqslant N^\circ(S) = C^\circ(S)$. Thus $S \leqslant N(Q_0) = Q_0$. Now as Q_0 and Q are conjugate, our claim follows. $\square$

4. Central involutions

4.1. The setup

We take up the proof of Theorem 1. We dispose of one case in the present section, and the other will occupy us to the end of the paper.

So G is a minimal connected simple group of finite Morley rank and odd type, with M definable and strongly embedded, and we assume that

$(\dagger)$ $\qquad\qquad\qquad\qquad\qquad$ G has Prüfer 2-rank at least two.

As explained in Section 2, we then have $M = N(B)$ strongly embedded, with B a standard Borel subgroup.

In the present section we take up the first of our two cases, namely:

(**Case I**) $\qquad\qquad\qquad\qquad\qquad$ All involutions in B are central in B.

In particular $C^\circ(i) = B$ for each involution i in B.

We will derive a contradiction in this case by constructing two disjoint generic subsets of G. That is, in Case I the Prüfer rank of G is at most 1.

The two generic subsets in question will be BI_1 and $BC(\sigma)B$ where $I_1 = I(G) \setminus I(M)$ and $\sigma \in M \setminus B$. We must deal with the following issues: that the ranks of the two sets in question are the same as the ranks of the corresponding Cartesian products $B \times I_1$ and $B \times C(\sigma) \times B$; that these ranks coincide with the rank of G; and that the sets in question are disjoint. We must also

produce a suitable element σ, but that is immediate since the involutions of B form an elementary abelian subgroup which by hypothesis has 2-rank at least two, and they are conjugate under the action of M, by strong embedding, and are central in B.

That the set BI or BI_1 turns out to be generic in G is perfectly natural, but in the case of the set $BC(\sigma)B$ it is surprising.

We record this notation.

Notation 4.1.

(1) $I = I(G)$ and $I_1 = I \setminus M = I \setminus B$.
(2) $\sigma \in M \setminus B$ (*fixed*).

4.2. The first generic subset

Recall that an element a of $G^{\#}$ is said to be *strongly real* if it is a product of two involutions, in which case a is inverted by these two involutions.

The following fundamental fact will be used in both of our cases.

Fact 4.2. *(See [BN94, 10.19].) In a group of finite Morley rank with a definable strongly embedded subgroup M, if a is a strongly real element commuting with an involution in M, then every involution which inverts a lies in M.*

In Case I, this applies to every strongly real element of B, so we arrive at the following under our present assumptions.

Lemma 4.3. *The strongly real elements of B are its involutions.*

Proof. By Fact 4.2, if $b \in B$ is strongly real and j is an involution inverting b, then we have $j \in B$. By our case hypothesis j commutes with b, so b is an involution.

Conversely, any involution in B lies in $Z(B)$ and is the product of two involutions by the hypothesis (†).   □

Lemma 4.4. *The set BI_1 is generic in G, and the multiplication map*

$$B \times I_1 \to G$$

is injective.

Proof. Notice first that $I(B)$ is finite since M/B operates transitively on this set. Hence $\mathrm{rk}(I) = \mathrm{rk}(I_1)$.

Let $i \in I(B)$. Recall that $C^{\circ}(i) = B$, and $\mathrm{rk}(I) = \mathrm{rk}(G/C(i))$ as all involutions of G are conjugate. So we have $\mathrm{rk}(G) = \mathrm{rk}(C(i)) + \mathrm{rk}(G/C(i)) = \mathrm{rk}(B) + \mathrm{rk}(I) = \mathrm{rk}(B) + \mathrm{rk}(I_1)$. It suffices therefore to check that the multiplication map $B \times I_1 \to G$ is injective.

Supposing the contrary, we have a nontrivial intersection $bI_1 \cap I_1$ with $b \in B^{\#}$, which yields an equation $b = jk$ with $j, k \in I_1$. Thus $b \in B$ is strongly real. By Lemma 4.3, b is an involution, and j centralizes b, hence lies in M, a contradiction.   □

From this genericity result, others of the same type can be deduced, for sets of the following form.

Notation 4.5. *For $g \in G$, we set $I_g = (BgI) \cap I_1$.*

Lemma 4.6. *For any $g \in G$, the set BgI is generic in G, and the set I_g is generic in I.*

Proof. The set BI is generic in G. Conjugating by g, the set $B^g I$ is generic in G. Translating on the left by g, the set BgI is generic in G. Hence also $(BgI) \cap (BI_1)$ is generic in G.

Now $BI_g = (BgI) \cap (BI_1)$ is generic in G. Hence $\mathrm{rk}(G) = \mathrm{rk}(BI_g) = \mathrm{rk}(B) + \mathrm{rk}(I_g)$ as the multiplication map restricted to $B \times I_g$ is injective (Lemma 4.4), and thus $\mathrm{rk}(B) + \mathrm{rk}(I_g) = \mathrm{rk}(B) + \mathrm{rk}(I)$ and $\mathrm{rk}(I_g) = \mathrm{rk}(I)$. □

4.3. Disjointness and centralizers

The rest of the argument requires some more structural information.

Lemma 4.7. *The intersection of two distinct conjugates of B is finite.*

Proof. Let S be a Sylow 2-subgroup of B, and $V = \Omega_1(S)$. Then $V \leqslant Z(B)$. Hence V centralizes $B \cap B^g$.

Suppose the group $(B \cap B^g)^\circ$ is nontrivial. Then it has a nontrivial Carter subgroup Q. As V centralizes Q, it normalizes $N^\circ(Q)$. By Lemma 2.4, $N^\circ(Q) \leqslant B$. Hence $N^\circ_{B^g}(Q) \leqslant (B \cap B^g)^\circ$ and thus $N^\circ_{B^g}(Q) = Q$. By Fact 3.9, Q is a Carter subgroup of B^g. Then by Lemma 3.11, Q contains a Sylow 2-subgroup of B^g. In particular $B \cap B^g$ contains an involution i and hence $B = C^\circ(i) = B^g$. □

Lemma 4.8. *The intersection of any two distinct conjugates of B is trivial.*

Proof. Supposing the contrary, let $B \cap B^g$ be finite and nontrivial, and let $x \in B \cap B^g$ have prime order p. Let P be a Sylow p-subgroup of B containing x. By Lemma 3.4, P is a central product of the form $U * T$ with U connected, nilpotent of bounded exponent and T divisible abelian. Here $d(U)$ is also a connected nilpotent p-group centralizing T, so $P = d(U) * T$ and $U = d(U)$ is definable.

If $U \neq 1$, let $U_0 = C^\circ_U(x)$. Then $U_0 \neq 1$ [BN94, 6.20]. Now U_0 is contained in a unique Borel subgroup by Lemma 3.3, and hence $C^\circ(x) \leqslant B$. But as B and B^g are conjugate, a Sylow p-subgroup of B^g has the same form, and hence this argument shows $C^\circ(x) \leqslant B^g$ as well. But then $U_0 \leqslant B^g$ and $B = B^g$.

This contradiction shows that $U = 1$ and hence P is a p-torus. Then Lemma 3.11 shows that there is a Carter subgroup Q of B containing P. Furthermore, the same lemma shows that Q contains a Sylow 2-subgroup S of B. As Q is nilpotent, it follows that x (in P) commutes with S. Similarly, x commutes with a Sylow 2-subgroup of B^g.

Let V be a four-group contained in S. Then V normalizes $C^\circ(x)$ and hence by Lemma 2.4 we have $C^\circ(x) \leqslant B$, and in particular B contains a Sylow 2-subgroup of B^g, forcing $B = B^g$, a contradiction. □

Lemma 4.9. *Let $x \in N(B) \setminus B$. Then the centralizer $C_B(x)$ is finite.*

Proof. Suppose $C_B(x)$ is infinite. Replacing x by a power, we may suppose that the order p of x modulo B is a prime.

Consider the subset X of xB defined as

$$\left\{ x' \in xB \colon x' \in \left(\langle x \rangle B \right)^g \text{ for some } g \in G \setminus N(B) \right\}.$$

This is generic in xB by Fact 3.6. For $x' \in X$ we have $x'^p \in B \cap B^g = 1$ for some $g \in G \setminus N(B)$. The same applies to any coset in $\langle x \rangle B$ other than B.

By Lemma 3.8 and Fact 3.2, $U_p(B) > 1$. Then for $x' \in X$, with $x' \in N(B^g)$, $g \in G \setminus N(B)$, consider the action of x' on $U_p(B)$ and $U_p(B^g)$. It follows [BN94, 6.20] that the groups $U_p(C_B(x'))$ and $U_p(C_{B^g}(x'))$ are nontrivial. Letting B_1 be a Borel subgroup containing $C^\circ(x')$, it follows by Lemma 3.3 that $B = B_1 = B^g$, a contradiction. $\quad\square$

This yields strong information concerning the coset $B\sigma$ (which is in fact an arbitrary coset of B in M, other than B).

Lemma 4.10.

(1) $B\sigma = \sigma^B$.
(2) *The elements of $B\sigma$ are all strongly real, and lie outside $\bigcup_{g \in G} B^g$.*

Proof. (1) Since $\sigma \in N(B)$ we have $\sigma^B \subseteq B\sigma$. Furthermore, as $C_B(\sigma)$ is finite, we have $\text{rk}(\sigma^B) = \text{rk}(B) = \text{rk}(B\sigma)$. As $B\sigma$ has Morley degree one, it consists of a single B-conjugacy class.

(2) As the set I_σ is generic in I by Lemma 4.6, it is nonempty, and we have an equation $xi = j$ with $x \in B\sigma$ and $i, j \in I$. Hence x is strongly real, and since $B\sigma$ is a single B-conjugacy class, all of its elements are strongly real.

Now the strongly real elements of B are involutions. Hence no element of $B\sigma$ can be conjugate to an element of B. $\quad\square$

Lemma 4.11. *Let $g \in G$ be strongly real, inverted by the involution i, with g not an involution. Then i acts on $C^\circ(g)$ by inversion.*

Proof. We may suppose $i \in B$. It suffices to show that $C^\circ(i, g) = 1$ [BN94, p. 79, Ex. 13, 15]. We have $C^\circ(i, g) \leqslant C^\circ(i) = B$, and similarly $C^\circ(i, g) \leqslant B^g$. So if $C^\circ(i, g) \neq 1$ then $g \in N(B)$, and we contradict Lemma 4.9 or Lemma 4.3. $\quad\square$

4.4. The second generic set

We now take up the proof that $BC(\sigma)B$ is generic in G and disjoint from BI_1. Let us begin with the latter point.

Lemma 4.12. *M/B has odd order.*

Proof. If not, we may choose σ of order 2 modulo B. By Lemma 3.5 we may choose σ to be a 2-element. Let i be an involution in the cyclic group $\langle \sigma \rangle$. Then $i \in B$.

By Lemma 4.10 the element σ is strongly real. Let j be an involution inverting σ. As $j \in C(i)$, we have $j \in M$, so $j \in B$. But $j\sigma$ is another involution in M, so $j\sigma \in B$. Hence $\sigma \in B$, a contradiction. $\square$

Lemma 4.13. $C(\sigma) \cap I_1 = \emptyset$.

Proof. In view of Lemma 4.10, there is an involution i inverting σ. Suppose toward a contradiction that there is also some $j \in C(\sigma) \cap I_1$, and let M^g be the conjugate of M containing j. As σ is strongly real, Fact 4.2 implies that $i \in M^g$, hence $i \in B^g$. But σ normalizes $B^g = C^\circ(j)$, so $\sigma \in M^g$. Now $\sigma^2 = [i, \sigma] \in B^g$, so $\sigma \in B^g$ by Lemma 4.12, and this contradicts Lemma 4.10(2). $\square$

Lemma 4.14. $BC(\sigma)B$ and BI_1 are disjoint.

Proof. Supposing the contrary, we have $b_1cb_2 \in I_1$ for some $b_1, b_2 \in B$ and $c \in C(\sigma)$, and conjugating by b_1 gives

$$(1) \qquad\qquad\qquad cb \in I_1$$

with $b \in B$. Conjugating by σ gives

$$(2) \qquad\qquad\qquad cb^\sigma \in I_1.$$

It follows that $b^{-1}b^\sigma = (cb)^{-1}(cb^\sigma)$ is inverted by an element j of I_1.

If $b \in C(\sigma)$, then $cb \in C(\sigma) \cap I_1$, contradicting Lemma 4.13. So $b^{-1}b^\sigma$ is nontrivial, and is a strongly real element of B. So $j \in M$ by Fact 4.2, hence $j \in B$, which is a contradiction since $j \in I_1$. $\square$

Now we compute the rank of $BC(\sigma)B$, getting the expected value.

Lemma 4.15. *The set $BC(\sigma)B$ has rank $2\operatorname{rk}(B) + \operatorname{rk}(C(\sigma))$.*

Proof. Let C_σ be $C(\sigma) \setminus N(B)$. As $C_B(\sigma)$ is finite by Lemma 4.9, the set C_σ differs from $C(\sigma)$ by a finite set. On the other hand, the centralizer $C_G(\sigma)$ is infinite, as otherwise the conjugacy class σ^G would be generic in G and hence meet $\bigcup_{g \in G} B^g$, contradicting Lemma 4.10. So $\operatorname{rk}(C_\sigma) = \operatorname{rk}(C(\sigma))$.

It now suffices to check that the multiplication map

$$\mu : B \times C_\sigma \times B \to G$$

has finite fibers.

If $g = bcb'$ with $b, b' \in B$, $c \in C_\sigma$, and $\mu^{-1}(g)$ is infinite, then the same applies to $c = b^{-1}gb'^{-1}$. So we consider $\mu^{-1}(c)$ with $c \in C_\sigma$ fixed. That is, we examine the solutions to the equation

$$(*) \qquad\qquad\qquad b_1c'b_2 = c$$

with $b_1, b_2 \in B$ and $c' \in C_\sigma$. Applying σ, we get

$$b_1^\sigma c' b_2^\sigma = c.$$

Combining these two equations yields

$$c' = b_1' c' b_2'$$

with $b_1' = b_1^{-1} b_1^\sigma$, $b_2' = b_2^\sigma b_2^{-1}$. So $(b_1')^{c'} = b_2'^{-1} \in B \cap B^{c'}$, and as $c' \notin N(B)$ we have $b_1' = b_2' = 1$, and $b_1, b_2 \in C_B(\sigma)$, which allows finitely many possibilities, by Lemma 4.9. $\square$

It remains now to recompute the rank of G. We have already noticed that $\mathrm{rk}(G) = \mathrm{rk}(B) + \mathrm{rk}(I)$, but we now need $\mathrm{rk}(G) = 2\,\mathrm{rk}(B) + \mathrm{rk}(C(\sigma))$, or in other words

$$\mathrm{rk}(I) = \mathrm{rk}(B) + \mathrm{rk}\big(C(\sigma)\big).$$

We know $\mathrm{rk}(I_\sigma) = \mathrm{rk}(I)$, so it will suffice to show that the rank of I_σ is $\mathrm{rk}(B) + \mathrm{rk}(C(\sigma))$.

Lemma 4.16. $\mathrm{rk}(G) = 2\,\mathrm{rk}(B) + \mathrm{rk}(C_G(\sigma))$.

Proof. We have $\mathrm{rk}(G) = \mathrm{rk}(B) + \mathrm{rk}(I) = \mathrm{rk}(B) + \mathrm{rk}(I_\sigma)$, and we claim $\mathrm{rk}(I_\sigma) = \mathrm{rk}(B) + \mathrm{rk}(C(\sigma))$.

For any element $i \in I_\sigma$ we have $i = yj$ for some $y \in B\sigma$ and $j \in I$, and hence i inverts some $y \in B\sigma$. This element y is *unique*: if i inverts y and by with $b \in B^{\#}$, then $y^{-1}b^{-1} = (by)^i = b^i y^{-1}$ and thus $b^{-1} \in B^{iy^{-1}}$, $iy^{-1} \in N(B)$, and $i \in N(B)$, contradicting the choice of $i \in I_1$. So we have a definable function

$$\beta : I_\sigma \to B\sigma$$

defined by $\beta(i)^i = (\beta(i))^{-1}$.

As $B\sigma$ is a single conjugacy class under the action of B, the rank of the fibers $\beta^{-1}(y)$ is a constant f, and hence $\mathrm{rk}(I_\sigma) = \mathrm{rk}(B\sigma) + f = \mathrm{rk}(B) + \mathrm{rk}(\beta^{-1}(\sigma))$.

It suffices therefore to show $\mathrm{rk}(\beta^{-1}(\sigma)) = \mathrm{rk}(C(\sigma))$. Fix i in $\beta^{-1}(\sigma)$. Then $i = \sigma i'$ with $i' \in I$ inverting σ.

We claim first

$$(*) \qquad\qquad\qquad i C^\circ(\sigma) \subseteq \beta^{-1}(\sigma).$$

Observe first that $C(\sigma)$ contains no involutions, by Fact 4.2, bearing in mind Lemma 4.12.

For $g \in C^\circ(\sigma)$ and $j = ig$, we have $j \in I$ by Lemma 4.11. Furthermore $j = \sigma \cdot i'g$, and $i'g \in I$ by Lemma 4.11. So $j \in B\sigma I \cap I$. By construction j inverts σ. It remains to check that $j \notin M$. If $j \in M$ then $j \in B$, and since j inverts σ, and σ is of odd order modulo B, this gives a contradiction. So $j \in I_1$ and $\beta(j) = \sigma$. This gives $(*)$, and in particular $\mathrm{rk}(\beta^{-1}(\sigma)) \geqslant \mathrm{rk}(C^\circ(\sigma)) = \mathrm{rk}(C(\sigma))$.

Conversely, for $j \in \beta^{-1}(\sigma)$, since j inverts σ we have $ij \in C(\sigma)$, that is $i \cdot \beta^{-1}(\sigma) \subseteq C(\sigma)$, and thus $\mathrm{rk}(\beta^{-1}(\sigma)) \leqslant \mathrm{rk}(C(\sigma))$. Our claim follows. $\square$

With this our analysis is complete. $BC(\sigma)B$ and BI_1 are disjoint generic subsets of G by Lemmas 4.4, 4.15 and 4.16, and 4.14, which is a contradiction. Thus Case I cannot occur in Prüfer 2-rank two or more.

5. Case II

The remaining case will require a longer analysis, and some more theoretical preparation. We begin afresh.

Our standing hypotheses and notation are as follows.

Notation 5.1.

(1) *G is a minimal connected simple group of finite Morley rank and of odd type.*
(2) *B is a Borel subgroup of G.*
(3) $M = N(B)$ *is strongly embedded in G.*

We suppose

(†) *G has Prüfer 2-rank at least two.*

The operative assumption for the remainder of the article will be as follows.

(**Case II**)   For i an involution of B, we have $C^\circ(i) < B$.

It will be convenient to state this in a slightly stronger form.

Lemma 5.2. $F(B)$ *contains no involutions.*

Proof. Let S be a Sylow 2-subgroup of $F(B)$. Then $A = \Omega_1(Z(S))$ is characteristic in $F(B)$ and normal in B. As G is of odd type, the group A is finite as well as B-invariant, and hence central in B. By our case assumption (II), $A = 1$ and hence $S = 1$.   □

5.1. Tameness and Fact 1.1

Let us first indicate why Case II disappears quickly if we assume tameness. First, by an easy argument (Lemma 5.5 below) our case assumption produces an involution w outside M such that $B \cap B^w$ is infinite. By strong embedding $M \cap M^w$ contains no involutions and hence the same applies to $H = (B \cap B^w)^\circ$. It then follows by an application of tameness that H is contained in the Fitting subgroup of both B and B^w, and this contradicts Fact 1.1.

We must make a distinction between the two applications of tameness in the foregoing argument. As mentioned earlier, we will make use of a weak analog of Fact 1.1 which does not require tameness. However the claim that a connected group without involutions must lie in the Fitting subgroup is a direct application of tameness with no obvious analog in general. So before entering into the general case, let us first give an argument which uses only Fact 1.1 and therefore serves as a reasonable point of departure for the general case. This will necessarily be more elaborate than the argument just given, and will only be sketched, since in any case it will need to be redone afterward at a greater level of generality.

We begin again with an involution $w \notin M$ for which $B \cap B^w$ is infinite, and more particularly

$$T(w) = \left\{ a \in B \colon a^w = a^{-1} \right\}$$

is infinite. Observe that $T(w) \subseteq B \cap B^w$, and under the assumption that Fact 1.1 applies, the group $B \cap B^w$ is abelian (as the natural map $B \cap B^w \to B/F(B) \times B^w/F(B^w)$ is injective). So $T(w)$ is a group in this case. Let $H = (B \cap B^w)^\circ$ and consider a maximal definable connected subgroup $\hat{H}$ containing H of the form $(B \cap B_1)^\circ$, with $B_1 \neq B$ a Borel, not necessarily standard. Again, Fact 1.1 implies that $\hat{H}$ is abelian. Note that $\hat{H} \leqslant N^\circ(H)$ and $N^\circ(H)$ is w-invariant. We will take $B_1 \geqslant N^\circ(H)$, and with some effort we may make B_1 w-invariant as well.

Observe that $\hat{H}$ cannot be a Carter subgroup of B: otherwise, it contains a Sylow 2-subgroup of B by Lemma 3.11, forcing $B_1 = B$. By the maximality of $\hat{H}$ it then follows that $\hat{H}$ is a Carter subgroup of B_1. On the other hand, by a Frattini argument w normalizes a Carter subgroup of B_1 and hence some conjugate w_1 of w (under the action of B_1) normalizes $\hat{H}$. As w_1 normalizes $N^\circ(\hat{H})$, it follows easily that w_1 normalizes, and hence lies in, the group B. This is the fundamental setup that will be reached below, in general,

$$w_1 \in I(B) \text{ normalizes } B_1, \text{ and is conjugate to } w \text{ under the action of } B_1.$$

We now use the characteristic zero unipotence theory introduced in [Bur04,Bur04a], and in particular the notation $U_0(T(w))$, which represents a kind of unipotent radical, and the notion of "reduced rank." We also use the associated generalized Sylow theory of [Bur04,Bur05a]. (This machinery will be presented in more detail below.)

The group $T(w)$ is a nontrivial definable abelian group, inverted by w, and it can be shown to be torsion free, hence connected. Since $T(w)$ is nontrivial, solvable, and torsion free, it has a nontrivial "0-unipotent radical" $T_w := U_0(T(w))$. Let the maximal reduced rank associated with T_w be r; then $T_w = U_{0,r}(T(w))$. One can extend T_w to a w-invariant Sylow $U_{0,r}$-subgroup P_r of B_1, and any two such are conjugate in $C_{B_1}(w)$ (Lemma 7.9, below). As P_r contains a nontrivial $U_{0,r}$-subgroup inverted by w, the same applies to every Sylow $U_{0,r}$-subgroup of B_1 which is normalized by w. Since w and w_1 are conjugate under the action of B_1, the involution w_1 satisfies an analogous condition: any w_1-invariant Sylow $U_{0,r}$-subgroup of B_1 contains a nontrivial $U_{0,r}$-subgroup inverted by w_1.

On the other hand, if $Q_r = U_{0,r}(\hat{H})$, then $N_B^\circ(Q_r) \geqslant N_B^\circ(\hat{H}) > \hat{H}$ and hence by maximality $N^\circ(Q_r) \leqslant B$, from which it follows that Q_r is a Sylow $U_{0,r}$-subgroup of B_1, and of course w_1-invariant. So there must be a nontrivial $U_{0,r}$-subgroup A_r of $\hat{H}$ inverted by w_1.

Now one can show easily that for any s, $F_s(B_1) := U_{0,s}(F(B_1))$ is either contained in $\hat{H}$, or meets $\hat{H}$ trivially (cf. the proof of Lemma 7.8, claim (∗), below). In either case, A_r commutes with $F_s(B_1)$: if $F_s(B_1) \leqslant \hat{H}$, this holds because $\hat{H}$ is abelian, and if $F_s(B_1) \cap \hat{H} = 1$, then w_1 inverts $F_s(B_1)$, and consideration of the action of w_1 on the group $F_s(B_1)A_r$ leads to the desired conclusion. From all of this it follows that A_r centralizes $F(B_1)$, and hence lies in $F(B_1)$. But $A_r = [w_1, A_r] \leqslant F(B)$ as well, and by Fact 1.1 we find $A_r = 1$, a contradiction.

We will argue in the remainder of the paper that some of the applications of Fact 1.1 can be avoided and others replaced by a more complicated, but general, form of that result. One major alternative that arises in general is that the group $\hat{H}$ in question is nonabelian, but this produces a rather well-defined configuration which turns out to have a good deal in common with the abelian case. In fact as we will see the analysis follows much the same lines whether $\hat{H}$ is abelian or nonabelian.

5.2. *The setup*

Now dropping the assumption that Fact 1.1 applies and returning to the general case, we begin by showing that the setup with which we started our analysis above can in fact be reached.

Notation 5.3.

(1) $I = I(G)$.
(2) *For* $w \in I$, *let* $T[w]$ *be* $\{a \in B: a^w = a^{-1}\}$.
(3) *Let* $I^* = \{w \in I \setminus N(B): \mathrm{rk}(T[w]) \geqslant \mathrm{rk}(I(B))\}$.

The ungainly notation $T[w]$ is intended to reflect the ungainly nature of the set involved, which in general need not be a group. Our main concern is that I^* should be nonempty, and this is afforded by Lemma 5.5 below.

We use the following general fact.

Fact 5.4. *(See [CJ04, 2.36].) Let G be a connected simple group of finite Morley rank, let M be a proper definable subgroup of G, and let X be a conjugacy class in G. Then* $\mathrm{rk}(X \cap M) < \mathrm{rk}(X)$.

Lemma 5.5. I^* *is generic in* I.

Proof. Let $i \in I(B)$. Then we have $\mathrm{rk}(I) = \mathrm{rk}(G) - \mathrm{rk}(C(i))$ and thus $\mathrm{rk}(I(B)) = \mathrm{rk}(B) - \mathrm{rk}(C(i)) = \mathrm{rk}(I) - \mathrm{rk}(G/B)$.

By Fact 5.4, $I \setminus N(B)$ is generic in I. It will suffice to prove that the set I' defined as

$$\{w \in I \setminus B: \mathrm{rk}(T[w]) < \mathrm{rk}(I(B))\}$$

is nongeneric in I.

Now $T[w] = w \cdot (wB \cap I)$, so $\mathrm{rk}(T[w]) = \mathrm{rk}(wB \cap I)$. For $w \in I'$, it follows that $\mathrm{rk}(wB \cap I) < \mathrm{rk}(I(B))$. Hence $\mathrm{rk}(I' \cap X) < \mathrm{rk}(I(B))$ for X any left coset of B in G. As $I' = \bigcup_{X \in G/B} (I' \cap X)$, we find

$$\mathrm{rk}(I') < \mathrm{rk}(G/B) + \mathrm{rk}(I(B)) = \mathrm{rk}(I),$$

as claimed. □

Now we can fix our notation for the remainder of the argument.

Notation 5.6.

(1) *Fix* $w \in I^*$. *Set* $H = (B \cap B^w)^\circ$.
(2) *Let* $B_1 \neq B$ *be a Borel subgroup containing H, chosen so as to maximize* $(B \cap B_1)^\circ$.
(3) *Let* $\hat{H} = (B \cap B_1)^\circ$.

There is some latitude in the choice of $\hat{H}$, and with $\hat{H}$ fixed there is some latitude in the choice of B_1, which will be examined more closely subsequently. Our initial goal is to show that B_1 can be chosen to be w-invariant. Along the way we will acquire other useful information.

6. w-Invariance

We begin our analysis of Case II. We have the notations G, B, w, H, $\hat{H}$, B_1 as laid out in the previous section (and also $T[w]$, which will be needed at a later stage, when the configuration is clearer). In particular $H \leqslant \hat{H} = (B \cap B_1)^\circ$ with B_1 a Borel subgroup. As always, the Prüfer 2-rank is assumed to be at least two.

Our goal in the present section is to show that B_1 can be chosen to be w-invariant.

6.1. Unipotence theory

We use the 0-unipotence theory of [Bur04] (cf. [Bur04a,FrJa]). We use the notation $\bar{r}_0(K)$ for the *maximal reduced rank* of a group of finite Morley rank K, which is the largest integer r for which $U_{0,r}(K) \neq 1$ (or 0 if there is no such r). If K is a group of finite Morley rank then we set $U_0(K) = U_{0,\bar{r}_0(K)}(K)$.

We make use of the following from the general 0-unipotence theory.

Fact 6.1. *Let K be a connected solvable group of finite Morley rank.*

(1) $U_0(K) \leqslant F(K)$.
(2) *If K is nilpotent and $K = U_{0,r}(K)$, then $K' = U_{0,r}(K')$.*
(3) *If $1 \to K \to H \to \overline{H} \to 1$ is a short exact sequence of definable groups with $U_{0,r}(K) = K$ and $U_{0,r}(\overline{H}) = \overline{H}$, then $U_{0,r}(H) = H$. Conversely, if $U_{0,r}(H) = H$ then $U_{0,r}(\overline{H}) = \overline{H}$.*
(4) $\bar{r}_0(K) = \bar{r}_0(Z(F(K)))$.
(5) *If $K = U_0(K)$ and $r = \bar{r}_0(K)$, then for any proper definable subgroup $K_0 < K$ we have $U_{0,r}(N_K(K_0)) > K_0$.*
(6) *If K is nilpotent, then $K = B * T$ with B, T definable, B of bounded exponent, and T divisible; B is the central product of the finitely many subgroups $U_p(K)$, with p prime, for which $U_p(K) \neq 1$; T is the central product of the finitely many groups $U_{0,s}(K)$ for which $U_{0,s}(K) \neq 1$, together with the group $K_\infty = d(T_{\text{tor}})$, the definable closure of the torsion subgroup of T.*

The first three points are given in [Bur04a, 2.16, 2.17, 2.11], and the fourth follows from the first two. The last two are given in [Bur05a, Lemma 2.4, Corollary 3.5]. These points are also in [Bur04] as Theorem 2.21, Lemmas 2.23, 2.12, 2.26, and 2.28, and Theorem 2.31, respectively.

We note that the first five points are close analogs of more elementary properties of U_p for p prime, which we will use without special comment.

The next result has a similar character, but is used less often.

Fact 6.2. *(See [Bur04, 4.9], [Bur05a, Lemma 4.4].) Let $G = H_1 \cdot H_2$ be a group of finite Morley rank with H_i a definable nilpotent U_{0,r_i}-subgroup of G for $i = 1$ or 2, where H_1 is normal in G, and $r_2 \geqslant r_1$. Then G is nilpotent.*

In the case that interests us, $r_1 = r_2$.

The utility of this abstract theory of unipotence often depends on the following.

Fact 6.3. *(See [Bur04a, 2.15], [Bur04, 2.19].) Let H be a connected solvable group of finite Morley rank such that $U_p(H) = 1$ for all primes, as well as for $p = 0$. Then H is a good torus;*

that is, a divisible abelian group in which every definable subgroup is the definable closure of its torsion subgroup.

The following generation principle can be very useful.

Fact 6.4. *(See [Bur04, 2.41], [Bur05a, Theorem 2.9].) If a nilpotent $U_{0,r}$-group is generated by a family $\mathcal{F}$ of definable subgroups, then it is generated by the family $\mathcal{F}_r = \{U_{0,r}(X) \colon X \in \mathcal{F}\}$.*

We have alluded also to a Sylow theory. By definition a *Sylow $U_{0,r}$-subgroup* is a maximal definable nilpotent $U_{0,r}$-subgroup. The conjugacy theorem applies, at least in a solvable context.

Fact 6.5. *(See [Bur04, 4.16, 4.18], [Bur05a, Theorem 5.5].) Let K be a solvable group of finite Morley rank. Then for each r, its Sylow $U_{0,r}$-subgroups are conjugate.*

Some further connections between the Sylow theory and the Carter theory will be recalled when needed.

6.2. A uniqueness lemma

As we have noticed previously, when Fact 1.1 applies, the intersections of distinct Borel subgroups are abelian. The following is an indication of the tension that arises when the latter condition fails. We give it in a general setting.

Lemma 6.6. *(See [Bur05b, Theorem 4.3].) Let G be a minimal connected simple group of finite Morley rank, let B_1, B_2 be distinct Borel subgroups of G, and $H = (B_1 \cap B_2)^\circ$. Suppose that H is nonabelian. Then the following conditions are equivalent.*

(1) *B_1 and B_2 are the only Borel subgroups of G containing H.*
(2) *If B_3 and B_4 are distinct Borel subgroups containing H, then $(B_3 \cap B_4)^\circ = H$.*
(3) *If $B_3 \neq B_1$ is a Borel subgroup containing H, then $(B_1 \cap B_3)^\circ = H$.*
(4) *$C^\circ(H')$ is contained in B_1 or B_2.*
(5) *$\bar{r}_0(B_1) \neq \bar{r}_0(B_2)$.*

Note that clauses (2), (3) express the maximality of H in two different senses: in (2) we vary B_1 and B_2, while in (3) we hold B_1 fixed and only vary B_2. The first clause is an even more extreme form of maximality, while the last two clauses provide remarkably simple criteria for identifying such pairs B_1, B_2. They both follow fairly readily from clause (2), but the converse is more subtle.

We note the following consequence.

Lemma 6.7. *Let G be a minimal connected simple group of finite Morley rank, and H a proper connected definable nonabelian subgroup of G. Then $HC^\circ(H')$ is contained in a unique Borel subgroup of G, and in particular $N^\circ(H')$ is contained in a unique Borel subgroup of G.*

Proof. If B_1, B_2 are distinct Borel subgroups of G containing $HC^\circ(H')$, let $H_1 = (B_1 \cap B_2^\circ)$. Then $H_1 C^\circ(H_1')$ is also contained in B_1 and B_2. So by the previous result, $\bar{r}_0(B_1) \neq \bar{r}_0(B_2)$; we

may suppose $\bar{r}_0(B_1) > \bar{r}_0(B_2)$. Then as $H'_1 \leqslant F(B_1)$ [BN94, Corollary 9.9], we have $U_0(B_1) \leqslant C(H'_1)$ by Fact 6.1(6), and hence $U_0(B_1) \leqslant B_2$, contradicting $\bar{r}_0(B_1) > \bar{r}_0(B_2)$. □

Now let us return to the case at hand, in which the pair (B, B_1) plays the role of (B_1, B_2) above, so that our $\hat{H}$ corresponds to the H of our lemma. Then the third clause expresses exactly the maximality condition that we have imposed on our pair (B, B_1), and hence when $\hat{H}$ is nonabelian all of these conditions apply. This fact will play a leading role throughout the rest of the analysis, coming into play whenever $\hat{H}$ is assumed nonabelian. The main point here is that we find ourselves in the situation described by the third clause, while the second clause is the one most conveniently adopted as a point of departure for the detailed analysis undertaken in [Bur05b]; since we need this only when $\hat{H}$ is nonabelian, we can cite [Bur05b] freely in such cases. For the record, we state this in the slightly stronger form afforded by Lemma 6.6, part (1).

Corollary 6.8. *If $\hat{H}$ is nonabelian, then B and B_1 are the only Borel subgroups containing $\hat{H}$.*

Furthermore, applying Lemmas 6.6 and 6.7, we now find the desired w-invariant Borel subgroup in one important case.

Lemma 6.9. *If H is nonabelian, then B_1 can be chosen to be w-invariant.*

Proof. Take $\tilde{B}_1$ to be a Borel subgroup containing $N^\circ(H')$. By Lemma 6.7, $\tilde{B}_1$ is the only such Borel subgroup, and is therefore w-invariant, and in particular $\tilde{B}_1$ is not B. Let $\tilde{H} = (B \cap \tilde{B}_1)^\circ$. Then $C^\circ(\tilde{H}') \leqslant C^\circ(H') \leqslant \tilde{B}_1$. So by Lemma 6.6, the groups B and $\tilde{B}_1$ are the only Borel subgroups containing $\tilde{H}$, and in particular $\tilde{H}$ is maximal. All of our conditions are met with $B_1 = \tilde{B}_1$. □

6.3. Extension lemma

We recall that the notation and operative hypotheses were established at the end of the previous section in Notation 5.6. We will insist somewhat on this point for the remainder of the present section, because some of the work takes place at a sufficient level of generality to allow for its reuse in the next section, and goes beyond the immediate needs of the moment. We will of course have to track carefully what additional hypotheses are imposed in particular results.

Lemma 6.10. *With our current hypotheses and notations concerning G, B, and w, suppose $K \leqslant G$ is a maximal proper definable connected w-invariant subgroup of G. Then K is nonabelian.*

Proof. Suppose on the contrary that K is abelian, and let B_w be the conjugate of B containing w.

By maximality of K, it follows that $N^\circ(K_0) = K$ for any nontrivial w-invariant definable subgroup K_0 of K.

In particular, if $K_0 = (K \cap B_w)^\circ \neq 1$, then $N^\circ(K_0) = K$. Then $N^\circ_{B_w}(K_0) = K_0$, so K_0 is a Carter subgroup of B_w, in view of Fact 3.9. Then K_0 contains a Sylow 2-subgroup S_w of B_w by Lemma 3.11. It follows from Lemma 2.4 that $K \leqslant B_w$, and hence by maximality that $K = B_w$. Thus B_w is abelian, so B is abelian, contradicting our current case hypothesis, namely $C^\circ(i) < B$.

So suppose that $K \cap B_w$ is finite, and in particular $C_K(w)$ is finite. Then w inverts K [BN94, p. 78, Ex. 13]. In particular, every subgroup of K is w-invariant. It follows that distinct conjugates of K are disjoint: if $K \cap K^g \neq 1$, then $N^\circ(K \cap K^g)$ is equal to both K and K^g, and $K = K^g$.

As K has finite index in its normalizer, and has pairwise disjoint conjugates, the union $\bigcup_{g \in G} K^g$ is generic in G by Fact 2.5, and hence meets $\bigcup_{g \in G} B^g$ nontrivially by Lemma 2.6. Let B_0 be a standard Borel subgroup that meets K nontrivially.

Now $K \cap B_0$ is w-invariant and nontrivial, and hence $N^\circ(K \cap B_0) \leqslant K$. It follows by Fact 3.9 that $K \cap B_0$ is a Carter subgroup of B_0, and hence contains a Sylow 2-subgroup of B_0. It then follows from Lemma 2.4 that $K \leqslant B_0$. Then again, $B_0 = K$ should be abelian, contradicting our case assumption. $\quad\square$

Corollary 6.11. *Under the assumptions of the present section, if K is a nontrivial proper connected definable w-invariant subgroup of G, then K is contained in a w-invariant Borel subgroup of G.*

Proof. We may take K to be maximal under the stated conditions, and hence nonabelian by the preceding lemma. Then by Lemma 6.7, $N^\circ(K')$ is contained in a unique Borel subgroup of G, which is again w-invariant. $\quad\square$

Using this result, we deal easily with the case in which $\hat{H}$ is abelian.

Lemma 6.12. *Suppose that $\hat{H}$ is abelian. Then B_1 can be chosen to be w-invariant.*

Proof. We have $\hat{H} \leqslant N^\circ(H)$ and the latter is w-invariant, So $N^\circ(H)$ can be extended to a w-invariant Borel subgroup B_1. So $B_1 \geqslant \hat{H}$, and as B_1 is w-invariant, we have $B_1 \neq B$. $\quad\square$

6.4. Maximal pairs

We have seen that B_1 can be chosen w-invariant if $\hat{H}$ is abelian, or if H is nonabelian, and we now consider the remaining possibility: H is abelian, while $\hat{H}$ is nonabelian. As we have seen in Lemma 6.6, if $\hat{H}$ is fixed then the Borel subgroup B_1 is uniquely determined by the conditions $\hat{H} \leqslant B_1 \neq B$. However, there is still some latitude in the choice of $\hat{H}$.

According to Fact 1.1, this situation cannot arise in the tame case. More generally, while not visibly contradictory, this configuration is tightly constrained in general, as described in [Bur04, 9.2] and [Bur05b].

Definition 6.13. Let G be a group of finite Morley rank, and B, B_1 two Borel subgroups of G. We call the pair (B, B_1) a *maximal pair* if $(B \cap B_1)^\circ$ is maximal, among all connected components of intersections of distinct Borel subgroups of G; in other words, clause (2) of Lemma 6.6 applies.

When the intersection in question is nonabelian, we need not be very particular about the notion of maximality invoked, but the definition adopted here coincides with the one given in [Bur05b], which is well suited also to analysis in some abelian cases.

In the first place we have the following. We will only apply this when our group $\hat{H}$ is non-abelian, but we take note of the slightly greater generality achieved in [Bur05b].

Fact 6.14. *(See [Bur04, 9.2, (2, 4d)], [Bur05b, Theorem 4.5, (1, 5)].) Let G be a minimal connected simple group of finite Morley rank, and (B, B_1) a maximal pair of Borel subgroups of G. Let $H = (B \cap B_1)^\circ$ and let Q be a Carter subgroup of H. Suppose that H is nonabelian, or more generally that $F(B_1) \cap F(B_2)$ is nontrivial. Then*

(1) $\bar{r}_0(B) \neq \bar{r}_0(B_1)$;
(2) *if $\bar{r}_0(B) < \bar{r}_0(B_1)$, then*
   (a) $\bar{r}_0(H) = \bar{r}_0(B)$ *and*
   (b) Q *is a Carter subgroup of B_1.*

Let us apply this now to the case at hand, in which our pair (B, B_1) plays the role of the given pair (B, B_1), and we suppose that $\hat{H}$, which plays the role of H, is nonabelian, so that we do indeed have a maximal pair in view of Lemma 6.6. Let us make the relevant conclusions explicit in this case. Note that one hypothesis of the foregoing fact becomes a conclusion in our context.

Lemma 6.15. *Under our present hypotheses and with our present notation, if Q is a Carter subgroup of $\hat{H}$, then Q is not a Carter subgroup of B. If in addition $\hat{H}$ is nonabelian, then*

(1) $\bar{r}_0(B) < \bar{r}_0(B_1)$.
(2) Q *is a Carter subgroup of B_1.*

Proof. If Q is a Carter subgroup of B, then by Lemma 3.11, the group Q contains a Sylow 2-subgroup S of B. Thus $S \leqslant B_1$, and by Lemma 2.4 we find $B_1 = B$, a contradiction. This proves the first point.

Now take $\hat{H}$ to be nonabelian. Then by Fact 6.14, the group Q is a Carter subgroup of whichever group, B or B_1, has the larger reduced rank. As this cannot be B, it must be B_1, and our claims follow. $\square$

The Carter subgroups of $\hat{H}$ play a central role in what follows, largely because of Lemma 6.19 below, which treats both the case in which $\hat{H}$ is abelian and in which it is not. For the nonabelian case, we will need some additional information from [Bur05b], particularly bearing on the commutator subgroup $\hat{H}'$ (Fact 6.18 below).

Definition 6.16. Let H be a solvable group of finite Morley rank. Then H is *rank-homogeneous* if it satisfies the following conditions.

(1) H is torsion free.
(2) For $r < \bar{r}_0(H)$, $U_{0,r}(H) = 1$.

Note that these two clauses force $H = U_0(H)$, and hence H is nilpotent by Fact 6.1(1), which clarifies the meaning of the definition, particularly if the structure theory of Fact 6.1(6) is kept in mind.

Notation 6.17. *Let H be a group of finite Morley rank, and $r \geqslant 0$. Then $F_r(H)$ denotes $U_{0,r}(F(H))$. (One prefers $r > 0$: $F_0(H) = 1$.)*

Fact 6.18. *(See [Bur04, 9.2, (5a, 1, 5b, 3, 5d, 5c)], [Bur05b].) Let G be a minimal connected simple group of finite Morley rank, and (B, B_1) a maximal pair of Borel subgroups of G. Let*

$H = (B \cap B_1)^\circ$ *and let Q be a Carter subgroup of G. Suppose that H is nonabelian, and let the notation be chosen so that $\bar{r}_0(B) < \bar{r}_0(B_1)$. Then we have the following.*

(1) *H' is rank-homogeneous.*
(2) *If $r' = \bar{r}_0(H')$, then $U_{0,r'}(H) = F_{r'}(H)$.*
(3) *$F(H)$ is abelian.*
(4) *Q is abelian.*
(5) *$Q_{r'} \leqslant Z(H)$ is nontrivial.*
(6) *For any nontrivial definable subgroup X of H which is contained in H', we have $N^\circ(X) \leqslant B_1$.*
(7) *$F_{r'}(H) < F_{r'}(B)$.*

The first four points are covered in [Bur05b] with the encyclopedic Theorem 4.5. More precisely: since $H' \leqslant F(B_1) \cap F(B_2)$, (1) is contained in 4.5(6); (2) is contained in 4.5(3); both (3) and (4) are contained in 4.5(2);

The fifth and sixth points are given as Lemma 3.23 and Corollary 3.29 of [Bur05b] respectively. The last point combines part of 4.5(3) with Lemma 3.13 of [Bur05b].

This fact also generalizes to the case in which the group $F(B_1) \cap F(B)$, which may well be larger than H', is nontrivial, and one can replace H' by that larger group for these purposes. We do not need this refinement.

We proceed now with our analysis. While we would be free at this point to assume $\hat{H}$ is nonabelian, we need some of these results in the following section in a broader context, so any assumptions needed on H or $\hat{H}$ will be stated explicitly as required; and otherwise we will take note of their absence.

Lemma 6.19. *Under our present hypotheses and notations, but without additional assumptions on H or $\hat{H}$, let Q be a Carter subgroup of $\hat{H}$, and for any r let $Q_r = U_{0,r}(Q)$. Then the following hold.*

(1) *$N(Q) \leqslant N(B)$.*
(2) *If $\hat{H}$ is nilpotent and $Q_r > 1$, then $N(Q_r) \leqslant N(B)$.*

Proof. First we claim

$$(*) \qquad\qquad N_B^\circ(Q) \not\leqslant \hat{H}.$$

If $N_B^\circ(Q) \leqslant \hat{H}$ then $N_B^\circ(Q) \leqslant N_{\hat{H}}(Q) = Q$ and thus Q is a Carter subgroup of B by Fact 3.9, contradicting Lemma 6.15. So $(*)$ holds.

Now we divide into two cases, according as $\hat{H}$ is or is not nilpotent.

Suppose first

$$(1) \qquad\qquad \hat{H} \text{ is nilpotent.}$$

Then $Q = \hat{H}$ and by $(*)$ we have $N_B^\circ(Q) > Q$. By maximality of $\hat{H}$, the group B is the only Borel subgroup containing $N_B^\circ(Q)$. Hence B contains $N^\circ(Q)$, and is the only Borel subgroup containing $N^\circ(Q)$. From the last point it follows that $N(Q)$ normalizes B. Now if $Q_r > 1$, then as $N^\circ(Q) \leqslant N(Q_r) < G$, it also follows that B is the unique Borel subgroup containing $N^\circ(Q_r)$. Hence $N(Q_r) \leqslant N(B)$.

Now suppose on the contrary

$$(2) \qquad\qquad \hat{H} \text{ is nonnilpotent}$$

and in particular nonabelian. We set $\hat{r}' = \bar{r}_0(\hat{H}')$.

Let $Q_{\hat{r}'} = U_{0,\hat{r}'}(Q)$. By Fact 6.18(5) we have $Q_{\hat{r}'} \leqslant Z(\hat{H})$, so $\hat{H} \leqslant N(Q_{\hat{r}'})$, and since $N^\circ(Q) \leqslant N^\circ(Q_{\hat{r}'})$ we find $\hat{H} < N^\circ_B(Q_{\hat{r}'})$. Since $Q_{\hat{r}'}$ is nontrivial by Fact 6.18(5), it follows by maximality of $\hat{H}$ that the group $N^\circ(Q_{\hat{r}'})$ is contained in B and in no other Borel subgroup. In particular as $N(Q)$ normalizes $Q_{\hat{r}'}$, it follows that $N(Q)$ normalizes B. $\square$

Recall now that $H = (B \cap B^w)^\circ$. We insert a lemma which simplifies matters somewhat as far as H is concerned.

Lemma 6.20. *Under our present hypotheses and notations, but without requiring $\hat{H}$ to be non-abelian, the intersection $B \cap B^w$ is torsion free.*

Proof. Suppose on the contrary $x \in B \cap B^w$ has prime order p. Let $P \leqslant B$ be a Sylow° p-subgroup of B containing x. As B is solvable, the group P is locally finite and we can use the structure theory of [BN94, 6.20]; since in addition [BN94, 9.39] P is connected, it follows that either $U_p(P) > 1$, or P is a P-torus.

If $U_p(B) \neq 1$, then $U_p(C^\circ_{U_p(B)}(x))$ is a nontrivial p-unipotent group and hence is contained in a unique Borel subgroup of G (Lemma 3.3), which must be B. So $C^\circ(x) \leqslant B$. But if $U_p(B) \neq 1$ then $U_p(B^w) \neq 1$, so similarly $C^\circ(x) \leqslant B^w$, and now the uniqueness statement yields $B = B^w$, a contradiction.

So $U_p(B) = 1$ and P is a p-torus. Then by Lemma 3.11, P is contained in a Carter subgroup R of B, and by the same lemma R also contains a Sylow 2-subgroup S of B. So as R is nilpotent, P and S commute, and it follows that $C(x)$ contains S. Similarly $C(x)$ contains a Sylow 2-subgroup S_1 of B^w. Then $\Omega_1(S)$ normalizes $C^\circ(x)$ and hence by Lemma 2.4, we have $C^\circ(x) \leqslant B$ and $S_1 \leqslant B$, forcing $B = B^w$, a contradiction. $\square$

In consequence we have $H = B \cap B^w$, H is torsion free, and $T[w] \subseteq H$. When H is abelian, it follows that $T[w]$ is a subgroup of H inverted by w. Note also that $T[w]$ contains some infinite definable abelian subgroups inverted by w, which is a small start in the right direction. Eventually we will arrive at the case in which $T[w]$ is itself an abelian group.

6.5. Invariance of B_1

Throughout this subsection we deal with the case in which $\hat{H}$ is nonabelian and H is abelian, though our preparatory work is more general. Recall that we still have some latitude in the choice of $\hat{H}$.

Notation 6.21.

(1) $\hat{r}' = \bar{r}_0(\hat{H}')$.
(2) Set $H^- = d(T[w])$, the smallest definable subgroup containing $T[w]$.
(3) Set $r^- = \bar{r}_0(H^-)$.

In the present subsection, with H abelian, H^- is just another name for $T[w]$. Later on, however, we will use the same notation in a more general setting, where it must be taken more seriously.

The next lemma is fundamental, and is the only one in which the precise choice of the involution w is fully exploited. It will be applied repeatedly.

Lemma 6.22. *Under our present hypotheses and notation, but without requiring $\hat{H}$ to be non-abelian, there is no involution $i \in B$ inverting H^-.*

Proof. Supposing on the contrary that the involution $i \in B$ inverts H^-, it follows in particular that H^- is abelian.

As H contains no involutions, it also follows that $H^- = [i, H^-] \leqslant F^\circ(B)$. Now by the choice of w, $\mathrm{rk}(H^-) \geqslant \mathrm{rk}(B/C(i)) \geqslant \mathrm{rk}(F^\circ(B)/C_{F^\circ(B)}(i))$. Furthermore, $F(B)$ is a $2^\perp$-group (Lemma 5.2).

So as i acts on $F^\circ(B)$, the latter decomposes definably as a product (of sets) as $F^\circ(B) = C_{F^\circ(B)}(i) \cdot F^\circ(B)^-$ where $F^\circ(B)^- = \{a \in F^\circ(B): a^i = a^{-1}\}$ [BN94, Ex. 14, p. 73]. As $F^\circ(B)$ is connected, each factor has Morley degree one. Considering the ranks, it follows that H^- is a generic subset of $F^\circ(B)^-$, and is therefore the unique definable subgroup of this rank contained in $F^\circ(B)^-$. It follows that $C(i)$ normalizes the group H^-, and in particular some Sylow 2-subgroup S of B normalizes H^-, so $N^\circ(H^-) \leqslant B$. By conjugation also S^w normalizes H^-. Hence $S^w \leqslant B$, and this gives a contradiction. $\quad\square$

Next we will give a companion lemma that goes in the opposite direction: there are involutions in B inverting large pieces of H^- (when H^- is abelian). This is Lemma 6.25 below. We prepare the way with a very general lemma, which depends on the following fact from the theory of generalized Sylow subgroups.

Fact 6.23. *(See [Bur04, 4.19, 4.20, 4.22], [Bur05a, Theorem 6.7, Corollaries 6.8, 6.9].) Let H be a connected solvable group of finite Morley rank, and $r > 0$. Then the Sylow $U_{0,r}$-subgroups U of H are of the following form*

$$U = Q_r \cdot H_r'$$

where Q is a Carter subgroup of H, $Q_r = U_{0,r}(Q)$, and $H_r' = U_{0,r}(H')$. In particular, U is normalized by a Carter subgroup, and if $U_{0,r}(H') = 1$ then U is contained in a Carter subgroup.

Lemma 6.24. *Let H be a group of finite Morley rank with H° solvable and $U_2(H) = 1$, and let Q be either a Carter subgroup, or a Sylow $U_{0,r}$-subgroup, of H°. Then $N_H(Q)$ contains a Sylow 2-subgroup of H. In particular, every involution of H is conjugate under H° to one which normalizes Q.*

Proof. We suppose first that Q is a Carter subgroup of H°. By Lemma 3.11, $N_H(Q)$ contains a Sylow 2-subgroup of H°. By the Frattini argument, $N_H(Q)$ covers H/H°. If S is a Sylow 2-subgroup of $N_H(Q)$, then by Lemma 3.5, the group S covers a Sylow 2-subgroup of H/H°. It follows easily that S is a Sylow 2-subgroup of H.

Now suppose that Q is a Sylow $U_{0,r}$-subgroup of H°. Then $Q = Q_1 Q_2$ where $Q_1 = U_{0,r}([H^\circ]')$ and $Q_2 = U_{0,r}(C)$ for some Carter subgroup C of H° (Fact 6.23). Hence $N(C) \leqslant N(Q)$ and the second claim follows.

The final claim is then immediate. $\quad\square$

We need the next lemma, at the moment, under the hypothesis that H is abelian, but it will be applied more generally in the following section.

Lemma 6.25. *With our usual hypotheses and notations, but with no additional assumptions on $\hat{H}$ or H, suppose that H^- is abelian. Let $P = U_{0,r}(H^-)$ for some r. Then there is an involution w_P in B which inverts P.*

Proof. We may suppose that P is nontrivial.

Applying Corollary 6.11, let $\tilde{B}_0$ be a w-invariant Borel subgroup containing $N^\circ(P)$. Let $\tilde{B}_1$ be a Borel subgroup distinct from B with $(B \cap \tilde{B}_0)^\circ \leqslant \tilde{B}_1$, chosen so as to maximize $(B \cap \tilde{B}_1)^\circ$. Let $\tilde{H} = (B \cap \tilde{B}_1)^\circ$.

Suppose $\tilde{H}$ is *abelian*. Let $U = U_{0,r}(\tilde{H})$. Then $N(U) \leqslant N(B)$ by Lemma 6.19(2). Hence $U_{0,r}(N_{\tilde{B}_0}(U)) \leqslant U_{0,r}(\tilde{H}) = U$, and by Fact 6.1(5) we find that U is a Sylow $U_{0,r}$-subgroup of $\tilde{B}_0$. Accordingly U is also a Sylow $U_{0,r}$-subgroup of $C^\circ(P)$. By Lemma 6.24, it follows that U is normalized by an involution w_P conjugate to w under the action of $C^\circ(P)$. Hence w_P inverts P as well. Since $N(U) \leqslant N(B)$, the involution w_P normalizes B, and hence lies in B.

Suppose now that $\tilde{H}$ is *nonabelian*.

By Lemma 6.6 we can apply Fact 6.18 freely, which requires also bearing in mind Lemma 6.15.

Let $\tilde{r}' = \bar{r}_0(\tilde{H}')$. Suppose first that $r \neq \tilde{r}'$. Then by Fact 6.23, P is contained in a Carter subgroup Q of $\tilde{H}$. Now B contains $N^\circ(Q)$ by Lemma 6.19. It follows that $N_{\tilde{B}_0}^\circ(Q) \leqslant (B \cap \tilde{B}_0)^\circ \leqslant (B \cap \tilde{B}_1)^\circ$, so $N_{\tilde{B}_0}^\circ(Q) \leqslant N_{\tilde{H}}(Q) = Q$. Hence Q is a Carter subgroup of $\tilde{B}_0$ by Fact 3.9, and hence also of $C^\circ(P)$, as Q is abelian (Fact 6.18). So there is an involution w_P conjugate to w under the action of $C^\circ(P)$ such that w_P normalizes Q. Since w inverts P, also w_P inverts P. By Lemma 6.19, w_P normalizes B. Thus in this case we have our claim.

Now suppose that $r = \tilde{r}'$. Then $U_{0,r}(\tilde{H})$ is abelian by Fact 6.18, and hence is contained in $C^\circ(P) \leqslant \tilde{B}_0$. On the other hand, $\tilde{H} < N_B^\circ(F_r(\tilde{H}))$ by Fact 6.18(7), and thus $N^\circ(F_r(\tilde{H}))$ is contained in B, and in no other Borel subgroup. So $N_{\tilde{B}_0}^\circ(F_r(\tilde{H})) \leqslant (B \cap \tilde{B}_0)^\circ \leqslant \tilde{H}$, and thus $F_r(\tilde{H}) = U_{0,r}(\tilde{H})$ (Fact 6.18) is a Sylow $U_{0,r}$-subgroup of $\tilde{B}_0$, and hence also of $C^\circ(P)$. Now by Lemma 6.24 it follows that w is conjugate under the action of $C^\circ(P)$ to an involution w_P normalizing $F_r(\tilde{H})$.

As B is the only Borel subgroup containing $N^\circ(F_r(\tilde{H}))$, the involution w_P normalizes B, and hence w_P lies in B. So again we have our claim. □

Now we can wrap up the first phase of our analysis. We will make use of another two points from the theory of maximal pairs, from [Bur05b].

Fact 6.26. *(See [Bur04, 9.2, (5d, 5b, 5c)], [Bur05b, Lemmas 3.12, 3.13].) Let G be a minimal connected simple group of finite Morley rank, and (B, B_1) a maximal pair of Borel subgroups of G. Let $H = (B \cap B_1)^\circ$. Suppose that H is nonabelian, and that $\bar{r}_0(B) < \bar{r}_0(B_1)$. Then the following hold.*

(1) $F_r(B) \leqslant Z(H)$ *for $r \neq \bar{r}_0(H')$.*
(2) $F_{\bar{r}_0(H')}(B)$ *is nonabelian.*

In particular, the group $F_r(B)$ is abelian if and only if $r \neq \bar{r}_0(H')$.

Item (2) was not noted explicitly in [Bur04, 9.2], but follows readily, and is given in [Bur05b, Theorem 4.5, (4)].

Lemma 6.27. *Under our standing hypotheses, the Borel subgroup B_1 can be chosen to be w-invariant.*

Proof. In view of Lemma 6.9, we may suppose that

$$H \text{ is abelian}$$

We will show that in this case there is a choice of B_1 for which $(B \cap B_1)^\circ$ is abelian, and thus by Lemma 6.12 we may also choose such a B_1 which is w-invariant. Recall that H is torsion free.

Suppose toward a contradiction that for all suitable choices of B_1,

$$(B \cap B_1)^\circ \text{ is nonabelian}$$

We first consider any $\tilde{B}_1$ arbitrarily which meets our basic conditions, and set $\tilde{H} = (B \cap \tilde{B}_1)^\circ$. As this group is assumed nonabelian, Fact 6.26 applies in view of Lemma 6.6. In particular, the value of $\bar{r}_0(\tilde{H}')$ is determined by the structure of B, as the value of r for which $F_r(B)$ is nonabelian. We will denote this value by $\hat{r}'$, as usual, but we emphasize that its value depends only on B.

Now if

$$U_{0,r}(H^-) = 1 \quad \text{for all } r \neq \hat{r}'$$

then in particular $H^- = U_0(H^-)$, and by Lemma 6.25 there is an involution $w_P \in B$ inverting H^-. This contradicts Lemma 6.22. So, in fact,

$$U_{0,r}(H^-) \neq 1 \quad \text{for some } r \neq \hat{r}'.$$

In this case, let $P = U_{0,r}(H^-)$ and let B_0 be a w-invariant Borel subgroup containing $N^\circ(P)$ (Corollary 6.11). Note that $B_0 \neq B$. Let $H_0 = (B \cap B_0)^\circ$, and choose B_1 distinct from B and containing H_0 so that $\hat{H} = (B \cap B_1)^\circ$ is maximal. By our hypothesis, $\hat{H}$ is nonabelian.

By Lemma 6.25, there is an involution $w_P \in B$ which inverts P. As P is torsion free it is 2-divisible, so $P = [w_P, P] \leqslant F(B)$. As $r \neq \hat{r}'$, it follows from Fact 6.1(6) that P and $F_{\hat{r}'}(B)$ commute. Hence $F_{\hat{r}'}(B) \leqslant C^\circ(P) \leqslant B_0$ and thus $F_{\hat{r}'}(B) \leqslant \hat{H}$, and this contradicts Facts 6.18(2), (7), since we assumed that $\hat{H}$ is nonabelian. $\quad\square$

At this point, we can take up the analysis afresh. As noted, we will reuse some of the auxiliary information found along the way (which has been stated in sufficient generality to allow this), and from this point on the logic of the argument is completely linear.

7. Case II, conclusion

We recall the notations: B is a standard Borel subgroup with $N(B)$ strongly embedded in G, and w is an involution outside $N(B)$. $T[w]$ is the set of elements of B inverted by w, and by hypothesis has rank at least $\mathrm{rk}(B/C(i))$ for i any involution of B. $H = (B \cap B^w)$ (which is torsion free, and in particular connected). The group B_1 is a Borel subgroup distinct from B,

containing H, and chosen to maximize $\hat{H} = (B \cap B_1)^\circ$. The group B_1 is also w-invariant. We fix a Carter subgroup Q of $\hat{H}$. Then Q is also a Carter subgroup of B_1, and $N(Q) \leqslant N(B)$ by Lemma 6.19. The Prüfer 2-rank is assumed to be at least two.

We have not determined whether or not $\hat{H}$ is abelian, and we will frequently have to argue according to cases. When $\hat{H}$ is nonabelian, we use the structural information afforded by [Bur05b], specifically Facts 6.14, 6.18, 6.26. This is justified by Lemma 6.6. When $\hat{H}$ is abelian we will have to argue directly. In either case we will arrive at much the same conclusions.

7.1. The involution w_1

There is one more essential ingredient in this configuration, as follows.

Lemma 7.1. *There is an involution $w_1 \in B$ which normalizes B_1 and Q, and which is conjugate to w under the action of B_1.*

Proof. As Q is a Carter subgroup of B_1 and w normalizes B_1, by Lemma 6.24 there is w_1 conjugate to w under the action of B_1 which normalizes Q. Then w_1 normalizes B_1, and as $N(Q) \leqslant N(B)$ also w_1 normalizes B, and hence lies in B. $\square$

We will make use of the following general principle.

Fact 7.2. *(See [Bur04, 3.18], [Bur04a, 3.6].) Let H be a nilpotent group of finite Morley rank and P a group of definable automorphisms of H. Suppose that P is a finite p-group and H is a $U_{0,r}$-group with no elements of order p. Then $C_H(P)$ is a $U_{0,r}$-group.*

Lemma 7.3.

(1) $\bar{r}_0(B_1) > \bar{r}_0(\hat{H})$.
(2) *The involution w_1 inverts $U_0(B_1)$.*

Proof. (1) If $\hat{H}$ is nonabelian, the first claim is given by Lemma 6.15.

Suppose now that $\hat{H}$ is abelian. Let $U = U_0(\hat{H})$. Recall that H is torsion free and hence $\bar{r}_0(\hat{H}) \geqslant \bar{r}_0(H) > 0$. By Lemma 6.19, $N^\circ(U) \leqslant B$. So $N^\circ_{B_1}(U) \leqslant \hat{H}$. It follows that $\bar{r}_0(B_1) > \bar{r}_0(\hat{H})$ as Fact 6.1(5) would apply to $U_0(B_1)$ in the contrary case, implying $U = U_0(B_1)$ and $B_1 \leqslant N^\circ(U) \leqslant B$.

(2) Let $P = U_0(B_1)$. We consider the action of w_1 on P. By Fact 7.2, the centralizer in P of w_1 is a U_{0,r_1}-group with $r_1 = \bar{r}_0(B_1)$. This centralizer is contained in B, by strong embedding, and since $r_1 > \bar{r}_0(\hat{H})$, it must be trivial. Now P is a $2^\perp$-group, as otherwise its Sylow 2-subgroup is central in B_1, and thus $C(i)$ is a Borel subgroup for each involution i. Thus w_1 inverts P [BN94, p. 78, Ex. 13]. $\square$

7.2. H^- and $\hat{H}^-$

We now consider more closely the action of w on H, and the action of w_1 on $\hat{H}$.

Notation 7.4.

(1) *Fix an involution $w_1 \in B$ normalizing B_1 and Q.*
(2) *Set $\hat{H}^- = \langle\{a^2\colon a \in \hat{H},\ a^{w_1} = a^{-1}\}\rangle$.*

Lemma 7.5. *The group $\hat{H}^-$ is an abelian subgroup of $F(B)$, and contains no involutions.*

Proof. For $a \in \hat{H}$ with $a^{w_1} = a^{-1}$, we have $a^2 = [w_1, a] \in F(B)$. Thus $\hat{H}^- \leqslant F(B)$.

So $\hat{H}^- \leqslant F(B) \cap \hat{H} \leqslant F(\hat{H})$, which is an abelian group, applying Fact 6.18 if $\hat{H}$ is not itself abelian.

So $\hat{H}^-$ is an abelian subgroup of $F(B)$, inverted by w_1. As $F(B)$ contains no involutions, the same applies to $\hat{H}^-$. $\quad\square$

It follows in particular that any definable subgroup of $\hat{H}^-$ is 2-divisible, and that

$$\hat{H}^- = \left\{a^2\colon a \in \hat{H},\ a^{w_1} = a^{-1}\right\};$$

in other words, this set is already a group.

The structure of $\hat{H}^-$ is clarified by Lemma 7.8 below. As preparation, we insert an additional lemma about our maximal pair (B, B_1).

Lemma 7.6. *For any r, and any nontrivial Sylow $U_{0,r}$-subgroup P of $\hat{H}$, $N^\circ(P) \leqslant B$.*

Proof. Extend P to a Sylow $U_{0,r}$-subgroup U of B.

We show first:

(1) $$N_B^\circ(P) \not\leqslant \hat{H}.$$

If $P < U$ then the claim is clear by Fact 6.1(5).

If $P = U$, then P is normalized by a Carter subgroup of B (Fact 6.23), and a Carter subgroup of B cannot be contained in $\hat{H}$ (Lemma 6.15). This proves (1).

Now if $\hat{H}$ is abelian then $N_B^\circ(P) > \hat{H}$ and the lemma follows by the maximality of $\hat{H}$. So we suppose

(2) $$\hat{H} \text{ is nonabelian.}$$

In this case Facts 6.14, 6.18, and 6.26 apply. We set $\hat{r}' = \bar{r}_0(\hat{H}')$.

If $r \neq \hat{r}'$ then $F_r(B) \leqslant P$ (Fact 6.26). We know that U is normalized by a Carter subgroup of B, and in particular (Lemma 3.11) by a Sylow 2-subgroup S of B. In particular $\Omega_1(S)$ normalizes U. For i an involution in S, we have $[i, U] \leqslant F(B) \cap U$, that is i acts trivially on $U/(F(B) \cap U)$ and $F(B) \cap U$ is a $2^\perp$-group. By [Bur04a, 3.2], the centralizer $C_U(i)$ then covers the quotient, that is we have $U = C_U(i)(F(B) \cap U)$, and so by Fact 6.4, we have $U = C_U(i)F_r(B)$. Hence $[i, P] \leqslant [i, U] \leqslant F_r(B) \leqslant P$ and i normalizes P. Thus $\Omega_1(S)$ normalizes P, and as the Prüfer 2-rank is at least two it follows that $N^\circ(P) \leqslant B$ (Lemma 2.4), as claimed.

If $r = \hat{r}'$ then $P = U_{0,r}(\hat{H}) = F_r(\hat{H})$ by Fact 6.18(2), so $\hat{H} < N_B^\circ(P)$ by Facts 6.18(7) and 6.1(5), and our claim follows again by maximality of $\hat{H}$. $\quad\square$

Now we can control the structure of $\hat{H}^-$. We will make use of the following very general result.

Lemma 7.7. *Let G be a group, and $H, K \leqslant G$ subgroups with K normalizing H. Let $t \in G$ act on H and K, inverting both groups. If K is 2-divisible, then $[K, H] = 1$.*

Proof. For $h \in H$ and $k \in K$ we have

$$\left(h^{-1}\right)^k = \left(h^k\right)^{-1} = \left(h^k\right)^t = \left(h^t\right)^{k^t} = \left(h^{-1}\right)^{k^{-1}}$$

and thus $(h^{-1})^{k^2} = h^{-1}$. $\quad\square$

Lemma 7.8. *If $U_{0,r}(\hat{H}') = 1$ then $U_{0,r}(\hat{H}^-) = 1$.*

Proof. Let $P = U_{0,r}(\hat{H}^-)$. Suppose $P > 1$.

The involution w_1 inverts $U_0(B_1)$ and P; the latter is 2-divisible. It follows by Lemma 7.7 that P centralizes $U_0(B_1)$.

Suppose first that $\hat{H}$ is noncommutative. Then by Fact 6.26, since $P \leqslant F_r(B) \leqslant Z(\hat{H})$, the group P commutes with $F(B)$ in view of the structure theory of Fact 6.1(6). So $C^\circ(P)$ contains $F(B)$, $U_0(B_1)$, and $\hat{H}$, and hence either $F(B) \leqslant B_1$ or $U_0(B_1) \leqslant B$. But $\bar{r}_0(B_1) > \bar{r}_0(B)$, so the second possibility is excluded, and the first possibility is excluded by Fact 6.18(2), (7).

So we may now suppose that

$$\hat{H} \text{ is commutative.}$$

We claim:

$(*)$ *For any s, either $F_s(B_1) \leqslant \hat{H}$, or $U_{0,s}(\hat{H}) = 1$.*

Suppose that $F_s(B_1) \not\leqslant \hat{H}$, and let U be a Sylow $U_{0,s}$-subgroup of $\hat{H}$. Then the group $F_s(B_1)U$ is nilpotent by Fact 6.2. So $N^\circ_{B_1}(U) \not\leqslant \hat{H}$ and by Lemma 7.6 we have $U = 1$. Our claim $(*)$ follows.

Now $\hat{H}$ is a Carter subgroup of B_1, by Lemma 6.19(1). So $B_1 = F(B_1)\hat{H}$ [Wag94]. Furthermore, as $\hat{H}$ is commutative, H^- centralizes any factor $F_s(B_1)$ which lies in $\hat{H}$.

Consider a factor $F_s(B_1)$, where $U_{0,s}(\hat{H}) = 1$. The centralizer in $F_s(B_1)$ of w_1 is a $U_{0,s}$-group, by Fact 7.2, and is contained in $\hat{H}$, hence is trivial. As $F(B_1)$ is a $2^\perp$-group, it follows that w_1 inverts $F_s(B_1)$ [BN94, Ex. 14, p. 73]. As w_1 and w are conjugate under the action of B_1, also w inverts $F_s(B_1)$. But w also inverts H^-, and as $H^- \leqslant H$ is 2-divisible it follows that H^- commutes with $F_s(B_1)$ by Lemma 7.7.

So H^- commutes with every factor $F_s(B_1)$ for $s \geqslant 1$.

Furthermore, since H^- is torsion free, it commutes with every $U_p(B_1)$ for p a prime, as a consequence of the main result of [Wag01]; this is given in [AC05, 3.13] for actions on abelian unipotent p-groups, and the general case has the same proof.

Since the divisible part of the torsion subgroup of $F(B_1)$ is central in B_1, it follows that H^- centralizes $F^\circ(B_1)$. Hence H^- centralizes $F^\circ(B_1)\hat{H} = B_1$, and $H^- \leqslant Z(B_1)$.

Now w and w_1 are conjugate under the action of B_1, and therefore w_1 inverts H^-. This contradicts Lemma 6.22. $\quad\square$

7.3. Invariants attached to w

Now Lemma 7.8 produces a peculiar situation. If, for example, $\hat{H}$ is abelian, then it follows that $U_{0,r}(\hat{H}^-) = 1$ for all r. On the other hand, this is certainly not the case for H^-, which is torsion free. Furthermore w and w_1 are conjugate under the action of B_1. Of course, this conjugation need not preserve $\hat{H}$, or carry H^- to $\hat{H}^-$, so we are still short of a contradiction. However, it is possible to attach certain invariants to involutions acting on solvable groups, which will be preserved by conjugation, and in this way arrive at a contradiction by comparing the actions of w and w_1.

This is based on the following considerations.

Lemma 7.9. *Let H be a solvable group of finite Morley rank with $U_2(H) = 1$, and w an involution in H. Fix $r \geqslant 1$. Then the following hold.*

(1) *Any w-invariant nilpotent $U_{0,r}$-subgroup P of H is contained in a w-invariant Sylow $U_{0,r}$-subgroup of H.*
(2) *If a Sylow 2-subgroup of H contains a unique involution, then any two w-invariant Sylow $U_{0,r}$-subgroups of H are conjugate under the action of $C_H(w)$.*

Proof. (1) We may suppose that P is a maximal w-invariant $U_{0,r}$-subgroup of H. We claim that P is a Sylow $U_{0,r}$-subgroup of H. It suffices to show that P is a Sylow $U_{0,r}$-subgroup of $N_H^\circ(P)$, or in other words we may suppose that H normalizes P. In this case, replacing H by H/P, we may suppose that $P = 1$. In this case, our claim reduces to Lemma 6.24.

(2) Let P and P^g (with $g \in H$) be two w-invariant Sylow $U_{0,r}$-subgroups of H.

Then w and w^g are in $N_H(P^g)$, and the latter group has a unique involution in each Sylow 2-subgroup. Hence w and w^g are conjugate in $N_H(P^g)$. If $w^{gh} = w$ with $h \in N_H(P^g)$, then $P^g = P^{gh}$ and $gh \in C_H(w)$. $\quad\square$

Definition 7.10. Let w be an involution acting definably on a solvable group H of finite Morley rank, and let $r \geqslant 1$. Then we define $\iota_r(w, H)$ as the maximal rank of a $U_{0,r}$-subgroup of H inverted by w.

Note that one would normally require the $U_{0,r}$-subgroups involved to be nilpotent, but here they are in any case abelian.

What makes this invariant manageable is the following.

Lemma 7.11. *Let w be an involution acting definably on a solvable group H of finite Morley rank with $U_2(H) = 1$, and let $r \geqslant 1$. Suppose that a Sylow 2-subgroup of $H\langle w \rangle$ contains a unique involution. Let U be a w-invariant Sylow $U_{0,r}$-subgroup of H. Then $\iota_r(w, H) = \iota_r(w, U)$.*

Proof. Let P be a $U_{0,r}$-subgroup of H inverted by w, and of maximal rank. Extend P to a Sylow $U_{0,r}$-subgroup Q of H. Then Q and U are conjugate under the action of $C(w)$, and our claim follows. $\quad\square$

Let us now specialize this to the case at hand.

Lemma 7.12. *With our usual hypotheses and notation, let U_1 be a w_1-invariant Sylow $U_{0,r}$-subgroup of B_1. Then $\iota_r(w_1, U_1) = \iota_r(w, B_1)$.*

Proof. Let $\hat{B}_1 = B_1\langle w \rangle = B_1\langle w_1 \rangle$. We show first that

$(*)$ a Sylow 2-subgroup of $\hat{B}_1$ contains a unique involution.

If B_1 is a $2^\perp$-group, then the Sylow subgroups of $\hat{B}_1$ are cyclic of order two.

Suppose B_1 contains an involution. Then B_1 meets a conjugate M_1 of M.

If $\hat{B}_1$ is contained in M_1, then B_1 is conjugate to B. But this case may be ruled out as follows. By Lemma 7.3, w_1 inverts $U_0(B_1)$. It follows that some involution of B inverts $U_0(B)$, and hence all involutions of B invert $U_0(B)$, which is impossible.

So $\hat{B}_1$ meets M_1 in a proper subgroup of $\hat{B}_1$, which is therefore strongly embedded in $\hat{B}_1$, and in particular all involutions of $\hat{B}_1$ are conjugate, and lie in B_1. So if $\hat{B}_1$ contains an elementary abelian 2-group of order 4, then the same applies to B_1, and hence by Lemma 2.4, B_1 is conjugate to B, which we have just ruled out. So $(*)$ holds in all cases.

Now the general theory applies to w and w_1 in $\hat{B}_1$, and as they are conjugate we find

$$\iota_r(w, B_1) = \iota_r(w, \hat{B}_1) = \iota_r(w_1, \hat{B}_1) = \iota_r(w_1, U_1). \qquad \square$$

7.4. Structure of H^-

Now we take up the structure of H^-, about which we know very little at this point; recall that this group is the definable closure of the group generated by $T[w]$.

Lemma 7.13. *If $U_{0,r}(\hat{H}') = 1$ then $U_{0,r}(H^-) = 1$.*

Proof. Let P be a w-invariant Sylow $U_{0,r}$-subgroup of H^-, and suppose that P is nontrivial. We claim:

(1) w inverts P.

If H^- is abelian, then w inverts H^- and this is clear.

If H^- is nonabelian, then also $\hat{H}$ is nonabelian, and as usual we set $\hat{r}' = \bar{r}_0(\hat{H}')$. By assumption $r \neq \hat{r}'$, so $U_{0,r}(\hat{H}') = 1$ by Fact 6.18(1). Let $\overline{H^-} = H^-/(H^-)'$. Then $\overline{H^-}$ is an abelian group inverted by w, and $U_{0,r}([H^-]') = 1$.

Let $P_0 = C_P(w)$. Then P_0 is a $U_{0,r}$-group by Fact 7.2. So the image $\overline{P}_0$ of P_0 in $\overline{H^-}$ is a $U_{0,r}$-group which is both centralized and inverted by w. Hence $\overline{P}_0 = 1$ and $P_0 \leqslant (H^-)'$. Since P_0 is a $U_{0,r}$-group, it follows that $P_0 = 1$. So w inverts P, and (1) holds in either case.

Let P_1 be a w_1-invariant Sylow $U_{0,r}$-subgroup of $\hat{H}$. By Lemma 7.6, $N^\circ(P_1)$ is contained in B and hence P_1 is a Sylow $U_{0,r}$-subgroup of B_1. By Lemma 7.12, we have $\iota_r(w_1, P_1) = \iota_r(w, B_1) > 0$. This contradicts Lemma 7.8. $\quad \square$

Now the analysis of H^- produces a contradiction.

Lemma 7.14. *This configuration is inconsistent.*

Proof. If $\hat{H}$ is abelian, then the preceding lemma shows that $U_{0,s}(H^-) = 1$ for all s, and as H^- is torsion free and nontrivial this is a contradiction.

So $\hat{H}$ is nonabelian, and we may apply Fact 6.18. As usual we set $\hat{r}' = \bar{r}_0(\hat{H}')$. Then by Lemma 7.13, $U_{0,r}(H^-) = 1$ for $r \neq \hat{r}'$. As $H^- \leqslant H$ is torsion free, it follows that $H^- = U_{0,\hat{r}'}(H^-) \leqslant U_{0,\hat{r}'}(\hat{H}) = F_{\hat{r}'}(\hat{H})$, which is abelian.

As H^- is abelian, we can apply Lemma 6.25, and there is an involution in B which inverts H^-, which contradicts Lemma 6.22. $\square$

With this contradiction, the proof of Theorem 1 in the second case is finally complete.

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