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Universal graphs with a forbidden subtree

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Abstract

We show that the problem of the existence of universal graphs with specified forbidden subgraphs can be systematically reduced to certain critical cases by a simple pruning technique which simplifies the underlying structure of the forbidden graphs, viewed as trees of blocks. As an application, we characterize the trees T for which a universal countable T-free graph exists.

Keywords: Graph; Universal graph; Forbidden subgraph; Tree; Model theory

1. Introduction

The systematic investigation of countable universal graphs with "forbidden" subgraphs was initiated in [10], followed by [11]. If C is a finite connected graph, then a graph G is *C-free* if it contains no subgraph isomorphic to C. A countable C-free graph G is *weakly universal* if every countable C-free graph is isomorphic to a subgraph of G, and *strongly universal* if every such graph is isomorphic to an *induced* subgraph of G. Such universal graphs, in either sense, are rare. Graph theorists tend to use the term "universal" in the weak sense, while model theorists tend to use it in the strong sense. We will use the term in the graph-theoretical sense here: "universal" means "weakly universal," though we sometimes include the adverb for emphasis. Similarly, while model theorists may sometimes use the term "subgraph" for "induced subgraph," we avoid such usage here.

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We deal here with the problem of determining the finite connected constraint graphs C for which there is a countable universal C-free graph. We introduce a new inductive method and use it to settle the case in which C is a tree, confirming a long-standing conjecture of Tallgren. The existence of such a countable universal graph says something about the class of all finite C-free graphs, and something about C itself, and the problem ultimately is to determine what, exactly, it does say.

This has been partially elucidated in [1]. Associated to any constraint graph C there is a natural notion of *algebraic closure*: loosely speaking, a vertex is in the algebraic closure of a given set if the number of vertices of the same type must be finite in any C-free graph. For example, if C is a star consisting of a vertex with a set of neighbors, so that the C-free graphs are those with a fixed bound on the vertex degrees, the algebraic closure of a set is simply the union of its connected components. For another example, if we consider P-free graphs where P is a path of length 3 (thus, of order 4), then one may get a universal P-free graph by taking the disjoint union of infinitely many triangles and infinitely many stars of infinite degree, in which case the algebraic closure of a vertex will consist of that vertex alone if its degree is infinite, or of the vertex and its neighbors if the vertex has finite degree (as do all the vertices in this case, with the exception of the centers of the stars). In general, an algebraically closed set of vertices will contain all neighbors of any of its vertices which are of finite degree, but may contain other vertices as well.

In general, if the algebraic closure of a set is the union of the algebraic closures of its elements, we say that the operation is *unary*. This is usually not the case: if for example C is a circuit of length 4, then the associated algebraic closure operation is generated by a (partial) binary operation: adjoin the unique common neighbor of any pair of points, if it exists; and iterate. At the opposite extreme, the algebraic closure operation may be trivial: for example, if C is a complete graph then the algebraic closure of a set is the set itself.

Now it turns out that the following three conditions are intimately related at both a theoretical and empirical level.

(1) The algebraic closure operation associated to C is (*uniformly*) *locally finite* in the sense that the algebraic closure of a set of size n in any C-free graph is finite (and then necessarily bounded in size by a function of n).
(2) There is a strongly universal C-free graph.
(3) There is a weakly universal C-free graph.

These conditions are successively weaker, and not much different in practice. We know of no case which separates the second from the third condition, but there are trivial examples falling under the last two cases and not the first: for graphs of maximal vertex degree 2 there is a universal graph made up of infinitely many cycles of all lengths and infinitely many two-way infinite paths. (Throughout, all cycles and paths are *simple*: that is, paths are trees of maximal vertex degree at most two, possibly infinite, and cycles are finite connected regular graphs of degree two.)

As the algebraic closure of a single vertex is its connected component, local finiteness fails, but what happens in this case is that the algebraic closure operation is very tightly structured. More generally, the same phenomenon occurs, and the situation as a whole is very much the same, if the constraint graph is a *near-path*, that is a tree which is not a path, but is obtained by attaching one edge with one additional vertex to a path.

Our goal is to arrive at a more concrete understanding of the exceptional constraints allowing a weakly universal C-free graph. It makes good sense to state the problem in full generality as follows: *is there an effective procedure to decide whether a given finite constraint C allows a corresponding universal (countable) graph?*

The problem is very much open and has been attacked from two directions. Some encoding results are known which aim in the direction of a proof of undecidability [3]. But the bulk of the research, like our present line, aims in the opposite direction, developing general tools to settle problems of this type: indeed, it is plausible at this stage that it may be possible to work out the list of exceptional constraints C allowing a universal graph in a completely explicit way. This would be the strongest form of a positive solution to the decision problem, though "softer" approaches are also available. While the condition of local finiteness is essentially a *halting problem* for a specific computation, the computations in question tend in the vast majority of cases to diverge.

The present paper has two goals: to present a simple inductive style of argument which suggests that if the list of exceptional "favorable" constraints is in fact as simple as we are suggesting it should be, then we should be able to prove that fact, and to buttress this claim by establishing Tallgren's Tree Conjecture. While we do not have a conjectured list of favorable constraints C in general, in the case of trees Tallgren conjectured the simple answer for this case explicitly many years ago. One might expect that knowing the answer would be the major ingredient in finding a proof, and as far as that goes it probably is, but nonetheless the question has remained open, and it seems to need our inductive method in order to be reduced to a finite number of individually treatable minimal cases.

The result is as follows.

Theorem 1 *(Tree Conjecture). If T is a finite tree, then the following are equivalent*:

(1) *There is a weakly universal T-free graph.*
(2) *There is a strongly universal T-free graph.*
(3) *T is a path or a near-path.*

We remark that the algebraic closure operation associated to a path is locally finite, as follows from the relatively explicit analysis of the models given in [11]; this can also be seen using the analysis of the algebraic closure operation given in [1], which gives less explicit information about the structure of models. On the other hand, as we have indicated, near-paths fall into the exceptional class, behaving much like the star of order four. They may indeed exhaust the exceptional class for which on the one hand the associated algebraic closure operation is *not* locally finite, while on the other hand, there *is* an associated universal graph.

While all of this might suggest that only the most obvious examples of universal graphs can exist, this is not so. Komjáth showed, unexpectedly, that the 2-bouquet formed by joining two triangles over a common vertex provides another constraint graph B allowing a (strongly) universal B-free graph [9]; more generally, 2-bouquets $B_{m,n}$ formed by joining complete graphs K_m and K_n over a single common vertex have been thoroughly analyzed in [4]: there is a (weakly or strongly) universal $B_{m,n}$-free graph if and only if the parameters satisfy the following conditions:

$$\min(m,n) \leqslant 5 \quad (m,n) \neq (5,5).$$

This is visibly a delicate condition, and requires a close combinatorial analysis to achieve. Examples of this type continue to hold open the possibility that the final list may be more delicate than anything we have seen to date.

There has been prior work on the case of tree constraints. First, taking a path or a near-path as forbidden subgraph does allow a universal graph [4,11].

In the other direction, the nonexistence of universal graphs has been treated in the following cases: (1) arrows, which are trees consisting of a path with two more edges adjoined to either endpoint, a case treated in [8]; (2) trees with a unique vertex of maximal degree $d \geqslant 4$, which is moreover adjacent to a leaf, treated in [7]; and (3) "bushy" trees, that is trees with no vertex of degree 2, treated in [5]. Of these, the case treated in [7] now seems the most suggestive. Indeed, we will show that by combining the case treated in [7] with a simple inductive idea, we deduce that for trees C having a unique vertex of maximal degree $d \geqslant 4$, there is no strongly universal C-free graph. The argument is prototypical for the general problem.

What we do here is motivated to a degree by a false conjecture in [1], the Monotonicity Conjecture: if a constraint C allows a universal graph (in either sense) then for any *induced* subgraph C_0 of C, the tighter constraint C_0 also allows a universal graph. It turns out that the close analysis of 2-bouquets $B(m,n)$ refutes this, as the graph $B(5,5)$ is an induced subgraph of $B(5,6)$ and the latter allows a strongly universal graph while the former does not allow a weakly universal graph. But there is enough truth to the conjecture to make it useful: if one passes to an induced subgraph by the operation of *pruning* introduced here (removing certain 2-blocks), the monotonicity principle is valid. So after explaining this, in taking up the Tree Conjecture we will deal with *critical* trees, which by definition are the trees which are not paths or near-paths, but become paths or near-paths when pruned—which, incidentally, is nothing but the removal of leaves in this case. The reader can see for himself that the structure of these trees is very simple. What we need to prove is that (a) the algebraic closure operation is not locally finite in these cases; (b) this leads to the nonexistence of strongly or weakly universal T-free graphs in all cases. Now (a) is easier than (b) and is a prerequisite for the latter, and the constructions used to accomplish (a) serve as templates for the more delicate constructions used to accomplish (b). In fact there are three layers of constructions. We found it fairly easy to decorate the constructions used in case (a) to refute the existence of strongly universal T-free graphs for T a critical tree T, and rather troublesome to convert the latter into refutations in the weakly universal case, which is the problem which was originally posed. Naturally we suppress all of these intermediate steps except in some illustrative cases. The result is that one will see various "bells and whistles" in the constructions, and arguments that to a certain extent the graphs that interest us are maximal in the sense that an embedding into a larger T-free graph does not create new edges. This is not literally the case: it would be more accurate to say that certain critical vertices acquire no new neighbors, and even this overstates the matter.

We make one further remark about these constructions. With the exception of the first cases treated (called monarchy and stardom), these constructions do not leap to mind; perhaps with better insight they should, and in any case as they are all variations on one theme, this theme can be added to the toolbox for future reuse. But at one point we doubted the truth of the Tree Conjecture, and computed the algebraic closure operator for the case of a specific tree on 14 vertices, the "most likely" counterexample to the conjecture. This turned out to be so tightly constrained that it only allowed one type of construction, which is the one used here throughout. Similarly, in working with bouquets, it is doubtful that one would find the construction used to refute the existence of a universal $B(5,5)$-graph on an ad hoc basis, but again one computes the behavior of the algebraic closure operator (expecting local finiteness, in fact) and the relevant construction simply appears. In the case of bouquets, where there are many cases of local finiteness, one can actually see these computations in [4], where they are necessary for the main results. Here they can and should be suppressed, as what interests us are the necessary constructions. But as the paper represents unfinished business, indeed merely the initial step of what could be a very long process, these methodological points should be noted.

Furthermore, we have drastically oversimplified our discussion in one crucial respect. All of these problems make equally good sense—more sense, in fact—when the constraint C is replaced by a finite set $\mathcal{C}$ of finite, connected, constraint graphs. Some things become clearer in the process: the case of complete constraint graphs generalizes to the case of sets $\mathcal{C}$ closed under homomorphism, where the common feature is that the algebraic closure operation is *trivial*: $\operatorname{acl}(A) = A$ for all A. Furthermore examples of *mixed type* occur: one may take any constraint C for which the algebraic closure operator is locally finite, and combine it with any further finite set of constraints closed under homomorphism, without altering the algebraic closure operator. These phenomena remain invisible when one considers only single constraints, and a number of the more general examples are exceedingly natural (universal graphs omitting all cycles of odd order up to some fixed bound [2]; or universal graphs omitting a path and any further set of constraints).

In this context, the pruning operation makes equally good sense, and the corresponding monotonicity principle is valid. The decision problem makes more sense in that context, and is equally open. But it is only at this level of generality that encoding arguments make sense, so that one can envision a "soft" proof of undecidability. At the same time, the possibility of a complete "list" of favorable constraint sets is viable. All known examples consist of a combination of some very special constraints with a set closed under homomorphism. Whether this merely reflects our inexperience remains to be seen. In any case, one can define a notion of "critical set" in general: practically speaking, this would be a set which after pruning produces a known example allowing a universal graph. In general, by determining the critical sets which allow universal graphs, and are not in the database of known examples, and iterating, one could arrive at the correct answer in general, together with a proof of it. It will be clear from our treatment here of an initial special case arising in the case of a single constraint that this is a large task. But the work of [6], for example, is encouraging. It follows from the arguments given there that a finite set $\mathcal{C}$ of 2-connected graphs allows a universal $\mathcal{C}$-free graph if and only if the set is closed under homomorphism. Since any graph is built up in a reasonable way from trees and 2-connected graphs, we are off to a decent start.

It would also be interesting to pass on quickly to the case of a finite set of trees. This may be entirely reasonable.

The paper is organized as follows.

In Section 2, we discuss the structural analysis of a general finite connected graph C as a "tree of blocks" (or 2-connected components), a standard topic of graph theory which has a great deal to do with the practical analysis of universality problems, and we introduce the new idea which allows an inductive analysis of universality problems according to the complexity of the underlying tree. Quite generally, universality problems can be reduced by this method to canonical "minimal" cases, which we call "critical." Here the constraint C can be any finite connected graph, or in fact any finite set of finite connected graphs. For our applications here, we will take C to be a tree in later sections.

In Section 3 we show that the Tree Conjecture holds for trees with a unique vertex of maximal degree. The method of Section 2 reduces this to the case treated in [7], and this provides a nice illustration of the force of the reduction, as well as disposing of a case that is best treated in isolation. This serves as a template for more elaborate constructions.

In the following sections we prove the Tree Conjecture by making a very coarse division of the critical cases into subcases according to the maximal vertex degree and the structure of the "external" vertices of maximal degree (those nearest the leaves).

We make little mention of algebraic closure in the remainder of the paper, apart from an occasional observation. While the ability to compute this operation is important when investigating a new example, it would contribute little or nothing to the exposition. But the notion will nonetheless be quite visible in all of our constructions, in the form of infinite paths of very tightly linked elements. If we were only interested in the issue of local finiteness we could shorten both our constructions and our analysis considerably, and also reduce the number of distinct cases considered.

Note that all graphs dealt with here are finite or countable. We may mention their countability for emphasis, on occasion.

There are many other universality problems involving infinite forbidden subgraphs, infinite sets of finite forbidden subgraphs, and uncountable graphs, but none of our methods apply in such cases, except possibly to the case of infinite sets of finite forbidden subgraphs (which includes such natural cases as graphs without circuits) where at the least new and mysterious phenomena arise, and the decidability problem is ill-posed.

2. Pruning trees, and other graphs

Our main objective in this section is to give a general inductive method for treating universality problems involving a finite set of finite connected constraints. It is based on the decomposition of a graph into *blocks*, or 2-*connected components*, and the underlying tree structure that results. First, we recall the definitions, which are standard.

Let C be a graph, which more often than not will be taken to be connected and nontrivial. We will assume in any case that C contains no isolated vertices: every vertex lies on an edge. Define an equivalence relation on the edge set $E(C)$ as follows. First, for $e, f \in E(C)$ write $e \sim f$ if e and f are either equal or lie on a (simple) cycle in C. Then extend this relation to an equivalence relation $\approx$, the transitive closure of $\sim$. A *block* of C is the graph induced on the set of vertices lying on the edges in a single equivalence class in $E(C)$; this can consist of two vertices lying on a single edge. Blocks are 2-connected, that is they remain connected after deletion of any vertex. Now a pair of blocks intersects in at most one vertex, and we associate to the graph C its *reduction* $\tilde{C}$ whose vertices are the blocks of C, as well as the "cut" vertices common to more than one block, with edges (v, B) and (B, v), where v is a cut vertex belonging to the block B. The underlying structure of $\tilde{C}$ is a forest, and as we will be taking C to be connected, the reduction $\tilde{C}$ is even a tree. We call this *the underlying tree* of C. We believe this analysis is highly relevant to our problem of universality. In fact, we believe the following.

Conjecture 1 *(Solidity Conjecture). If there is a C-free universal graph* (*in either the weak or strong sense*)*, then the blocks of C are complete.*

We call such a graph *solid*. One could conjecture in general that the algebraic closure operation should be unary. For the case of one constraint this becomes the solidity conjecture, but for the case of multiple constraints we do not know its precise content in graph theoretic terms. For example if $\mathcal{C}$ is the class of all trees of order $n + 1$ then the algebraic closure operation is unary, and locally finite, since each connected component of the graph has order at most n.

Conjecture 2 *(Reduction Conjecture). If there is a C-free universal graph* (*in either of the two senses*)*, then there is a $\tilde{C}$-free universal graph, where $\tilde{C}$ is the underlying tree of C.*

This is an instance of the ill-fated Monotonicity Conjecture discussed in the introduction, which will be partially rehabilitated in the present section. But it lacks any theoretical support, and is merely plausible (and testable, fortunately).

Combining this with the Tree Conjecture, one gets a fairly precise sense of what is expected, namely that beyond Komjáth's 2-bouquet, similar bouquets, and some further substantial generalizations of that example, the class of exceptional constraints allowing universal graphs (weakly or strongly) should run out fairly soon. In the background there is also the expectation, as noted earlier, that a constraint allowing a weakly universal graph also allows a strongly universal one, though again not for any theoretical reason.

We move on from idle conjecture to something more rigorous.

Definition 2.1. Let C be a connected graph consisting of more than one block.

(1) A pair (B,u), where B is a block and $u \in V(B)$, is called a *pointed block*.
(2) A pointed block (B,u) is an *attached leaf* of the graph C if there is a block B' of C which represents a leaf in the underlying tree of C, and a vertex $u' \in B'$ belonging to another block of C, such that the pointed block (B',u') is isomorphic to (B,u).
(3) A *minimal* attached leaf of C is an attached leaf (B,u) such that there is no embedding of any other attached leaf (B_1,u_1) into (B,u) as a proper subgraph (that is, such an embedding must be an isomorphism).

Observe that in the above, any block B of C which represents a leaf of $\tilde{C}$ in fact meets exactly one other block of C and hence has a unique vertex of attachment. Furthermore, any such pointed block containing a minimal number of edges will be a minimal attached leaf, and similarly there are minimal attached leaves among those with a minimal number of vertices.

What we wish to consider are the operations of *pruning* or *attaching* leaves of a particular minimal type, which we give in a slightly more general form.

Definition 2.2. Let C be a finite connected graph, $\mathcal{C}$ a finite set of finite connected graphs, G an arbitrary (in practice, countable) graph, and (B,u) a pointed block.

(1) C^- is the graph obtained from C by *pruning* (B,u): this means, for every attached leaf (B',u') which can be embedded isomorphically into (B,u), we delete $V(B') \setminus \{u'\}$, and take the induced graph on the remaining vertices. Note that vertices lying in more than one block remain.
(2) G° is the graph induced by G on the set of those vertices v of G such that G contains infinitely many copies of (B,u), disjoint over u, with u identified with v.
(3) G^+ is the graph obtained from G by freely attaching infinitely many disjoint copies of (B,u) to each vertex v of G, with u identified with v.
(4) For sets $\mathcal{C}$ of constraints, $\mathcal{C}^-$ is the set of pruned graphs C^- for $C \in \mathcal{C}$.

If greater precision is needed, we may write $C^-(B,u)$, $\mathcal{C}^-(B,u)$, $G^\circ(B,u)$, and $G^+(B,u)$ instead.

Now we come to the point.

Proposition 2.3. *Let $\mathcal{C}$ be a finite set of finite connected graphs and suppose there is a $\mathcal{C}$-free graph which is universal, either in the weak or strong sense. Let (B,u) be an attached leaf of* *some graph C in $\mathcal{C}$, and $\mathcal{C}^- = \mathcal{C}^-(B,u)$ the result of pruning. Then there is a universal $\mathcal{C}^-$-free graph (in the same sense). In fact, if G is a universal $\mathcal{C}$-free graph, then $G^\circ = G^\circ(B,u)$ is a universal $\mathcal{C}^-$-free graph.*

Proof. We have $\mathcal{C}$, (B,u), G, and $\mathcal{C}^-$, G° as described, and we observe first that G° is $\mathcal{C}^-$-free. As we have accumulated a number of definitions at this point, we will walk through this point.

Suppose toward a contradiction that G° contains a graph C^- as a subgraph, where $C \in \mathcal{C}$. In G, every vertex v of C^- lies on infinitely many disjoint copies of (B,u) with v identified with u. Therefore we can extend the given embedding of C^- into G° to an embedding of C into G by mapping each pruned block (B',u') of C into G over u', with the extended map still 1–1, as it is only necessary to avoid finitely many vertices at each stage. Thus we find that C embeds into G, and we have the desired contradiction.

Now let Γ be $\mathcal{C}^-$-free and consider $\Gamma^+ = \Gamma^+(B,u)$. Then we claim

(1) Γ^+ is $\mathcal{C}$-free.

Indeed, in any embedding of some $C \in \mathcal{C}$ into Γ^+ as a subgraph, C will map into a single connected component, and each block of C will map into a block of that component. The blocks of C^- which do not correspond to leaves of the associated tree $\tilde{C}$ go into Γ; and the blocks of C^- which do correspond to leaves of $\tilde{C}$ also go into Γ, as none of them embeds into (B,u) over the attaching vertex. So an embedding of C into Γ^+ would induce an embedding of C^- into Γ, and (1) follows.

Now Γ^+ must embed in G, by (1), either as a subgraph or as an induced graph, as the case may be. Under such an embedding, $\Gamma \subseteq [\Gamma^+]^\circ$ will embed into G°, either as a subgraph or as an induced graph, correspondingly. Our claim follows. □

In view of the importance of this result for our analysis, we make the following definition in the case of a single constraint.

Definition 2.4. A finite connected graph C is *critical* if the underlying tree $\tilde{C}$ is neither a path nor a near-path, but for any type of attached leaf of C, the tree $\tilde{C}^-$ associated with the corresponding pruned graph is a path or near-path.

Corollary 2.5. *Suppose that C is a finite connected graph whose underlying tree $\tilde{C}$ is neither a path nor a near-path, and that there is a weakly or strongly universal C-free graph. Then there is an induced subgraph C' of C, which is critical, for which, correspondingly, a weakly or strongly universal C'-free graph exists.*

For the proof, one prunes C repeatedly until it becomes critical.

We have conjectured that there are no graphs with the properties of the corollary; and we see that it suffices to consider critical ones. We prove the Tree Conjecture in this framework by considering critical trees. In this case attached leaves are essentially just leaves, or rather edges connecting a leaf to its point of attachment, and pruning amounts to the removal of the leaves (or to put it another way, shortening all the external branches).

3. Monarchy and stardom

In the present section we will prove the following.

Theorem 2. *Let T be a tree with a unique vertex of maximal degree. Then the following are equivalent*:

(1) *There is a strongly universal T-free graph.*
(2) *There is a weakly universal T-free graph.*
(3) *The tree T is a path or near-path.*

A further equivalence would be: algebraic closures of points are either finite, or are two-way infinite paths without additional edges. But as we have remarked, there is no need to bring in the notion of algebraic closure explicitly in such cases.

The implication (3) $\Rightarrow$ (1) requires argument, and is treated in [4]. This has a completely different character from anything we do here, lying on the positive side; all of our work here fills in the gap on the negative side. We need to show $(\neg 3) \Rightarrow (\neg 2)$.

We distinguish two cases. Let d be the maximal vertex degree in the tree T. Then either $d \geqslant 4$ or $d = 3$. One might expect this distinction to be significant, since the case $d = 3$ includes the case of a near-tree, which at some point has to be singled out as an exception, but there are other reasons for the case distinction as well.

In the present section, what will be important is the behavior of a regular tree with vertex degree $d - 1$, and more precisely of its approximations, namely regular graphs of vertex degree $d - 1$ and large girth. There is certainly a distinction to be observed here between the case $d = 3$ and $d \geqslant 4$. Later on, the issue will be somewhat different. We will need to construct infinite graphs from finite pieces while controlling the vertices of degree d, and it is difficult to avoid introducing new vertices of degree 3.

While this case distinction is not *always* essential, it tends to play a role, and the case $d = 3$ is the more complicated of the two. On the other hand, as we work with critical trees there is some compensation in the form of improved control of the structure of the tree in this case.

3.1. Monarchs

We begin with the generic case, $d \geqslant 4$, and while dealing with this case we will encounter all the issues that arise in any of the cases, as well as most of the strategies for dealing with them.

Proposition 3.1. *Let T be a tree with a unique vertex of maximal degree d, with $d \geqslant 4$. Then there is no weakly universal T-free graph.*

Proof. Suppose toward a contradiction that T is a counterexample of minimal order. Then there is a weakly universal T-free graph, and hence for the graph T' derived from T by pruning (removal of its leaves) there is also a universal T'-free graph (Section 2). Now if T' also has a vertex of degree d, then this violates the choice of T as a minimal counterexample. So the vertex v of degree d in T must have at least one leaf as a neighbor in T.

Now this turns out to be precisely the situation considered in [7]: a tree with a unique vertex v of maximal degree d, which is adjacent to a leaf v', and with $d \geqslant 4$. $\square$

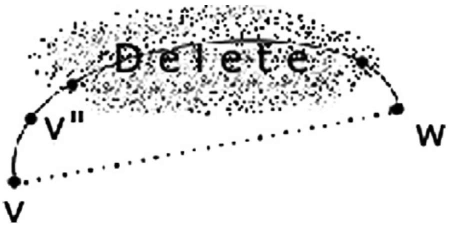

Fig. 1.

Let us expand on this, as the same type of construction and analysis is needed in general with a host of minor complications. The following construction applies in the critical case of [7].

Let Γ be a regular tree of degree $d-1$, or more generally a regular graph of degree $d-1$ which is *tree-like* in the sense that the girth is large (larger than $2n$ with $n=|T|$).

As Γ is regular of degree $d-1$ it is T-free (this part of the argument blows up considerably as soon as we leave the domain of monarchy), and we may vary the construction of Γ, and in particular the cycle lengths occurring in Γ, to give $2^{\aleph_0}$ graphs of this type. Here (and only here) we exploit the hypothesis $d-1>2$.

The key property is the following.

(∗) If Γ is a subgraph of a T-free graph G, then Γ is a connected component of G, and is an induced subgraph.

We will check this. Another way to phrase this claim is that the vertices of Γ can acquire no new neighbors. If our aim is to refute only *strong* universality, then most of this argument drops out of the picture, along with any preparations which may have been made for it. There are no such preparations in the present case, but usually there will be.

It is immediate that Γ is a connected component of G, because as soon as one adjoins a new neighbor u' to a vertex u in Γ, which is not already a vertex of Γ, one gets an embedding of T into the extended graph by identifying v with u, a leaf adjacent to v with u', and the rest of T with a suitable part of Γ, which locally (near u) looks like a regular tree of degree $d-1$.

Similarly, there can be no new edge between vertices v, w whose distance in Γ is greater than $n=|T|$. Local connections require more attention; this is also a characteristic feature of the more complicated constructions later.

If we adjoin a new edge (v,w) between two nonadjacent vertices of Γ which lie at distance at most n, then we may embed T into the resulting graph as follows (see Fig. 1). Let P be the path linking v to w in Γ, and let v'' be the neighbor of v on P. Then v'' may play the role of v', and the part of Γ remaining after deleting the component of $\Gamma\setminus\{v,w\}$ containing v'', and with the edge (v,w) adjoined, again will be regular of degree $d-1$, and girth greater than n, so the remainder of T can be embedded over v, v''.

Now it is impossible that all of these graphs Γ could occur inside a single countable graph among its connected components, so no countable weakly universal C-free graph exists in this case.

For such constructions we require T-free graphs to which the adjunction of a single edge will in many cases produce an embedding of T, without having a detailed knowledge of the structure of T. In general the main mechanism for keeping the necessary control involves paying attention to the distribution of vertices of degree d in Γ (as there are none in this case, the distribution is particularly transparent), and building up the minimal vertex degrees to $d-1$. On the other hand, the idea of making the graph closely resemble a tree will have to be severely curtailed in general, and instead by considering critical trees we will find we need much less control of Γ to ensure the necessary embeddings become available. Fortunately we do not need to prevent the addition of arbitrary edges: it will be sufficient if we can recover some invariants of Γ once we know the restriction of the embedding of Γ to a suitable finite set. When Γ is a connected component of the ambient graph, we can recover Γ itself from the image of any vertex, but we need much less than that.

3.2. Stardom

The method of the previous section clearly will not work with $d=3$. So we now deal separately with this case, reducing to the critical case and making use of a construction that looks closely at the structure of the constraint tree. This is very reasonable, since we still need to distinguish the exceptional cases.

We will refer to vertices of degree at least 3 in a tree as *branch vertices*.

Definition 3.2.

(1) A *star* is a tree with a unique branch vertex, called its *center*.
(2) For $n \geqslant 1$ and $d_1 \geqslant \cdots \geqslant d_n \geqslant 1$, the tree $S(d_1,\ldots,d_n)$ is the star formed by attaching paths of length $d_1,\ldots,d_n$ to a central vertex.

We will always take $n \geqslant 3$ here, to get a proper star. In this case the star has a well-defined *center* and the maximal vertex degree is n. Near-paths are stars $S(d_1,d_2,1)$. Stars with $n \geqslant 4$ have been dealt with in the preceding subsection.

Proposition 3.3. *If $S = S(d_1,\ldots,d_n)$ is a star and is not a near-path, then there is no weakly universal countable S-free graph.*

Proof. Since the case $n \geqslant 4$ is covered by the previous proposition, we will take $n=3$, and we will also take S critical in the sense of Section 2, which means that we take

$$d_3 = 2,$$

so that pruning produces a near-path with the same center.

Let H_0 and H_1 be the following graphs (see Fig. 2). First, fix two vertices u_0,u_1. To form H_0, adjoin two common neighbors v_0,v_1 to u_0 and u_1, with v_0 and v_1 adjacent; this is K_4 with one edge deleted. To form H_1, adjoin infinitely many common neighbors v_i to u_0 and u_1, and add an edge (u_0,u_1), with no further adjacencies.

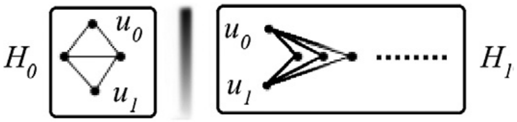

Fig. 2.

Now for $\epsilon \in 2^{\mathbb{Z}}$ a bit string, form a graph Γ^ϵ as follows. Begin with an infinite independent set A of vertices a_i ($i \in \mathbb{Z}$). For each i, attach to the pair a_i, a_{i+1} a copy of $H_{\epsilon(i)}$ with a_i and a_{i+1} corresponding to u_0 and u_1. Then Γ^ϵ is $S(2,2,2)$-free and in particular S-free.

Now we have to think about "decoding" Γ^ϵ when it is embedded in a larger S-free graph as a subgraph. As usual this involves getting some control over at least some of the additional edges adjoined in such an extension.

Define a relation $R(u,v)$ on the vertices of any graph G by the following condition: *u, v lie in a copy of H_0 or H_1, with u and v playing the roles of u_0 and u_1 respectively.* By construction successive pairs (a_i, a_{i+1}) satisfy this relation in any graph G into which Γ embeds. We must show that this relation is not much affected by embedding into a larger S-free graph. Our claim is as follows:

(1) If $\Gamma^\epsilon \subseteq G$ and G is S-free, then for each vertex $a_i \in V(\Gamma^\epsilon)$, and each $v \in V(G)$, if $R(a_i, v)$ holds then $v = a_{i\pm 1}$.

Let P be a path in Γ^ϵ containing all a_i, with $d_P(a_i, a_{i+1}) = 2$ for all i.

Now either a_i and v have infinitely many common neighbors, or a_i and v play the roles of u_0 and u_1 in H_0 (or both).

If v has infinitely many neighbors, then v must lie on the path P, as otherwise we may embed S into the extension of Γ by one of the new edges attached to v. But there is some freedom in the choice of the path P, and if v cannot be pushed off it by altering the path, then v must in fact be some a_j. Now it is easy to see that if $|j - i| > 1$ then there is an embedding of S into G, a contradiction.

So v has finite degree in G and thus a_i and v must play the role of u_0 and u_1 in H_0. So they have common neighbors w, w' which are adjacent.

If $w \in A$, that is $w = a_j$ for some j, then a_i and a_j are adjacent in G, and easily $j = i \pm 1$. It follows easily that w and w' are not both in A. So we may assume that w is not in A, and choose P so that w is not on P. Then v must be on P. If $v = a_j$ for some j, then again by inspection $j = i \pm 1$ as claimed. So we may suppose that v is not in A, but lies on P, and that P cannot be chosen to avoid both v and w. This means that v and w are the common neighbors of some pair (a_j, a_{j+1}), and are the only such common neighbors, as otherwise the path P could still be moved. But w' is adjacent to v and w, hence by considering a_j, a_{j+1} we see that w' is also forced onto P, and hence must be a_k for some k. However, looked at from the point of view of a_k, this is also impossible: a_k becomes the center of a copy of S.

So (1) holds. We can now deduce the nonexistence of a weakly universal S-free graph. Suppose toward a contradiction that G is a weakly universal S-free graph, and, of course, countable. For each ϵ choose an embedding f_ϵ of Γ^ϵ into G. Choose a pair ϵ, ϵ' for which these embeddings agree on the successive vertices a_0, a_1 in A. It follows from (1) that the restriction of f_ϵ to A coincides with the restriction of $f_{\epsilon'}$ to A. But for some i, we have $\epsilon(i) \neq \epsilon'(i)$, and thus the vertices v_0, v_1 in G which correspond to a_i, a_{i+1} in both Γ_ϵ and in $\Gamma_{\epsilon'}$ must occur on copies of both H_0 and H_1 in G. This immediately provides an embedding of S into G, and a contradiction. □

The final argument is typical and will occur in some form in all cases. As long as the essential invariant ϵ of Γ can be recovered from the embedding (after fixing some points to get rid of shifts and reflections along A), we can argue in this fashion. On the other hand, if we are dealing with strong universality, there would be little to check at this point. Still, even when dealing with induced subgraphs one has to check for example that the relation $R(a_i, v)$ is not satisfied by new elements of G, and indeed without this one would not even know that the algebraic closure operation is nontrivial. So in this decoding phase, the issues are similar regardless whether we deal with local finiteness, strong universality, or weak universality, though the degree of control needed to effect the decoding varies considerably.

4. Toward the Tree Conjecture

Our goal now is the following.

Theorem 3. *If T is a finite tree with maximal vertex degree $d \geqslant 3$, and if T has more than one vertex of degree d, then there is no weakly universal T-free graph.*

In view of the result of the previous section, it suffices to prove this theorem in the critical case. So we record it in this form.

Theorem 3′. *If T is a critical finite tree with maximal vertex degree $d \geqslant 3$, and if T has more than one vertex of degree d, then there is no weakly universal T-free graph.*

Recall that in the critical case the pruned tree T' is either a path or a near-path.

4.1. A special case

We first prove a considerably weaker result in which the basic construction can be seen most simply, and without invoking the criticality hypothesis.

Proposition 4.1. *Let T be a tree with maximal vertex degree $d \geqslant 5$, and suppose that every vertex of degree d is adjacent to a leaf of T. Then there is no strongly universal T-free graph.*

We describe the construction of an uncountable family of countable T-free graphs, and show that they cannot all be simultaneously embedded into a T-free countable graph. There are three phases to this argument: (a) construction; (b) T-freeness; (c) decoding (i.e., analysis of the image of such a graph under embedding into a larger T-free graph). The assumption that $d \geqslant 5$ simplifies the construction, and the fact that we deal with strong universality simplifies the decoding process by limiting the class of embeddings considered. The arguments for T-freeness amount to saying that the metric structure induced by T on its vertices of degree d does not embed into the metric structure induced by our graphs on their vertices of degree d or more, and is typical of the analysis in general.

The extra hypothesis on the vertices of degree d is much stronger than what we actually require below, and much weaker than what one has if one restricts attention to critical trees. Some form of this condition is helpful in stage (c).

Definition 4.2.

(1) $V_1(T)$ is the set of vertices of degree d in T, construed as a metric space with the induced metric. From this one can recover the tree structure induced on the convex hull of this set in T.

(2) If $v \in T$, then a *v-component* of T is a connected component of the graph resulting from deletion of v in T.
(3) A vertex v in $V_1(T)$ is an *external* vertex of maximal degree if it is a leaf in the convex hull of $V_1(T)$ in T. Equivalently, at most one v-component of T contains vertices of degree d.

For the proof of the proposition we may assume that there are at least two vertices in T of maximal degree, as otherwise we apply Theorem 2. All we really require for the proof of this proposition is a single external vertex of degree d adjacent to a leaf.

Construction 1. Let v_1 be an external vertex of T of maximal degree and let C be the v_1-component of T containing all other vertices of T of maximal degree. Let H be the graph induced by T on $C \cup \{v_1\}$.

Let P be a $(d-2)$-regular 2-connected graph of very large girth. For each vertex $u \in P$, attach a copy H_u of H to u with u corresponding to v_1. Call the resulting graph Γ_0^P. For any vertex u of degree less than $d-1$ in Γ_0^P, raise its degree to $d-1$ by adjoining suitable trees (regular of degree $d-1$ except at the root, where the degree is $d-1-\deg(u)$). Call the resulting graph Γ^P.

Lemma 4.3. *The graph Γ^P is T-free.*

Proof. Let $V_1(\Gamma^P)$ denote the set of vertices of Γ^P of degree at least d construed as a metric space with the induced metric. These vertices in fact have degree exactly d and lie in the subgraphs H_u, with $V_1(H_u)$ isometric to $V_1(T) \setminus \{v_1\}$. It will suffice to show that there is no embedding of $V_1(T)$ into $V_1(\Gamma^P)$ as metric spaces which is semicontractive in the sense that distances do not increase. Note that this metric structure is the same in Γ_0^P and in Γ^P, so for the rest of this argument one may as well think in terms of Γ_0^P.

Call a subspace A of $V_1(T)$ *isolated* if it satisfies the following condition:

Iso For every subspace A_0 of $V_1(T)$ isometric with A and every embedding f of $V_1(T)$ into $V_1(\Gamma^P)$, the image $f(A_0)$ is contained in some single $V_1(H_u)$.

A better term might be "indecomposable" but we wish to emphasize here that the subgraph P is avoided, something which will be less clear in subsequent constructions.

Now if there are no such embeddings f then $V_1(T)$ itself is isolated, but if there are any such embeddings then $V_1(T)$ is not isolated. So let us assume there are such embeddings and let A be an isolated subspace of $V_1(T)$ of maximal order. By our assumption $A \neq V_1(T)$, and as points are isolated, A is nonempty.

Choose a pair (B, v) with B a subspace of $V_1(T)$ isometric with A, and with $v \in V_1(T) \setminus B$, and furthermore with

$$\delta = d\big(f(v), f(B)\big)$$

minimized over all such pairs, and all semicontractive embeddings f of $V_1(T)$ into $V_1(\Gamma^P)$. By the maximality of A, $B' = B \cup \{v\}$ is not isolated, and thus there is an embedding $f : V_1(T) \to V_1(\Gamma^P)$ such that the image $f(B')$ meets at least two distinct sets of the form $V_1(H_u)$ with $u \in P$. But $f(B)$ is contained in one such set $V_1(H_u)$, and thus $f(v)$ is contained in another. Now let $\tilde{B}$ be the subspace of $V_1(T)$ corresponding to $f(B)$ under the identification of H with H_u, and observe that $\tilde{B}$ is isometric with A, that $v_1 \notin \tilde{B}$, and that $d(f(v_1), f(\tilde{B})) \leqslant d(v_1, \tilde{B}) =$

$d(u, f(B)) < d(f(v), f(B))$, the latter point in view of the structure of the metric on Γ^P. So $d(f(v_1), f(\tilde{B})) < \delta$, contradicting our choice of δ as minimal. $\square$

In later arguments we will use some of this metric terminology while formulating the main argument directly in terms of graph embeddings. We simply wished to emphasize here that the obstruction really is captured by the metric structure on the vertices of high degree. But the argument in general will depend a little more on the graph structure, particularly near vertices of P.

The second question to take up is a kind of rigidity (or decoding) for Γ^P when considered as an induced subgraph of a general T-free graph.

Lemma 4.4. *Let G be a T-free graph containing Γ^P as an induced subgraph. Then for any vertex $u \in P$, the neighbors of u in G are its neighbors in Γ^P.*

Proof. Here we recall the assumption that the girth of P is large, and thus the local structure of P near the vertex u is exactly that of a $(d-2)$-regular tree. We also use the assumption that v_1 has a neighbor which is a leaf.

What we need to show is the following: the graph $\Gamma^P_{u,v}$ obtained by adjoining one new neighbor v to u contains a copy of the tree T. One begins the construction of a suitable embedding by taking the map identifying H and H_u, in which v_1 corresponds to u. Now some leaf v_1' adjacent to v_1 may correspond to v. It remains to embed the remaining $d-2$ v_1-components of T into Γ^P, making use of P and some of the trees attached at the end of the construction.

Now each of the remaining $(d-2)$ v_1-components of Γ^P contains a tree closely resembling a $(d-1)$-regular tree (except for its root, adjacent to v_1, whose degree in the component is one less); any cycles will be the cycles of large girth allowed in P. So it is easy to see that there is no obstruction to the completion of our embedding. $\square$

Our two lemmas prove the proposition. This is a general principle; let us check it in this case.

Proof of Proposition 4.1. Suppose toward a contradiction that G is a countable strongly universal T-free graph. Consider embeddings $f_P : \Gamma^P \to G$. We have uncountably many isomorphism types of P available, as we may control the cycle lengths that appear. So there must be at least two nonisomorphic graphs P, Q whose images $f_P(P)$ and $f_Q(Q)$ meet in a vertex u.

Now consider the subgraph G_0 of G induced on the vertices of degree exactly $d-1$ in G. This contains $f(P)$ and $f(Q)$, by our lemma. In particular the connected component of u in G_0 contains $f(P)$ and $f(Q)$. Now observe that the connected component G_u of u in G_0 is contained in $f(\Gamma^P)$. Otherwise, we would have an edge $v, v' \in G$, with v in $f(\Gamma^P)$ and $v' \notin f(\Gamma^P)$, and with v of degree $(d-1)$ in G. But v is already of degree at least $d-1$ in Γ^P and thus $v' \in f(\Gamma^P)$, a contradiction.

So G_u is contained in both $f(\Gamma^P)$ and $f(\Gamma^Q)$. Now the only nontrivial block in $f(\Gamma^P)$ is $f(P)$ and similarly for Q, so $f(P) = f(Q)$ and as these are induced subgraphs of G, the graphs P and Q must be isomorphic, a contradiction. $\square$

To put the matter briefly, our second lemma proves that P is recoverable from finite data, and any countable graph contains only countably many candidates for such data, so if we have uncountably many candidates for P then can be no universal graph (strongly or weakly, depending on the strength of the recoverability lemma).

We can see that this argument is going to require significant adaptation as we remove the simplifying hypotheses. For $d = 4$ the graph P becomes a path (this is why we call it P, actually) and for $d = 3$ we will again use a path P, but we will have to look considerably farther into the structure of T to find a suitable way to extend P without creating a copy of T. The most extreme case was treated earlier: the case of stars. There the construction looks very little like the one just given, though the path P is still visible.

In the decoding phase most of the weight was borne above by the fact that we dealt only with strong universality. So in most cases we will have to modify the construction to "block" the adjunction of at least some potential new edges to our graphs Γ.

The method used in the proof of this proposition lies at the core of most of our subsequent constructions and proofs.

Criticality will be important to ensure an adequate supply of leaves, and it also simplifies the embedding argument made in the decoding phase. In the critical case the embedding argument in the proof of our decoding lemma would work in Γ_0^P as well as in Γ^P, but on the other hand the full decoding argument was only given in the proof of the proposition and this argument actually needs Γ^P rather than Γ_0^P, so the saving is not very great here.

In some delicate cases criticality may also help in checking that our graphs Γ are T-free. Most of our constructions place additional vertices of degree d on the graph corresponding to P here, so the analysis must become more precise.

4.2. Amalgamation and the parameter ℓ

Before entering into a detailed consideration of how the foregoing construction may be adapted to deal with the question of weak universality, we may consider some general points that are relevant to the decoding process and give some indication of what additional structural features of the constraint tree T are relevant in general. This leads to a certain proliferation of cases, handled by a unified method but with considerable variation from case to case.

One such parameter, and an important one, is the maximal degree d. The case $d = 4$ turns out not to be much more troublesome, in comparison with $d \geqslant 5$. While P is just a 2-way infinite path in this case it turns out that there are some variations available in the "attachment" procedure that passes from P to Γ^P and we can again arrive at uncountably many variations on each theme. Ultimately the same will apply when $d = 3$ but not so simply.

But there is a second parameter which comes into play in the decoding phase. When our graph Γ embeds into a larger T-free graph G it may acquire new edges between its own vertices, and this "noise" threatens to make recovery of Γ from G impossible. However, what will be true is that the vertices of P will remain of finite degree, and that just as we considered the vertices of degree $d - 1$ in the previous subsection, consideration of the vertices of finite degree is generally useful.

The critical observation is the following: if $\hat{T}$ is the result of amalgamating two copies, or even infinitely many copies, freely over the vertex set $V_1(T)$ then $V_1(\hat{T}) = V_1(T)$ as a metric space (but here $V_1(\hat{T})$ means all vertices of degree at least d, not exactly d).

Thinking back to the "attachment graph" H of the previous subsection, if $\hat{H}$ is the corresponding subgraph of $\hat{T}$, then this suggests the idea of using $\hat{H}$ in place of H and going on as before. This would be sound apart from one fatal flaw: all the vertices along P are likely to acquire infinite degree in the process. This makes it extremely likely that T will embed in the graph so constructed, and also makes the recovery of P highly improbable. In short, everything needed is destroyed.

There is one case in which this flaw is not actually present: if the external vertex v_1 of degree d is adjacent to a vertex v_0 of degree d then v_1 will have no new neighbors in $\hat{H}$. Really what we are doing in this case is working with the subgraph H_0 of H obtained by deleting v_1 and the corresponding amalgam $\hat{H}_0$, then attaching $\hat{H}_0$ to v_1 by an edge.

In general, we must consider the parameter

$$\ell = d(v_1, v_0),$$

where v_0 is the closest vertex of degree d to v_1. We want to treat the part of H based at v_0 as the "attachment graph," and take further pains to deal with the path from v_1 to v_0, which we think of as potentially running along P. It turns out that the relevant case division is as follows: $\ell \geqslant 3$; $\ell = 2$; $\ell = 1$ with the first two cases similar and the last of a different character.

Again, once one enters into this kind of more precise construction, the structure of the tree T between v_0 and v_1 plays a major role; as the pruned tree T' will be a path or near-path one hopes that the corresponding part of T' will be just a path, though a few exceptional configurations must be treated separately.

So our case division comes out something like the following:

I $\ell \geqslant 2$:

A $\ell \geqslant 3$; B $\ell = 2$.

II $\ell = 1$:

A $d \geqslant 4$; B $d = 3$ (with various subcases).

III Left-over near-paths

A $d \geqslant 4$; B $d = 3$.

We will ultimately list the cases differently for reasons of convenience, but the logical structure is properly reflected above. The full list of cases actually used is recapitulated at the end.

5. Case I: $\ell \geqslant 2$

We take up the proof of Theorem 3′. We deal with an external vertex v_1 of maximal degree, and a closest vertex v_0 of maximal degree, with $\ell = d(v_0, v_1)$.

5.1. *Case IA: $d \geqslant 4$, $\ell \geqslant 3$*

Case IA. T has a vertex v_0 of maximal degree $d \geqslant 4$ such that some v_0-component C of T contains a unique vertex v_1 of degree d, and $\ell = d(v_0, v_1) \geqslant 3$. Either T' is a path, or else T' is a near-path whose center does not lie in the v_0-component C.

We allow v_0 to be the center of T'.

Construction 2. Let $H = T \setminus C$. Let P be a $(d-1)$ regular graph of large girth, and let Γ^P be the result of adjoining a vertex b adjacent to all vertices of P and then attaching a copy H_b of H with b corresponding to v_0.

Lemma 5.1. *Γ^P is T-free.*

Proof. Since the number of vertices of degree d in T is greater than the number of vertices in H_b which have degree at least d in Γ^P, any embedding f of T into Γ^P has to carry at least one vertex of degree d into P.

If exactly one vertex u of degree d in T corresponds to a vertex of P, then b must also correspond to a vertex of degree d in T under the embedding f. Now the vertex $f(u) \in P$ has degree exactly d in Γ^P and hence b must be one of the neighbors of u in $f(T)$. As $d(u,b)=1$ it follows that the diameter of the set of vertices of degree d in $f(T)$ is less than its diameter in T, so this is not an isomorphism.

Thus there are at least two vertices u, v of degree d in T whose images under f lie on P. Again, the vertex b must occur as a neighbor of u and v in the image $f(T)$ and therefore $d(u,v)=2$. Since $\ell \geqslant 3$ there must be other vertices of degree d in T (or it would be enough for our purposes to assume this, if $\ell = 2$). It follows easily that T must be a near-path with center corresponding to b. Then it is easy to see that the diameter of the set of vertices of degree d in T is greater than the diameter of the corresponding set in $f(T)$, a contradiction. □

Lemma 5.2. *Suppose that G is a T-free graph containing Γ^P and that u is a vertex of P. Then any neighbor v of u in G is either a neighbor of u on P, or a vertex of H_b.*

Proof. In view of the structure of T', v_1 is adjacent to a leaf of T. If $v \notin \Gamma^P$ then we look for an embedding of T into G in which u represents v_1, v represents such a leaf, and H_b represents H with b corresponding to v_0. The path from v_0 to v_1 can run along P. As P has locally the structure of a $(d-1)$-regular tree, the extension to the remainder of T is possible.

If $v \in \Gamma^P$ then we may suppose $v \in P$. Suppose that v is far from u in the metric on P. Then we proceed as in the case when v is not in Γ^P. Now suppose v is close to u, and the girth of P is large relative to the distance $d(u,v)$. Then there is a unique shortest path L from v to u. Let u' be the neighbor of u on L. Consider the graph obtained from P by deleting the rest of L (between u' and v) and adjoining the edge (u,v). If P had been a $(d-1)$-regular tree then this new graph would also be a $(d-1)$-regular tree, but in fact it is a $(d-1)$-regular graph of high girth. In any case, using u' to represent a leaf adjacent to u we may again proceed as in the first case to embed T into G. □

Proposition 5.3. *In Case* IA *there is no* (*countable*) *universal T-free graph.*

Proof. Otherwise we find ourselves considering embeddings $f : \Gamma^P \to G$, $g : \Gamma^Q \to G$ with P and Q nonisomorphic, and with the images $f[H_b^P]$ and $g[H_b^Q]$ identical, and also $f(P)$ meets $g(Q)$. Now we look at the graph G_0 obtained by deleting $f[H_{b^P}]$. In this graph, $f(P)$ and $g(Q)$ are connected components, and the induced structure from G_0 is the original structure on P or Q. As $f(P)$ meets $g(Q)$, the images coincide and the graphs are isomorphic. □

5.2. Case IB: $d = 3$, $\ell \geqslant 3$

Case IB. T has a vertex v_0 of maximal degree $d=3$ such that some v_0-component C of T contains a unique vertex v_1 of degree d, and $\ell = d(v_0, v_1) \geqslant 3$. Either T' is a path, or else T' is a near-path whose center is not v_1.

Construction 3. Take an infinite path P and partition it into successive (alternating) finite intervals P_i, Q_i for $i \in \mathbb{Z}$ of lengths p_i, q_i respectively, satisfying the following conditions:

(1) $p_i \geqslant 3\ell - 3$;
(2) $q_i = \ell$.

Let C be the v_0-component of T containing v_1 and $H^0 = T \setminus C$. Let H be the graph obtained by amalgamating two copies of H^0 freely over the vertices of H^0 of degree d in T (this includes v_0), and adjoining additional vertices to bring up the degree of any vertex in $V_1(H)$ to ∞.

Adjoin vertices b_i adjacent to all vertices in P_i, for all i, and attach a copy H_i of H to b_i with b_i corresponding to v_0. Call the result $\Gamma = \Gamma^\epsilon$ where ϵ is the sequence $(p_i)_{i\in\mathbb{Z}}$.

Lemma 5.4. *Γ is T-free.*

Proof. Let X_i be the set of vertices in $P_i \cup H_i$ of degree at least d in Γ, and let Y_i be the set of vertices of H_i of degree at least d in Γ.

Call a subspace A of $V_1(T)$ *indecomposable* if under every embedding of T into Γ, the image of any subspace of $V_1(T)$ isometric with A lies in one set X_i; call A *isolated* if every such image lies in one set Y_i.

We will write V_1 for $V_1(T)$ throughout.

(1) There is a nonempty isolated metric subspace of V_1.

Consider a geodesic path $P^* = (p_0, \ldots, p_n)$ in V_1 of maximal length subject to the condition $d(p_i, p_{i+1}) \leqslant \ell$ for $i < n$, and among all such maximize $d(p_0, p_1) + d(p_{n-1}, p_n)$.

We claim that this geodesic path is isolated. First, as the H_i are widely separated ($q_i = \ell$) this path is indecomposable.

Now consider any $A \subseteq V_1$ isometric to P^* and embedded into X_i by an embedding f of T into Γ. We claim that $f[A]$ lies in Y_i. If not, writing $A = (a_0, \ldots, a_n)$ we have $u = f(a_i) \in P_i$ for some i.

Now u has degree d in Γ and hence the neighbors of a_i in T map onto the neighbors of u in Γ, including b_i. So any vertices of A which map into P_i share a common neighbor in T, and hence lie at distance 2 in T. As A is a geodesic path, and corresponds to points on a path in T, there are at most two such points in A.

If there are two such points in A then as b_i lies between their images, the other vertices of A map into P_i rather than H_i. But as this is impossible, we find that $|A| = 2$ in this case, and as $d(a_0, a_1)$ is maximized, that

$$\ell = 2$$

as well, contradicting our current assumptions.

So a unique point of A maps into P_i. Then the other points of A map into H_i and are linked to u by a path through b_i. In particular u must correspond to an endpoint of A. We may suppose $u = f(a_n)$. Now the path $(f(a_0), \ldots, f(a_{n-1}))$ corresponds to a path $(a'_0, \ldots, a'_{n-1})$ in $T \setminus C$, and $d(a'_{n-1}, v_0) = d(a_{n-1}, b_i) < d(a_{n-1}, u) \leqslant \ell$. Hence if $a'_{n-1} \neq v_0$, this path may be lengthened to a path $(a'_0, \ldots, a'_{n-1}, v_0, v_1)$ satisfying our conditions and contradicting the maximality of n. We conclude that $a'_{n-1} = v_0$. But then the path $(a'_0, \ldots, a'_{n-1} = v_0, v_1)$ satisfies our conditions with an increase in the distance between the last two vertices, again contradicting maximality. This last contradiction completes the proof of (1).

Now we consider a subspace $A \subseteq V_1$ which is isolated, and maximal. One possibility is that $A = V_1$, but as $|V_1| > |V_1(H)|$, this would mean that there are no embeddings of T into Γ, as we claim. So for the remainder of the argument we suppose that there are such embeddings, and we aim at a contradiction.

We claim that there is some subspace A' of V_1 isometric to A, and some embedding f of T into Γ such that for some i we have

(2) $b_i \in f[A']$.

We choose a pair $A' \subseteq V_1$ and $v \in V_1 \setminus A'$ so that A' is isometric to A and $d(v, A')$ is minimized, and we consider the space $B = A' \cup \{v\}$ inside V_1. By our choices of A, this is not isolated. Take $B' = A'' \cup \{v'\}$ isometric to B inside V_1 so that there is an embedding f of T into Γ for which $f[B']$ does not go into any H_i. As A' is isolated, $f[A'']$ goes into some H_i, and by hypothesis $f(v') \notin H_i$, so $d(b_i, A'') < d(v', A'')$. By the choice of v', we must have $b_i \in f[A'']$ and thus (2) is achieved.

Now we repeat the general thrust of the first part of the argument. We consider a geodesic path $P^* = (a_0, \ldots, a_n)$ which can be attached to A' at the point u_0 corresponding to b_i in $f[A']$, so that with the natural metric the extension $A^* = A' \oplus_{u_0} P^*$ is isometric with a subspace of V_1, and so that $d(a_i, a_{i+1}) \leqslant \ell$ for all $i < n$, and we first maximize n, then maximize $d(a_{n-1}, a_n)$. By condition (2) one possibility is to take $P^* = (v_0, v_1)$ with v_0 corresponding to some point u_0 in A', so $|A^*| > |A'|$.

By the maximality of A', the space A^* cannot be isolated. It is certainly indecomposable since A' is indecomposable, in view of the metric structure of A^* and Γ. So there is an embedding f of T into Γ taking a copy of A^* (which we continue to call A^*) into X_i for some i, but not into H_i. Here A' goes into H_i and the geodesic path P^* does not, though the endpoint a_0 does. Arguing as in the first instance we see that one end of P^* goes into P_i and the rest of A^* goes into H_i. But then we adjust P^* as before and obtain a final contradiction: if b_i is not in the image of P^* we lengthen the path P^*, while if b_i is in the image of P^* we move a_n farther away.

Retracing our steps from this contradiction, we see that in fact the maximal isolated space A must be V_1, and thus Γ is T-free. □

We can now enter the decoding phase.

Lemma 5.5. *If Γ is contained in the T-free graph G, then for any vertex u of P we have the following*:

(1) *The degree of u in G is finite.*
(2) *Any neighbor of u of finite degree in G lies on P, and is one of the neighbors of u in Γ.*
(3) *If i is chosen so that $d(u, P_i)$ is minimal, then any neighbor of u in G off P lies in H_i.*

Proof. As $d = 3$ the v_0-component containing v_1 consists of a path with one edge adjoined at v_1.

Suppose (u, v) is an edge with $u \in P$, $v \in G$ and $v \notin \Gamma$. Then we embed T into G with u corresponding to v_1 and v corresponding to a leaf of T adjacent to v_1. It suffices to notice that if i is chosen to minimize $d(u, P_i)$ then there is a path of length ℓ from u to b_i, and b_i can play the role of v_0. The same applies if $v \in H_j$ with $j \neq i$. It remains to consider vertices of finite degree in H_i and vertices on P.

Now as H_i is obtained by free amalgamation of two copies of H^0 over the vertices of degree d, and the vertices originally of degree d are transformed into vertices of infinite degree, the vertices of finite degree in H_i may be treated just like vertices outside Γ.

Suppose therefore that $v \in P$. Then we must consider the possibility that v lies along our intended path L from u to b_i. If $u \notin P_i$ then we may substitute for L a path beginning with (u, v). The main point to consider is the possibility $u, v \in P_i$, but u, v are nonadjacent.

Then after deleting u, v from P_i there remain $p_i - 2 \geqslant (3\ell - 5) > 3(\ell - 2)$ vertices, and at least one of the three resulting subintervals in P_i contains at least $\ell - 1$ vertices. Furthermore as the roles of u and v are now symmetric, we may suppose that u is an endpoint of such an interval. As we have a path of length ℓ from u to b_i which does not pass through v, we may take u, b_i to correspond to v_1, v_0 respectively, and use v to represent a leaf adjacent to v_1. One may also use an additional vertex of P_i to represent a leaf adjacent to b_i as

$$p_i \geqslant 3\ell - 3 > \ell + 1.$$

(For $\ell = 2$ one would just add this inequality as a restriction on p_i, but there are other difficulties in that case.)

This proves the lemma in all cases. $\square$

Proposition 5.6. *In Case* IB *there is no* (*countable*) *universal T-free graph.*

Proof. If G is a countable universal T-free graph we can find embeddings $f : \Gamma \to G$ and $f' : \Gamma' \to G$ with $\Gamma = \Gamma^{\epsilon}$, $\Gamma' = \Gamma^{\epsilon'}$ nonisomorphic and with $f(P)$ meeting $f'(P')$, where P' is the copy of P associated with Γ'.

We look at the graph G_0 induced by G on its vertices of finite degree. This contains $f(P)$ and $f'(P')$. Furthermore the connected component of their intersection coincides with both, as an induced subgraph. So the path $f(P) = f'(P')$ is an induced subgraph of G. Call this path P^G.

Now we consider the following relation $R(a, b)$ in G:

a, b are vertices of P^G with a common neighbor in G which is not in P_G.

Writing $a = f(a_0)$ and $b = f(b_0)$ with $a_0, b_0 \in P$, choose i and j to minimize $d(a_0, P_i)$ and $d(b_0, P_j)$ respectively. By our lemma, if $R(a, b)$ holds in G then $i = j$ (and in particular i, j are uniquely determined).

Now consider the equivalence relation generated by the relation R on P^G. If A is an equivalence class with representative $a = f(a_0)$, and i is chosen to minimize $d(a_0, P_i)$, then A is contained in the image of the set $\{v\colon\ d(v, P_i) \leqslant \ell/2\}$. Furthermore A either contains the image of P_i or is disjoint from it. If we consider equivalence classes of order at least $3\ell - 3$, these will contain the corresponding set $f(P_i)$ as otherwise the size of A would be bounded by $\ell < 3\ell - 3$. So we can now identify the equivalence classes containing the $f(P_i)$. These must also be the equivalence classes containing the $f'(P'_i)$ and thus one finds $|p_i - p'_i| \leqslant \ell$, up to a shift or and possible reflection of indices. By restricting the allowed sequences (p_i) somewhat one may ensure that this forces $p_i = p'_i$ for all i, and thus a contradiction. $\square$

5.3. *Case IC: $\ell = 2$*

Case IC. T has a vertex v_0 of maximal degree $d \geqslant 3$ such that some v_0-component C of T contains a unique vertex v_1 of degree d, and $\ell = d(v_0, v_1) = 2$. Either T' is a path, or else T' is a near-path whose center is not v_1.

Construction 4. Let H^0 be $T \setminus C$ and let H^1 be the amalgam of two copies of H^0 over the set $V_1(H^0)$ of vertices in H^0 corresponding to vertices of degree d in T. Extend H^1 freely so as to raise the degree of vertices of degree at least d to ∞, and to raise all vertex degrees to at least $d-1$. Call the result H.

Let P be either a $(d-1)$-regular graph of large girth, if $d>3$, or else a two-way infinite path, if $d=3$, and suppose that there is a family of paths P_i contained in P, of order p_i, satisfying:

(1) $p_i = 2$ or 3 all i;
(2) Every vertex of P is either on some P_i or adjacent to one of its vertices;
(3) No two vertices on distinct paths P_i, P_j are adjacent.

Attach to each interval P_i a vertex b_i adjacent to its vertices, and attach a copy H_i of H to b_i with b_i playing the role of v_0.

Call the resulting graph Γ.

Remark 5.7. One has uncountably many possibilities for the structure of P if $d \geqslant 4$, and for the sequence p_i if $d=3$.

Let us verify this in case $d \geqslant 4$. In this case first choose a path P^* and intervals P_i on P meeting our conditions. Then add edges to P^* whose endpoints lie outside all the P_i, keeping the girth high, and raising the vertex degrees to $d-1$.

Lemma 5.8. *Γ is T-free.*

Proof. This is the usual metric argument. We will sketch the main points.

We begin by considering a maximal geodesic path $A=(a_0,\dots,a_n)$ embedding into $V_1 = V_1(T)$ with successive distances at most ℓ, and suitably maximized. We have $n \geqslant 1$ by the case assumption.

We claim that this path is isolated relative to Γ in our usual sense.

Note that as $p_i \leqslant 3$ and the vertices of P_i have degree d, with a common neighbor b_i, no embedding of T into Γ can carry two vertices of degree d into the same interval P_i. So as in the proof of Lemma 5.4 it follows that the path A is isolated, and after that the argument is relatively formal. One considers a maximal isolated subspace B of $V_1(T)$ which may be supposed proper, and one finds that there must be some embedding in which some b_i is in the image of an isometric copy of B, after which one can attach another such geodesic path to B and arrive at a contradiction; the gap between distinct intervals P_i becomes relevant again when the geodesic path argument is repeated at the end. □

Lemma 5.9. *Let Γ be embedded in the T-free graph G. Then any vertex $u \in P$ is of finite degree in G, and its neighbors in G of finite degree are exactly its neighbors in P.*

Proof. Suppose first that (u,v) is an edge of G and v does not occur in Γ. Then u will play the role of v_1 in T, with v an adjacent leaf. One easily finds a path of length 2 connecting u to some b_i, which will play the role of v_0, and one extends this to an embedding of T into Γ, with H_i absorbing $T \setminus C$ while the component C itself, apart from one leaf attached to v_1, embeds into P.

The same construction applies whenever the vertex v is not needed to complete the embedding of T into Γ, and in particular only finitely many vertices v require attention, so the vertices of P certainly continue to have finite degree.

We now need to consider only the case in which v is in Γ and has finite degree (in G, and in particular in Γ). If $v \in H^1$ then $v \notin V_1(H^0)$ and hence by the amalgamation process used to construct H^1 such a choice of v cannot block anything. It is also possible that v lies in H_i but off H^1, but this is essentially the same situation; indeed, we could have extended H^0 before amalgamating, and then H would be just the result of the final amalgam!

So all that really concerns us is the possibility that v is on P. But as we have seen previously, we can delete most of the path from u to v along P, just retaining the neighbor v' of u along that path, and then v' acts as a "new" vertex with respect to the revised version of P. □

Proposition 5.10. *In Case* IC*, there is no weakly universal T-free graph.*

Proof. We need to show that we can recover information, either about the sequence (p_i) or the structure of P (if $d \geqslant 4$), from an embedding of Γ into a larger T-free graph G, given the image of a vertex in P. For any $u \in P$, the neighbors of u of finite degree in G are its neighbors in P, and thus at least the graph P can be recovered from G. If $d \geqslant 4$ there is sufficient flexibility in the structure of P to complete the argument.

Suppose therefore that $d = 3$, and the induced structure on P is an ordinary two-way infinite path. We must decode some information about the numbers p_i. Consider the graph P^* on the path P whose edges are the edges of P whose endpoints have a common neighbor in G. Then vertices adjacent in Γ to distinct intervals P_i, P_j are not adjacent in P^*, and each nontrivial connected component of P^* consists of an interval P_i with possibly one or both of its neighbors on P adjoined.

If one looks at a long interval L in P, one can use P^* to count accurately the number of intervals P_i which meet L, and the number of vertices involved, and find the average value of p_i over the interval. This is sufficient to discriminate between substantially different parameter sequences, taken to be constant over long intervals. □

6. Case II: $\ell = 1$, $d \geqslant 4$

In this case we have an external vertex of maximal degree adjacent to another vertex of maximal degree.

6.1. $\ell = 1$, $d \geqslant 5$

Case IIA. T has a vertex v_0 of maximal degree $d \geqslant 5$ such that some v_0-component C of T contains a unique vertex v_1 of degree d, and v_0, v_1 are adjacent. Either T' is a path, or T' is a near-path whose center does not lie in the v_0-component C.

This case is essentially the same as the illustrative example treated in Proposition 4.1.

Construction 5. Let $H^0 = T \setminus C$ and let H^1 be the amalgam of two copies of H^0 with over its vertices of degree d in T. Adjoin vertices adjacent to the vertices of degree at least d in H^1 in order to make their degrees infinite. This yields an attachment graph H.

Take a two-way infinite path P and attach a copy H_i of H to a neighbor b_i of each vertex $a_i \in P$, with b_i playing the role of v_0.

Finally, bring up the vertex degrees along P to exactly $d-1$ (initially these are of degree 3, and $d>4$). Do this by adding additional edges to P, but keep the girth of the graph induced on P very large.

The result is called Γ.

Lemma 6.1. *Γ is T-free.*

Proof. As there are no vertices of degree d on P, the notions of indecomposability and isolation coincide in this case.

Take a maximal indecomposable subspace A of V_1. If $A \neq V_1$, take a pair A', v with A' isometric to A and contained in V_1, $v \in V_1 \setminus A'$, and $d(v, A')$ minimized. Let $A_1 = A' \cup \{v\}$ and as A_1 is decomposable take an isometric copy $A'_1 = A'' \cup \{v'\}$ of A_1 in V_1 and an embedding $f: T \to \Gamma$ which witnesses this. Then $f[A'']$ will be contained in some H_i and $f(v')$ will lie in a different H_j, and farther than b_i. By the minimality of $d(v', A'')$ we have b_i in the image of A''. Pulling this back into T, we have an isometric copy A^* of A in $T \setminus C$ containing v_0. So $A^* \cup \{v_1\}$ is also a subspace of V_1, and as A^* is indecomposable and v_1 is adjacent to a vertex of A^*, $A^* \cup \{v_1\}$ is also indecomposable. This however contradicts the maximality of A.

Thus $A = V_1$ is indecomposable. However $|V_1(T)| = |V_1(H)| + 1$ and thus $V_1(T)$ cannot embed in a copy of H. So there are no such embeddings, and Γ is T-free. □

Lemma 6.2. *Let G be a T-free graph containing Γ, and $u \in P$. Then any neighbor v of u of finite degree in G is on P, and is a neighbor of u in Γ.*

Proof. Once the vertex u acquires degree d, we extend to the neighboring copy of H_i and a path along P, together with suitable neighbors (all distinct by our restriction on the girth).

If v lies in the copy H_i of H associated with u, and has finite degree, then it is a vertex duplicated in the construction of H^1 (or one of the additional neighboring vertices added at the end, which present no problems). Such a vertex cannot block the embedding of T.

There remains the possibility that the vertex v lies on P and is not a neighbor of u in P. Then as in the proof of Proposition 4.1 we use the neighbor of u on the path toward v to represent a leaf adjacent to v_1, and use the additional edge (u, v) to replace P by a similar $(d-1)$-regular graph of large girth. □

Proposition 6.3. *In Case* IIA *there is no weakly universal T-free graph.*

Proof. Given a T-free graph extending Γ and the image of a point in P we recover the set P and the graph induced on it by G, which is the same as the graph induced on P by Γ. As $d>4$ there is some latitude in the structure of this graph (in particular, in the lengths of circuits in the graph) and thus we can recover uncountably many different invariants. □

6.2. $\ell = 1$, $d = 4$

Case IIB. T has a vertex v_0 of maximal degree $d = 4$ such that some v_0-component C of T contains a unique vertex v_1 of degree d, and v_0, v_1 are adjacent. Either T' is a path, or T' is a near-path whose center does not lie in the v_0-component C.

Construction 6. We vary the preceding construction. With the same attachment graph H, we take an infinite path P and divide it into consecutive intervals P_i of length $p_i = 1$ or 2, where furthermore $p_i = 1$ with rare, and widely spaced, exceptions.

The vertex b_i is attached to the interval P_i and the graph H_i is appended to it. There is no final decoration phase since $d = 4$ and all vertices of P have degree $d - 1 = 3$ already. The only variability is in the sequence (p_i).

This is T-free as before; any sequence (p_i) with $p_i = 1$ or 2 would be suitable at this stage, as the duplication of the neighbor of b_i on P has no substantial effect.

Lemma 6.4. *If G is a T-free graph containing Γ and $u \in P$, then u has finite degree in G, and every neighbor v of u of finite degree in G is on P.*

Proof. The "widely spaced" condition on the p_i is used here.

The argument goes as before except that one should pay some attention to vertices u lying on or near an interval P_i of length 2, as the neighbor u' of u in P_i may be unsuitable for our purposes, having degree $d - 1$ but sharing a neighbor with u. As the v_0-component C may continue on past v_1, and the next vertex after v_1 may have degree $d - 1$, this constrains us to working on P on one definite side of u.

If the vertex v lies on P on this preferred side of u, but beyond the neighbor of u according to Γ, it potentially blocks the embedding of T. But then we can use the neighbor of u on that side to represent a leaf of T and take a path through u, v and along P to complete the construction. □

Proposition 6.5. *In Case* IIB *there is no weakly universal T-free graph.*

Proof. The path P is recoverable and has no extra structure. Adjacent points belonging to distinct intervals P_i can have no common neighbors, as there are then two attachment graphs H_i available and the common neighbor could lie in at most one of them. Therefore the intervals P_i and the numbers p_i are also visible in any T-free graph containing $\Gamma = \Gamma^\epsilon$, and the usual argument applies. □

7. Case III: $\ell = 1, d = 3$

Since we now deal with the case $d = 3$ all branch vertices of T have maximal degree. These cases require a slightly finer consideration of the structure of T, bringing in the location of a third branch vertex, assuming there is one.

7.1. A special case

Case IIIA. T contains exactly two branch vertices v_0 and v_1, which are adjacent, and of degree 3.

Construction 7. Take a two-way infinite path P and divide it into intervals P_i, Q_i which are alternately of length $p_i = 3$ and $q_i = 1$ or 2. Adjoin a common neighbor b_i to each interval P_i. Call the result Γ.

Proposition 7.1. *In Case* IIIA*, there is no weakly universal T-free graph.*

Proof. The graph Γ is T-free, and under any embedding into a larger T-free graph G, Γ will be a connected component of G. Hence no countable T-free graph contains all possible variants of Γ. $\square$

This is important, because we need to move somewhat further away from near-paths before we can make suitable constructions of any generality.

7.2. *Three adjacent branch vertices*

Here is one case which is sufficiently far from the near-path case to be handled uniformly.

Case IIIB. The maximal vertex degree is 3. T contains a sequence of three adjacent branch vertices v_1, v_0, v_2 with v_1 external and adjacent to a leaf. Some v_0-component of T is a path attached to v_0.

Construction 8. Let the v_0-components of T be C, C_1, C_2, where $v_i \in C_i$ for $i = 1, 2$. By hypothesis C is a path. Let H be the graph obtained from C_2 by freely amalgamating infinitely many copies of C_2 over the subset $V_1(C_2)$ consisting of its branch vertices in T.

Take an infinite path P partitioned into intervals P_i, Q_i of lengths $p_i = 2$ or 3 and $q_i = 1$ and adjoin a vertex b_i adjacent to the vertices of P_i. Attach a copy H_i of H to b_i with b_i playing the role of v_2. This yields Γ.

Lemma 7.2. *Γ is T-free.*

Proof. Let P^* be a path of maximal length consisting of adjacent branch vertices of T. We claim that P^* is isolated with respect to embeddings of T into Γ. Certainly P^* is indecomposable, and for any of embedding of T into Γ which takes a vertex a of P^* into P_i, as these vertices have degree 3 and a common neighbor, only one of them can be in $f[P^*]$. Hence a must be an endpoint of P^* and its neighbor b in P^* must correspond to b_i. In particular if P_0 is the path with a deleted, then $f[P_0]$ is a path in H_i terminating at b_i, which corresponds to a path of adjacent branch vertices in C_2 terminating at v_2. Such a path can be extended by v_0, v_1 and contradicts the maximality of the length of P^*. So P^* is isolated.

Now let A be a maximal isolated subspace of $V_1 = V_1(T)$. Assuming that there is in fact some embedding of T into Γ, then $A \neq V_1$, and then on formal grounds as we have seen in earlier arguments, there is an embedding of T into Γ which carries an isometric copy of A, which we will continue to call A, into some H_i with b_i included in the image. But then looking at this inside T it gives an isometric copy of A, say A', containing v_2 but not v_0. So consider the longest path $\tilde{P}$ consisting of adjacent vertices which can be attached to the metric space A at the corresponding vertex v, subject to the restriction that the extended space $A \oplus_v \tilde{P}$ with its natural metric embed into V_1. This is visibly indecomposable and easily seen to be isolated by the same sort of analysis with which we began. This then contradicts the maximality of A and completes the analysis. $\square$

Lemma 7.3. *For any T-free graph G containing Γ, and any vertex $u \in P$, the degree of u is finite in G, and the neighbors of u of finite degree in G and in Γ coincide. If $u \in Q_i$ for some i then its neighbors in G are on P.*

Proof. Let us first see how to embed T in G if u has a neighbor v not in Γ. We take u' adjacent to u and lying in one of the P_i. We use the sequence b_i, u', u to represent the sequence v_2, v_0, v_1. The graph H_i disposes of any need to think about the component C_2. There is room for the path C on the far side of u' along P. The vertex v represents a leaf adjacent to v_1, and the rest of T consists of a path attached to v_1, which can lie along P.

From this it follows that the vertices of P have finite degree in G. Now suppose v is a neighbor of u of finite degree in G, and in particular v is a vertex of Γ. As usual if $v \in H_i$ then this does nothing. So we may suppose $v \in P$, and v is nonadjacent to u.

If v and u' lie on opposite sides of u then we use the neighbor of u on the side of v to represent a leaf adjacent to v_1, and use the continuation of the path (u, v) along P to complete the embedding with no further interference.

If u is on Q_i for some i then there are two choices for u' so we can fall directly into the previous case, and the analysis applies to any neighbor v of u in this case.

So we may suppose u is on P_i. If v and u' are on the same side of u, the only obstruction arises if v is adjacent to u'. If $v \in Q_{i\pm1}$ we can interchange u and v, so there remains only the case in which v, u', u are the three points of some P_i. In this case we may take b_i, v, u to represent the sequence v_2, v_0, v_1 and use u' as a neighbor of u. □

Proposition 7.4. *In Case* IIIB *there is no weakly universal T-free graph.*

Proof. It follows at once from the preceding lemma that in any T-free graph G containing one of our graphs Γ, we can recover P as well as enough information about the neighbors of P to determine the sequence (p_i) up to reflection and translation from an element of P. So the customary argument applies. □

Under the assumption that T' is a path we have seen that we may suppose that external branch vertices are adjacent to branch vertices. With the last two cases out of the way there must be at least four branch vertices, with each outer pair adjacent.

7.3. Two adjacent branch vertices, $\ell \geqslant 3$

Case IIIC. The maximal vertex degree is 3. T contains a sequence of three successive branch vertices v_0, v_1, v_1' with v_1' external and adjacent to a leaf, v_1 adjacent to v_1', and $\ell = d(v_0, v_1) \geqslant 3$, where v_0 is the closest branch vertex to v_1 other than v_1'. Either the pruned tree T' is a path, or a near-path with the center not in the v_0-component containing v_1. All external branch vertices of T are adjacent to branch vertices.

Construction 9. Let v_2 be the vertex lying between v_1 and v_0 at distance 2 from v_1. Let C be the v_2-component of T containing v_1. Let H be the result of amalgamating infinitely many copies of $T \setminus C$ over the set consisting of its branch vertices together with the path from v_2 to v_0, extended to give each vertex strictly between v_2 and v_0 infinite degree.

Take a path P broken into intervals P_i, Q_i of lengths $p_i = 3$ and $q_i = 1$ or 2. Adjoin a vertex b_i adjacent to the vertices of P_i, adjoin a vertex c adjacent to all b_i, and attach H to c with c playing the role of v_2. Call the result Γ.

Lemma 7.5. *Γ is T-free.*

Proof. We look first at how an adjacent pair of branch vertices can be embedded into Γ. Consider a subtree of Γ consisting of two adjacent branch vertices v, v' of degree three with their neighbors.

Suppose $v \in P$. Then $v \in P_i$ for some i and by inspection $v' = b_i$, with c belonging to the subtree as one of the neighbors of v'.

Now consider an embedding of T into Γ and more particularly the images of a pair of branch vertices consisting of an external branch vertex and its neighboring branch vertex. If all such images miss P, then the diameter of the convex hull in T of the branch vertices is larger than the diameter of the graph into which they can embed.

Similarly if one of these pairs of adjacent branch vertices maps to vb_i with $v \in P_i$ and the other pair maps into H, the diameter is still slightly too small as the distance from v_0 (in H) to b_i is $\ell - 1$.

Finally if both pairs of adjacent branch vertices correspond to pairs of the form vb_i and $v'b_j$ then c occurs in the embedding as a neighbor of both b_i and b_j and the whole tree has only four branch vertices, with the interior pair lying at distance $\ell = 2$, which contradicts our case hypothesis. □

Lemma 7.6. *For any T-free graph G containing Γ, and $u \in P$, the neighbors of u of finite degree in G are its neighbors in P, and possibly a vertex b_i if u is either in P_i, or else in Q_i or Q_{i-1} and adjacent to a vertex of P_i, with $q_i = 2$ or $q_{i-1} = 2$ respectively.*

Proof. Bearing in mind that any pair of vertices on P with one in P_i and the other adjacent to it are candidates for the role of v_1 and v_1' respectively in an embedding of T, we see first that we cannot adjoin any new vertices as neighbors of u, secondly that the vertices in H which are of finite degree are not available to serve as neighbors as they were duplicated in the amalgamation process, and thirdly that for $q_i = 1$ as there are two vertices adjacent to u and lying in P_i or P_{i+1} respectively, there can be no new neighbors of u in that case. Of course if $u \in Q_i$ or Q_{i-1} with $q_i = 2$ our statement allows for b_i as a new neighbor and the only other vertex which would be a plausible candidate for a new neighbor of u would be the next vertex beyond the immediate neighbor of u, but in that case this new neighbor of u in P_i could serve as an alternate candidate to play the role of v_1.

After all this there remains the possibility that $u \in P_i$ and that u has a new neighbor along P, not already adjacent to it in Γ—notably, u and this new neighbor could be endpoints of P_i. However here one uses the new neighbor of u in the role of v_1' and the old neighbor of u in P_i represents a leaf in the embedding. □

Proposition 7.7. *In Case* IIIC *there is no weakly universal T-free graph.*

Proof. We have freedom in the choice of the q_i and so we need only check that we have possibilities for decoding.

Given an embedding of Γ into a T-free graph G we look at the graph G_0 induced on vertices of finite degree in G and then we look at the graph G_1 induced on vertices of degree at most three in G_0.

The vertices of P occur on a path in G_1. Their neighbors in G_1 consist of P and those vertices b_i which are of finite degree in G and have no neighbors in G of finite degree other than those in P_i. In particular the connected component of G_1 containing P is a subgraph of Γ. One cannot necessarily recover the path P itself since the midpoint of P_i and b_i have similar properties, but we claim that we may recover the sequence q_i, which is sufficient.

The 2-connected blocks of G_1 consist of certain edges of P together with the induced subgraph on $P_i \cup \{b_i\}$ whenever $b_i \in V(G_1)$. In the latter case the endpoints of P_i can be recovered from the 2-block. Let Q be the subgraph of P obtained by deleting those vertices occurring as the midpoint of an interval P_i for which $b_i \in V(G_1)$. Then the graph Q can be recovered from G_1, with those pairs consisting of endpoints of some interval P_i distinguished. From this one can recover the sequence (p_i) (up to shift and reflection). □

7.4. *Two adjacent branch vertices, $\ell = 3$*

Case IIID. The maximal vertex degree is 3. T contains a sequence of three successive branch vertices v_0, v_1, v_1' with v_1' external and adjacent to a leaf, v_1 adjacent to v_1', and $\ell = d(v_0, v_1) = 2$, where v_0 is the closest branch vertex to v_1 other than v_1'. Either the pruned tree T' is a path, and both external branch vertices of T are adjacent to branch vertices, or it is a near-path with the center not in the v_0-component containing v_1.

In most cases the construction for the previous case will work. With $\ell = 2$, the vertex c in the previous construction becomes identified with v_0.

But in the proof that Γ is T-free, we may encounter an exception precisely in the case when the external vertices correspond to vertices of P_i, P_j for some i, j and their neighbors correspond to b_i, b_j. In this case $c = v_0$ occurs as a common neighbor of both and the structure of the tree is determined: it has exactly four branch vertices separated by one vertex of degree 2. The only uncertain point concerns the precise lengths of the paths emanating from the external path vertices.

Since the previous construction definitely fails in this case we use a variant.

Construction 10. Suppose T is as described (four branch vertices, adjacent in pairs, separated by one vertex of degree 2). Take a two-way infinite path P divided into intervals P_i, Q_i of lengths $p_i = 2$ and $q_i = 1$ respectively. Attach vertices b_i adjacent to the vertices of P_i.

Attach a path (b_i, c_i, c_i') to each b_i and attach an infinite family of infinite rays to each of the vertices c_i, c_i' (really what interests us is to have two infinite rays at c_i' and to ensure that the vertices c_i, c_i' have infinite degree).

Now take a maximal subset S of $\mathbb{Z}$ containing no adjacent pairs in $\mathbb{Z}$; in other words, if S is arranged as a sequence $(n_i\colon i \in \mathbb{Z})$ then $n_{i+1} - n_i$ is 2 or 3 for all i. Note that there are many such sets. Give each vertex b_i $(i \in S)$ infinite degree.

Call the result Γ.

Lemma 7.8. *Γ is T-free.*

Proof. Consider subtrees of Γ consisting of two adjacent branch vertices and their neighbors. Such a tree either has its branch vertices off P entirely, or has branch vertices of the form v, b_i with $v \in P_i$ and $i \in S$.

As the distance (in $\mathbb{Z}$) between distinct elements of S is greater than 1, the distance between such pairs in Γ is greater than 2, unless they lie in the part of Γ attached to a single P_i. But here the diameter is too small. □

Lemma 7.9. *For any T-free graph G containing Γ, and $u \in P$, the neighbors of u in G of finite degree are its neighbors in P together with b_i if $u \in P_i$ and $i \notin S$.*

Proof. Observe that b_i has finite degree if $i \notin S$ since for some adjacent $j = i \pm 1$ we have $j \in S$ and it follows easily that giving b_i infinite degree produces an embedding of T.

The rest is clear by inspection; adjoining a new vertex as a neighbor of u produces an embedding of T into Γ directly, and as usual there are no serious candidates of finite degree in Γ. □

Lemma 7.10. *For any T-free graph G containing Γ, and $u = b_i$, with $i \notin S$, the neighbors of u in G of finite degree are its neighbors in P.*

Proof. Taking $j = i \pm 1$ in S, so that b_j has infinite degree, then as noted in the previous argument adjoining a new neighbor of b_i would give an embedding of T directly involving b_i and b_j.

Therefore the only new neighbors which b_i might acquire are those lying along the image of this embedding. Now the embedding passes in part along rays which have been duplicated and since these rays have been duplicated none of their elements can be a neighbor of b_i. The remaining elements in the image of the embedding either have infinite degree or lie on P and therefore are either irrelevant or excluded already in the previous lemma. □

Proposition 7.11. *In Case* IIID *there is no weakly universal T-free graph.*

Proof. The path P with its neighbors b_i ($i \in S$) can be recovered from any T-free graph containing Γ together with one vertex of P, so the set S can be determined up to a shift (or exactly if two specific vertices of P are fixed).

As usual there are uncountably many possibilities for Γ and only countably many realized in any particular countable graph. □

8. The Tree Conjecture

8.1. Taking stock

We review the analysis from the very beginning. A minimal counterexample T to the Tree Conjecture will have the following properties:

(1) T is neither a path nor a near-path.
(2) The pruned tree T' is a path or a near-path.

Let d be the maximal vertex degree. Then $d \geqslant 3$. If there is a unique vertex of degree d then Theorem 2 applies. So we suppose the contrary.

(3) There are at least two vertices in T of degree d.

Suppose first

A $\quad$ T' is a path.

Let v_0, v_1 be a pair of vertices of degree d with v_1 external and with v_0 the closest vertex of degree d to v_1. Let $\ell = d(v_0, v_1)$.

If $\ell \geqslant 2$ then one of Cases IA–C applies. Hence we suppose

A1. Any external vertex of maximal degree is adjacent to a vertex of maximal degree.

Then if $d \geqslant 4$ one of Cases IIA–B applies. So we suppose

A2. $d = 3$.

Then the four Cases IIIA to IIID cover the remaining possibilities, Case IIIA when there are exactly two branch vertices and one of Cases IIIB–D otherwise.

Now suppose

B T' is a near-path with center v_*.

One can adapt the foregoing to this case, more or less, by treating v_* as if it has degree d. But let us first isolate the cases not already explicitly covered by our constructions.

First, suppose

B_1 There is an external vertex of degree d which is not adjacent to v_*.

Let C be a v_*-component containing a vertex of degree d not adjacent to v_*, and let $\hat{C}$ be the induced subgraph of T on C with the vertex v_* adjoined.

If there are two nonadjacent vertices of degree d in $\hat{C}$ then one of the foregoing cases applies. Otherwise, considering the vertices of degree d in $\hat{C}$ together with v_* we have one of the following possibilities:

$B_1.1$ There is a unique vertex v_1 of degree d in $\hat{C}$, at distance $\ell \geqslant 2$ from v_*.
$B_1.2$ There are two adjacent vertices v_1, v_1' of degree d in $\hat{C}$.

In either case, if v_* has degree d, with $d \geqslant 4$, we will fall into one of our cases with $v_0 = v_*$. We need to adapt our constructions when v_* does not have degree d.

The effect of this is that in each case we need to reexamine the proof that the resulting graph is T-free.

We will discuss these cases further below.

We have also the following possibility to consider.

B_2 T' is a near-path with center v_*. Every vertex of degree d other than v_* is adjacent to v_*.

Here v_* may or may not have degree d itself.

This case escapes from those treated earlier and must be handled separately.

8.2. *The case $d = 3$*

It will be convenient to clear away the case in which the maximal degree d is 3. In particular in this case the center v_* has degree d.

If some external branch vertex has distance at least 2 from the closest branch vertex in T then Case I or II applies. So we may suppose that every external branch vertex is adjacent to a branch vertex.

If some v_*-component C of T contains at least two branch vertices then taking v_1' the external branch vertex of C, v_1 the neighboring branch vertex, and v_0 the next branch vertex in T, that is either the next one in C or v_* itself, we fall into Cases IIIB, IIIC, or IIID.

So we may suppose that

$(*)$ All branch vertices other than v_* are adjacent to v_*.

Case IVA. T' is a near-path, $d = 3$, and there are exactly two branch vertices in T, namely the center v_*, and an adjacent vertex v_1.

This case is highly reminiscent of Case IIIA, and we may adapt that construction as follows.

Construction 11. Take a two-way infinite path P and divide it into intervals P_i, Q_i which are alternately of lengths $p_i = 4$ and $q_i = 1$ or 2. Adjoin a common neighbor b_i to each interval P_i. Call the result Γ.

Lemma 8.1. *Γ is T-free.*

Proof. Consider any subgraph of Γ containing two adjacent branch vertices. These must be of the form u, b_i with $u \in P_i$. Suppose that this graph is part of a subgraph of Γ isomorphic with T. Then either u or b_i corresponds to v_* and hence is an endpoint of three disjoint paths of length 2. The vertex b_i as well as any endpoint of P_i has this property, but the three paths involved must completely cover P_i and b_i. So if the other vertex is to represent a branch point of the subgraph, it cannot be b_i. Thus b_i must play the role of v_* and then the three paths must embed in such a way as to cover the neighbors of both endpoints of P_i as well and again u cannot be a branch vertex of the image. $\square$

Lemma 8.2. *If $\Gamma \subseteq G$ and G is T-free, then the graph induced on $V(\Gamma)$ by G is a connected component of G, and consists of Γ with any additional edges (u, v) involving vertices $u, v \in P$ at distance* 2*, of the following types*:

(1) *For some $q_i = 2$, if $u \in Q_i$ is adjacent to $u' \in P_{i\pm1}$ then possibly (u, v) is an edge, with v the neighbor of u' in $P_{i\pm1}$.*
(2) *Two vertices of some P_i at distance* 2 (*one an endpoint, one an interior point*) *may be adjacent.*

Proof. First we exclude edges between Γ and $G \setminus \Gamma$. If (u, v) is an edge with $u \in \Gamma$ and $v \notin \Gamma$ we look for an embedding of T into Γ in which u plays the role of v_1, and we need to select an appropriate vertex to play the role of v_*.

If u is adjacent to an endpoint u' of P_i then u' can play the role of v_*, though there are two cases to be distinguished here: $u = b_i$ or $u \in P$. We leave further inspection of this case to the reader.

Now suppose u is not adjacent to an endpoint of P_i. Then u is an endpoint of P_i itself, and b_i can play the role of v_*.

So the graph G_0 induced on the vertices of Γ by G is a connected component of G and the question remains as to its precise structure.

Suppose first that (b_i, v) is an edge in G but not in Γ. Then easily $v \in P$ and since $v \notin P_i$ we can take v to play the role of v_* and b_i to play the role of v_1, taking as leaf adjacent to b_i an interior vertex of P_i lying on the side away from v.

So the only edges that come into consideration are edges (u, v) joining vertices of P. If $d(u, v) \geqslant 3$ and $u \in P_i$ then we let u play the role of v_1 with b_i representing a leaf adjacent to u, and v may play the role of v_*.

If $d(u, v) \geqslant 3$ and $u \in Q_i$ with u adjacent to $u' \in P_{i\pm1}$ then we let u play the role of v_* with u', b_i representing a path attached to u and with v in the role of v_1.

So we have

$$u, v \in P; \quad d(u, v) = 2.$$

Now if $u \in Q_i$ and $q_i = 1$ then easily as u is adjacent to endpoints in P_{i-1} and P_{i+1}, there can be no new neighbors of u in P.

If $u \in Q_i$ with $q_i = 2$ is adjacent to u' in P_{i+1} then there can be no edge (u, v) with v the endpoint of P_i closest to u, as then v could play the role of v_1 and b_i could play the role of v_*.

So for $u \in Q_i$ we have only the case mentioned in the statement of the lemma. □

Proposition 8.3. *In Case* IVA*, there is no weakly universal T-free graph.*

Proof. One needs to decode some information from an embedding of Γ into a T-free graph G. This can be simplified by taking $q_i = 1$ over large intervals of fixed size, with occasional values of $q_i = 2$ inserted optionally at regular intervals.

Let G_0 be the graph induced on $V(\Gamma)$ by G. Viewing G_0 as a collection of 2-connected blocks which are connected in a tree structure, we see that the 2-connected blocks have approximately the same vertices as they do in Γ, with some possible variation involving Q_i when $q_i = 2$. The vertices with four neighbors in their 2-block are the b_i and possibly some interior vertices of P_i.

Most of the nontrivial 2-connected blocks have order 5, and their points of attachment are the endpoints of the intervals P_i. The exceptions may occur when $q_i = 2$ and these occurrences will be signalled either by the presence of a 2-connected block of order 6, or by two successive 2-connected blocks with a gap of size 2. From this rudimentary analysis one cannot recover the exact placement of the exceptional values of q_i, but one can localize it with an error of ± 1, which is good enough. □

The next case to consider would have T' a near-path, exactly three branch vertices consisting of the center v_* and two neighbors of v_*, but Case IIIB covers this one.

So in fact there is just one more case with $d = 3$.

Case IVB. T' is a near-path, $d = 3$, and there are exactly four branch vertices in T, namely the center v_*, and three adjacent vertices v_1, v_2, v_3, in distinct v_*-components. We may suppose that v_3 is adjacent to two leaves (and the same may possibly apply to one or both of v_1, v_2).

Construction 12. Begin with a two-way infinite path P divided into intervals P_i, Q_i of lengths $p_i = 6$ and $q_i = 1$ or 0. Attach a vertex b_i adjacent to the vertices of P_i and if Q_i contains a vertex, give it infinite degree. Call the result Γ.

Lemma 8.4. *Γ is T-free.*

Proof. Consider the subgraph Γ_0 of Γ with the same vertices, and with edges between any pair of vertices u, v of Γ which lie in a subgraph of Γ for which u, v are adjacent branch vertices of degree 3 with no common neighbor. Then all the edges containing b_i in Γ are retained, but the only edges along P which are retained are the ones involving a vertex of some Q_i, and the vertices of Q_i have degree 2 in Γ_0.

Under an embedding of T into Γ, v_* must correspond to a vertex of degree at least 3 in Γ_0, thus a vertex b_i for some i. But there is no such embedding as one of the three branch vertices neighboring v_* must correspond to an interior point of P_i, so that together with its neighbors this v_*-component requires at least 3 vertices of P_i, and each of the others requires at least 2 vertices of P_i. $\square$

Lemma 8.5. *If Γ is contained in the T-free graph G, and if G_0, Γ_0 are the subgraphs of G induced on the branch vertices of G and of Γ respectively, then Γ_0 is a connected component of G_0.*

Proof. The vertices of Γ_0 are the vertices of P together with the vertices b_i.

We claim first

(1) There is no edge (u, v) in G with $u \in P_i$, $v \notin \Gamma_0$.

We may suppose that u is the first, second, or third vertex of P_i. If u is the first or third vertex then we embed T into Γ with b_i playing the role of v_* and with the extra vertex v playing the role of a leaf attached to one of its neighbors. If u is the second vertex of P_i we let the first vertex of P_i play the role of v_*. So (1) holds.

Now we claim

(2) There is no edge (u, v) in G with $u \in Q_i$, $v \notin \Gamma_0$, and v a branch vertex of G.

Let u, u', u'' be three neighbors of v in G. We attempt to embed T into G with u playing the role of v_*. This can be blocked if u' or u'' lies on P or coincides with b_i or b_{i+1}. Suppose therefore that u' is of one of these two forms.

If $u' = b_{i+1}$ then we may let b_{i+1} play the role of v_* with u' as one of its neighboring branch vertices. This could only be blocked by having $u'' \in P$ in which case $u'' \in Q_j$ for some j. In this case u'' could play the role of v_* instead. Similarly the case $u' = b_i$ may be excluded.

So we may suppose that the vertices u', u'' which lie in Γ lie in P, and in each such case in some Q_j. Furthermore we may suppose that among those vertices of u, u', u'' lying on P, u is the first in order. Then we use u to represent v_*, getting a contradiction. So (2) holds.

So our lemma is proved as far as edges involving vertices of P are concerned. Suppose finally that (u, v) is an edge with $u = b_i$ for some i and v a branch vertex of G not in Γ_0. Let u', u'' be additional neighbors of v. By the cases already treated, these vertices do not lie on P. It is then easy to embed T into G with b_i playing the role of v_*, arriving at a contradiction. $\square$

Lemma 8.6. *If Γ is contained in the T-free graph G, and Γ_0 is the graph induced on the branch vertices of Γ by G, then any edge (u, v) of Γ_0 which is not an edge of Γ involves two vertices of P, at distance at most 3, and of one of the following two forms*:

(1) *u or v is in Q_i for some i*;
(2) *u and v are in successive intervals P_{i-1}, P_i, with $q_i = 0$, and adjacent to endpoints of these intervals*; $d(u, v) = 3$.

Proof. First, one may eliminate the possibility $u = b_i$, as a new edge of this type leads to an embedding of T into G with b_i playing the role of v_*. So we may suppose $u, v \in P$.

Now suppose $d(u, v) \geqslant 4$ where all distances will be measured in P. Then we let u play the role of v_* and we let the neighbor u' of u along P in the direction of v play the role of a branch vertex adjacent to v_*, whose further neighbors are leaves of T. As $d(u, v) \geqslant 4$ this embedding can be completed to an embedding of T.

If u is an endpoint of P_i, say a left endpoint, and v lies farther to the left along P, embed T into G with b_i representing v_* and with the immediate neighbor of u to its left representing a leaf adjacent to u.

If v lies to the right of P, and at distance at most 3, then let u play the role of v_* with v an adjacent branch vertex. Here b_i will also play the role of an adjacent branch vertex.

In the remaining cases, we may choose notation so that u is adjacent to an endpoint u' of P_i for some i. Leaving aside the cases mentioned in the statement of the lemma, we may suppose $v \in P_i$ as well. We let u' represent v_* and u represents a branch vertex adjacent to u' with a neighbor on P and v as its adjacent leaves. □

Proposition 8.7. *In Case* IVB*, there is no weakly universal T-free graph.*

Proof. If Γ is contained in a T-free graph G then the graph Γ_0 induced on the branch vertices of Γ by G can be recovered from one of its vertices.

We examine the vertices of degree 6 in Γ_0. These include the vertices b_i, and for these vertices the graph induced on its neighbors is connected.

If $q_i = 1$ and u is the unique vertex of Q_i, then u can have a maximum of 6 neighbors in Γ_0, which would then be all of its neighbors in P up to distance 3. In this case the graph induced on the neighbors of u in Γ_0 is disconnected.

If u lies in some interval P_i, then in addition to its 3 neighbors in Γ, there can be at most one more in Γ_0. So these do not come into consideration.

As we may distinguish the b_i by the structure of the graph induced on their neighbors, we can also recognize the path P and the intervals P_i, which remain paths in Γ_0. While Γ_0 may have some additional edges we can then detect the vertices in Q_j for $q_j = 1$, as well as their locations relative to the P_i. □

8.3. The case $d \geqslant 4$

T' is a near-path with center v_*. If the degree of v_* is d, or if one of the v_*-components of T contains two vertices of degree d, then one of the Cases I, II applies.

So we suppose

(1) $\deg(v_*) < d$.
(2) Each v_*-component of T contains at most one vertex of degree d.

As we have disposed of the case in which there is a unique vertex of degree d in T, there are either two or three vertices of degree d.

Case IVC. T' is a near-path, $d \geqslant 4$, and there are exactly two vertices v_0, v_1 of degree d in T, lying in distinct v_*-components.

If $d(v_0, v_1) = 2$ then Case IC applies, so we suppose $\ell = d(v_0, v_1) \geqslant 3$.

Construction 13. Take a $(d-1)$-regular tree P and find disjoint intervals P_i in P of lengths $p_i \geqslant 3\ell - 3$ such that every vertex of P not in one of the P_i lies at distance less than ℓ from at least two of the intervals P_i, but no vertices of distinct P_i, P_j lie at distance less than ℓ of each other.

This is done inductively. At each stage finitely many intervals P_i have been selected, no two at distance less than ℓ, so that the subgraph induced on the vertices within distance $\ell - 1$ of some P_i is connected. Then a vertex at minimal distance ℓ from some P_i is selected and put into a new interval P_i. With some housekeeping one may ensure that the whole tree P is exhausted by this process. For any vertex v of P which is not in the P_i, there is a first stage at which v falls within distance $\ell - 1$ of one of the P_i, and at that stage v lies outside the convex hull of the P_i selected up to that point. Consider the tree T_v rooted at v obtained by deleting the v-component of P containing the convex hull of the P_i. If the distance from v to the nearest P_i (so far chosen) is k, then the vertices of T_v lying at distance ℓ from the closest P_i are those at distance $\ell - k$ from v. As the construction proceeds, and new intervals P_i are selected, one of two things will occur. Possibly the distance from v to the nearest P_i will be diminished at some point, in which case v is close to at least two such intervals. In the contrary case, the distances of the vertices in T_v to the nearest P_i will also be unaltered until one of them, lying at distance ℓ from the nearest P_i and at distance $\ell - k$ from v, is selected as an endpoint of a new interval P_i, at which point v will lie within $\ell - 1$ of at least two such intervals.

Now adjoin a vertex b_i adjacent to the vertices of P_i for each i, and attach to b_i infinitely many $(d-1)$-regular trees—but with its root of degree $(d-2)$, so that after attachment to b_i the degree of the root is also $(d-1)$.

Call the result Γ.

Lemma 8.8. *The graph Γ is T-free.*

Proof. The only vertices of degree d in Γ are the b_i and the vertices in the intervals P_i. As these are widely spaced, the only way to embed T into Γ is to send the two vertices of degree d into $P_i \cup \{b_i\}$ for some i.

The vertices of P_i have degree d and a common neighbor, so at most one of these vertices can serve as the image of a vertex of degree d in T. Therefore the two images must be of the form b_i, u with $u \in P_i$. But then the edge (u, b_i) is not in the image, and u cannot have degree d in the image. □

Lemma 8.9. *If Γ is contained in a T-free graph G, then*

(1) *The vertices of P have finite degree in G, and their neighbors all lie in Γ.*
(2) *For $u \in P$, the neighbors of u of finite degree in G are its neighbors in P.*

Proof. First, if one adjoins an edge (u, v) linking a vertex $u \in P$ to a vertex $v \notin \Gamma$, then T embeds into the extended graph by finding a path of length ℓ along P to some b_i. Note that in view of the structure of T the vertices of degree d have at least one adjacent leaf, so the new vertex v can serve to represent one such leaf, u can represent a vertex of degree d, and b_i can represent the other vertex of degree d. As there are only two vertices of degree d in T, the embedding may be completed.

The same applies if the vertex v lies in one of the trees attached to a b_i. So we may suppose that v lies on P or among the vertices b_i, and not too far from u. So the first point follows.

If we now require v to have finite degree then we are no longer concerned with the b_i, and we may suppose $v \in P$.

Now if v is not adjacent to u in P there are various possibilities. Let us fix a leaf $\tilde{v}$ adjacent to v_0 and an embedding of $T \setminus \tilde{v}$ into Γ taking one vertex of degree d to u and the other to some b_i. We can extend this to an embedding of T into G unless v lies either on the path from u to b_i along P, or in the remaining part of the neighborhood of u in Γ used to embed the other v_0-components of T.

In the second case, we can examine the path from u to v along P and use the neighbor u' of u along this path to represent $\tilde{v}$, and the tree originating with the edge from u through v and continuing along P to replace the u-component of P containing u'.

So suppose that v lies along the path L from u to a neighbor of b_i in P_i, of length $\ell - 1$. One or both of the vertices u, v may lie in the interval P_i. Removing such vertices, P_i is divided into at most three intervals, of total length at least $3\ell - 5$, and hence one of these intervals contains at least $\ell - 1$ vertices. Such an interval may or may not be separated from u by v. If it is not separated, we can make use of it and possibly other vertices of P to find a path of length ℓ from u to b_i avoiding v, and complete the construction. If it is separated, we can make use of the edge (u, v) to find a suitable replacement path, and use the neighbor of u on P in the direction of v as a representative for the leaf $\tilde{v}$. □

Proposition 8.10. *In Case* IVC*, there is no weakly universal T-free graph.*

Proof. We have considerable latitude in the choice of the size p_i of P_i. It suffices to decode the set of p_i involved in the construction after Γ is embedded into a T-free graph G (whereas the "sequence" is not that well defined at this point).

By the preceding lemma, we can recover the graph structure on P from one of its vertices, in G. We would like to recover the intervals P_i by considering vertices of P with a common neighbor in G lying at the root of an infinite system of $(d-1)$-regular trees. By the preceding lemma this common neighbor would have to lie in Γ and it only be some b_i. So we have to deal with the possibility that a vertex u outside the interval P_i might be connected to b_i. But then by our construction, this vertex would lie within $(\ell - 1)$ of some second interval P_j, leading to an embedding of T into G. □

Finally we come to the case of three vertices of maximal degree, with the center v_* of lower degree. At least one of these three vertices must be adjacent to v_*. We can unify the treatment of these cases, but we prefer to first treat the case in which all vertices of degree d are adjacent to v_*, and then discuss the modification of our construction suitable for other cases.

Case IVD. T' is a near-path, $d \geqslant 4$, and there are three vertices v_0, v_1, v_2 of degree d in T, all of which are adjacent to v_*.

Construction 14. Let C_1, C_2 be the v_*-components of T containing v_1, v_2 respectively. Let H be the graph obtained by amalgamating the induced graph on $\{v_*\} \cup C_1 \cup C_2$ with itself over the vertices v_*, v_1, v_2 and then giving v_1, v_2, v_* infinitely many new neighbors.

Take a two-way infinite path P partitioned into intervals P_i of lengths $p_i = 1$ or 2. Adjoin a vertex b_i adjacent to each vertex of P_i, and attach H to b_i with b_i playing the role of v_*.

Raise the degrees of the vertices on P to $d - 1$ by adding additional edges between pair on P, keeping the girth of the induced graph on P extremely large (it will resemble a $(d-2)$-regular tree locally).

Then take any remaining vertices of degree less than $d - 1$ and attach trees to them so as to raise all such vertex degrees up to $d - 1$.

Call the result Γ.

Lemma 8.11. *The graph Γ is T-free.*

Proof. There are no vertices with three neighbors of degree d. □

Lemma 8.12. *If Γ is contained in a T-free graph G then the vertices of P have finite order in G, and if $u \in P$ then the neighbors v of u of finite order in G are its neighbors in P.*

Proof. A vertex $u \in P$ can have no new neighbor v in $G \setminus \Gamma$ as this immediately produces an embedding of T into Γ.

If $u \in P_i$ then the only candidates for a vertex $v \in \Gamma$ which could serve as a new neighbor without producing an embedding of T into Γ are the vertices adjacent to b_i which correspond to v_1 or v_2, and these have infinite degree in Γ. So the vertices of P have finite degree, and if we restrict our attention to neighbors of finite degree then there are none available other than the $(d-1)$ neighbors we have already selected. □

Proposition 8.13. *In Case* IVD*, there is no weakly universal T-free graph.*

Proof. If Γ is contained in the T-free graph G then we can recover the graph P from G and one vertex of P. Now if $d \geqslant 5$ there is enough variability in the structure of P itself to yield the desired conclusion, so we may suppose that $d = 4$ and P is a path.

If a pair (u, u') of vertices of P has a common neighbor, then that neighbor is not on P and could only be some b_i or some vertex adjacent to b_i, and then only if $u, u' \in P_i$. Thus the sets P_i can be recovered, and thus the sequence p_i can be recovered up to a shift and reversal. □

Case IVD′. T' is a near-path, $d \geqslant 4$, and there are three vertices v_0, v_1, v_2 of degree d in T, in distinct v_*-components of T.

Construction 15. We proceed much as in the previous case but with a different treatment for the v_*-components C_i containing v_i nonadjacent to v_* ($i = 1$ or 2, possibly): these components we allow to be freely amalgamated over v_* (without fixing the vertex v_i). Otherwise, we proceed as in the previous construction.

Call the result Γ.

Lemma 8.14. *The graph Γ is T-free.*

Proof. If v_* is adjacent to n vertices of degree d in T, where $1 \leqslant n \leqslant 3$, then no vertex of Γ is adjacent to more than $n-1$ vertices of degree d. □

This is actually the main point, since we have loosened the construction of H in a way that in other contexts could easily lead to a violation of this first step.

The rest of the analysis is as before since the construction of H is if anything even freer than it was in the previous case.

Proposition 8.15. *In Case* IVD′*, there is no weakly universal T-free graph.*

With this, the proof of Theorem 3′ is complete, and thus also the proof of Theorem 3. Together with Theorem 2 this gives the full Tree Conjecture.

We believe that these methods can be applied to the complete identification of all finite connected graphs C for which there is a countable universal (weakly or strongly) C-free graph, in part because we expect the list of exceptional C allowing such a universal graph to be fairly limited. In particular we now think it quite likely that this problem is decidable in the case of a single constraint, and very possibly more generally.

Based on the results of [4] it would appear that the "generic" case corresponds roughly to the case in which there is some block of order at least 6, and that the nongeneric case is therefore inconveniently complicated.

We remark that a proof of decidability for the case of a single constraint may be achievable without actually working through all the critical cases. Once the set of unsolved cases is reduced to a well-quasiordered set relative to the "pruning" relation, one knows that the remaining minimal cases not allowing a universal graph form a finite set, and the problem is therefore algorithmically decidable. This style of argument does not necessarily provide any further indication as to what the relevant finite subset might be, any bound on its size, or a fortiori any explicit algorithms. We hope to return to this topic.

9. List of cases

9.1. Paths and some near-paths

Case I. $\ell \geqslant 2$.

Case IA. T has a vertex v_0 of maximal degree $d \geqslant 4$ such that some v_0-component C of T contains a unique vertex v_1 of degree d, and $\ell = d(v_0, v_1) \geqslant 3$. Either T' is a path, or else T' is a near-path whose center does not lie in the v_0-component C.

Case IB. T has a vertex v_0 of maximal degree $d = 3$ such that some v_0-component C of T contains a unique vertex v_1 of degree d, and $\ell = d(v_0, v_1) \geqslant 3$. Either T' is a path, or else T' is a near-path whose center does not lie in the v_0-component C.

Case IC. T has a vertex v_0 of maximal degree d such that some v_0-component C of T contains a unique vertex v_1 of degree d, and $\ell = d(v_0, v_1) = 2$. Either T' is a path, or else T' is a near-path whose center is not v_1.

Case II. $\ell = 1$.

Case IIA. T has a vertex v_0 of maximal degree $d \geqslant 5$ such that some v_0-component C of T contains a unique vertex v_1 of degree d, and v_0, v_1 are adjacent. Either T' is a path, or T' is a near-path whose center does not lie in the v_0-component C.

Case IIB. T has a vertex v_0 of maximal degree $d = 4$ such that some v_0-component C of T contains a unique vertex v_1 of degree d, and v_0, v_1 are adjacent. Either T' is a path, or T' is a near-path whose center does not lie in the v_0-component C.

Case IIIA. T' is a path and T contains exactly two branch vertices v_0 and v_1, which are adjacent, and of degree 3.

Case IIIB. The maximal vertex degree is 3. T contains a sequence of three adjacent branch vertices v_1, v_0, v_2 with v_1 external and adjacent to a leaf. Some v_0-component of T is a path.

Case IIIC. The maximal vertex degree is 3. T contains a sequence of three successive branch vertices v_0, v_1, v_1' with v_1' external and adjacent to a leaf, v_1 adjacent to v_1', and $\ell = d(v_0, v_1) \geqslant 3$, where v_0 is the closest branch vertex to v_1 other than v_1'. Either the pruned tree T' is a path, with both external branch vertices of T are adjacent to branch vertices, or a near-path with the center not in the v_0-component containing v_1.

Case IIID. The maximal vertex degree is 3. T contains a sequence of three successive branch vertices v_0, v_1, v_1' with v_1' external and adjacent to a leaf, v_1 adjacent to v_1', and $\ell = d(v_0, v_1) = 2$, where v_0 is the closest branch vertex to v_1 other than v_1'. Either the pruned tree T' is a path, and both external branch vertices of T are adjacent to branch vertices, or it is a near-path with the center not in the v_0-component containing v_1.

9.2. The remaining near-paths

Case IVA. T' is a near-path, $d = 3$, and there are exactly two branch vertices in T, namely the center v_*, and an adjacent vertex v_1.

Case IVB. T' is a near-path, $d = 3$, and there are exactly four branch vertices in T, namely the center v_*, and three adjacent vertices v_1, v_2, v_3, in distinct v_*-components. We may suppose that v_3 is adjacent to two leaves (and the same may possibly apply to one or both of v_1, v_2).

Case IVC. T' is a near-path, $d \geqslant 4$, and there are exactly two vertices v_0, v_1 of degree d in T, lying in distinct v_*-components.

Case IVD. T' is a near-path, $d \geqslant 4$, and there are three vertices v_0, v_1, v_2 of degree d in T, and all are adjacent to v_*.

Case IVD′. T' is a near-path, $d \geqslant 4$, and there are three vertices v_0, v_1, v_2 of degree d in T, in distinct v_*-components of T.

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