GOOD TORI IN GROUPS OF FINITE MORLEY RANK

GREGORY CHERLIN*
DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY
HILL CENTER, PISCATAWAY, NEW JERSEY 08854, U.S.A

1. A conjugacy theorem

In affine algebraic groups, maximal tori are conjugate. We will prove a similar conjugacy theorem in the larger category of groups of *finite Morley rank*. Groups of finite Morley rank occur in model theory, both pure and applied, as an appropriate generalization of the class of algebraic groups (over algebraically closed fields). They are groups equipped with a rudimentary, yet nonetheless powerful, notion of dimension applying to all definable subsets; in the algebraic case, a suitable notion of dimension is obtained by taking the geometric dimension of the Zariski closure.

While the class of groups of finite Morley rank is broader than the class of algebraic groups, being for example closed under the formation of finite products (which allows examples of "mixed characteristic"), the two classes seem to be closely related; the question raised by Zilber and the present author as to whether every *simple* group of finite Morley rank is algebraic remains open. A considerable body of work has developed with regard to this *Algebraicity Conjecture*, in the course of which certain abstract notions of unipotence and semisimplicity have come to play an increasing role. By general consent, a definable divisible abelian group is referred to as a *torus* in this context; however, under this definition the additive group of a field of characteristic zero is a torus, and for this and related reasons complications arise. A better behaved notion of torus runs as follows.

Definition 1. *A* good torus *is a definable divisible abelian group with the property that every definable subgroup is the definable closure of its torsion subgroup; or equivalently, every definable section contains torsion.*

Every algebraic torus (with algebraically closed base field) is a good torus in this sense; so are abelian varieties, but we tend to concentrate on the affine category. An outstanding open problem is whether the multiplicative group of any field of finite Morley rank is a good torus. This is widely believed to be false, a possibility which enormously complicates our theory. However, it does hold in *positive characteristic*, as a consequence of work of Frank

* Supported by NSF Grant DMS 0100794; the hospitality of the Institut Girard Desargues, Lyon, is gratefully acknowledged.

Wagner [4], cf. [1, Lemma 3.11]. For this reason, the theory of good tori has found substantial application in work on the Algebraicity Conjecture for groups of "even type" (which corresponds to characteristic two in the algebraic case). They have a number of agreeable properties, notably a number of rigidity properties. The simplest of these is the following, which follows simply from the fact that the torsion subgroup of each order in T is finite.

Lemma 1 (Rigidity I)**.** *Let T be a good torus, and H a connected definable group normalizing T. Then H centralizes T.*

The following more substantial rigidity property was introduced and applied in [1], where however the formulation was more technical (cf. §§2,3 below).

Lemma 2 (Rigidity II)**.** *Let T be a good torus of finite Morley rank, and $\mathcal{F}$ a uniformly definable family of subgroups of T. Then $\mathcal{F}$ is finite.*

Here, a uniformly definable family is the natural generalization of an "algebraic family" to a model theoretic context. We will introduce a third rigidity property here, which may be combined with the two foregoing to prove the conjugacy theorem mentioned at the outset.

Theorem 1 (Conjugacy Theorem)**.** *Let G be a group of finite Morley rank. Then any two maximal good tori in G are conjugate.*

The reader who is familiar with algebraic groups and unfamiliar with groups of finite Morley rank may possibly wish to review the formal definitions given as axioms in [3]; alternatively, he may simply note that the proof given here is valid in algebraic groups and uses only the notion of good torus and the most rudimentary properties of the notion of dimension. Such conjugacy theorems are valuable in the study of groups of finite Morley rank, in much the same way they are used in finite group theory. In this context we have several conjugacy theorems similar to those used in finite group theory: a Sylow theorem for the prime two, and Hall and Carter theorems for connected solvable groups. It would be very useful to have conjugacy of maximal tori, and Borel subgroups, as well, but these are unfortunately tied up with the problematic side of the theory.

The proof of the Conjugacy Theorem runs as follows. We begin with the second rigidity lemma, which we prove for the reader's convenience. We then derive another rigidity lemma, with a very similar flavor.

Lemma 3 (Rigidity III)**.** *Let G be a group of finite Morley rank, and let H and T be definable sections of G with T a good torus. Then any uniformly definable family of homomorphisms from H to T is finite.*

Using this, we can derive a "generic covering" result; methodologically this is close to, but weaker than, a conjugacy theorem, and results of this type are often used in model theory either as a step toward a conjugacy theorem or as a substitute for such a theorem.

Lemma 4 (Generic Covering)**.** *Let G be a connected group of finite Morley rank, and $T \leq G$ a good torus. Let $H = N_G^\circ(T)$ be the connected component of the normalizer of T. Then the union of the conjugates of H in G is generic in G (i.e., the dimension of its complement in G is less than the dimension of G).*

There is a curious, though simple, interplay between non-genericity and genericity in the proof of such results. As G is connected, it is sufficient to prove in the above that $\bigcup H^G$ has full dimension in G. Now the *disjoint union* of the conjugates of H (indexed by cosets $N(H)\backslash G$) has the correct dimension, so it is necessary that the overlap between conjugates is not too large, and this is expressed by the following, which is really the proof of the Generic Covering Lemma.

Lemma 5 (Nongenericity)**.** *Let G be a group of finite Morley rank and $T \leq G$ a maximal good torus in G. Let $H = N_G^\circ(T)$. Then the intersection*

$$H \cap \bigcup_{g \in G \setminus N(H)} H^g$$

is nongeneric in H.

Now we may sketch the proof of the Conjugacy Theorem.

We begin with two maximal tori $T_1, T_2 \leq G$, and we proceed by induction on the dimension of G. If G is centerless, we consider the associated groups $H_i = N^\circ(T_i)$ (or what is the same, $C^\circ(T_i)$, by the first Rigidity Lemma).

By generic covering, H_1 and H_2 have conjugates which meet nontrivially, so we may as well suppose H_1 and H_2 meet nontrivially, and fix a nontrivial h in the intersection. Then $T_1, T_2 \leq C(h) < G$ and we can conclude by induction.

If G has a center, we can make a reduction of the problem. We distinguish two cases. If the center is infinite, we factor it out, lowering the dimension, and apply induction, which reduces our claim to the nilpotent case. If it is finite, we also factor it out, after which our first case applies.

2. The Details

The previous section was written in a language which was intended to be taken in a model theoretic sense, but could be specialized without harm to the language of algebraic geometry. In the present section, where we will be more precise, we adopt the language of model theory. The general reference [3] serves as a guide to that language in the context of groups of finite Morley rank. The following points should be noted at the outset. They are all covered in [3], and illustrate but do not exhaust the elementary properties on which we rely.

(1) The rank (in Morley's sense) of a definable set X is denoted $rk(X)$; this is what we have called the "dimension" in the previous section.

(2) A definable group is said to be *connected* if it has no proper definable subgroup of finite index. Every definable group X has a unique definable connected subgroup X° of finite index.
(3) A definable subset X of a definable *connected* group G is said to be *generic* if $rk(X) = rk(G)$, or equivalently $rk(G \setminus X) < rk(G)$. If G is not connected, the two conditions are not equivalent, and genericity would be defined by the former condition, but it is best to avoid the term in that case.
(4) Associated to a definable set X of rank r is a *degree* d, defined as the largest number such that there are d disjoint definable subsets of X, all of rank r. A set of degree 1 may be thought of as "irreducible in the top dimension"; and the degree is the number of irreducible components in this sense. The number of components is well-defined, but the components themselves are not (they are, however, well-defined up to sets of lower rank).
(5) Any subset X, definable or not, in a group of finite Morley rank G, is contained in a unique smallest definable subgroup of G. This group is called the "definable hull" of X and is denoted $d(X)$.
(6) If H is a definable subgroup of a group G of finite Morley rank, and moreover the rank and degree of H coincide with the rank and degree of G, then $H = G$.

One subtlety that has been glossed over in the previous section is the possibility of replacing a particular group of finite Morley rank by an elementary extension, just as algebraic groups are really functors rather than individual groups. The point requires some attention, as it is not clear a priori that a formula defining a good torus in one group of finite Morley rank will necessarily define a good torus in an elementary extension. One might call a torus *absolutely good* if it remains good under elementary extensions. A technical lemma shows that good tori are absolutely good; we will give this later. If one wishes to avoid such technicalities, one should take "good" to mean "absolutely good" throughout.

From the general theory of good tori, the following points will be helpful.

Fact 1. *Let G be a connected group of finite Morley rank, and T a good torus, also of finite Morley rank.*
(1) *The torsion subgroup of T is countable (and the torsion of fixed order is finite; in other words, T has finite Prüfer p-rank for each p).*
(2) *G has a unique minimal definable normal subgroup K for which G/K is a good torus.*

The second point is not usually stated in that precise form. It is a combination of the descending chain condition for definable subgroups of G with an elementary property of good tori: a definable connected subgroup of a finite product of good tori is a good torus.

We begin with a proof of the second Rigidity Lemma, as the method is instructive.

Rigidity II. *Let G be a group of finite Morley rank, T a good torus in G, and $\mathcal{F}$ a uniformly definable family of subgroups of T.*
Then $\mathcal{F}$ is finite.

Proof. The torsion subgroup of T is countable and each definable subgroup of T is determined by its intersection with the torsion. Hence $|\mathcal{F}| \leq 2^{\aleph_0}$, and by saturation $\mathcal{F}$ is finite. □

This saturation argument is a form of the Compactness Theorem; in the algebraic context it says that any algebraic set which remains of bounded cardinality after arbitrary extension of the base field is 0-dimensional. A combinatorial argument that avoids saturation is given in §3.

Rigidity III. *Let G be a group of finite Morley rank, and H and T definable sections, with T a good torus. Let $\mathcal{F}$ be a uniformly definable family of homomorphisms from H to T.*
Then $\mathcal{F}$ is finite.

Proof. We may suppose that the group H is connected, as any homomorphism $h : H \to T$ is determined up to finitely many possibilities by its restriction to H°.

Let K be the minimal normal definable subgroup of H such that H/K is a good torus. Then K is contained in all the kernels $\ker(h)$ for $h \in \mathcal{F}$. So we may suppose $K = 1$ and H is a good torus. Then any $h \in \mathcal{F}$ is determined by its restriction to the torsion of H. Hence $|\mathcal{F}| \leq 2^{\aleph_0}$ and by saturation it follows that $\mathcal{F}$ is finite. □

The following is a more abstract version of the nongenericity theorem stated in §1, where we write $\mathcal{F}$ in place of the family explicitly mentioned at that point. (Compare the proof of the Generic Covering Theorem, following.)

Nongenericity. *Let H be a connected group of finite Morley rank, T a maximal good torus of H. Suppose T is central in H. Let $\mathcal{F}$ be a uniformly definable family of subgroups of H, none of which contain T.*
Then the union $\bigcup \mathcal{F}$ is not generic in H.

Proof. We may suppose without loss of generality that the intersections $X \cap T$ for $X \in \mathcal{F}$ are independent of X. Then after passing to a quotient, we may suppose that $X \cap T = 1$ for $X \in \mathcal{F}$.

Now suppose $\bigcup \mathcal{F}$ is generic in H. Let V be a coset of T in H such that $V \cap \bigcup \mathcal{F}$ is generic in V. We may suppose that $g \in V \cap \bigcup \mathcal{F}$ is chosen to minimize the rank and degree of the definable closure $d(g)$.

Then for $gt \in V \cap \bigcup \mathcal{F}$, we have $d(gt) \leq d(g) \times T$, and $d(gt) \cap T = 1$. Hence the projection $\pi_1 : d(gt) \to d(g)$ is injective, and by the choice of g, also surjective. It follows that the group $d(gt)$ is the graph of a homomorphism $h_t : d(g) \to T$. Furthermore, if $gt \in X \in \mathcal{F}$, and we set $\tilde{X} = X \cap (d(g) \times T)$, then as $d(gt) \leq \tilde{X}$ and $\tilde{X} \cap T = 1$, the same considerations show that

$d(gt) = \tilde{X}$. Thus the family of homomorphisms $\{h_t : gt \in \bigcup \mathcal{F}\}$ is uniformly definable, and hence finite.

On the other hand, for $X \in \mathcal{F}$, we have $|V \cap X| \leq 1$, and hence $V \cap \bigcup \mathcal{F}$ is finite, contradicting the genericity. □

Generic Covering. *Let G be a connected group of finite Morley rank, T a good torus in G, and $H = C^\circ(T)$. Then $\bigcup H^G$ is generic in G.*

Proof. We may suppose that T is a maximal good torus in G.

We show first that H is almost self-normalizing in G (that is, $N^\circ(H) = H$). As $T \leq Z(H)$, T is the unique maximal good torus of H. So $N^\circ(H) \leq N^\circ(T) = C^\circ(T) = H$.

Now let $\mathcal{F} = \{H \cap H^g : g \in G \setminus N(H)\}$. It suffices to show that $\bigcup \mathcal{F}$ is not generic in H, as the conjugates of $H \setminus \bigcup \mathcal{F}$ are pairwise disjoint. We apply the previous lemma. Suppose $T \leq H \cap H^g$. Then as T^g is central in H^g and is a maximal good torus of H^g, we have $T \leq T^g$, hence $T = T^g$ and $H = H^g$. So the previous lemma applies to $\mathcal{F}$. □

Conjugacy Theorem. *Let G be a group of finite Morley rank. Then any two maximal good tori of G are conjugate.*

Proof. We proceed by induction on the rank of G. We may suppose that G is connected. Let T_1, T_2 be two maximal good tori of G.

Suppose first that G is centerless. Let $H_i = C^\circ(T_i)$. As $\bigcup H_i^G$ is generic in G for $i = 1, 2$, we may suppose $H_1 \cap H_2 \neq 1$. Let $h \in (H_1 \cap H_2)^\#$. Then $T_1, T_2 \leq C(h)$ and we may conclude by induction.

Suppose that G has a finite center. Then $G/Z(G)$ is centerless [3, elementary], and the first case applies. Furthermore, T_1 and T_2 map to maximal good tori in the quotient $G/Z(G)$. So after conjugating we may suppose $T_2 \leq (T_1 \cdot Z(G))^\circ = T_1$, and thus $T_2 = T_1$.

Finally, suppose that G has an infinite center, and let $\bar{G} = G/Z(G)$. Let $\hat{T}_1$ be the preimage in G of a maximal good torus of $\bar{G}$ containing $\bar{T}_1$. Then by induction we may suppose that $T_2 \leq \hat{T}_1$. But $\hat{T}_1$ is nilpotent, and from the structure of nilpotent groups of finite Morley rank it follows that T_1 and T_2 are central in $\hat{T}_1$, so by maximality $T_2 = T_1$. □

The following corollary is useful [2].

Corollary. *Let G be a group of finite Morley rank, and $\mathcal{F}$ a uniformly definable family of good tori in G. Then the tori in $\mathcal{F}$ fall into finitely many conjugacy classes under the action of G.*

Proof. Let T be a maximal good torus in G. Every torus in $\mathcal{F}$ is conjugate to a subtorus of T, so we may as well suppose that every torus in $\mathcal{F}$ is a subtorus of T. In that case $\mathcal{F}$ is finite by the second Rigidity Lemma. □

This last result, and the main result of the paper, simplify the analysis of [2], which in its first form involved similar arguments applied in more special cases. The first author of that paper has also contributed a number of useful comments to the present paper.

3. Appendix: Degrees of virtue

In its original formulation, Lemma 2 was stated for good tori in $\aleph_0$-saturated groups. This is sufficient for applications as one may work with an $\aleph_0$-saturated elementary extension of any given group, but in this case we are dealing with the tori which we have called *absolutely good*, that is, good in any elementary extension. This issue can be eliminated entirely by showing that good tori remain good in elementary extensions; oddly, in order to prove this we have to reverse the natural sequence of ideas, and first prove the second rigidity lemma *without saturation.*

Lemma 6. *Let T be a definable subtorus of the group G of finite Morley rank, with T good relative to G. Let $\mathcal{F}$ be a uniformly definable family of subgroups of T, defined in G. Then $\mathcal{F}$ is finite.*

Proof. We work without saturation, and hence proceed more combinatorially. Furthermore we generalize the hypotheses slightly: we will not require T to be connected, just abelian, with T° a good torus, and we allow T to interpreted in G, not necessarily a subgroup.

Let T be chosen of minimal rank and degree allowing a counterexample $\mathcal{F}$. We may also suppose that the Morley rank r and Morley degree d of the elements of $\mathcal{F}$ is fixed. Note that $T \notin \mathcal{F}$, and that the elements of $\mathcal{F}$ are infinite (otherwise, they are all finite, and then of bounded order, which forces $\mathcal{F}$ to be finite).

We choose $T_0 \leq T$ definable and maximal so that T_0 is contained in infinitely many groups in $\mathcal{F}$. This is done as follows. For any $T_0 \leq T$ contained in infinitely many groups in $\mathcal{F}$, let $\hat{T}_0$ denote the intersection of those groups. Then $\hat{T}_0$ belongs to $\hat{\mathcal{F}}$, the collection of intersections of elements of $\mathcal{F}$, and is itself contained in infinitely many elements of $\mathcal{F}$. Again, $\hat{\mathcal{F}}$ is a uniformly definable family, and hence satisfies a bounded chain condition, in particular an ascending chain condition. Let $T_0 \in \hat{\mathcal{F}}$ be chosen maximal so that T_0 is contained in infinitely many members of $\mathcal{F}$. Then this will do.

Now replace T by T/T_0. So we may suppose that any nontrivial subgroup is contained in only finitely many groups in $\mathcal{F}$. Now for $A \in \mathcal{F}$ as $A \neq T$ it follows from our initial minimization that the family $\mathcal{F}_A = \{X \cap A : X \in \mathcal{F}\}$ is finite. On the other hand, only finitely many elements of $\mathcal{F}$ meet A in a fixed nontrivial subgroup. So all but finitely many elements of $\mathcal{F}$ meet A trivially.

Fix $A_1 \in \mathcal{F}$ and let $\mathcal{F}_1 = \{X \in \mathcal{F} : X \cap A_1 = 1\}$. Recall that A_1 is infinite and hence $rk(T/A_1) < rk(T)$. By induction, the family $\bar{\mathcal{F}}_1 = \{(X+A_1)/A_1 : X \in \mathcal{F}_1\}$ of subgroups of T/A_1 is finite, so there is an infinite uniformly definable subfamily $\mathcal{F}_2 \subseteq \mathcal{F}_1$ so that $X + A_1$ is independent of X for $X \in \mathcal{F}_2$; notice that the sum $X + A_1$ is direct.

Now fix $A_2 \in \mathcal{F}_2$ and let $A = A_1 \oplus A_2$. Then for $B \in \mathcal{F}_2$ we have $A = A_1 \oplus B$. Let $\mathcal{F}_2' = \{X \in \mathcal{F}_2 : X \cap A_2 = 1\}$, which contains all but finitely many subgroups in $\mathcal{F}_2$. Then for $B \in \mathcal{F}_2'$, the projection maps from B to A_1 and A_2 are injective, and in view of the equality of ranks and degrees, they are isomorphisms. So B is the graph of an isomorphism $\sigma_B : A_1 \to A_2$.

Now consider the family $\Sigma = \{\sigma_{B_1}^{-1}\sigma_{B_2}\}$ of automorphisms of A_1 for $B_1, B_2 \in \mathcal{F}_2'$ with $B_1 \neq B_2$. For $\sigma \in \Sigma$ let A_σ be the fixed point set of σ on A_1. Then A_σ is a uniformly definable set of subgroups of A_1, so by our initial minimization there are finitely many such subgroups A_σ.

Now for any finite subgroup $C \leq A_1$, there are only finitely many homomorphisms from C to A_2, so there must be some $\sigma \in \Sigma$ which acts trivially on C. It follows that one of the sets A_σ contains all the torsion of A_1, and as T is good in G, $A_\sigma = A_1$, in other words, σ acts trivially on A_1, contradicting the definition of Σ. This contradiction completes the argument. □

Proposition 1. *Let G be a group of finite Morley rank, G^* an elementary extension, T a torus defined in G and good in G, and T^* its canonical extension to G^*. Then the following hold.*

(1) *Any G^*-definable subgroup of T^* is definable with parameters in G.*
(2) *T^* is a good torus in G.*

Proof.
1. Let $A \leq T^*$ be a definable subgroup of T^* in G^*, and let $\phi(x, \mathbf{g})$ be a definition of A, where $\mathbf{g} \in G^*$. Let $X_{\mathbf{g}} = \phi[G, \mathbf{g}]$ for $\mathbf{g} \in G$, and let $\mathcal{F}_\phi$ be the uniformly definable family

$$\{X_{\mathbf{g}} : \mathbf{g} \in G,\ X_{\mathbf{g}} \text{ is a subgroup of } T\}$$

As T is good in G, the family $\mathcal{F}_\phi$ consists of finitely many groups, and this fact is expressed by a first order sentence with parameters in G. Interpreted in G^*, it says that the corresponding family in G^* also consists of finitely many groups, with the same definitions. In particular our original group A is one of these, and our claim follows.

2. Let $A^* \leq T^*$ be definable in G^*, and suppose that B^* is any definable subgroup of A^* which contains the torsion of A^*. We claim that $B^* = A^*$.

By the first point, A^* and B^* are the extensions to G^* of definable subgroups A and B of T in G, respectively. Now as G^* is an elementary extension of G and the torsion of fixed exponent in A is finite, the groups A and A^*, as well as B and B^*, have the same torsion. Thus A and B have the same torsion, and as A is good, we have $A = B$, hence in the elementary extension $A^* = B^*$. □

In another direction, Borovik has suggested the following.

Definition 2. *A torus T of finite Morley rank is* decent *if its quotient by the Frattini subgroup is a good torus.*

Here the Frattini subgroup $\Phi(T)$ is the intersection of the connected maximal proper definable subgroups, and is characterized by the condition: if $T = \Phi(T)A$ with A connected, then $T = A$.

Some conditions equivalent to decency are the following: (1) T is the definable closure of its torsion subgroup; (2) Any nontrivial definable quotient of T contains torsion.

While decent tori may not necessarily have the same rigidity properties as good tori, they do satisfy Rigidity I, and they also satisfy the nongenericity lemma, because it can be reduced to the case of good tori:

Extended Nongenericity. *Let H be a connected group of finite Morley rank, and T a maximal decent torus of H. Suppose T is central in H. Let $\mathcal{F}$ be a uniformly definable family of subgroups of H, none of which contain T.*

Then the union $\bigcup \mathcal{F}$ is not generic in H.

For the proof, pass to $H/\Phi(T)$ and apply the previous version.

The derivation of the conjugacy theorem from this result proceeds as in the case of good tori. This refinement may possibly be useful in the study of groups of degenerate type, as the absence of decent tori would restrict the torsion considerably.

References

[1] T. Altınel and G. Cherlin. Simple L^*-groups of even type with strongly embedded subgroups. *J. Algebra* 272:95–127, 2004.

[2] T. Altınel and G. Cherlin. Limoncello. Submitted.

[3] A. V. Borovik and A. Nesin. *Groups of Finite Morley Rank.* Oxford University Press, 1994.

[4] F. Wagner Fields of finite Morley rank. *J. Symbolic Logic,* 66:703–706, 2001.