Simple L^*-groups of even type with strongly embedded subgroups

Tuna Altınel
Institut Girard Desargues, Université Claude Bernard Lyon-1
Domaine de la Doua, Bâtiment Braconnier
21 avenue Claude Bernard
69622 Villeurbanne CEDEX, France
e-mail: `altinel@desargues.univ-lyon1.fr`

Gregory Cherlin
Department of Mathematics, Rutgers University
Hill Center, Piscataway, New Jersey 08854, U.S.A
e-mail: `cherlin@math.rutgers.edu`

June 10, 2003

1 Introduction

This paper is about the classification of infinite simple groups of finite Morley rank. It has been conjectured that these are linear algebraic groups over algebraically closed fields (*algebraicity conjecture*). Various approaches to the algebraicity conjecture have been developed over the years from both model theoretic and group theoretic sides. On the group theory side, the presence of a well-developed Sylow 2-theory and various finiteness conditions permitted the use of finite group theoretic ideas, and it has been possible to adapt techniques from the classification of the finite simple groups. The context where these techniques have been the most efficient has been the context of simple *K^*-groups*. The notion of K^*-group, a group of finite Morley rank all of whose infinite definable simple and proper sections are algebraic groups over algebraically closed fields, was introduced to set firm grounds for an analogue of the *revisionist approach* to the classification of the finite simple groups, and has made it possible to carry out a systematic analysis of various classes of simple groups of finite Morley rank. In this vein, a project to verify the algebraicity conjecture for simple K^*-groups of *even type* has been completed, and the nonexistence of simple K^*-groups of *mixed type* has been proven in [21] (see Section 2 for definitions).

A weakness of the notion of K^*-group is the strength of its definition. In order to arrive at a classification statement free of an inductive hypothesis one needs to verify that the algebraicity conjecture holds for *all* the simple K^*-groups of finite Morley rank. A broader inductive notion, that of an L^*-group, seems to be relevant in this context. This is a group of finite Morley rank in which every proper infinite definable simple section is either an algebraic group over an algebraically closed field, or of odd or degenerate type. The notion of L^*-group is a general one, which is most relevant in the context of the class of groups of finite Morley rank of even type, or more generally the class of groups that have infinite 2-subgroups of bounded exponent. Indeed, the verification of the algebraicity conjecture for the simple L^*-groups of even type would prove the algebraicity conjecture for the simple groups of finite Morley rank of even type completely. Moreover, it has been proven that such a classification would also eliminate the possibility of simple groups of mixed type in general. These issues have been addressed in [4], to which we refer the reader for more detail.

A natural approach to the study of simple L^*-groups is to try to generalize what has already been accomplished in analogous contexts for K^*-groups. In this line, an analysis of simple L^*-groups of even type with *strongly/weakly embedded sugbroups* was initiated in [4]. A proper definable subgroup M of a group G of finite Morley rank is said to be strongly embedded if M contains involutions and for any $g \in G \setminus M$, the intersection $M \cap M^g$ has no involutions (see Section 5 for further definitions and properties). In this paper, we continue along the line initiated in [4] and prove the following result:

Theorem 1 *Let G be a simple L^*-group of even type with a strongly embedded subgroup M. Assume*

(∗) $C_G(A_1, A_2)$ *is finite whenever A_1 and A_2 are two distinct conjugates of $\Omega_1(M)$.*

Then $G \cong \mathrm{PSL}_2(F)$, where F is an algebraically closed field of characteristic 2

In an arbitrary group G, $\Omega_1(G)$ denotes the subgroup generated by the involutions of G. A priori, in the statement of the hypothesis (*) of Theorem 1, the potential differences among $\Omega_1(M^\circ)$, $\Omega_1^\circ(M)$ and $\Omega_1(M)$ should be taken into account. However Fact 5.7 below, which is an immediate consequence of basic fusion properties of groups of finite Morley rank with strongly embedded subgroups, shows that these three subgroups are the same in our context, which involves infinite Sylow 2-subgroups. Moreover, Fact 5.12 implies that this subgroup is a definable, connected, elementary abelian 2-subgroup of M.

The classification of simple K^*-groups of even type with weakly embedded subgroups – as PSL_2 in even characteristic – was the first step in the classification of simple K^*-groups of even type. In the context of groups of even type, weak embedding offers a generalization of strong embedding which is more frequently encountered in practice. Thus it will be desirable to obtain a weakly embedded analogue of Theorem 1, towards which we have recently made substantial progress by completing the necessary Sylow 2-subgroup analysis, along the lines of [3]. This work being still in progress, here we are content with Theorem 1, which already illustrates both the challenges presented by, and the new methods required for, the study of L^*-groups.

Our hypothesis (*) is a strong one. Theorem 1 corresponds to Section 3 of [22]. The nature of the complications encountered by Eric Jaligot under the contrary hypothesis in the classification of simple K^*-groups of even type with weakly embedded subgroups show that the complete classification of simple L^*-groups of even type with weakly embedded subgroups remains a substantial challenge.

In the balance of this introduction we will indicate how and why some arguments change when one replaces the K^* assumption by the L^* assumption. In §4 one will find specific technical results that support such changes.

It immediately follows from algebraic group theory and elementary properties of groups of finite Morley rank that the definable connected sections of degenerate type in K^*-groups are solvable. This is no longer the case for L^*-groups. Using a finite group theoretic analogy, our classification can be compared to classifying the finite simple groups (of sufficiently large 2-rank) without having the Feit-Thompson theorem available. Evidently this poses major difficulties. Indeed, very little is known about nonsolvable groups of finite Morley rank of degenerate type, which are potential counterexamples to the algebraicity conjecture.

Weakening the inductive hypotheses from K^* to L^*, we must analyze hypothetical simple groups that are not of degenerate type but may, a priori, contain simple degenerate sections. In the K^* context one treats degenerate (hence *solvable*) sections using the well-developed theory of solvable groups of finite Morley rank: this includes powerful conjugacy results (Hall theory, Carter subgroups, etc) and the Schur-Zassenhaus theorem. To cope with degenerate nonsolvable sections one must change tack completely. The main technical ingredient of our new approach is a fundamental result by Frank Wagner on fields of finite Morley rank in nonzero characeristic, and some of its more concrete consequences, including a result of Bruno Poizat on linear groups in positive characteristic. Whenever a definable connected degenerate section acts nontrivially and definably on an infinite elementary abelian p-group (in our context, p will be 2 in practice), these theorems, together with linearization techniques by Ali Nesin and Boris Zil'ber, are used to carry out various *genericity arguments* that replace the conjugacy theorems for solvable groups (Sections 3, 4). Such arguments have been met with in the past, beginning with the treatment of bad groups, but they take on a very different form in our even type context.

We emphasize that genericity arguments are not only used to prove conjugacy results for Borel subgroups and the like. The first case of Theorem 2 provides an important example of another line of genericity argument which can be interpreted as control of the Weyl group in a degenerate environment. This goes back to the analysis of groups of Morley rank 2 in [13], and was also very effective in the analysis of *bad groups* as well as in [14].

2 Preliminaries

In this section we present some definitions and facts which will be needed in the sequel, and which will be exploited in the present context in the same way that quite similar material has been exploited in the K^*-context in prior publications. Our terminology and notation is consistent with those earlier publications on simple K^*-groups and with [11], which is our main reference on groups of finite Morley rank. This section presents no innovations. However, in the L^*-context more model theoretic arguments are frequently encountered, and we will begin to develop these in the next section. The most useful references for the more model theoretic aspects of groups of finite Morley rank are [28] and [31].

Let us begin by recalling the structure of the Sylow 2-subgroups of a group of finite Morley rank:

Fact 2.1 ([12]) *Let G be a group of finite Morley rank. Then the Sylow 2-subgroups of G are* *conjugate. If S is a Sylow 2-subgroup of G then $S^\circ = B * T$ where B is a definable connected group of bounded exponent, T is divisible abelian, $*$ denotes the central product and $B \cap T$ is finite.*

Definition 2.2

1. *A* unipotent *subgroup is a connected definable solvable subgroup of bounded exponent.*
2. *A* p-torus *is a divisible abelian p-group. It is the direct sum of copies of the quasicyclic group $\mathbb{Z}_{p^\infty}$. The* Prüfer p-rank *of a p-torus is the number of these factors.*

   In a group of finite Morley rank a torus *is a definable divisible abelian subgroup. Since it is divisible it is connected. The Prüfer p-rank of a torus is the Prüfer p-rank of its maximal p-torus. By exercise 9 on page 93 of [11], this is finite. A nontrivial p-torus is not definable, but its definable closure is a torus.*
3. *A group of finite Morley rank is of* even type *if the connected component of a Sylow 2-subgroup is unipotent and nontrivial.*
4. *A group of finite Morley rank is of* odd type *if the connected component of a Sylow 2-subgroup is a nontrivial 2-torus.*
5. *A group of finite Morley rank is of* mixed type *if the connected component of a Sylow 2-subgroup is the central product of a nontrivial unipotent subgroup and a nontrivial 2-torus.*
6. *A group of finite Morley rank is of* degenerate type *if the connected component of a Sylow 2-subgroup is trivial (that is, the Sylow 2-subgroups are finite).*

Definition 2.3

1. *An* L-group *is a group of finite Morley rank in which every infinite definable simple section is either an algebraic group over an algebraically closed field, or of odd or degenerate type; in other words, we exclude definable simple sections of mixed type, and we require definable simple sections of even type to be algebraic.*
2. *An* L^*-group *is a group of finite Morley rank in which every* proper *infinite definable simple section is either an algebraic group over an algebraically closed field, or of odd or degenerate type.*

Fact 2.4 ([26]) *Let H be a nilpotent group of finite Morley rank. Then $H = D * B$, where D and B are definable characteristic subgroups, with D divisible and B of bounded exponent. Moreover, $D \cap B$ is finite and B is the direct sum of its maximal unipotent p-subgroups.*

Fact 2.5 $\mathrm{Aut}(\mathbb{Z}_{p^\infty})$ *has no elements of order p when $p \neq 2$.*

Proof. This endomorphism group of $\mathbb{Z}_{p^\infty}$ is isomorphic to the ring of p-adic integers $\mathbb{Z}_p$, so the automorphism group is its group of units U_p, while the cyclotomic polynomial ϕ_p is irreducible over $\mathbb{Z}_p$, by Eisenstein's criterion applied to $\phi_p(1+x)$. □

Fact 2.6 ([11, Exercise 10, p. 98]) *Let G be a nilpotent connected group of finite Morley rank and ϕ a definable automorphism of G with finitely many fixed points. Then $G\phi = \phi^G$.*

Fact 2.7 ([11, Exercise 14, p. 73]) *Let G be a group of finite Morley rank without involutions. If α is a definable involutive automorphism of G then $G = C_G(\alpha)G^-$, where $G^- = \{g \in G : g^\alpha = g^{-1}$. Moreover, for $c \in C_G(\alpha)$ and $g \in G^-$, the map $(c, g) \longmapsto cg$ is a definable bijection. In particular, G is connected if and only if $C_G(\alpha)$ is connected and G^- is of Morley degree 1.*

Fact 2.8 ([11, Theorem 8.4]) *Let $\mathcal{G} = G \rtimes H$ be a group of finite Morley rank where G, H and the action of H on G are definable, G is an infinite simple algebraic group over an algebraically closed field, and $C_H(G) = 1$. Then, viewing H as a subgroup of $\mathrm{Aut}(G)$, we have $H \leq \mathrm{Inn}(G)\Gamma$ where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G and Γ is the group of graph automorphisms.*

Remark 2.9 We will frequently use the special case of Fact 2.8 with $G = \mathrm{PSL}_2$. Here as there are no nontrivial graph automorphisms, all definable actions induce inner automorphisms.

Definition 2.10 *A* Borel subgroup *of a group of finite Morley rank is a definable connected solvable subgroup which is maximal with respect to these properties.*

Remark 2.11 Since infinite groups of finite Morley rank contain infinite definable abelian subgroups, a Borel subgroup is of finite index in its normalizer.

For a group G of finite Morley rank, $F(G)$ is the *Fitting subgroup* of G. This is the subgroup of G generated by its normal nilpotent subgroups; it is definable and nilpotent [25].

Fact 2.12 ([24]) *Let G be a connected solvable group of finite Morley rank. Then $G/F^\circ(G)$ is divisible and abelian.*

Corollary 2.13 *A unipotent group U is nilpotent.*

Proof. $U/F^\circ(U)$ is divisible and of bounded exponent. □

Fact 2.14 (Borovik) *Let $G = UX$ be a group of finite Morley rank, where U, X, and the action of X on U are definable. Let p be a prime number. Assume also that U is a normal unipotent p-subgroup of G, X is connected, solvable and does not contain elements of order p. If the action of X on U is faithful then X is divisible and abelian.*

Proof. G is connected solvable, and $F(G) = U \cdot (F(G) \cap X)$ by Corollary 2.13. Since $X \cap F(G)$ does not contain nontrivial p-elements, Fact 2.4 shows that $F(G) \cap X$ acts trivially on U. As X acts faithfully, we find that $F(G) = U$, and hence $X \cong G/F(G)$ is divisible abelian, by Fact 2.12. □

Fact 2.15 ([11, Exercises 10, 11, 12, p. 72]) *Let G be a group of finite Morley rank without involutions. Then every element in G has a unique square root.*

Fact 2.16 ([11, Exercise 11, p. 93]) *Let G be a group of finite Morley rank and H a normal definable subgroup. If $x \in G$ is such that $\overline{x} \in G/H$ is a p-element, then the coset xH contains a p-element.*

Lemma 2.17 *If G is a nontrivial connected $2^\perp$-group of finite Morley rank then $C_G(x)$ is infinite for every $x \in G$.*

Proof. If $C_G(x)$ is finite for some $x \in G$ then x^G and x^{-G} are generic in G and hence x and x^{-1} are conjugate. This forces G to have nontrivial 2-elements (Fact 2.16), a contradiction. □

Fact 2.18 ([12]) *Let T be a p-torus in a group G of finite Morley rank. Then $|N_G(T) : C_G(T)| < \infty$. Moreover there exists a natural number c such that $|N_G(T) : C_G(T)| < c$ for any p-torus $\leq G$.*

Fact 2.19 ([11, Theorem 9.29];[17]) *The Hall π-subgroups of a connected solvable group of finite Morley rank are connected.*

Remark 2.20 It follows from Fact 2.19 that a connected solvable group of finite Morley rank of degenerate type does not have involutions.

There are various versions of the Schur-Zassenhaus theorem in the context of solvable groups of finite Morley rank. The following has been of crucial importance in the context of simple groups of even type.

Fact 2.21 ([8, Theorem 2];[7, Proposition C]) *Let G be a solvable group of finite Morley rank and H a normal Hall π-subgroup of G of bounded exponent. Then any subgroup K of G with $K \cap H = 1$ is contained in a complement to H in G, and the complements of H in G are definable and conjugate to each other.*

The following facts are corollaries of Fact 2.21.

Fact 2.22 ([3, Proposition 2.43]) *Let $G = H \rtimes Q$ be a group of finite Morley rank where H, Q and the action of Q on H are definable. Let $H_1 \lhd H$ be a solvable Q-invariant definable π-subgroup of bounded exponent in G. Assume that Q is a solvable $\pi^\perp$-subgroup. Then $C_H(Q)H_1/H_1 = C_{H/H_1}(Q)$.*

Fact 2.23 *[[3, Cor. 2.44]] Let $G = H \rtimes Q$ be a solvable group of finite Morley rank, with H and Q definable. Assume H is a π-group of bounded exponent, and Q is a $\pi^\perp$-group. Then $H = [H, Q]C_H(Q)$.*

Fact 2.24 ([3, Corollary 2.45]) *Let Q and X be definable subgroups of a group of finite Morley rank with Q a unipotent 2-group, X a $2^\perp$-group, and X acting on Q, and suppose that X acts trivially on the factors Q_i/Q_{i-1} of a definable normal series for Q. Then X acts trivially on Q.*

We will also use the following corollaries of *Zil'ber's indecomposability theorem*:

Fact 2.25 ([11, Corollaries 5.28 and 5.29]) *Let G be a group of finite Morley rank*

1. *If H is a definable connected subgroup of G and X is any subset in G, then $[H, X]$ is definable and connected.*
2. *The subgroup of G generated by any family of definable connected subgroups is again definable and connected, and it is the setwise product of finitely many of them.*

We recall some notions from the theory of permutation groups.

Definition 2.26 *A doubly transitive permutation group G acting on a set X with at least 3 elements is called a* Zassenhaus group *if the stabilizer of any three distinct points is trivial. For x, y, two distinct points in X, if $B = G_x$ and $T = G_{x,y}$, then G is said to be* split *if B has a normal subgroup U such that $B = U \rtimes T$.*

The following result is crucial in order to finish the proof of Theorem 1:

Fact 2.27 ([16]) *Let G be an infinite split Zassenhaus group of finite Morley rank. Assume that the subgroup U contains a central involution, where U is as in Definition 2.26. Then either G is sharply 2-transitive of characteristic different from 2 (that is, the one-point stabilizer contains an involution), or $G \cong \mathrm{PSL}_2(K)$ for some algebraically closed field F of characteristic 2.*

We will also use the following standard group theory notation: for any group G and $X \subseteq G$, $I(X)$ will denote the set of involutions in X, and if G is a 2-group then $\Omega_1(G)$ will denote the subgroup of G generated by its involutions.

The following classical result from number theory turns out to be very useful in the proof of Theorem 2.

Fact 2.28 (Dirichlet's theorem on arithmetic progressions) *Any arithmetic progression $a + kd$, where a and d are relatively prime positive integers and $k \in \mathbb{N}$, contains infinitely many prime numbers.*

3 Fields and good tori

As was mentioned in the introduction, a recent result by Wagner will play a crucial role in our proof of Theorem 1. In this section we show how one can extract relatively concrete group theoretic information from Wagner's result on fields of finite Morley rank. We start with the two theorems that constitute the *fons et origo* of this approach.

Fact 3.1 ([23, Macintyre]) *A field interpretable in a structure of finite Morley rank is either finite or algebraically closed.*

Definition 3.2 *Let G be a group acting definably on a group H. Then H is* G-minimal *if H is infinite and has no proper infinite definable G-invariant normal subgroup.*

Fact 3.3 (Zil'ber) *Let $G = A \rtimes T$ be a group of finite Morley rank where A, T and the action of T on A are definable. Assume that T and A are abelian, $C_T(A) = 1$ and A is T-minimal. Then $A \cong F_+$, where F is an algebraically closed field and T is isomorphic to a subgroup of $F^\times$. The action of T on A is by scalar multiplication.*

Definition 3.4 *A structure $(F, +, 0, 1, \cdot, T)$ of finite Morley rank where F is an algebraically closed field and T is a predicate for an infinite proper subgroup of the multiplicative group $F^\times$ is called a* bad field.

Fact 3.5 ([27, 30]) *Let F be a field of finite Morley rank and T a definable subgroup of the multiplicative group $F^\times$ containing the multiplicative group of an infinite subfield of F. Then $T = F^\times$.*

The following striking result by Wagner concerns bad fields in nonzero characteristic. Although the existence of bad fields in any characteristic is a longstanding open problem in model theory, Fact 3.8 goes a long way toward taming bad fields in characteristic $p \neq 0$. Thus, for example, while we cannot prove that the distinguished subgroup of the multiplicative group necessarily contains all possible torsion (that is, l-torsion for $l \neq p$), we will be able to prove for example that it contains *some* torsion.

Before stating the result we require a model theoretic definition, and an observation.

Definition 3.6 *Let F be an arbitrary structure, A a subset. Then the* algebraic closure *of A in F, denoted* $\mathrm{acl}(A)$*, is the union of the* finite *A-definable subsets.*

The following remark is a reformulation of Proposition 7 and Corollary 8 in [32]. Note that the assumption on finite Morley rank is used only to show that the Frobenius automorphism is not only a field automorphism but also an automorphism of the field with its additional structure.

Remark 3.7 *Let F be a field of finite Morley rank of characteristic $p > 0$, possibly equipped with additional structure; but assume that this additional structure consists of certain subgroups of a Cartesian power $(F^\times)^n$. Then $\mathrm{acl}(\emptyset)$, in the model theoretic sense, is F_{alg}, the field theoretic algebraic closure of the prime field.*

Proof. The critical point is that the Frobenius automorphism $\mathrm{Fr}(x) = x^p$ acts as an automorphism of the field F together with its additional structure. This is due to the fact that if T is a group contributing to the additional structure then $\mathrm{Fr}(T) = T$ by Morley rank and degree considerations. As a result, any $\emptyset$-definable set is invariant under the action of Fr. So any element of $\mathrm{acl}(\emptyset)$ lies in a *finite orbit* of the Frobenius automorphism, hence is in F_{alg}. The converse inclusion, $F_{\mathrm{alg}} \subseteq \mathrm{acl}(\emptyset)$, is clear. □

Fact 3.8 ([32]) *If F is a field of finite Morley rank then*

1. $\mathrm{acl}(\emptyset)$ *(with the structure inherited from F) is an elementary substructure of F;*
2. *If F has positive characteristic, and all additional structure on F consists of multiplicative subgroups of Cartesian products $(F^\times)^n$ for various n, then F_{alg} is an elementary substructure of F.*

The first point is very subtle, and the second is an immediate consequence, as pointed out in the preceding remark. The following notion will prove most useful in bringing Fact 3.8 to bear in a group theoretic context, and will provide a crucial tool in various genericity and conjugacy arguments.

Definition 3.9 *A definable divisible abelian group T of finite Morley rank is a* good torus *if every definable subgroup of T is the definable closure of its torsion.*

We record some formal properties of the notion.

Lemma 3.10

1. *If T is a good torus and $T_0 \leq T$ is definable and connected, then T_0 is a good torus.*
2. *A finite product of good tori is a good torus.*

Proof. The first point is clear. For the second, we deal with the product $T_1 \times T_2$ of two good tori, and a definable subgroup $T \leq T_1 \times T_2$. Note that $T \leq T^\circ \mathrm{Tor}(T)$ by Fact 2.16, so we may suppose that T is connected; we may then suppose further that T projects onto T_2.

Let T_0 be the definable closure of $\mathrm{Tor}(T)$. Then $\mathrm{Tor}(T)$ projects onto the torsion of T_2, by Fact 2.16, so T_0 projects onto T_2 and $T \leq T_1 T_0$, $T = T_0(T \cap T_1)$. As T_1 is a good torus, $T \cap T_1 \leq T_0$ as well. □

Lemma 3.11 *Let F be a field of finite Morley rank and nonzero characteristic. Then $F^\times$ is a good torus.*

Proof. Let T be a definable subgroup of $F^\times$. We will argue that it is the definable closure of its torsion. Let T_1 be the definable closure of $\mathrm{Tor}(T)$. Then the structure (F, T, T_1) is a field of finite Morley rank. Fact 3.8 shows that the induced structure $(F_{\mathrm{alg}}, T \cap F_{\mathrm{alg}}, T_1 \cap F_{\mathrm{alg}})$ is an elementary substructure of (F, T, T_1). But $T \cap F_{\mathrm{alg}} = T_1 \cap F_{\mathrm{alg}}$, and hence passing to the elementary extension, $T = T_1$. □

Lemma 3.12 *Let D be a good torus in an $\aleph_0$-saturated structure. Then every uniformly definable collection of subgroups of D is finite.*

Proof. Let $\mathcal{F} = \{\phi(x, \overline{a}) : \overline{a} \in D^{l(\overline{a})}\}$ be a family of subgroups of D. We argue that there exists a natural number n such that:

(1)
For any $T_1, T_2 \in \mathcal{F}$, whenever their elements of order at most n are the same, then $T_1 = T_2$.

Suppose that there is no such bound n. Add to the language constants $\overline{a_1}$ and $\overline{a_2}$ and define the following set of sentences:

$$\begin{aligned}
\Phi \quad = \quad & \mathrm{Th}(D) \cup \{\text{“}\phi(x, \overline{a_i}) \text{ is a subgroup”} : i = 1, 2\} \cup \\
& \forall x (x^n = 1 \to (\phi(x, \overline{a_1}) \leftrightarrow \phi(x, \overline{a_2}))) \quad (n \geq 1) \cup \\
& \exists x ((\phi(x, \overline{a_1}) \wedge \neg\phi(x, \overline{a_2})) \vee (\neg\phi(x, \overline{a_1}) \wedge \phi(x, \overline{a_2})))
\end{aligned}$$

By saturation this set is satisfiable, and as a result D is not good, a contradiction.

Given (1), the fact that two given elements of $\mathcal{F}$ are distinct is witnessed within a fixed finite set of elements, as the Prüfer ranks in a torus of finite Morley rank are finite. This implies that $\mathcal{F}$ is finite. □

Lemma 3.13 *Let $A \rtimes B$ be a solvable group of finite Morley rank where A, B, and the action of B on A are definable. Assume that A and B are connected and B acts on A faithfully. If A is an elementary abelian p-group for some prime p, and B has no nontrivial p-elements, then B is a good torus.*

Proof. We first note that by Fact 2.14, B is divisible abelian. We may form a series of subgroups $(0) = A_0 < A_1 < \ldots < A_n = A$ for which each A_i/A_{i-1} is B-minimal. Let B_i be the subgroup of $\mathrm{Aut}(A_i/A_{i-1})$ induced by the action of B. By Fact 3.3 each B_i is a definable subgroup of the multiplicative group of some field of finite Morley rank, hence is a good torus by Lemmas 3.10 1 and 3.11. We have a canonical definable map $B \to \prod_i B_i$, whose image is therefore a good torus (Lemma 3.10). Furthermore the kernel of this map is trivial by Fact 2.24, so B is a good torus. □

The following result by Poizat is also crucial for many arguments below. Its proof uses Fact 3.8 before invoking the classification of simple locally finite groups of finite Morley rank, which in turn uses the classification of the finite simple groups [5, 6, 9, 10, 19].

Fact 3.14 ([29]) *If F is a field of finite Morley rank of characteristic $p \neq 0$, then every simple definable section of $\mathrm{GL}_n(F)$ is definably isomorphic to an algebraic group over F.*

Poizat's theorem, together with the following linearization result, will eliminate various configurations involving a nontrivial definable action of a connected nonsolvable group of degenerate type on an elementary abelian 2-group.

Fact 3.15 ([11, Theorem 9.5]) *Let $A \rtimes G$ be a connected group of finite Morley rank where G is definable, A is abelian and G-minimal, and $C_G(A) = 1$. Assume further that G has a definable infinite abelian normal subgroup H. Then $C_A(G) = 1$, H is central in G, $F = \mathbb{Z}[H]/ann_{\mathbb{Z}[H]}(A)$ is an interpretable algebraically closed field, A is a finite dimensional F-vector space, and the action of G on A is by vector space automorphisms; so $G \leq \mathrm{GL}_n(F)$ via this action, where n is the dimension. Furthermore, $H \leq Z(G) \leq Z(\mathrm{GL}_n(F))$.*

Poizat states his result only for simple subgroups of $\mathrm{GL}_n(K)$. For the reduction of the general result to that case, see [4] and Remarque 3 in [29].

4 Genericity

As was mentioned in the introduction, two types of genericity arguments are encountered in the sequel. In this section we provide some tools for both of them. The first half, up to Lemma 4.6, is used to understand the structure of the normalizers of Borel subgroups. The second half makes use of the notion of good torus to obtain a conjugacy statement for the Borel subgroups of groups of degenerate type. This will be our main conjugacy theorem in the rest.

Fact 4.1 ([14]) *Let G be a connected group of finite Morley rank and B a definable subgroup of G of finite index in its normalizer. Assume that there is a definable subset X of B, not generic in B, such that $B \cap B^g \subseteq X$ whenever $g \in G \setminus N_G(B)$. Then $\cup_{g\in G} B^g$ is generic in G.*

Fact 4.2 ([14]) *Let G be a group of finite Morley rank and B a definable divisible abelian subgroup of G such that $B \cap B^g$ is finite for every $g \in G \setminus N_G(B)$. Then there exists B_0, a finite subgroup of B, such that $B \cap B^g \leq B_0$ for every $g \in G \setminus N_G(B)$.*

Fact 4.3 ([14]) *Let H be a group of finite Morley rank such that H° is abelian. If x is an element in $H \setminus H^\circ$ such that the elements of the coset xH° are generically of order n for some fixed integer $n > 1$, then every element of xH° is of order n.*

Fact 4.4 ([14]) *Let G be a connected group of finite Morley rank and B be a proper definable connected subgroup of finite index in its normalizer in G such that $\cup_{g\in G}B^g$ is generic in G. Assume that $x \in N_G(B) \setminus B$ is of order $n > 1$ modulo B. Then the definable subset*

$$\{x_1 \in xB : x_1 \in (\langle x\rangle B)^g \text{ for some } g \in G \setminus N_G(B)\}$$

of xB is generic in xB.

Lemma 4.5 *Let G be a connected group of finite Morley rank with a conjugacy class of definable divisible abelian subgroups that are of finite index in their normalizers. Assume that any two distinct elements of this family have finite intersection. If B is a subgroup in this family and $x \in N_G(B) \setminus B$, then $C_B(x)$ is finite.*

Proof. The proof is a blend of ideas and results from [14]. Fact 4.2 implies that B has a finite subgroup B_0 such that for any B^g distinct from B, $B \cap B^g \leq B_0$. Fact 4.1 then implies that the set $\cup_{g\in G}B^g$ is generic in G.

Now suppose towards a contradiction that there exists $x \in N_G(B) \setminus B$ with $C_B(x)$ infinite. By Fact 4.4 and the last paragraph, we conclude that

$$\mathcal{B} = \{x_1 \in xB : x_1 \in (\langle x\rangle B)^g \text{ for some } g \in G \setminus N_G(B)\}$$

is generic in xB. Let m be the order of x in $N_G(B)/B$.

If $x_1 \in \mathcal{B}$, then there exists $g \in G \setminus N_G(B)$ such that $x_1^m \in B \cap B^g \leq B_0$. Thus xB has a generic subset such that if x_1 is in this subset then $x_1^m \in B_0$. It follows that there exists n such that $x_1^n = 1$ generically on xB. By Fact 4.3 all elements in xB are of order n. In particular for any $c \in C_B^\circ(x)$, $c^n = x^n c^n = (xc)^n = 1$. This contradicts the structure of B. □

Lemma 4.6 *Let G be a connected group of finite Morley rank. Assume that B is a good torus which is of finite index in $N_G(B)$. Then the set $\mathcal{B} = \cup_{g\in G}B^g$ is generic in G.*

Proof. By Lemma 3.12, there exist $g_1, \ldots, g_m \in G$ such that for any $g \in G$, $B \cap B^g = B \cap B^{g_i}$ for some $1 \leq i \leq m$. As a result, $B \cap (\cup_{g\in G\setminus N_G(B)}B^g)$ is a definable subset of B not generic in B. Fact 4.1 implies that $\cup_{g\in G}B^g$ is generic in G. □

Lemma 4.7 *Let G be a connected group of finite Morley rank. Assume that B is a good torus which is of finite index in $N_G(B)$. Assume also that B_1 is a definable connected subgroup of G such that $\mathcal{B}_1 = \cup_{g\in G}B_1^g$ is a generic subset of G. Then B is conjugate to a subgroup of B_1.*

Proof. Let $\mathcal{B} = \cup_{g\in G}B^g$. Let $X = B \setminus \mathcal{B}_1$ and $Y = B \cap \mathcal{B}_1$. To prove the statement it suffices to show that Y is generic in B. Indeed, if Y is generic in B, then, since by Lemma 3.12 there exist $g_1, \ldots, g_m \in G$ such that $B \cap \mathcal{B}_1 = B \cap (\cup_{i=1}^m B_1^{g_i})$, $B \cap B_1^{g_i}$ is generic in B for some g_i. As B is connected, we conclude that $B = B \cap B_1^{g_i}$.

Now suppose towards a contradiction that Y is not generic in B. We define the map

$$\begin{array}{rccc} \Psi & : & (G/B) \times Y & \longrightarrow & \cup_{g\in G}Y^g \\ & & (Bg, y) & \longmapsto & y^g \end{array}$$

As B is abelian and $Y \subseteq B$, the map Ψ is well-defined. Since it is definable and surjective, it follows that

$$\mathrm{rk}\,(\cup_{g\in G}Y^g) \leq \mathrm{rk}\,(G) - \mathrm{rk}\,(B) + \mathrm{rk}\,(Y).$$

The nongenericity of Y in B implies that $\mathrm{rk}\,(Y) < \mathrm{rk}\,(B)$ and therefore $\mathrm{rk}\,(\cup_{g\in G}Y^g) < \mathrm{rk}\,(G)$. Since $\mathcal{B}$ is generic in G by Lemma 4.6, and $\mathcal{B} = (\cup_{g\in G}X^g) \cup (\cup_{g\in G}Y^g)$, it follows that $\cup_{g\in G}X^g$ is generic in G. Since G is connected, it follows that $(\cup_{g\in G}X^g) \cap \mathcal{B}_1 \neq \emptyset$. But by the definition of X, $(\cup_{g\in G}X^g) \cap \mathcal{B}_1 = \emptyset$, a contradiction. □

Corollary 4.8 *Let $A \rtimes G$ be a group of finite Morley rank where G, A and the action of G on A are definable. Assume that A is connected and elementary abelian of exponent 2, that G is connected of degenerate type, and that G acts faithfully on A. If B is a Borel subgroup of G then $B \cap (\cup_{g \in G \setminus N_G(B)} B^g)$ is not generic in B, and the Borel subgroups of G are conjugate in G.*

Proof. By Lemma 3.13, and Fact 2.19, any Borel subgroup of G is a good torus. Now the conclusion follows from Lemma 3.12, Lemma 4.6 and Lemma 4.7. □

Corollary 4.9 *The same conclusion as that of Corollary 4.8 holds if the kernel of the action of G on A is solvable, in particular when this kernel is finite.*

Proof. Let K denote the kernel of the action of G on A. $A \rtimes (G/K)$ satisfies the hypotheses of the above theorem. Moreover if B_1 and B_2 are Borel subgroups of G then B_1K/K and B_2K/K are Borel subgroups of G/K. By Corollary 4.8 there exists $g \in G$ such that $B_1^g K = B_2 K$. But $B_1 = (B_1K)^\circ$ and $B_2 = (B_2K)^\circ$. Hence $B_1^g = B_2$. □

5 Groups with strongly/weakly embedded subgroups: preliminaries

In this section we review some basic facts about groups of finite Morley rank with strongly and weakly embedded subgroups. The emphasis will be on strongly embedded subgroups. We recall the definitions of these two important notions:

Definition 5.1

1. *Let G be a group of finite Morley rank. A proper definable subgroup M of G is said to be* strongly embedded *if $I(M) \neq \emptyset$ and for any $g \in G \setminus M$ $I(M \cap M^g) = \emptyset$.*
2. *Let G be a group of finite Morley rank. A proper definable subgroup M of G is said to be* weakly embedded *if M has infinite Sylow 2-subgroups and for $g \in G \setminus M$ $M \cap M^g$ has finite Sylow 2-subgroups.*

The notion of weak embedding is a weakening of strong embedding if the ambient group G has infinite Sylow 2-subgroups, that is if G is not of degenerate type.

The following theorem by Eric Jaligot is the strongest result proven to this day about groups of finite Morley rank with weakly embedded subgroups, and plays a major role in the classification simple K^*-groups of even type. Evidently its generalization to the context of L^*-groups would be an important breakthrough in the analysis of simple L^*-groups of even type.

Fact 5.2 ([22]) *A simple K^*-group of even type with a weakly embedded subgroup is isomorphic to $\mathrm{PSL}_2(F)$ where F is an algebraically closed field of characteristic 2.*

Now we go over some properties of strongly embedded subgroups. We start with some elementary general properties.

Fact 5.3 ([18, Theorem 9.2.1]) *Let G be a group of finite Morley rank with a proper definable subgroup M. Then the following are equivalent:*

1. *M is a strongly embedded subgroup.*
2. *$I(M) \neq \emptyset$, $C_G(i) \leq M$ for every $i \in I(M)$, and $N_G(S) \leq M$ for every Sylow 2-subgroup of M.*
3. *$I(M) \neq \emptyset$ and $N_G(S) \leq M$ for every nontrivial 2-subgroup S of M.*

Before we go any further we recall, only for the sake of comparison, an analogous characterization in the case of weak embedding. This result will not be used in the sequel but it gives useful insight into the changes in the arguments when results in the context of groups of even type with strongly embedded subgroups are generalized to the context of groups with weakly but not necessarily strongly embedded subgroups.

Fact 5.4 ([2]) *Let G be a group of finite Morley rank, M a proper definable subgroup of G. M is weakly embedded if and only if the following hold:*
1. M has infinite Sylow 2-subgroups.
2. For any nontrivial unipotent 2-subgroup U and nontrivial 2-torus T in M, $N_G(U) \leq M$ and $N_G(T) \leq M$.

Arguments involving *strongly real* elements will frequently be encountered in the sequel.

Definition 5.5 *Let G be a group.*

1. *For $x \in G$, $C^*_G(x) = \{g \in G : x^g = x \text{ or } x^{-1}\}$.*
2. *An element of G is said to be* strongly real *if it is the product two involutions.*

Now we take up the review of elementary general properties of groups of finite Morley rank with strongly embedded subgroups.

Fact 5.6 ([18, Theorem 9.2.1];[11, Theorem 10.19]) *Let G be a group of finite Morley rank and M be a strongly embedded subgroup of G. Then the following hold:*

1. $\mathrm{Syl}_2(M) \subseteq \mathrm{Syl}_2(G)$.
2. *$I(G)$ is a single conjugacy class.*
3. *The involutions in M are conjugate in M.*
4. *If $i \in I(M)$ and x is a nontrivial strongly real element in $C_G(i)$, then $C^*_G(x) \leq M$.*

As was mentioned in the introduction, the following simple consequence of Fact 5.6 3 shows that $\Omega_1(M^\circ) = \Omega_1^\circ(M) = \Omega_1(M)$ when M is strongly embedded with infinite Sylow 2-subgroups, which is the general context of this paper.

Fact 5.7 ([4]) *Let G be a group of finite Morley rank with a strongly embedded subgroup M, and X a normal subgroup of M with an infinite Sylow 2-subgroup. Then $I(M) \subseteq X^\circ$. In particular, if G is of even type then $I(M) = I(M^\circ)$.*

Fact 5.8 ([18, Theorem 9.2.1 (iii)];[1, Lemma 3.8]) *Let G be a group of finite Morley with a strongly embedded subgroup M. Then there is an involution $w \in G \setminus M$ such that $\mathrm{rk}\,(I(wM)) \geq \mathrm{rk}\,(I(M))$.*

Let w be an involution with $\mathrm{rk}\,(I(wM)) \geq \mathrm{rk}\,(I(M))$. Then we define

$$Y = \{uw : u \in I(wM)\},\ K = d(Y)$$

$$Y_0 = \{y \in K^\circ : y^w = y^{-1}\},\ K_1 = d(Y_0)$$

Note that $K_1 \leq K^\circ$. The following conclusions can be proven:

Fact 5.9 ([1]) *Let G be a group of finite Morley rank with a strongly embedded subgroup. Then the group $K = d(Y)$ as defined above contains no involutions.*

Fact 5.10 ([1]) *Let G be a group of finite Morley rank with a strongly embedded subgroup M. Then for $i \in I(M)$, $M^\circ = C_G{}^\circ(i)K^\circ$.*

These results are variations, for finite Morley rank, on results from finite group theory ([18], chapter 9). Under stronger hypotheses we may refine them as follows:

Fact 5.11 ([4]) *Let G be a group of finite Morley rank with a strongly embedded subgroup M containing infinitely many involutions. Then $K^\circ = C_{K^\circ}(w)Y_0$ and $\mathrm{rk}\,(Y_0) = \mathrm{rk}\,(Y)$.*

Fact 5.12 ([4]) *Let G be a group of finite Morley rank with a strongly embedded subgroup M containing infinitely many involutions. Then* $\mathrm{rk}\,(Y_0) = \mathrm{rk}\,(I(M))$. *If in addition G is a simple L^*-group of even type then the group $A = \Omega_1(M) = \langle I(M) \rangle$ is a definable connected elementary abelian 2-subgroup such that $A \leq Z(B(M))$ and $A = I(M) \cup \{1\}$. Moreover, any subgroup of M° containing Y_0, in particular K_1, acts transitively on $I(M)$.*

The following two facts are very useful corollaries of Fact 5.12:

Fact 5.13 ([4]; [1, Corollary 4.6]) *Let G be a simple L^*-group of even type with a strongly embedded subgroup M. If $a, i, j \in G^\times$ and i and j are involutions, with i commuting with a and j inverting a, then a is also an involution.*

Fact 5.14 *Let G be a simple L^*-group of even type with a strongly embedded subgroup M. If $x, i \in M$ such that $i \in I(M)$ and $x^i = x^{-1}$, then $x^2 = 1$.*

Proof. If x and i are as in the statement then $(xi)^2 = 1$ and by Fact 5.12, both i and xi are in $\Omega_1^\circ(Z(O_2(M)))$. Thus $x = (xi)i \in \Omega_1^\circ(Z(O_2(M)))$. □

One can also prove suitable versions of the last two facts in the context of simple L^*-groups of even type with weakly embedded subgroups, but this is not necessary for our present purposes.

The next L-group fact will be a useful tool in the sequel:

Fact 5.15 ([4]) *Let H be a connected L-group of even type with a weakly embedded subgroup M. Then*

$$H \cong L \times D$$

where $L = B(H) \cong \mathrm{SL}_2(F)$, with F algebraically closed of characteristic 2, and $D = C_H(L)$ is a subgroup of degenerate type. $M^\circ \cap L$ is a Borel subgroup of L and $D \leq M$.

Our point of departure is the following result on simple L^*-groups of even type with a weakly embedded subgroup:

Fact 5.16 ([4]) *Let G be a simple L^*-group of even type with a weakly embedded subgroup M. Then $M^\circ/{O_2}^\circ(M^\circ)$ is of degenerate type.*

In the context of simple L^*-groups of even type, Fact 5.16 serves as a substitute for the result that the connected component of a weakly embedded subgroup of a simple K^*-group of even type is solvable. In this paper this result will be applied only to *strongly* embedded subgroups.

6 $M \cap M^w$

In this section we begin the proof of Theorem 1 below. Most of the proof of Theorem 1 follows the general line of argument in [1] and the third section of [22], whose computational aspects were strongly inspired by [15]. On the other hand, the proof of Theorem 2 represents a major deviation from those lines of argument. Its proof involves the ideas introduced in §§3, 4.

G will denote a simple L^*-group satisfying the hypotheses of Theorem 1 with a strongly embedded subgroup M and $A = \Omega_1(O_2(M))$. Note that by Fact 5.12, A is the largest elementary abelian 2-subgroup in M, it contains all the involutions in M and it is connected. Thus $A = \Omega_1(M) = \Omega_1^\circ(M) = \Omega_1(M^\circ)$.

Definition 6.1 *Let G be a group of finite Morley rank with a strongly embedded subgroup M.*

1. *For $w \in I(G) \setminus M$, set $T(w) = \{x \in M^\circ : x^w = x^{-1}\}$.*
2. *$X_1 = \{w \in I(G) \setminus M : \mathrm{rk}\,(T(w)) < \mathrm{rk}\,(A)\}$.*
3. *$X_2 = \{w \in I(G) \setminus M : \mathrm{rk}\,(T(w) \geq \mathrm{rk}\,(A)\}$.*

Note that Facts 5.8 and 5.12 show that $X_2 \neq \emptyset$ in a simple L^*-group of even type with a strongly embedded subgroup. We will occasionally refer to the elements of X_2 as X_2-involutions. For the rest of the article we fix an involution $w \in X_2$. As the first major step in the proof of Theorem 1, the action of w on the intersection $M \cap M^w$ will be analyzed. The main result will be that w inverts $T = (M \cap M^w)^\circ$ under the assumption $(*)$ of Theorem 1; this will be given as Theorem 2 below. We recall that since M is strongly embedded, we have $I(T) = \emptyset$.

Notation 6.2 *As indicated, we keep the following notation throughout this section.*

$$w \in X_2; T = (M \cap M^w)^\circ$$

Lemma 6.3 *The Borel subgroups of T are good tori.*

Proof. Let B be a Borel subgroup of T. Then B acts on A and on A^w, giving us two maps $B \to B_1, B_2$ onto the corresponding subgroups of $\mathrm{Aut}(A)$ and $\mathrm{Aut}(A^w)$. By the assumption $(*)$, the kernel of the induced map into $B_1 \times B_2$ is finite, hence central. As B_1 and B_2 are good tori (Lemma 3.13), so is the image of B (Lemma 3.10), and hence so is B. □

As preparation for the proof of Theorem 2, we will establish a number of results concerning the case in which $C_T^\circ(A) = 1$. The case in which this centralizer is infinite will be handled separately, and with less difficulty, in the proof of Theorem 2.

Proposition 6.4 *If $C_T^\circ(A) = 1$ then we may assume, after modifying the choice of w appropriately, without however altering the choice of coset wM or the intersection $M \cap M^w$, that w normalizes some Borel subgroup B of T.*

Proof. Since $C_T(A)$ is assumed to be finite, Corollary 4.9 implies that the Borel subgroups of T are conjugate. By the Frattini argument $N(T) = TN_{N(T)}(B)$, where B is a Borel subgroup of T. Hence $w = w't$ for some $t \in T$ and $w' \in N(B) \cap N(T)$. It follows that $1 = (w't)^2 = {w'}^2 t^{w'} t$ and ${w'}^2 \in T$. Since $I(T) = \emptyset$, there exists $t' \in T \cap d(w')$ such that $w't'$ is an involution. Note that $t' \in N(B)$ since $d(w') \subseteq N(B)$. So $w't'$ is an involution in $wT \subseteq wM$, and, in particular, $w't'$ is also an X_2-type involution. Since $M^w = M^{w't'}$, we can replace w with $w't'$. □

Lemma 6.5 *Let $G = A \rtimes T$ be a group of finite Morley rank, with A, T and the action of T on A definable. Assume A is an elementary abelian p-group and T is a connected solvable $p^\perp$-group. Then $A = [A, T] \oplus C_A(T)$.*

Proof. We proceed by induction on the rank and degree of A. We may assume the action is faithful. By Lemma 3.13, T is a good torus. Being nontrivial, T contains a nontrivial element t of finite order n. Since T has no elements of order p, we have $(n, p) = 1$. By Fact 2.23, $A = [A, t]C_A(t)$. The intersection $[A, t] \cap C_A(t)$ is trivial, since for $\gamma = [a, t] \in C_A(t)$ we find $1 = [a, t^n] = \gamma^n$, with $(n, p) = 1$. So $A = [A, t] \oplus C_A(t)$.

Let $A_0 = C_A(t)$. Since the action is faithful, we have $A_0 < A$. Since A_0 is normalized by T, inductively, $A_0 = [A_0, T] \oplus C_{A_0}(T)$ and our claim follows. □

Proposition 6.6 *If $C_T^\circ(A) = 1$ then the intersection of two distinct Borel subgroups of T is finite.*

Proof. Let B_1 and B_2 be two distinct Borel subgroups of T. These are good tori by Lemma 6.3. Suppose $B_0 = (B_1 \cap B_2)^\circ \neq 1$. As $C_T^\circ(A) = 1$, applying w we find $C_T^\circ(A^w) = 1$ as well. Thus $C_{B_0}^\circ(A) = C_{B_0}^\circ(A^w) = 1$.

Let $X = C_T^\circ(B_0)$. Note that X is not solvable since it contains two distinct Borel subgroups, namely B_1 and B_2. Let A_1 be an X-minimal subgroup of $[A, B_0]$. Since $[A, B_0] \neq 1$, B_0 acts nontrivially on A_1 by Lemma 6.5. Let K_1 denote the kernel in X of this action. By Facts 3.15 and 3.14, X/K_1 is solvable. In the rest of the proof we will show that K_1° is solvable, so that X is solvable, yielding a contradiction. We may assume that K_1 is infinite.

We show:

(1) $C_{A^w}(K)$ is finite for any infinite definable nonsolvable connected subgroup of K_1.

Suppose $K \leq K_1$ is infinite, definable, and nonsolvable, but that $C_{A^w}(K)$ is infinite. Since $K \leq K_1$, $[K, A_1] = 1$ and $C_A(K)$ is infinite. This, together with the hypothesis that $C_{A^w}(B)$ is infinite, implies $L = B(C(K)) \cong \mathrm{PSL}_2$ in characteristic 2 (Fact 5.15). Let S_1 and S_2 be the two Sylow 2-subgroups of L such that $S_1 \leq A$ and $S_2 \leq A^w$. By Fact 5.3 3, $N_L(S_1) \cap N_L(S_2) \leq M \cap M^w$, and thus $M \cap M^w$ contains a maximal torus T_1 of L. By Fact 5.13 no nontrivial element of this torus commutes with an involution in A. Let $(0) = V_0 < \ldots < V_m = A$ be a definable KT_1-invariant series for A with KT_1-minimal quotients. The torus T_1 acts on each factor freely by Fact 2.22. It follows from Facts 3.15 and 3.14 that $KT_1/C_{KT_1}(V_{i+1}/V_i)$ is solvable for each $0 \leq i < m$, and thus by Fact 2.24 we have that $KT_1/C_{KT_1}(A)$ is solvable. But $C_{KT_1}(A)$ is finite, and therefore central in KT_1, which forces KT_1 to be solvable. So K is solvable, a contradiction. Therefore (1) holds. In particular, $C_{A^w}(K_1{}^\circ)$ is finite.

Now we consider a definable $K_1^\circ B_0$-minimal subgroup V of $[A^w, B_0]$. Since $[A^w, B_0] \neq 1$, B_0 acts nontrivially on V by Lemma 6.5. Let K_2 be the kernel of the action of $K_1^\circ B_0$ on V. Then $K_1{}^\circ B_0/K_2$ is solvable by Facts 3.15 and 3.14. In particular $K_1/(K_1 \cap K_2)^\circ$ is solvable. Hence $(K_1 \cap K_2)^\circ$ is nonsolvable. This contradicts (1). This final contradiction finishes the proof. □

Proposition 6.7 *If $C_T^\circ(A) = 1$ and B is a Borel subgroup of T, then $[w, B] \neq 1$.*

Proof. Suppose towards a contradiction that $[w, B] = 1$ for some Borel subgroup B of T. By Fact 5.13, B contains no strongly real elements.

Let y be a strongly real element in T inverted by w. Let $K = C_T^\circ(y)$. Since $I(T) = \emptyset$, Lemma 2.17 implies that $K \neq 1$. Then w normalizes K. Moreover, by the assumption $C_T^\circ(A) = 1$ and Corollary 4.9 applied to K, we conclude that the Borel subgroups of K (there may be only one, namely K) are conjugate. An application of the Frattini argument as in Proposition 6.4 shows that K has a Borel subgroup B_1 which is normalized by an involution w' inverting y. B_1 is contained in a Borel subgroup C of T. By Proposition 6.6, w' and y normalize C. Since C is a T-conjugate of B by Corollary 4.9, C does not have nontrivial strongly real elements. Thus the involutions w' and $w'y$ centralize C. It follows that y centralizes C. Since y is strongly real, $y \notin C$. But such a setup cannot exist by Lemma 4.5. □

Theorem 2 *Let G be a simple L^*-group of even type with a strongly embedded subgroup M, and $A = \Omega_1(O_2(M))$. Assume that G satisfies the hypothesis $(*)$ of Theorem 1. Then w inverts $T = (M \cap M^w)^\circ$.*

Proof. The proof is by contradiction. We suppose that w does not invert T. By Fact 2.7, $T = C_T(w)T^-$, where $T^- = \{t \in T : t^w = t^{-1}\}$, and $C_T(w)$ is infinite and connected. The argument divides into two cases according to whether $C_T(A)$ is finite or not.

Case 1: $C_T(A)$ is finite

Then Corollary 4.9 applies to all definable, connected subgroups of T.

Since $C_T(w)$ is infinite, it has a nontrivial Borel subgroup, which can be extended to a Borel subgroup B of T. By Proposition 6.6, w normalizes B. By Proposition 6.7, this action is not trivial. So $B^- = \{x \in B : x^w = x^{-1}\}$ is infinite.

By Fact 5.12, T acts on $I(A)$ transitively. Since $C_T(w)$ is infinite and $\mathrm{rk}\,(T^-) = \mathrm{rk}\,(A)$ (Facts 5.12 and 5.11), we have $\mathrm{rk}\,(T) > \mathrm{rk}\,(A)$. As a result, for every $u \in I(A)$, $C_T(u)$ is infinite. Let B_0 be a Borel subgroup of $C_T(u)$ where $u \in I(A)$. By Corollary 4.9, B_0 is T-conjugate to a subgroup of B. Replacing u by a T-conjugate accordingly, we may assume that $B_0 \leq B$. Fact 5.13 and the fact that $B^- \neq \{1\}$ imply that $B > B_0$. As B centralizes B_0 and B_0 centralizes u, it follows that $C_A(B_0)$ is infinite. Let $A_0 = C_A^\circ(B_0)$. Note that $A_0 < A$ since $C_T(A)$ is finite. We may assume that $u \in A_0$.

We will prove that for at least one prime p:

(1) B_0 has nontrivial p-torsion, and the Prüfer p-rank of B is at least 2.

By the case assumption and Lemma 3.13, B_0 is a good torus. The p-torsion in B_0 will be denoted by $\mathrm{Tor}_p(B_0)$. We suppose towards a contradiction that the Prüfer p-rank of B is 1 for any prime p such that $\mathrm{Tor}_p(B_0) \neq 1$. By Fact 2.24 and the case assumption, B_0 acts nontrivially on some definable connected section of A. Since w normalizes B and $B_0 \leq B$, both B_0 and B_0^w are subgroups of B. Thus our Prüfer rank assumption on B implies that $\mathrm{Tor}_p(B_0) = \mathrm{Tor}_p(B_0^w)$. Therefore, $B_0 \cap B_0^w$ is an infinite group with a nontrivial p-torus. Moreover, $B_0 \cap B_0^w$ is centralized by $\langle A_0, A_0^w \rangle$. By Fact 5.15, $C^\circ(B_0 \cap B_0^w) = L \times C_{C^\circ(B_0 \cap B_0^w)}(L)$ where $L = B(C(B_0 \cap B_0^w)) \cong \mathrm{PSL}_2$ in characteristic 2. Let $H = N_L(A) \cap N_L(A^w)$. Then $H \leq T$ and there is a Borel subgroup of T which contains both H and $(B_0 \cap B_0^w)^\circ$. Since $B_0 \leq B$, this Borel is B by Proposition 6.6. Hence $HB_0 \leq B$. The structure of L implies that H is a full torus, i.e. H contains a copy of $\mathbb{Z}_{p^\infty}$ for every prime $p \neq 2$. Since B_0 centralizes u, Fact 5.13 implies that $H \cap B_0 = 1$. It follows that B has Prüfer p-rank 2, a contradiction. So (1) holds.

We fix a definable B-invariant subgroup A_1 of M containing A_0, with A_1/A_0 B-minimal. It follows from (1) and Fact 3.3 that $C_B(A_1/A_0)$ is infinite. Let $K_0 = C_B^\circ(A_1/A_0)$. Then by Fact 2.22, $A_1/A_0 = C_{A_1}(K_0)A_0/A_0$. Let $A_2 = C_{A_1}^\circ(K_0)$ and $B_1 = C_B^\circ(A_2)$. Since $K_0 \leq B$, $B_1 \neq 1$. Since A_2 covers A_1/A_0, $A_2 \not\subseteq A_0$. By the transitive action of T on $I(A)$ (Fact 5.12), there exists $g \in T$ such that $u^{g^{-1}} \in A_2 \setminus A_0$. Since $u \in A_0$, $g \notin C_T(u)$. Note that $u \in C_A^\circ(B_1^g)$. By the conjugacy of the Borel subgroups of $C_T^\circ(u)$ in $C_T^\circ(u)$ (Corollary 4.9), there exists $g' \in C_T(u)$ such that $B_1^{gg'} \leq B_0$. Since $B_0 \leq B$, Proposition 6.6 implies that $gg' \leq N_T(B)$. We have $u^{(gg')^{-1}} = u^{g'^{-1}g^{-1}} = u^{g^{-1}} \in A_2 \setminus A_0$ while $u \in A_0$. But A_0 is B-invariant, thus $gg' \notin B$. As $gg' \in N(B) \setminus B$, we have:

(2) There exists an element $\sigma \in N_T(B) \setminus B$ such that $\sigma^p \in B$ for some prime p.

By Fact 2.16, we may assume σ is a p-element.

We claim that B has a nontrivial Sylow p-subgroup. Suppose towards a contradiction that B has no p-torsion. In any case, since T has no involutions and is connected, $C_T(\sigma)$ is infinite by Lemma 2.17. Thus $C_T(\sigma)$ has a nontrivial Borel subgroup C_0, which is contained in a Borel subgroup C of T. By Proposition 6.6, σ normalizes C. Moreover since C is conjugate to B by Corollary 4.9 and B is assumed not to contain p-torsion, $\sigma \notin C$. This contradicts Lemma 4.5.

By Lemma 6.3 we know that B is divisible abelian. Hence the Sylow p-subgroup of B is the direct sum of finitely many copies of $\mathbb{Z}_{p^\infty}$. If the Prüfer p-rank of B is 1, then we contradict Lemma 4.5 using Fact 2.5. As a result, the Prüfer p-rank of B is at least 2.

We let $R = C_B(w)$. As has already been mentioned, R is infinite and connected. We define $V = C_{A_w}(R)$, where A_w is the conjugate of A containing w. We claim that $\langle V^B \rangle \not\leq A_w$. Suppose that $\langle V^B \rangle \leq A_w$. Since B normalizes $\langle V^B \rangle$, $B \leq N(A_w) = M_w$, where M_w is the conjugate of M containing A_w. Then $B^- \leq M_w$, which contradicts Fact 5.14.

As V contains w, $V \neq 1$. However, V can in principle be finite. We eliminate this possibility first. In this case, since A_w is conjugate to A, M° contains a conjugate R_1 of R such that $C_A(R_1)$ is finite and nontrivial. Let $\overline{M^\circ} = M^\circ/C_{M^\circ}(A)$. Corollary 4.8 implies that the Borel subgroups of $\overline{M^\circ}$ are conjugate. Since $\overline{R_1}$ and $\overline{B^-}$ are contained in Borel subgroups of $\overline{M^\circ}$, we may assume they are in the same Borel subgroup. It follows that $\overline{B^-}$ normalizes $C_A(\overline{R_1})$. Since $\overline{B^-}$ is connected, we conclude that $\overline{B^-}$ centralizes $C_A(\overline{R_1})$, that is B^- centralizes $C_A(\overline{R_1})$. $C_A(\overline{R_1})$ is nontrivial, but on the other hand B^- contains strongly real elements, and these elements cannot centralize involutions in A (Fact 5.13), a contradiction.

The last two paragraphs show that $C(R)$ is a definable subgroup of G with a strongly embedded subgroup, namely $C(R) \cap M_w$, and that $B(C(R)) \neq 1$. It follows using Fact 5.15 that $B(C(R)) \cong \mathrm{PSL}_2$ in characteristic 2. Note that since R is infinite, the Sylow 2-subgroups of $B(C_G(R))$ are strictly contained in the conjugates of A which contain them by the hypothesis $(*)$. We claim that $w \in B(C(R))$. w normalizes R and by Remark 2.9 it acts on $B(C(R))$ as an element of $B(C(R))$, say a. Since $B(C(R)) \cap A_w \neq 1$, $a \in A_w$. Hence, wa is an element of

A_w whose centralizer contains $B(C(R))$. Facts 5.3 2 and 5.16 imply that this is possible only if $w = a$.

We let $L = B(C(R))$. Since B centralizes R, B normalizes L, and by Remark 2.9, $B \leq LC(L)$. It follows that $R \leq C(L)$. In fact, R normalizes L and centralizes w, and there exists no nontrivial noninvolutory inner automorphism of L with this property. Now we show that $B^- \leq L$. Let x be a nontrivial element of B^-. By Remark 2.9 there exists an element $t \in L$ such that $xt^{-1} \leq C(L)$. Since $x^w = x^{-1}$, we have $x^{-1}t^{-w} = (xt^{-1})^w = xt^{-1}$. Therefore, $x^2 = t^{-w}t = [w,t] \in B^- \cap L$. If $x \notin L$ then Fact 2.16 implies that B^- has nontrivial 2-elements, which is not true. Hence $x \in L$.

We claim

(3) For any prime l, the Prüfer l-rank of B is at most 2.

We have shown in the above paragraph that B^- is a subgroup of L. It is contained in a maximal torus of L. Since $L \cong \mathrm{PSL}_2$, the Prüfer l-rank of B^- is at most 1. Thus it suffices to show that the Prüfer l-rank of R is at most 1, because $RB^- = B$. Let A_1 be a conjugate of A such that $L \cap A_1$ is a Sylow 2-subgroup of L normalized by B. Such a conjugate exists because $B \leq LC(L)$ (Remark 2.9). We define $A_{11} = L \cap A_1$. By the hypothesis $(*)$, $A_{11} < A_1$. Since B normalizes A_{11}, it acts on A_1/A_{11}. Let A_{12}/A_{11} be an R-minimal subgroup of A/A_{11}. Now suppose towards a contradiction that the Prüfer l-rank of R is at least 2 for some odd prime l. Then Fact 3.3 implies that $R_0 = C_R(A_{12}/A_{11})$ is infinite. By Fact 2.22, $A_{12}/A_{11} = C_{A_{12}/A_{11}}(R_0) = C_{A_{12}}(R_0)A_{11}/A_{12}$. It follows that $B(C(R_0)) > B(C(R))$. But since R centralizes R_0, R normalizes $B(C(R_0))$. Since R centralizes involutions in $B(C(R_0))$, Remark 2.9 and the fact that R has no involutions imply that R centralizes $B(C(R_0))$, and we have $B(C(R)) \geq B(C(R_0)) > B(C(R))$, a contradiction. Thus (3) holds.

In particular with $l = p$ where $p = |\overline{\sigma}|$, we find: the Prüfer p-rank of R is 1, and the Prüfer p-rank of B^- is 1. We will prove that two distinct conjugates of R under the action of $\langle\overline{\sigma}\rangle$ have trivial intersection, where $\overline{\sigma}$ denotes the coset of σ modulo B. It suffices to show $R \cap R^{\sigma^i} = 1$ for $1 < i < p$. Note that for such an i, $C_B(\sigma^i)$ is finite by Lemma 4.5. Suppose towards a contradiction that $x \in (R \cap R^{\sigma^i})^\times$. We then have $G > B(C(x)) \geq \langle B(C(R)), B(C(R))^{\sigma^i}\rangle$. Since $[R,x] = 1$, R normalizes $B(C(x))$. Moreover, $B(C(x)) \geq B(C(R))$ which implies using Fact 5.15 that $B(C(x)) \cong \mathrm{PSL}_2$ in characteristic 2. Remark 2.9 implies that R centralizes $B(C(x))$. As a result $B(C(x)) = B(C(R))$. A similar argument shows that $B(C(x)) = B(C(R))^{\sigma^i}$. Therefore σ^i normalizes L. Since $C_B(\sigma^i)$ is finite and the Prüfer p-rank of B^- is 1, Fact 2.5 implies that $B^- \cap (B^-)^{\sigma^i}$ does not contain a p-torus. Therefore, since $(B^-)^{\sigma^i} \leq B$, B is abelian and the Prüfer p-rank of B^- is 1, we conclude that $B^-(B^-)^{\sigma^i}$ is a group of Prüfer p-rank 2. But $B^-(B^-)^{\sigma^i} \leq LL^{\sigma^i} = L$ and PSL_2 does not contain a p-torus of Prüfer rank 2.

We claim that the elements of order p in B are partitioned by the conjugates of R under the action of σ, together with those of B^-. Indeed, since the Prüfer p-rank of B is 2, B contains p^2-1 elements of order p. By the above paragraph $p(p-1)$ of these are covered by the conjugates of R. Since B^- contains strongly real elements that cannot centralize involutions (Fact 5.13), it intersects trivially the conjugates of R. Moreover, B^- is of Prüfer p-rank 1. Hence the $p-1$ elements of order p that are not covered by the conjugates of R under the action of σ are contained in B^-. In particular, the only elements in B of order p that are centralized by σ are those in B^-.

Now we will show that B^- is a maximal torus of L. We first show that $L \cap A$ and $L \cap A^w$ are Sylow 2-subgroups of L. We showed above that the Prüfer p-rank of B is 2. Let A_0 be a B-minimal subgroup of A. Then, using Fact 3.3 we conclude that the Prüfer p-rank of $C_B(A_0)$ is at least 1. The last paragraph shows that the elements of order p in $C_B(A_0)$ are contained in a conjugate of R under the action of σ. We may assume they are contained in R. Let x be such an element. As usual one obtains $B(C(x)) = L$. In particular, $A_0 \leq L$. Since $x^w = x$, $A_0^w \leq L$ as well. Hence $L \cap A$ and $L \cap A^w$ are Sylow 2-subgroups of L. The maximal torus of L, which normalizes $L \cap A$ and $L \cap A^w$ is thus contained in T. But this maximal torus contains B^-. Since B is a Borel subgroup of T, B^- is exactly this torus.

Next we prove that $|N_T(B)/B| = 3$. By Lemma 4.5, $C_T(B) = B$. As a result $N_T(B)/B$

embeds in the automorphism group of B. Let $W = N_T(B)/B$ and $n = |W|$. Let l be an odd prime. Then the last paragraph shows that the Prüfer l-rank of B is at least 1. Since we already showed that the Prüfer l-rank of R cannot be higher than 1, we conclude that there are either $l-1$ or l^2-1 elements of order l in B. W acts on this set. By Lemma 4.5, $C_B(\overline{\rho})$ is finite for every $\overline{\rho} \in W$. Hence for every prime number l except finitely many, W acts regularly on the set of elements of B of order l. It follows that n divides l^2-1 for all primes l except finitely many. We will make a number theoretic argument to conclude that $n = 3$. Let p^α be a prime power divisor of n. Then p^α divides l^2-1 for almost all primes l. Equivalently, $l^2 - 1 \equiv 0 \pmod{p^\alpha}$. It is well-known (see for example [20] Theorem 4.19) that the units in $\mathbb{Z}/p^\alpha\mathbb{Z}$ form a cyclic subgroup when p is odd. Since this group is also of even order, it has a unique cyclic subgroup of order 2. Hence $l^2 - 1 \equiv 0 \pmod{p^\alpha}$ is equivalent to $l \equiv \pm 1 \pmod{p^\alpha}$. So for almost all primes l, we have $l \equiv \pm 1 \pmod{p^\alpha}$. On the other hand, by Fact 2.28, there are infinitely many primes l with $l \equiv 2 \pmod{p^\alpha}$. Thus $2 \equiv \pm 1 \pmod{p^\alpha}$. Hence $p^\alpha | 3$, and thus $n = 3$.

In particular $\sigma^3 \in B$. We will show that $\sigma^3 = 1$.

Suppose toward a contradiction that $|\sigma| = 3^i$ with $i > 1$. We let B_3 denote the Sylow 3-subgroup of B. We first prove that $\langle C_{B_3}(\sigma), \sigma\rangle = \langle\sigma\rangle$. As $C_B(\sigma)$ is a finite group, Fact 2.6 implies that all elements in the coset of σ modulo B are conjugate to σ. Thus all of these elements are of order strictly bigger than 3. The same argument yields the same conclusion for the coset of σ^{-1}. Since $\sigma^3 \in B$, we conclude that there are no elements of order 3 in $\langle C_{B_3}(\sigma), \sigma\rangle \setminus B$. On the other hand, we have proven above that the only elements of order 3 in B that are centralized by σ are in B^-. But B^- has been proven to be a group of Prüfer 3-rank 1. As a result, the elements of order 3 in $C_{B_3}(\sigma)$ generate a cyclic group. It follows that $\langle C_{B_3}(\sigma), \sigma\rangle$ is a cyclic group. Since no element of B can have σ as a power, we find $\langle C_{B_3}(\sigma), \sigma\rangle = \langle\sigma\rangle$.

Let U denote the copy of $\mathbb{Z}_3 \times \mathbb{Z}_3$ in B. The above discussion on the partition of elements of order 3 implies that $C_U(\sigma)$ is of order 3 and is generated by $\tau = \sigma^{3^{i-1}}$. Since σ acts on $U/C_U(\sigma)$ and does not centralize U, we conclude that $[\sigma, U] = C_U(\sigma)$. Thus there exists $u \in U$ such that $[\sigma, u] = \tau$. It follows that $\sigma^u = \sigma^{3^{i-1}+1}$ and in particular u normalizes $C(\sigma)$ and thus $C_T(\sigma)$. By Corollary 4.9, the Borel subgroups of $C_T^\circ(\sigma)$ are conjugate in $C_T^\circ(\sigma)$. The Borel subgroups of $C_T^\circ(\sigma)$ are contained in Borel subgroups of T, and Proposition 6.6 implies that σ normalizes each one of these Borel subgroups of T. Then Lemma 4.5 implies that σ is contained in each one of these Borel subgroups. The Frattini argument shows that there exists $h \in C_T^\circ(\sigma)$ such that uh normalizes one of these Borels, say C. The action of uh on C is the same as that of σ on B. Since h centralizes σ, we have $\sigma^{uh} = \sigma^{3^{i-1}+1}$. Comparing the actions of uh on C and σ on B, we conclude that there exists τ_1 in B such that $[\sigma, \tau_1] = \tau$ with τ_1 of order 3^i and $\tau_1^{3^{i-1}} = \tau$. In particular, $[\sigma, \tau_1^3] = 1$. Then comparing the orders of τ_1 and σ, and using the conclusion $\langle C_{B_3}(\sigma), \sigma\rangle = \langle\sigma\rangle$, we conclude that $C_{B_3}(\sigma) = \langle\sigma^3\rangle = \langle\tau_1^3\rangle$.

Now we consider the map

$$\begin{array}{cccc} \gamma_\sigma & : & \langle\tau_1, u\rangle & \longrightarrow & \langle\tau\rangle \\ & & x & \longmapsto & [\sigma, x] \end{array}$$

This is a surjective homomorphism whose kernel contains $\langle\tau_1^3\rangle$. As $\langle\tau_1, u\rangle/\langle\tau_1^3\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, this kernel properly contains $\langle\tau_1^3\rangle$. Equivalently $C_{\langle\tau_1,u\rangle}(\sigma) > \langle\tau_1^3\rangle$. On the other hand, we have proven that $C_{B_3}(\sigma) = \langle\tau_1^3\rangle$. These two conclusions are contradictory since $\langle\tau_1, u\rangle \leq B_3$. We have proven that $|\sigma| = 3$.

Now we can reach the final contradiction. The involution w normalizes $N_T(B)$. Since $N_T(B)$ does not contain involutions, Fact 2.7 implies that $N_T(B) = (C(w) \cap N_T(B))N_T(B)^-$, where $N_T(B)^-$ is the set of elements in $N_T(B)$ inverted by w. We first show that $C(w) \cap N_T(B) \leq B$. Suppose towards a contradiction that $x \in C(w) \cap (N_T(B) \setminus B)$. Then x normalizes R and thus acts on L. Since $d(x)$ does not contain involutions and $w \in L$, this action is trivial by Remark 2.9. In particular, x centralizes B^- which is an infinite subgroup of B. This contradicts Lemma 4.5. It follows that $N_T(B)^-$ covers $N_T(B)/B$. In particular, there exists σ_1 in the same coset modulo B as σ and inverted by w. Since σ_1 and σ are in the same coset modulo B, the preceding discussion on σ can be applied to σ_1 as well. Thus, we may replace σ by σ_1 and assume that

σ is inverted by w. Let τ be an element of order 3 in B centralized by σ. Then we know from the above that τ is inverted by w and $\langle \sigma, \tau \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. But $\langle \sigma, \tau \rangle$ cannot operate on a $\langle \sigma, \tau \rangle$-irreducible subgroup of A faithfully. This implies that for some $i, j \in \{1, 2\}$, $\sigma^i \tau^j$ centralizes an involution in A. But this element is also inverted by w, a contradiction to Fact 5.13.

Case 2: $C_T(A)$ is infinite.

We start by setting $K_1 = C_T^\circ(A)$ and $K_2 = C_T^\circ(A^w)$. Under the current case assumption, $K_1 \neq 1$ and $K_2 \neq 1$. Note that the group generated by K_1 and K_2 is their central product, $K_1 * K_2$. In fact, by the assumption $(*)$ $K_1 \cap K_2$ is a finite group. Since $K_1 \lhd T$ and $K_2 \lhd T$, $[K_1, K_2] \leq K_1 \cap K_2$. But $[K_1, K_2]$ is connected. Therefore, $[K_1, K_2] = 1$.

We let B be a Borel subgroup of K_1. Note that the assumption $(*)$ forces $C_{K_1}(A^w)$ to be finite. As a result the Borel subgroups of K_1 are conjugate by Corollary 4.9. Moreover by Fact 2.14 these Borel subgroups are divisible abelian. The same structural conclusion is valid for K_2 since $C_{K_2}(A)$ is finite by the assumption $(*)$ as well.

Before entering into the argument, we define $R = \{bb^w : b \in B\}$ and $T_0 = \{bb^{-w} : b \in B\}$. Since $[K_1, K_2] = 1$, R is a group; and since B is abelian, T_0 is a group as well. Note also that the mappings $b \longmapsto bb^w$ and $b \longmapsto bb^{-w}$ are isogenies and thus $\mathrm{rk}\,(R) = \mathrm{rk}\,(B) = \mathrm{rk}\,(T_0)$.

We first prove that $R = C_{BB^w}(w)$ and $T_0 = (BB^w)^- = \{x \in BB^w : x^w = x^{-1}\}$. Clearly, $R \leq C_{BB^w}(w)$ and $T_0 \leq (BB^w)^-$. By Fact 2.7, $BB^w = C_{BB^w}(w) \times (BB^w)^-$ and both subgroups are connected. On the other hand, since $\mathrm{rk}\,(R) = \mathrm{rk}\,(T_0) = \mathrm{rk}\,(B)$ and $\mathrm{rk}\,(BB^w) = 2\mathrm{rk}\,(B)$, it follows that $\mathrm{rk}\,(C_{BB^w}(w)) = \mathrm{rk}\,(R)$ and $\mathrm{rk}\,((BB^w)^-) = \mathrm{rk}\,(T_0)$. The connectedness of $C_{BB^w}(w)$ and $(BB^w)^-$ yield the desired equality. In particular, $BB^w = R \times T_0$.

Next we define $V = C_{A_w}(R)$, where A_w is the conjugate (in G) of A containing w. We claim that $\langle V^{BB^w} \rangle \not\leq A_w$. In fact, if $\langle V^{BB^w} \rangle \leq A_w$, then by Fact 5.3 $BB^w \leq N(A_w)$. In particular, T_0 belongs to the strongly embedded subgroup containing A_w, which contradicts Fact 5.14. Now it follows from the assumption $(*)$ that $\mathrm{rk}\,(V) < \mathrm{rk}\,(A)$. Indeed, since BB^w centralizes R, R centralizes $\langle V^{BB^w} \rangle$ and we have just seen that this last group has nontrivial intersection with at least two distinct conjugates of A.

In the remainder of the proof we will have to consider a further division into two cases, depending on whether V is finite or not. Either possibility will yield a contradiction in due course.

We first eliminate the case in which V is finite. In this case, since A_w is conjugate to A, M° contains a conjugate R_1 of R such that $C_A(R_1)$ is finite and nontrivial. Let $\overline{M^\circ} = M^\circ / C_{M^\circ}(A)$. Corollary 4.8 implies that the Borel subgroups of $\overline{M^\circ}$ are conjugate. Since $\overline{R_1}$ and $\overline{T_0}$ are contained in Borel subgroups of $\overline{M^\circ}$, we may assume they are in the same Borel subgroup. It follows that $\overline{T_0}$ normalizes $C_A(\overline{R_1})$. Since $\overline{T_0}$ is connected, we conclude that $\overline{T_0}$ centralizes $C_A(\overline{R_1})$, that is T_0 centralizes $C_A(\overline{R_1})$. $C_A(\overline{R_1})$ is nontrivial but on the other hand T_0 contains strongly real elements which cannot centralize involutions (Fact 5.13), a contradiction.

Now we embark on a longer argument, which will eliminate the remaining case, in which V is infinite. As $\langle V^{BB^w} \rangle \not\leq A_w$, $C(R)$ is a group with a strongly embedded subgroup $C(R) \cap M_w$, where M_w is the strongly embedded subgroup of G containing A_w. As V is infinite, Fact 5.15 shows that $L = B(C_G(R)) \cong \mathrm{PSL}_2(K)$ in characteristic 2.

Let C be a Borel subgroup of T containing BB^w. We recall that by Lemma 6.3, C is divisible abelian. Since C is abelian, $[C, R] = 1$ and it follows that C normalizes L. Since C acts by inner automorphisms on L (Fact 2.8) and is a divisible abelian group without involutions, $C/C_C(L)$ embeds in a maximal torus of L and C normalizes two distinct Sylow 2-subgroups of L. By Fact 5.3 3, C normalizes a Sylow° 2-subgroup of G. We call this group A_1. We will prove that $C_C(A_1)$ is finite.

Before we go any further we note that $C_C(L) \neq 1$ as $R \leq C$. Moreover, $B(C(C_C(L))) = L$. In fact, $B(C(C_C(L))) \geq L$ and thus $B(C(C_C(L))) \cong \mathrm{PSL}_2$ in characteristic 2 by Fact 5.15. Now, Fact 3.5 and a comparison of the maximal tori of $B(C(C_C(L)))$ and L shows that $B(C(C_C(L))) = L$.

Let $X = C_C^\circ(A_1)$ and suppose towards a contradiction that $X \neq 1$. Since X centralizes $C_C(L)$, X normalizes $B(C(C_C(L))) = L$. But X centralizes $A_1 \cap L$ which is a nontrivial

2-subgroup (indeed a Sylow 2-subgroup) of L. Since by Remark 2.9 X acts on L by inner automorphisms, we conclude that X centralizes L. The definition of X and the fact that A_1 is a conjugate of A contradict the hypothesis $(*)$. Hence $C_C(A_1)$ is finite. In particular, C is a good torus by Lemma 3.13.

We now show that $C_G^\circ(BB^w) \leq T$. The group $C_G(BB^w)$ normalizes $B(C_G(B))$. On the other hand, $B(C_G(B))$ contains A and is centralized by B, which is an infinite subgroup of G. Therefore the hypothesis $(*)$ implies that $B(C_G(B)) = A$. It follows from Fact 5.3 3 that $C_G(BB^w) \leq M$. Since w normalizes $C_G(BB^w)$, $C_G^\circ(BB^w) \leq T$.

We claim $N_G^\circ(C) = C_G^\circ(C) = C$. Since $C \geq BB^w$, $C_G^\circ(C) \leq C_G^\circ(BB^w) \leq T$. But C is a Borel subgroup of T and it is abelian, hence $C_G^\circ(C) = C$. On the other hand, we have proven that C is a good torus, so by Fact 2.18 $N_G^\circ(C) = C_G^\circ(C)$.

Now we will reach a contradiction, which will eliminate the case in which V is infinite, and thus complete the proof of Theorem 2. Let M_1 be the conjugate of M containing A_1. Then M_1 contains both C and a conjugate (in G) C_1 of C distinct from C such that $C_{C_1}(A_1)$ is infinite. Being a conjugate of C, C_1 is a good torus. It was proven above that $N_G^\circ(C) = C$. Thus $N_G^\circ(C_1) = C_1$ as well. By Lemma 4.6, $\cup_{g \in M_1^\circ} C^g$ and $\cup_{g \in M_1^\circ} C_1^g$ are both generic in M_1°. It follows from Lemma 4.7 that C and C_1 are conjugate in M_1. This is a contradiction, since $C_{C_1}(A_1)$ is infinite while $C_C(A_1)$ is finite. □

Theorem 2 allows us to obtain more precise information on the structure of M° and its interaction with w. We will continue to use the notation T and w defined as in the proof of Theorem 2. The first corollary is in fact equivalent to Theorem 2.

Corollary 6.8 $\mathrm{rk}\,(T) = \mathrm{rk}\,(A)$.

Proof. Let us prove the equivalence of this statement to Theorem 2. We remind that in the notation of Fact 5.12, $Y_0 \subseteq K^\circ \leq (M \cap M^w)^\circ = T$. If w inverts T as stated in Theorem 2, then we have $T = Y_0$. Then the equality $\mathrm{rk}\,(T) = \mathrm{rk}\,(A)$ follows from the same fact. On the other hand, if $\mathrm{rk}\,(T) = \mathrm{rk}\,(A)$, then by Fact 5.12 $\mathrm{rk}\,(T^-) = \mathrm{rk}\,(T)$, where T^- is the set of elements in T inverted by w. Theorem 2 follows from an application of Fact 2.7 to T and w. □

Corollary 6.9 $A \rtimes T \cong F_+ \rtimes F^\times$ *where F is an algebraically closed field of characteristic* 2.

Proof. By Theorem 2, T is abelian, and by Corollary 6.8, T acts regularly on A. As everything is definable in a structure of finite Morley rank, the field is algebraically closed. □

The following corollary will be useful in the final section:

Corollary 6.10 $C^\circ(A)^w \cap M$ *is finite.*

Proof. Since $(M \cap M^w)^\circ$ has no involutions, Fact 5.13 and Theorem 2 imply that $(C^\circ(A)^w \cap M)^\circ = 1$. □

Corollary 6.11 *For any $i \in I(A)$, $C_{M^\circ}(i) = C_{M^\circ}(A)$. $M^\circ = C_{M^\circ}(A) \rtimes T$.*

Proof. By Fact 5.12, T acts transitively on $I(A)$. As a result, $M^\circ = C_{M^\circ}(i)T$, where $i \in I(A)$. Since $I(T) = \emptyset$ and T is inverted by$w$, $C_T(i) = 1$ by Fact 5.13. Hence, we have $1 = \deg(M^\circ) = \deg(C_{M^\circ}(i))\deg(T)$. As a result $\deg(C_{M^\circ}(i)) = 1$ and $C_{M^\circ}(i) = C_{M^\circ}^\circ(i) = C_M^\circ(i)$. We therefore have $M^\circ = C_M^\circ(i)T$ as well.

Suppose towards a contradiction that $C_{M^\circ}(A) < C_{M^\circ}(i)$. Let $\overline{M^\circ} = M^\circ/C_{M^\circ}(A)$. By Corollary 4.8 the Borel subgroups of $\overline{M^\circ}$ are conjugate. As a result, in order to achieve a contradiction, it would suffice to prove that $\overline{T}$ is a Borel subgroup of $\overline{M^\circ}$. In fact, by assumption, $\overline{C_{M^\circ}(i)}$ is an infinite definable subgroup whose Borel subgroups are contained in those of $\overline{M^\circ}$, and thus the fact that no element of $\overline{T}$ (Fact 5.13, Theorem 2) can centralize an involution would yield a contradiction.

By Fact 2.14 it suffices to prove that $\overline{T}$ is a maximal definable connected abelian subgroup of $\overline{M^\circ}$. If $\overline{T} \leq \overline{T}_1$ where $\overline{T}_1$ is a definable abelian subgroup of $\overline{M^\circ}$, then the transitive action of $\overline{T}$ on A implies that $\overline{T}_1 = \overline{T}C_{\overline{T}_1}(A)$. But $C_{\overline{T}_1}(A) = 1$. □

Corollary 6.12 *For any $t \in T^\times$, w inverts $C^\circ(t)$.*

Proof. By Fact 5.13, $I(C^\circ(t)) = \emptyset$. Hence, by Fact 2.7, it suffices to prove that $C^\circ(t) \cap C(w) = 1$. Suppose $X = C^\circ(t) \cap C(w) \neq 1$. Then by Fact 2.7, X is connected and hence infinite. By Corollary 6.11, $X \leq C^\circ(w) = C^\circ(A_w)$ where A_w is the conjugate of A which contains w. Then by Fact 5.3 3 $X \leq M_w$, where M_w is the strongly embedded subgroup containing A_w. Note that $A \neq A_w$ and thus $M \neq M_w$ (Fact 5.12). The assumption $(*)$ implies that $C(X) \leq M_w$ as otherwise one could find distinct conjugates of A_w in $C(X)$ using elements of $C(X) \setminus M_w$. Since $[t, X] = 1$, we conclude that $t \in M_w$. By Fact 5.14, this is impossible since t is a nontrivial strongly real element inverted by w and $|t| > 2$. □

Corollary 6.13 *For any nontrivial subgroup $X \leq T$, $C^\circ_M(X) = T$. In particular, $C(A, T)$ is finite.*

Proof. By Corollary 6.11, $C_M{}^\circ(X) = C_{M^\circ}(AX) \rtimes T$; as this is connected, it follows that $C_{M^\circ}(AX)$ is connected. Thus if $T < C^\circ_M(X)$, then $C_M{}^\circ(AX)$ is nontrivial. By Corollary 6.12, w inverts $C_M{}^\circ(AX)$, and thus by Fact 5.13, $C^\circ_M(AX)$ is a connected elementary abelian 2-group which is centralized by w. But then Fact 5.3 3 implies that $w \in M$, a contradiction. □

Corollary 6.14 *Two distinct M°-conjugates of T have trivial intersection.*

Proof. Let $x \in M^\circ$ be such that $T \cap T^x \neq 1$. Then by Corollary 6.13, $T = C^\circ_M(T \cap T^x) = T^x$. □

Corollary 6.15 $N_{M^\circ}(T) = T$. *In particular, $C_{M^\circ}(A, T) = 1$.*

Proof. We first prove that $N^\circ_M(T) = T$. $N^\circ_M(T)$ centralizes the torsion subgroup of T by Fact 2.18. But by Corollary 6.9, T is the full multiplicative group of an algebraically closed field of characteristic 2. Then, by Fact 3.5 $N^\circ_M(T)$ centralizes T. But $C^\circ_M(T) = T$ by Corollary 6.13.

By Corollary 6.11, $N_{M^\circ}(T) = (C_{M^\circ}(A) \cap N_{M^\circ}(T))T$. Let $X = C_{M^\circ}(A) \cap N_{M^\circ}(T)$. Then X centralizes T by the semidirect product structure of M°. We will show that this forces $X = 1$. The last paragraph shows that T is of finite index in its normalizer in M°. Moreover, by Corollary 6.14, two distinct M°-conjugates of T have trivial intersection. Since T is divisible abelian, Lemma 4.5 can be applied to M° and T. Since $C_T(X)$ is infinite (namely T), we conclude that $X = 1$. □

Corollary 6.16 *If X is any nontrivial subgroup of T, then $C_{M^\circ}(X) = T$.*

Proof. Corollary 6.14 implies that $C_{M^\circ}(X)$ normalizes T. The conclusion follows from Corollary 6.15. □

Corollary 6.17 *If w is an X_2-involution then $M^\circ \cap M^{\circ w} = T$ where, as above, $T = (M \cap M^w)^\circ$.*

Proof. Clearly, $T \subseteq M^\circ \cap M^{\circ w}$. This inclusion, together with Corollary 6.11, implies that $M^\circ \cap M^{\circ w} = C_{M^\circ \cap M^{\circ w}}(A) \rtimes T$. But $C_{M^\circ \cap M^{\circ w}}(A)$ is a finite group by Corollary 6.10 and as a result T centralizes $C_{M^\circ \cap M^{\circ w}}(A)$. Then Corollary 6.15 implies that $C_{M^\circ \cap M^{\circ w}}(A) = 1$. □

Corollary 6.18 *For any X_2-involution w, $T(w) = (M \cap M^w)^\circ$.*

Proof. As in Theorem 2 and the preceding corollaries we let $T = (M \cap M^w)^\circ$. By its very definition $T(w) \subseteq M^\circ \cap M^{\circ w}$. Theorem 2 implies that $T \subseteq T(w)$. The conclusion follows from Corollary 6.17. □

7 Rank of G

As in the last section G will denote a simple L^*-group of even type with a strongly embedded subgroup M, which satisfies the hypothesis $(*)$ of Theorem 1. In this section we will obtain a formula for $\mathrm{rk}\,(G)$. The genericity arguments in the next section will make this sharper. We recall that if w is an X_2-involution then by Theorem 2, w inverts $(M \cap M^w)^\circ$. Moreover, Corollary 6.18 shows that $T(w) = (M \cap M^w)^\circ$. In particular $T(w)$ is a definable, connected subgroup of M.

The underlying ideas in this section stem from [15]. They were later taken up with slight modifications in [1], and in the third section of [22].

Proposition 7.1 $\mathrm{rk}\,(I(G)) = \mathrm{rk}\,(X_2)$.

Proof. The proof consists of showing that $\mathrm{rk}\,(X_1) < \mathrm{rk}\,(I(G))$. An equivalence relation $\sim$ is defined on X_1 as follows: for $u_1, u_2 \in X_1$, $u_1 \sim u_2$ if and only if $u_1 M^\circ = u_2 M^\circ$. This condition is equivalent to $u_2 u_1 \in T(u_1)$. Note that $\mathrm{rk}\,(X_1) \leq \mathrm{rk}\,(X_1/\!\sim) + m$ where m is the maximal fiber rank for the quotient map $X_1 \longrightarrow X_1/\!\sim$. By the definition of $\sim$ and X_1, $m < \mathrm{rk}\,(A)$. Moreover, the mapping from $X_1/\!\sim$ into G/M° which assigns to each equivalence class $u/\!\sim$ the coset uM° is an injection by the definition of $\sim$. Hence, $\mathrm{rk}\,(X_1) < \mathrm{rk}\,(G) - \mathrm{rk}\,(M) + \mathrm{rk}\,(A) = \mathrm{rk}\,(G) - \mathrm{rk}\,(C_G(i)) - \mathrm{rk}\,(T) + \mathrm{rk}\,(A)$ where $i \in I(A)$, using Corollary 6.11. But $\mathrm{rk}\,(G) - \mathrm{rk}\,(C_G(i)) - \mathrm{rk}\,(T) + \mathrm{rk}\,(A) = \mathrm{rk}\,(I(G))$ using Corollary 6.8. □

Lemma 7.2 *If w_1 and w_2 are two X_2-involutions such that $T(w_1) \neq T(w_2)$ then $T(w_1) \cap T(w_2) = 1$.*

Proof. If $T(w_1) \cap T(w_2) \neq 1$ then Corollary 6.13 implies that $T(w_1) = C_M^\circ(T(w_1) \cap T(w_2)) = T(w_2)$. □

Proposition 7.3 *If $w_1 \in X_2$, then $T(w)$ and $T(w_1)$ are $C^\circ(A)$-conjugate.*

Proof. By Lemma 7.2 and Corollary 6.15, $\cup_{x \in M^\circ} T(w)^x$ is generic in M°. If w_1 is another X_2-involution, then the connectedness of M° implies that for some $x \in M^\circ$, $T(w)^x \cap T(w_1) \neq 1$. Then Lemma 7.2 implies that $T(w)^x = T(w_1)$. The $C^\circ(A)$-conjugacy follows from the structure of M° as described by Corollary 6.11. □

Proposition 7.4 $\mathrm{rk}\,(G) = \mathrm{rk}\,(C(T)) + 2\mathrm{rk}\,(C(A))$.

Proof. The standard line of argument (introduced in [15] and also used in [1, 22]) to reach such a conclusion consists of defining a suitable mapping from X_2 into $w^{C(T)C^\circ(A)}$. We have the necessary tools, notably Corollary 6.15, to reproduce the same analysis.

By Proposition 7.3, for any X_2-involution w_1, there exists $f \in C^\circ(A)$ such that $T^f = T(w_1)$. It follows that $w_1^{f^{-1}}$ inverts T and thus $w_1^{f^{-1}} w$ centralizes T. Note also that by Corollary 6.15 f is unique.

Hence we can define the following definable map:

$$\begin{array}{cccc} \Phi & : & X_2 & \longrightarrow & w^{C(T)C^\circ(A)} \\ & & w_1 & \longmapsto & w^{w_1^{f^{-1}} w f} \end{array}$$

We show that Φ has finite fibers. If $w^{w_1^{f^{-1}} w f} = w^{w_1'^{f'^{-1}} w f'}$, then since this element inverts both T^f and $T^{f'}$, we have $T^f = T^{f'}$. Then Corollary 6.15 implies that $f = f'$. It follows that $w^{f w_1} = w^{f w_1'}$ and $(w_1 w_1')^{f^{-1}} \in C(T, w)$. But $C(T, w)$ is a finite group by Corollary 6.12, which proves the finiteness of the fibers.

The conclusion of the last paragraph implies that $\mathrm{rk}\,(X_2) \leq \mathrm{rk}\,(w^{C(T)C^\circ(A)})$. Since $\mathrm{rk}\,(X_2) = \mathrm{rk}\,(I(G))$ by Proposition 7.1, we have $\mathrm{rk}\,(X_2) = \mathrm{rk}\,(w^{C(T)C^\circ(A)})$

Next we show that $\mathrm{rk}\,(w^{C(T)C^\circ(A)}) = \mathrm{rk}\,(C(T)C^\circ(A))$. We define the following definable map:

$$\begin{array}{cccc} \Psi & : & C(T)C^\circ(A) & \longrightarrow & w^{C(T)C^\circ(A)} \\ & & cf & \longmapsto & w^{cf} \end{array}$$

The fibers of this map are finite because if $w^{cf} = w^{c'f'}$ then both w^{cf} and $w^{c'f'}$ invert $T^f = T^{f'}$. Then it follows from Corollary 6.15 that $f = f'$, thus $c'c^{-1} \in C(T, w)$, and this last group is finite by Corollary 6.12. Since Ψ is clearly surjective we have $\mathrm{rk}\,(C(T)C^\circ(A)) = \mathrm{rk}\,(w^{C(T)C^\circ(A)})$. The rank computations using Φ now yield $\mathrm{rk}\,(X_2) = \mathrm{rk}\,(C(T)C^\circ(A))$.

Since $C(T) \cap C^\circ(A) = 1$ by Corollary 6.15, it follows from Proposition 7.1 and Fact 5.6 that $\mathrm{rk}\,(C(T)) + \mathrm{rk}\,(C^\circ(A)) = \mathrm{rk}\,(X_2) = \mathrm{rk}\,(I(G)) = \mathrm{rk}\,(G) - \mathrm{rk}\,(C_G(i))$, where i can be taken to be in $I(A)$. Using Corollary 6.11, we have $\mathrm{rk}\,(G) = \mathrm{rk}\,(C(T)) + 2\mathrm{rk}\,(C(A))$. □

8 Centralizers of tori

We continue to use the same notation as in the previous sections. The main result in this section is that T is of finite index in its centralizer (Proposition 8.4). As in the last section we follow the line of approach introduced in [15], incorporating variations from [22] and keeping track of the L^*-structure of G. The shift from the K^*-context to the L^*-context becomes visible in the proof of Lemma 8.3, where we use an adaptation of the arguments in Lemme 4.25 of [22], a lemma about an analogous configuration.

Lemma 8.1 $\mathrm{rk}\,(X_2M^\circ) = \mathrm{rk}\,(G)$.

Proof. The following equivalence relation is defined on X_2: $w_1 \sim w_2$ if and only if $w_1M^\circ = w_2M^\circ$ (if and only if $w_2w_1 \in T(w_1)$). As $\mathrm{rk}\,(T(w_1)) = \mathrm{rk}\,(T)$, we conclude that $\mathrm{rk}\,(X_2) = \mathrm{rk}\,(X_2/\sim) + \mathrm{rk}\,(A)$. Since $\mathrm{rk}\,(X_2) = \mathrm{rk}\,(I(G))$ by Proposition 7.1, it follows using Facts 5.6 2, 5.12 and Corollary 6.11 that

$$\begin{aligned} \mathrm{rk}\,(G) &= \mathrm{rk}\,(C_G(A)) + \mathrm{rk}\,(X_2/\sim) + \mathrm{rk}\,(A) \\ &= \mathrm{rk}\,(M) - \mathrm{rk}\,(T) + \mathrm{rk}\,(X_2/\sim) + \mathrm{rk}\,(A) \\ &= \mathrm{rk}\,(M) + \mathrm{rk}\,(X_2/\sim) \\ &= \mathrm{rk}\,(X_2M^\circ) \end{aligned}$$

□

Lemma 8.2 *If $c \in C^\circ(T) \setminus M$ then $I(fcM^\circ) = \emptyset$ for any $f \in C^\circ(A)$.*

Proof. Suppose fcb is an involution for $b \in M^\circ$ and f, c as in the statement of the lemma. Using Corollary 6.11 we may assume that $b \in C^\circ(A)$. After conjugating fcb by f we conclude that cu is also an involution where $u = bf^{-1} \in C^\circ(A)$. If $t \in T$ then $(cu)^t = cu^t$ and $[u, t] = (cu)^{-1}(cu)^t \in T(cu) \cap C^\circ(A)$. By Fact 5.13, the set $T(cu) \cap C^\circ(A)$ contains elements of order at most 2. But $cu \in I(G) \setminus M$ and it cannot centralize involutions in M by Fact 5.3 2. Hence, $T(cu) \cap C^\circ(A) = 1$. As t is an arbitrary element of T, we conclude that u centralizes T. Hence $cu \in I(C(T))$. But by Fact 5.13, no involution can centralize a nontrivial element of T. □

Lemma 8.3 *If for $f_1,\ f_2 \in C^\circ(A)$, $c_1,\ c_2 \in C^\circ(T) \setminus M$, we have $f_1c_1M^\circ = f_2c_2M^\circ$, then $f_1 = f_2$ and $c_1T = c_2T$.*

Proof. Suppose $f_1c_1 = f_2c_2v$ for some $v \in M^\circ$. We may assume $v \in C^\circ(A)$ by Corollary 6.11 and $c_1 = uc_2v$ where $u = f_1^{-1}f_2$.

We claim that $X = [v, T] = 1$. X is a definable connected subgroup contained in $M \cap M^{c_1}$ as $T^v = T^{c_2v} = T^{u^{-1}c_1} \leq M^{c_1}$ and $T^v \leq M^v = M$. As T normalizes X, XT is a group. In fact it is definable and connected. Note also that $X \leq M^{\circ\prime} \leq C^\circ(A)$ by Corollary 6.11. Thus $C_X(A^{c_1})$ is finite by the assumption $(*)$ of Theorem 1 and the fact that $A^{c_1} \neq A$. Since T is inverted by w, it acts freely on A^{c_1} (Fact 5.13). Let $K = C^\circ_{XT}(A^{c_1})$. By Corollary 6.11 applied to M^{c_1} and the connectedness of XT, we have $XT = KT$. Then $[T, K] \leq (XT)' \leq X \cap K$ as T is abelian and both X and K are normal in XT. Since $C_X(A^{c_1})$ is finite and $[T, K]$ is connected, we conclude that $[T, K] = 1$. But then $K \leq C^\circ(T)$, and this last group is inverted by w (Corollary 6.12) and has no involutions. Using Fact 5.13 it follows that $K = 1$. Therefore we have $XT = T$, and $X \leq T$. Since T acts freely on A, $X = 1$.

The last paragraph shows that $v \in C_{M^\circ}(T)$. It follows that $u \in C_{M^\circ}(T, A)$. The conclusion follows using Corollary 6.15. □

Proposition 8.4 $C^\circ(T) = T$.

Proof. It suffices to prove that $C^\circ(T) \leq M$. Suppose not and let $Y = \bigcup\{fcM^\circ : f \in C^\circ(A),\ c \in C^\circ(T) \setminus M\}$. By Lemma 8.3, the fact that $C_{M^\circ}(T) = T$ (Corollary 6.16) and Proposition 7.4, $\mathrm{rk}\,(Y) = \mathrm{rk}\,(C(A)) + \mathrm{rk}\,(C(T)) - \mathrm{rk}\,(T) + \mathrm{rk}\,(M) = \mathrm{rk}\,(C(T)) + 2\mathrm{rk}\,(C(A)) = \mathrm{rk}\,(G)$. Since by Lemma 8.1 X_2M° is also generic in G, Y and X_2M° share a coset of M°. This contradicts Lemma 8.2. □

Corollary 8.5 $\mathrm{rk}\,(G) = \mathrm{rk}\,(T) + 2\mathrm{rk}\,(C(A))$.

Corollary 8.6 *For $g \in G \setminus N_G(T)$, $T^g \cap T = 1$.*

Proof. Suppose that $g \in G$ such that $T^g \cap T \neq 1$. Let $t \in (T \cap T^g)^\times$. $\langle T, T^g\rangle$ is a definable connected subgroup that centralizes t. Thus it is inverted by w by Corollary 6.12. Hence $\langle T, T^g\rangle$ is abelian. Then by Proposition 8.4, we have $T = T^g$. □

9 Double transitivity

In this section we will finish the proof of Theorem 1. We continue to use the notation fixed in the previous sections. G denotes a simple L^*-group of even type with a strongly embedded subgroup M which satisfies the assumption $(*)$ of Theorem 1. As before, w is an X_2-involution and $T = (M \cap M^w)^\circ$.

Before starting the argument, we remark that here, as in [1] and in the third section of [22], the final steps are also to show that G is a Zassenhaus group. Here the arguments will be more complicated because it is more difficult to describe the intersection of two distinct conjugates of M. This is mainly due to the fact that, unlike the situation in [1] or [22], M° is not necessarily solvable.

Lemma 9.1 *For any $g \in G \setminus M$, $\mathrm{rk}\,(M \cap M^g) \geq \mathrm{rk}\,(T)$. In particular, $M \cap M^g$ is infinite.*

Proof. This lemma summarizes the preceding section. $2\mathrm{rk}\,(C(A)) + 2\mathrm{rk}\,(T) - \mathrm{rk}\,(M \cap M^g) = 2\mathrm{rk}\,(M) - \mathrm{rk}\,(M \cap M^g) = \mathrm{rk}\,(MM^g) \leq \mathrm{rk}\,(G) = 2\mathrm{rk}\,(C(A)) + \mathrm{rk}\,(T)$. □

Lemma 9.2 *For $g \in G \setminus M$, $(M \cap M^g)^\circ$ is abelian.*

Proof. Let $X = (M \cap M^g)^\circ$ where $g \in G \setminus M$. Let $X_1 = C_X(A)$ and $X_2 = C_X(A^g)$. By the assumption $(*)$ of Theorem 1, $X_1 \cap X_2$ is a finite group. The structure of M° (thus that of $M^{\circ g}$ as well) as described by Corollary 6.11 forces $[X, X] \leq X_1 \cap X_2$. But $[X, X]$ is connected, thus trivial. □

Lemma 9.3 *For $x \in I(G) \setminus M$, x does not centralize $(M \cap M^x)^\circ$.*

Proof. Suppose towards a contradiction that $x \in I(G) \setminus M$ centralizes $(M \cap M^x)^\circ$. In particular x centralizes $(C(A)^x \cap M)^\circ$, and as x is an involution it follows that $(C(A)^x \cap M)^\circ = (C(A) \cap M^x)^\circ$. Thus, the assumption $(*)$ implies that $(C(A)^x \cap M)^\circ = 1$. In particular, $C(A)^x \cap M$ is a finite group.

Consider the map

$$\begin{array}{cccc} \theta & : & C^\circ(A) \times M & \longrightarrow & G \\ & & (f, m) & \longmapsto & fxm \end{array}$$

For $f,\ f' \in C^\circ(A)$ and $m,\ m' \in M$, $fxm = f'xm'$ if and only if $(f'^{-1}f)^x = m'm^{-1} \in C^\circ(A)^x \cap M$. Thus, θ has finite fibers as $C(A)^x \cap M$ is a finite group. It follows from Corollary 8.5 that $C^\circ(A)xM$ is a generic subset of G. On the other hand, Corollary 6.10 implies that this last argument can be carried out to conclude that $C^\circ(A)wM$ is generic as well. Since G is connected, we conclude that $C^\circ(A)xM = C^\circ(A)wM$. Hence, we also have $MxM = MwM$ and there exist $m,\ m' \in M$ such that $mx = wm'$. We then have $T^{mx} = T^{wm'} = T^{m'} \leq M$ and therefore $T^{mx} \leq (M \cap M^x)^\circ$. But this contradicts Fact 5.13 since x centralizes $(M \cap M^x)^\circ$ while T^{mx} is inverted by w^{mx}. □

Proposition 9.4 *For any $g \in G \setminus M$, $C^\circ(A) \cap M^g$ is finite.*

Proof. It suffices to prove that $(C^\circ(A) \cap M^g)^\circ = 1$. Suppose that $K = (C^\circ(A) \cap M^g)^\circ \neq 1$. Let $i \in I(A)$. Then $K \leq M^g \cap M^{gi}$. The assumption $(*)$ implies that $A \lhd C(K)$. Now, let $X = (M^g \cap M^{gi})^\circ$. By Lemma 9.2, $X \leq C(K)$ as well. It follows that $[i, X] \leq X \cap A = 1$. But an application of Lemma 9.3 to i and $M^g \cap M^{gi}$ shows that this is impossible. □

Proposition 9.5 *The action of G on G/M is doubly transitive.*

Proof. Let $g \in G \setminus M$. Proposition 9.4 implies that the mapping

$$\begin{array}{cccc} \theta & : & C^\circ(A) \times M & \longrightarrow & G \\ & & (f, m) & \longmapsto & fgm \end{array}$$

has finite fibers. As a result the set $C^\circ(A)gM$ is generic in G (Corollary 8.5). Evidently, so is MgM. Since g is an arbitrary element in $G \setminus M$ it follows as in the proof of Lemma 9.3 that if $x \notin M$ then $x \in MgM$. Therefore $G = M \sqcup MgM$. □

Corollary 9.6 *For any $g \in G \setminus M$, $M \cap M^g$ is conjugate to $M \cap M^w$.*

Proof. This follows from Proposition 9.5 and the fact that for $g \in G \setminus M$, $M \cap M^g$ is a 2-point stabilizer. □

The arguments of Proposition 9.5 yield sharper information in the special case $g = w$.

Proposition 9.7 *$C^\circ(A) \cap M^w = 1$ and M is connected. In particular, $C(A) = C^\circ(A)$ and $M = C(A) \rtimes T$.*

Proof. Suppose $x \in C^\circ(A) \cap M^w$. Since $C_G(A) \leq M$ by Fact 5.3 2, $x \in M^\circ$. It follows that $x \in N_{M^\circ}(T)$. Thus by Corollary 6.15, $x \in T$. But $C_T(A) = 1$.

Now the mapping

$$\begin{array}{cccc} \theta & : & C^\circ(A) \times M & \longrightarrow & G \\ & & (f, m) & \longmapsto & fwm \end{array}$$

of Proposition 9.5 becomes injective by the last paragraph's conclusion. As in the proof of Proposition 9.5, $C^\circ(A)wM$ is generic in G. In addition to this, the injectivity of θ implies that $\deg(C^\circ(A) \times M) = 1$. Hence $\deg(M) = 1$. □

Corollary 9.8 *$N_M(T) = T$. In particular, for any $g \in G \setminus M$, $M \cap M^g$ is connected.*

Proof. The first statement is a consequence of Corollary 6.15 and Proposition 9.7. As for the second statement, let $g \in G \setminus M$. By Corollary 9.6, we may assume that $M \cap M^g = M \cap M^w$. Then since $T = (M \cap M^w)^\circ$, the conclusion follows from the first part. □

Proof of Theorem 1. By Proposition 9.5, the action of G on G/M is doubly transitive. Moreover by Corollary 6.11 and Proposition 9.7, $M = C(A) \rtimes T$ where $T = M \cap M^w$ is a 2-point stabilizer. Moreover by Lemma 9.1, $T \neq 1$, i.e. G is not sharply 2-transitive. In order to conclude using Fact 2.27 it suffices to prove that 3-point stabilizers are trivial. In this vein consider M, wM and fwM where $f \in C(A)$. Suppose an element t stabilizes these three points. Then $t \in M \cap M^w = T$ and we have $fwM = tfwM = f^{t^{-1}}twM = f^{t^{-1}}wt^{-1}M = f^{t^{-1}}wM$. Thus $[f, t^{-1}]^w \in C(A)^w \cap M = 1$ (Proposition 9.7). As $fwM \neq wM$ by the choice of the three points, $f \neq 1$. Since two distinct M-conjugates of T have trivial intersection by Corollary 6.14 and Proposition 9.7, either $t = 1$ or $T^f = T$. Since $f \in C(A)^\times$ and $C_T(A) = 1$, Corollary 9.8 leaves only one possibility: $t = 1$. □

References

[1] T. Altınel. Groups of finite Morley rank with strongly embedded subgroups. *J. Algebra*, 180:778–807, 1996.

[2] T. Altınel, A. Borovik, and G. Cherlin. Groups of mixed type. *J. Algebra*, 192:524–571, 1997.

[3] T. Altınel, A. Borovik, and G. Cherlin. On groups of finite Morley rank with weakly embedded subgroups. *J. Algebra*, 211:409–456, 1999.

[4] T. Altınel and G. Cherlin. On groups of finite morley rank of even type. To appear in J. Algebra.

[5] V. V. Belyaev. On locally finite Chevalley groups. In *17th All Union Algebraic Conference*, page 17, Leningrad, 1981. In Russian.

[6] V. V. Belyaev. On locally finite Chevalley groups. In *Investigations in Group Theory*, pages 39–50. Urals Scientific Center, Sverdlovsk, 1984. In Russian.

[7] A. Borovik and A. Nesin. On the Schur-Zasssenhaus theorem for groups of finite Morley rank. *J. Symbolic Logic*, 57:1469–1477, 1992.

[8] A. Borovik and A. Nesin. Schur-Zassenhaus theorem revisited. *J. Symbolic Logic*, 59:283–291, 1994.

[9] A. V. Borovik. Periodic linear groups of odd characteristic. *Soviet Math. Dokl.*, 26:484–486, 1982.

[10] A. V. Borovik. Embeddings of finite Chevalley groups and periodic linear groups. *Siberian Math. J.*, 24:843–851, 1983.

[11] A. V. Borovik and A. Nesin. *Groups of Finite Morley Rank*. Oxford University Press, 1994.

[12] A. V. Borovik and B. Poizat. Tores et p-groupes. *J. Symbolic Logic*, 55:478–490, 1990.

[13] G. Cherlin. Groups of small Morley rank. *Ann. Math. Logic*, 17:1–28, 1979.

[14] G. Cherlin and E. Jaligot. Tame minimal simple groups of finite Morley rank. submitted.

[15] M. DeBonis and A. Nesin. On CN-groups of finite Morley rank. *J. London Math. Soc.*, (2) 50:532–546, 1994.

[16] M. DeBonis and A. Nesin. On split Zassenhaus groups of mixed characteristic and of finite Morley rank. *J. London Math. Soc.*, (2) 50:430–439, 1994.

[17] O. Frécon. Sous-groupes anormaux dans les groupes de rang de Morley fini résolubles. *J. Algebra*, 229:118–152, 2000.

[18] D. Gorenstein. *Finite Groups.* Chelsea Publishing Company, New York, 1980.

[19] B. Hartley and G. Shute. Monomorphisms and direct limits of finite groups of Lie type. *Quart. J. Math.*, 35:49–71, 1984.

[20] N. Jacobson. *Basic Algebra I.* W. H. Freeman and Company, 1989.

[21] E. Jaligot. Groupes de type mixte. *J. Algebra*, 212:753–768, 1999.

[22] E. Jaligot. Groupes de type pair avec un sous-groupe faiblement inclus. *J. Algebra*, 240:413–444, 2001.

[23] A. Macintyre. On ω_1-categorical theories of fields. *Fund. Math.*, 71:1–25, 1971.

[24] A. Nesin. On solvable groups of finite Morley rank. *Trans. Amer. Math. Soc.*, 321:659–690, 1990.

[25] A. Nesin. Generalized Fitting subgroup of a group of finite Morley rank. *J. Symbolic Logic*, 56:1391–1399, 1991.

[26] A. Nesin. Poly-separated and ω-stable nilpotent groups. *J. Symbolic Logic*, 56:694–699, 1991.

[27] L. Newelski. On type definable subgroups of a stable group. *Notre Dame J. Formal Logic*, 32:173–187, 1991.

[28] B. Poizat. *Groupes Stables.* Nur Al-Mantiq Wal-Ma'rifah, Villeurbanne, France, 1987.

[29] B. Poizat. Quelques modestes remarques à propos d'une conséquence inattendue d'un résultat surprenant de Monsieur Frank Olaf Wagner. *J. Symbolic Logic*, 66:1637–1646, 2001.

[30] F. Wagner. Subgroups of stable groups. *J. Symbolic Logic*, 55:151–156, 1990.

[31] F. Wagner. *Stable Groups.* Number 240 in London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, UK, 1997.

[32] F. Wagner. Fields of finite Morley rank. *J. Symbolic Logic*, 66:703–706, 2001.