Pushing up and $C(G, T)$ in groups of finite Morley rank of even type

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1 Introduction

According to a long-standing conjecture in model theory, simple groups of finite Morley rank should be algebraic. The present paper is part of a series aimed ultimately at proving the following:

Conjecture 1 (Even Type Conjecture) *Let G be a simple group of finite Morley rank of even type, with no infinite definable simple section of degenerate type. Then G is algebraic.*

An infinite simple group G of finite Morley rank is said to be of *even type* if its Sylow 2-subgroups are infinite and of bounded exponent. It is of *degenerate type* if its Sylow 2-subgroups are finite. If the main conjecture is correct, then there should be no groups of degenerate type. So the flavor of the Even Type Conjecture is that the classification in the even type case reduces to an extended Feit-Thompson Theorem. Those who are skeptical about the main conjecture would expect degenerate type groups to exist. The Even Type Conjecture confirms that this is the heart of the matter.

We believe that it is realistic to aim at a proof of the Even Type Conjecture with existing tools. In the present paper we obtain the following results:

Theorem 3.4 (Pushing Up)

Let G be a simple K^*-group of finite Morley rank and of even type, Q a unipotent 2-subgroup of G such that $Q = O_2(N^\circ(Q))$, with $B(N^\circ(Q)/Q) \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2. Then $N^\circ(Q)$ contains a Sylow° 2-subgroup of G.

Theorem 3.5 (C(G,T))

Let G be a simple K^*-group of finite Morley rank of even type with T a Sylow° 2-subgroup. If $C(G,T) < G$ then G has a weakly embedded subgroup.

The precise meaning of these results, and the relevant general definitions, will be given in the next section. They are natural analogs of results in finite group theory which were useful in the classification of the finite simple groups. In our context we view them as preparatory to an analysis of a minimal counterexample by the method of amalgams.

The key to all the results of the present paper is the following analog of a result of Baumann:

Theorem 3.2 part 1 (Baumann, [8])

Let G be a group of finite Morley rank of even type. Let M be a definable connected subgroup of G such that $\overline{M} = M/O_2(M) \cong \mathrm{SL}_2(K)$ with K an algebraically closed field of characteristic 2. Assume that $F^*(M) = O_2(M)$. If S is a Sylow 2-subgroup of M then it contains a nontrivial definable connected subgroup which is normal in M and $N_G{}^\circ(S)$.

Stellmacher showed in [27] that this result, or more exactly the structural analysis needed for this result, can be carried out by the "amalgam" method (cf. [15]), which by its nature goes over quite smoothly to our context. The proof of our analog of Stellmacher's theorem, which follows [27], is given in an Appendix, the main point being that certain issues of connectivity do not disrupt the argument significantly. In addition, a result of Timmesfeld ([29]) is very helpful as there is no general represention theory for representations of SL_2 of finite Morley rank.

The present paper is a sequel to [2, 20, 1, 3, 21, 4]; its results will be exploited in [10] to eliminate certain components in "parabolic subgroups" which are the main obstruction to undertaking a classification of even type groups (under a K^*-hypothesis) via the amalgam method. All we need from previous papers in the series are the main results of [21] and [4], reviewed in the next section. Modulo standard group theoretic facts and some general properties of groups of finite Morley rank the present paper is self-contained.

2 Preliminaries

In this section we will review the main facts required for the present paper. We use some of the basic facts and notions as given in [11] without explicit reference, but the more substantial points are all given explicitly below. We have included a first subsection on some basic model theoretic notions which are used in this paper. We hope that this will provide some background for the reader who wants to concentrate on the group theoretic aspects of the paper. The four subsections which follow the first one review some of the general theory of groups of finite Morley rank, while the last four address more specialized topics directly related to the concerns of the present paper.

2.1 Model theory

In model theory, one studies various *structures* by means of their *definable subsets*. Although the model theoretic definition of a structure corresponds to what a mathematician intuitively has in mind, it is appropriate to define it rigorously. A structure consists of an underlying set, called the *universe*, together with an indexed family of distinguished elements of the universe, an indexed family of relations on the universe, and an indexed family of functions with domain a cartesian power of the universe and range contained in the universe. Thus one can think of a group as a structure where the underlying set is the set of the elements of the given group together with a distinguished binary function, namely the group multiplication, and a constant, namely the identity of the group. Evidently this is not the only way to consider a group as a structure; we could include the inverse function as part of the group structure. In the first version, a group is given by its multiplicative structure (this is the point of view normally adopted when defining homomorphisms), while in the second version the inverse function is part of the structure as well (this is the point of view normally adopted when defining subgroups). In the long run one may switch freely between the various points of view available, but when setting up the foundations of model theory, it is convenient to work with structures of a definite type.

More substantial inclusions to a structure can be done in special cases. As an example, one can add to the structure consisting of the set of real numbers together with addition a unary relation denoting the positive numbers. This would be an example of a group with *additional structure*. In model theory the word group is generally used in this more general sense.

The *signature* of a structure consists of the three index sets involved, together with a function which specifies for each index i corresponding to a relation or function, the number of variables involved in the corresponding relation or function, its *arity*. With a class of structures with a fixed signature one can associate a *first-order language*. A first order language $\mathcal{L}$ is a set of symbols together with some rules which distinguish the strings of symbols which are acceptable from those which are not. The symbols can be divided into three categories: those which name the elements of the common signature; the logical symbols, i.e. equality ($=$), negation ($\neg$), disjunction ($\vee$), conjunction ($\wedge$), the universal quantifier $\forall$, the existential quantifier $\exists$; the variables. There are two main rules: no infinite conjunctions or disjunctions are allowed, and only variables are quantified. With these symbols and rules one can write *first-order formulas* (i.e. acceptable strings of symbols). The structures associated with a fixed first-order language $\mathcal{L}$ are called $\mathcal{L}$-structures.

An example of a language is that of monoids: $\mathcal{L} = \{., 1\}$. Here . denotes the binary multiplication function while 1 denotes the identity in the structure. One can expand this language to $\mathcal{L}_1 = \{.,\ ^{-1}, 1\}$ where $^{-1}$ is a symbol for the unary inversion function. This richer language can be seen as that of groups. The following example is a first-order formula in the language of monoids (or groups) which express the property of being central: $\forall x\,(y.x = x.y)$. In other words in a structure corresponding to this language (i.e. a monoid), any element *satisfying* this first-order formula would be central in the structure containing it. This is a *definition* for the central elements of a group.

In general a set S is said to be *definable* in an $\mathcal{L}$-structure $\mathcal{M}$ if $S \subseteq M^n$, where M denotes the underlying set of $\mathcal{M}$, and the elements of S are exactly those which satisfy a given first-order formula in the language $\mathcal{L}$. The above example shows that the center of a group is a definable set.

It is also possible to extend the language $\mathcal{L}$ by adding constant symbols which name elements of a fixed $\mathcal{L}$-structure. These are called *parameters*. The expansion of a language through the addition of parameters (or any expansion in general) can eventually allow more sets to become definable. An important example of this in the case of groups is the centralizer of a group element. If we restrict the language to that of groups then there is no reason why centralizers of elements should be definable but if g is a group element and one adds a constant symbol c_g to the language to name g, then $x.c_g = c_g.x$ is a first-order formula which defines the centralizer of g.

It is useful to note that a certain set can have more than one definition. For example if the group $\mathrm{SL}_2(\mathbb{C})$ is seen as an $\mathcal{L}$-structure with $\mathcal{L} = \{.,\ ^{-1}, 1\}$, then $Z(\mathrm{SL}_2(\mathbb{C}))$ is also defined by the formula $x.x = 1$. If K is a field of characteristic 2 and $B(K)$ denotes the group of 2×2 upper triangular matrices over K with determinant 1, then the centralizer of a nontrivial unipotent element can either be defined using an additional parameter naming this element, or just by the formula $x.x = 1$.

Once the notion of a definable set is established one can define definable relations and functions in the natural way: those relations or functions whose graphs are definable sets in the given structure. A well-known example of this is the equivalence relation of being in the same coset of a definable subgroup. The corresponding coset space is in fact an example of an important notion in model theory: an *interpretable set*. If $\mathcal{M}$ is an $\mathcal{L}$-structure then a set is said to be interpretable in $\mathcal{M}$ if it is obtained from a set definable in $\mathcal{M}$ after factoring out a definable equivalence relation.

On some structures it is possible to introduce a *rank function* rk from the set of sets interpretable in the given structure to the set of natural numbers which satisfies the following four axioms:

A If A is an interpretable set, then $\mathrm{rk}\,(A) \geq n+1$ if and only if there are infinitely many pairwise disjoint, nonempty, interpretable subsets of A whose ranks are at least n.

B If f is an interpretable function from A into B, then for each $n \in \mathbb{N}$, the set $\{b \in B : \mathrm{rk}\,(f^{-1}(b)) = n\}$ is interpretable.

C If f is an interpretable function from A onto B whose fibers have all the same rank n, then $\mathrm{rk}\,(A) = \mathrm{rk}\,(B) + n$.

D If f is an interpretable function from A into B then there is an integer m such that for any $b \in B$, the set $f^{-1}(b)$ has infinitely many elements as soon as it has at least m elements.

A structure which admits such a rank function is said to be a *ranked structure*. It is worth emphasizing that a ranked structure is *not* necessarily what is known as a structure of finite Morley rank in model theory. The above axiomatization (more precisely axioms A,B,C in some form) was introduced by the second author in order to concentrate on the algebraic properties of groups of finite Morley rank. Later Bruno Poizat made the necessary organization of the axioms and proved the equivalence of the notion of a ranked group (here the word group is taken in the more general model theoretic sense mentioned above) and that of a group of finite Morley rank ([25], Corollaire 2.14 and Théorème 2.15).

By a theorem of Macintyre ([23]), fields of finite Morley rank are algebraically closed. As for groups, finite groups are of Morley rank 0. In this article are studied the infinite ones of which algebraic groups over algebraically closed fields form one of the most important classes. Indeed the only known infinite simple groups of finite Morley rank are simple algebraic groups over algebraically closed fields, which gave rise to the conjecture in the introduction of this article.

Admitting Morley rank is a very strong condition on a group, which is inherited by its definable subgroups. Since the natural numbers are well-ordered, one cannot have infinite descending chains of definable subgroups (the *descending chain condition*). This allows one to define a robust notion of *connected component* of a group G of finite Morley rank: the intersection of the definable subgroups of finite index. This intersection, denoted by G°, is definable thanks to the descending chain condition. It is exactly the smallest definable subgroup of finite index in G. The descending chain condition allows one to define also the *definable closure* of an arbitrary subset X of G. Denoted by $d(X)$, this is the intersection of the definable subgroups of G which contain X. Again this is a well-defined and definable intersection. As a result one can talk about the connected component of an arbitrary subgroup H of G: H° is defined as $H \cap d(H)^\circ$.

Before finishing this review of relevant model theory, it is worth emphasizing that the above axiomatization of Morley rank in the context of groups has a very practical value in that the rank function is a useful computational tool. An illustration of this phenomenon is the following rank equality when G is a group of finite Morley rank and H is a definable subgroup:

$$\mathrm{rk}\,(G) = \mathrm{rk}\,(G/H) + \mathrm{rk}\,(H).$$

2.2 Generalities

The following two statements are the corollaries of Zil'ber's Indecomposability Theorem which we will need in the paper.

Fact 2.1 ([11], Corollaries 5.28 and 5.29) *Let G be a group of finite Morley rank*

1. *If H is a definable connected subgroup of G and X is any subset in G then $[H, X]$ is definable and connected.*
2. *The subgroup of G generated by any family of definable connected subgroups is again definable and connected, and it is the setwise product of finitely many of them.*

The following corollary will be useful in the appendix.

Corollary 2.2 *Let G be a group of finite Morley rank and A and B definable subgroups. If A is also connected then $\langle A, B\rangle$ is a definable subgroup of G.*

Proof. Note that $\langle A, B\rangle = \langle A^b : b \in B\rangle B$. Since $\langle A^b : b \in B\rangle$ is definable by Fact 2.1 2, $\langle A, B\rangle$ is also definable. □

The following definition contains some fundamental terminology which is used frequently in this paper and in many papers related to the classification project of which this paper is part.

Definition 2.3

1. *A* section *of a group G is a quotient of the form H/K where H and K are subgroups of G and $K \lhd H$. Such a section is said to be* definable *if H and K are definable.*
2. *A K-*group *is a group G of finite Morley rank such that every infinite definable simple section of G is isomorphic to an algebraic group over an algebraically closed field.*
3. *A K^*-*group *is a group G of finite Morley rank such that every infinite proper definable simple section of G is isomorphic to an algebraic group over an algebraically closed field. Equivalently, G is either a K-group, or a simple group all of whose definable subgroups are K-groups. As we are concerned here with techniques relevant to an inductive proof of the Even Type Conjecture, we confine ourselves in practice to the study of simple K^*-groups of even type.*

2.3 Sylow theory

There is a good Sylow theory for the prime 2 in our context:

Fact 2.4 ([12])

1. *The Sylow 2-subgroups of a group of finite Morley rank are conjugate.*

2. *If S is a Sylow 2-subgroup of a group of finite Morley rank then S is nilpotent-by-finite and its connected component is the central product of a definable, connected, nilpotent subgroup of bounded exponent and a divisible, abelian 2-group. Moreover, these two subgroups are uniquely determined.*

This provides a rather good analog to the general structure of the connected component of a Sylow 2-subgroup in an algebraic group, where depending on the characteristic we may be dealing with a maximal unipotent subgroup, or the 2-torsion in a torus (semisimple elements).

Accordingly we adopt the terminology suggested by the algebraic case:

Definition 2.5 1. *A* unipotent *subgroup is a connected definable subgroup of bounded exponent (in our context, typically a 2-group and hence nilpotent by Fact 2.4).*

2. *A* torus *is a definable divisible abelian group. For any prime p, a p-*torus *is a divisible abelian p-group. (A nontrivial p-torus is not definable, but its definable closure is a torus.)*

3. *A group of finite Morley rank is of* even type *if the connected component of a Sylow 2-subgroup is unipotent and nontrivial.*

4. *A group of finite Morley rank is of* odd type *if the connected component of a Sylow 2-subgroup is a nontrivial 2-torus.*

5. *A group of finite Morley rank is of* mixed type *if the connected component of a Sylow 2-subgroup is the central product of a nontrivial unipotent subgroup and a nontrivial 2-torus.*

6. *A group of finite Morley rank is of* degenerate type *if the connected component of a Sylow 2-subgroup is trivial (that is, the Sylow 2-subgroups are finite).*

The conjecture is that degenerate type and mixed type do not arise. The nonexistence of infinite simple groups of finite Morley rank of degenerate type would be a strong form of Feit-Thompson for this context. This is by far the hardest case to deal with. On the other hand the mixed type case can be eliminated *a priori* when working inductively:

Fact 2.6 ([20]) *A simple K^*-group of finite Morley rank is not of mixed type.*

Definition 2.7 *Let H be a group of finite Morley rank.*

1. *The connected components of Sylow 2-subgroups are called* Sylow° 2-subgroups.

2. *$B(H)$ denotes the subgroup generated by the unipotent 2-subgroups of H. ($B(H)$ is connected by Fact 2.1 2).*

3. *A subgroup of H is called a $2^{\perp}$-group if it contains no elements of order 2.*

Fact 2.8 ([3], Proposition 3.4) *Let $\mathcal{G} = GT$ be a connected K-group of even type with G and T definable and connected. Assume that T is a $2^{\perp}$-group which acts on G definably. Then T leaves invariant a Sylow° 2-subgroup of G.*

Corollary 2.9 *Let G be a connected K^*-group of even type and $T \leq G$ a torus. If U is a T-invariant nontrivial unipotent 2-subgroup of G then U is contained in a T-invariant Sylow° 2-subgroup of G.*

Proof. We consider $N^\circ(U)$ with U a maximal nontrivial T-invariant unipotent 2-subgroup. Then $N^\circ(U)/U$ is a K-group to which we can apply Fact 2.8 and the normalizer condition if U is not a Sylow° 2-subgroup of G in order to get a contradiction. □

2.4 Weak embedding

Definition 2.10 *Let G be a group of finite Morley rank. A proper definable subgroup M of G is said to be* weakly embedded *if it satisfies the following conditions:*

(i) Any Sylow 2-subgroup of M is infinite.

(ii) For any $g \in G \setminus M$, $M \cap M^g$ has finite Sylow 2-subgroups.

Jaligot has proved the following classification theorem:

Fact 2.11 ([21]) *Let G be a simple K^*-group of finite Morley rank and of even type, with a weakly embedded subgroup. Then G is of the form $\mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2.*

This yields:

Fact 2.12 ([3], Proposition 2.33; [2], Proposition 5.21; [20], Fact 3.9) *Let G be a K^*-group of finite Morley rank of even type and let $\Gamma(G)$ be the graph whose vertices are the unipotent 2-subgroups of G, with edges between any two distinct subgroups with infinite intersection. If $\Gamma(G)$ is disconnected then $B(G) \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2.*

Proof. We may assume $G = B(G)$ and in particular G is connected. By the definition of a K^*-group either G is a K-group or a simple group. If it is a K-group then Proposition 5.21 in [2] and Fact 3.9 in [20] prove the statement. If it is a simple group then the arguments used to prove Proposition 2.33 in [3] show that the stabilizer of a connected component of $\Gamma(G)$ is weakly embedded in G. Then Fact 2.11 yields the stated identification. □

Corollary 2.13 *Let G be a K^*-group of finite Morley rank of even type and let $\Gamma^*(G)$ be the graph whose vertices are the Sylow° 2-subgroups of G, with edges between any two distinct Sylow° 2-subgroups with infinite intersection. If $\Gamma^*(G)$ is disconnected then $B(G) \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2.*

Proof. If two vertices S, T of $\Gamma^*(G)$ are joined by a path in $\Gamma(G)$, then extending each vertex along that path to a vertex of $\Gamma^*(G)$, we see that S and T lie in the same connected component of $\Gamma^*(G)$. □

Definition 2.14 *Let G be a group of finite Morley rank.*

1. *A 2-local subgroup of G is the normalizer of a nontrivial definable 2-subgroup of G.*
2. *$O(G)$ is the largest connected definable normal $2^\perp$-subgroup of G.*
3. *$O_2(G)$ is the largest normal 2-subgroup of G. If G is of even type, this is definable and nilpotent.*

The following result was proved in [3] though stated somewhat differently; our formulation comes from [4].

Fact 2.15 ([3]) *Let G be a simple K^*-group of finite Morley rank of even type and H a 2-local subgroup of G with $O(H) \neq 1$. Then G has a weakly embedded subgroup.*

Since the stated configuration does not occur in groups of the form $\mathrm{SL}_2(K)$, by Jaligot's classification theorem we have:

Corollary 2.16 *Let G be a simple K^*-group of finite Morley rank of even type and H a 2-local subgroup of G. Then $O(H) = 1$.*

2.5 Miscellany

Definition 2.17 *Let G be a group of finite Morley rank of even type. $E(G)$ denotes the subgroup generated by the definable subnormal quasisimple (i.e., perfect and simple modulo center) subgroups of G. This is a finite central product of quasisimple groups; we are more interested in $E^\circ(G)$, the subgroup generated by the connected subnormal quasisimple subgroups of G.*

Note $E^\circ(G) \subseteq B(G)$. In addition $E^\circ(G)$ is a central product of connected definable quasisimple subgroups ([24]). If G is connected then $E(G) = E^\circ(G)$.

Definition 2.18 *Let P be a nilpotent p-group of bounded exponent. Then the* Frattini subgroup *$\Phi(P)$ is the subgroup generated by P' and $\{x^p : x \in P\}$. In the context of groups of finite Morley rank, if P is definable then $\Phi(P)$ is definable since on the one hand P' is definable, and on the other hand $\Phi(P)/P'$ is clearly definable in the quotient.*

$\mho^1$ and Ω_1 will have their usual group-theoretic meaning: if P is a p-group then $\mho^1(P) = \langle x^p : x \in P \rangle$ and $\Omega_1(P) = \langle x \in P : x^p = 1 \rangle$.

Fact 2.19 *Let N be a definable connected nilpotent p-group of bounded exponent. Then $\Phi(N)$ is definable and connected.*

Proof. N' is connected (Fact 2.1 1), and $\Phi(N)/N' = \mho^1(N/N')$. □

As usual, for any group G of finite Morley rank we write $\sigma(G)$ for the solvable radical of G, the largest normal solvable subgroup of G. It is definable and generated by the normal solvable subgroups of G (Theorem 7.3 in [11]).

Fact 2.20 ([1]) *Let G be a connected nonsolvable K-group of finite Morley rank. Then $G/\sigma(G)$ is isomorphic to a direct sum of simple algebraic groups over algebraically closed fields. In particular the definable connected $2^\perp$-sections of G are solvable.*

Fact 2.21 ([4]) *Let H be a connected K-group of finite Morley rank of even type such that $O_2(H) = 1$. Then $H = O(H) * E(H)$.*

Fact 2.22 ([5]) *Let G be a perfect group of finite Morley rank such that $G/Z(G)$ is a simple algebraic group. Then G is an algebraic group. In particular, $Z(G)$ is finite ([19] Section 27.5).*

Corollary 2.23 *If G is a K-group of finite Morley rank then $Z(E(G))$ is finite.*

Fact 2.24 ([4]) *Let L be a K-group of even type with $L = L_1 \times \ldots \times L_t$, where the L_i are simple algebraic groups. If X is a definable simple subgroup of L normalized by a Sylow 2-subgroup of L then $X = L_i$ for some i.*

We will apply this in the case in which L is a central product of quasisimple algebraic groups, and K is quasisimple, but this amounts to the same thing.

Fact 2.25 ([4]) *Let H be a connected K-group of finite Morley rank and even type, and L a definable quasisimple subgroup of H such that $N^\circ(L)$ contains a Sylow° 2-subgroup of H. Then $L \lhd H$.*

Proof. Let $S \leq N^\circ(L)$ be a Sylow° 2-subgroup of H. Let $\bar{H} = H/O_2(H)$. By Fact 2.21 $\bar{H} = E(\bar{H}) * O(\bar{H})$ and by Fact 2.24, or the remark following, $\bar{L}$ is normal in $E(\bar{H})$ and hence in $\bar{H}$. In terms of H we have $LO_2(H) \lhd H$. But $O_2(H) \leq S$ so $[L, O_2(H)] \leq L \cap O_2(H) \leq Z(L)$ as L is quasisimple. As L is perfect, $[L, O_2(H)] = 1$ by the three subgroups lemma. Thus $L = E(LO_2(H)) \lhd H$. □

Fact 2.26 *Let H be a connected K-group of even type. Then $O_2(H)$ is connected.*

Proof. By Fact 2.20 and Fact 2.22 $H/\sigma^\circ(H)$ is a central product of algebraic groups over algebraically closed fields of characteristic 2, hence the problem reduces to the case in which H is solvable and connected. In this case it is given in [11], Theorem 9.29. □

2.6 Borel-Tits

Fact 2.27 ([9], cf. [19], Corollary 30.3 A) *Let G be a reductive algebraic group and let U be a closed unipotent subgroup of G. Then $N_G(U)$ is contained in a parabolic subgroup $\mathcal{P}(U)$ of G such that $U \le R_U(\mathcal{P}(U))$, where R_U denotes the unipotent radical.*

Lemma 2.28 *Let X be a K-group of even type and Y a definable connected subgroup of X such that $Y = N_X^\circ(O_2(Y))$. Then Y contains a Sylow° 2-subgroup of X.*

Proof. We may suppose that X is connected. We set $Q = O_2(Y)$. By Fact 2.26, Q and $O_2(X)$ are connected. The subgroup $QO_2(X)$ is a connected 2-subgroup, hence nilpotent. Thus $N_{O_2(X)}{}^\circ(Q)$ is nontrivial. As this group is normalized by Y, it is a subgroup of Q. Hence, $N_{QO_2(X)}(Q) = Q$ and thus $O_2(X) \le Q$. Thus we may factor out $O_2(X)$ and assume that $O_2(X) = 1$. By Fact 2.21, $X = E(X) * O(X)$. We may therefore assume that $X = E(X)$.

As $Q = O_2(QZ(X))$, $N_X(QZ(X)) = Y$ and hence we may pass to $\bar{X} = X/Z(X)$, a direct product of simple algebraic groups over algebraically closed fields of characteristic 2. This is almost the situation to which Fact 2.27 applies, though as the base fields of the factors may vary one cannot say that this is literally so. While it would suffice to apply that result to each factor, we may argue more directly as follows.

Let X^* be an elementary extension of X in which each direct summand is uncountable, and of fixed cardinality. Then the base fields of the factors may be identified and X^* becomes an algebraic group over an algebraically closed field of characteristic 2. Thus after replacing X by X^* we may suppose that X is itself algebraic. Then the condition on Y implies that Q is Zariski closed and hence by Fact 2.27 Y is contained in a parabolic subgroup P of X whose unipotent radical U contains Q. Then $N_U(Q) \le O_2(Y) = Q$ so $U = Q$ and Y is a parabolic subgroup of X. □

2.7 $C(G,T)$

Definition 2.29 *Let G be a group of finite Morley rank, T a subgroup (typically a Sylow° 2-subgroup). Then $C(G,T)$ is the subgroup of G generated by all subgroups of the form $N_G{}^\circ(X)$ where $X \le T$ is definable, connected, and invariant under the action of $N_G{}^\circ(T)$.*

This is the notion to which we refer in the $C(G,T)$ classification theorem stated in §1. It would be somewhat more natural to replace this by the following.

Definition 2.30 *Let G be a group of finite Morley rank and T a definable subgroup of G.*

1. *A subgroup X of T is said to be* continuously characteristic *in T, relative to G, if it is invariant under the action of all connected groups of automorphisms of T which can be interpreted in G.*
2. *$C_0(G,T)$ is the subgroup of G generated by $N_G{}^\circ(X)$ as X varies over all definable, connected, continuously characteristic subgroups of T.*

Evidently if X is continuously characteristic in T relative to G then it is invariant under the action of $N_G{}^\circ(T)$. In particular $C_0(G,T) \le C(G,T)$. Thus the version of the $C(G,T)$-theorem based on $C_0(G,T)$ is stronger. Furthermore it is true, and can be proved by paying more attention to issues of definability in the proof of Stellmacher's theorem. However as the version given here covers all intended applications, we leave this point aside.

2.8 Abelian Sylow subgroups and standard components

The following two facts were proved in [4] under a restrictive hypothesis ("tameness") which can now be eliminated using Jaligot's classification (Fact 2.11). This is clear from the presentation in [4], where the theorems were given explicitly in the form of the existence of a weakly embedded subgroup in a minimal counterexample, in the absence of a tameness hypothesis.

Definition 2.31 *Let $A \leq B \leq G$ be three groups. Then A is said to be* strongly closed *in B (relative to the ambient group G) if for any elements $a \in A$, $g \in G$, if the conjugate a^g lies in B then it lies in A.*

Fact 2.32 ([4, 21]) *Let G be a simple K^*-group of finite Morley rank and of even type. Suppose that G contains an infinite definable abelian subgroup A which is strongly closed in a Sylow° 2-subgroup of G. Then $G \simeq \mathrm{PSL}_2(K)$ with K an algebraically closed field of characteristic 2.*

Definition 2.33 *Let G be a group of finite Morley rank, and L a quasisimple definable subgroup. Then L is said to be a* standard component *for G if:*

1. *$C(L)$ contains at least one involution;*
2. *For any involution $i \in C(L)$, L is a component of $C^\circ(i)$ (i.e., L is normal in $C^\circ(i)$, and is accordingly a factor of $E(C^\circ(i))$).*

Fact 2.34 ([4, 21]) *Let G be a simple K^*-group of finite Morley rank and of even type. Suppose that G has a standard component L of the form $\mathrm{SL}_2(K)$ for some algebraically closed field of characteristic 2. Let U be the connected component of a Sylow 2-subgroup of $C(L)$ and let A be a Sylow 2-subgroup of L. If U is nontrivial then AU is a Sylow° 2-subgroup of G.*

As far as Fact 2.32 is concerned, we will only make use of the case in which the Sylow° 2-subgroups of G are themselves abelian.

2.9 Properties of SL_2

The material of this subsection is needed only to carry out the proof of an analog of Stellmacher's pushing-up theorem in the finite Morley rank context. We need essentially the same facts that Stellmacher uses in the finite case, relating to representations of SL_2 over the prime field. In our case these will be infinite dimensional representations and some care must be taken on that account. The following result of Timmesfeld is helpful in this connection.

Definition 2.35 *Let G be a group and V an elementary abelian 2-group on which G acts, and A a subgroup of G. The action of A on V is said to be* quadratic *if $[V, A, A] = 0$.*

Fact 2.36 ([29], Proposition 2.7) *Let V be a $\mathbb{Z}X$-module where $X \simeq \mathrm{SL}_2(K)$ with K a field. Suppose the following:*

(1) $C_V(X) = 0$ and $[V, X] = V$

(2) $[V, A, A] = 0$, where A is a maximal unipotent subgroup of X.

Then for some field action on $\langle v^X \rangle$, the vector space $\langle v^X \rangle$ is a natural module for each $v \in C_V(A)^\times$.

Fact 2.37 ([26]) *If K is a field of finite Morley rank, every definable subgroup of $\mathrm{GL}_2(K)$ is either solvable-by-finite or contains $\mathrm{SL}_2(K)$.*

Corollary 2.38 *Let G be group of finite Morley rank which is isomorphic to $SL_2(K)$ as an abstract group with K an algebraically closed field. Suppose A is an infinite definable unipotent subgroup of G. Then for some conjugate B of A, $\langle A, B \rangle = G$.*

Proof. Let A be as in the statement and B be a conjugate of A which does not normalize A. Then $H = \langle A, B \rangle$ is a definable connected subgroup of G by Fact 2.1. If H is solvable then H is contained in a Borel subgroup of G, contradicting the choice of B. Thus Fact 2.37 applies and $H = G$. □

Lemma 2.39 *Let G be a group of finite Morley rank which is isomorphic to $\mathrm{SL}_2(K)$ as an abstract group with K an algebraically closed field of characteristic 2. Let $S \rtimes R$ be a Borel subgroup with S a Sylow 2-subgroup of G and R a maximal torus. Then the following hold:*

(1) G is generated by S together with any involution i not in S.

(2) Let V be an elementary abelian 2-group on which G acts faithfully so that (G,V) has finite Morley rank, and set $f = \mathrm{rk}\, K$. Then $\mathrm{rk}\, V \geq 2f$.

Proof.
Ad (1). This follows from Corollary 2.38 applied to $\langle S, S^i \rangle$.

Ad (2) Let V be as stated. We may assume that V is irreducible. If some nontrivial element $v \in V$ satisfies $\mathrm{rk}\,(C_G(v)) \leq f$ then $\mathrm{rk}\, V \geq \mathrm{rk}\,(v^G) \geq 2f$ and we are done. So we assume toward a contradiction that $\mathrm{rk}\,(C_G(v)) > f$ for all nontrivial $v \in V$.

Fix $v \in C_V(S)^\times$. As $\mathrm{rk}\,(C_G(v)) > f$, we have $C_G{}^\circ(v) > S$ and thus $C_G{}^\circ(v)$ has the form $S \rtimes R_0$ with R_0 a nontrivial torus, which is not necessarily algebraic. Let w be an involution that inverts R_0 and set $v_1 = v + v^w$. Note that $v_1 \neq 0$; in fact, if $w \in C_G(v)$ then by Corollary 2.38, we have $C_G(v) = G$. But by assumption, the action of G on V is irreducible.

Now, $\langle w, R_0 \rangle \leq C_G(v_1)$. As $\mathrm{rk}\,(C_G(v_1)) > f$, $C_G{}^\circ(v_1)$ has a nontrivial Sylow° 2-subgroup Q, which is normal in $C^\circ(v_1)$ and in particular is normalized by R_0 and by w. But there is no such 2-group Q in G since the only Sylow 2-subgroups normalized by R_0 are S and S^w. □

Proposition 2.40 *Let G be a group of finite Morley rank which, as an abstract group, is isomorphic to $\mathrm{SL}_2(K)$ with K an algebraically closed field of characteristic 2. Let A be an infinite definable 2-subgroup of G, V a connected elementary abelian 2-group which is a G-module such that (G,V) has finite Morley rank. Suppose $C_V(G) = 0$. Then:*

1. $\mathrm{rk}\,(A) \leq \mathrm{rk}\,(V/C_V(A))$;
2. *Equality holds only if A is a Sylow 2-subgroup of G, and V is a natural G-module.*

Proof. Let $f = \mathrm{rk}\, K$. By Corollary 2.38, $G = \langle A, B \rangle$ with B some conjugate of A. As $C_V(G) = 0$, the natural map $V \longrightarrow [V/C_V(A)] \times [V/C_V(B)]$ is injective and thus $\mathrm{rk}\, V \leq 2\mathrm{rk}\,(V/C_V(A))$. By Lemma 2.39 (2), $\mathrm{rk}\,(V/C_V(A)) \geq f \geq \mathrm{rk}\, A$. This proves the first point.

Now suppose $\mathrm{rk}\, A = \mathrm{rk}\,(V/C_V(A))$. Then $\mathrm{rk}\, A = f$ and A is a Sylow 2-subgroup of G. Furthermore $\mathrm{rk}\,(V/C_V(A)) = f$ so $\mathrm{rk}\, V \leq 2f$ and by Lemma 2.39 $\mathrm{rk}\, V = 2f$.

It remains to be seen that in this case V is a natural module. For this we use Timmesfeld's result, Fact 2.36. As $\mathrm{rk}\, V = 2f$, V is irreducible and thus $[V,G] = V$. The only point that needs to be checked is the quadratic action: $[V,A,A] = 0$ where A is a Sylow 2-subgroup of G.

Let $X = \bigcup\{C_V(A^g)^\times : g \in G\}$. Then X is the union of pairwise disjoint sets of rank f and hence $\mathrm{rk}\, X = 2f$, and X is generic in V. Thus a generic element of V is fixed by a Sylow 2-subgroup of G.

We claim that every element $v \in C_V(A)^\times$ has $C_G{}^\circ(v) = A$. Supposing the contrary, we proceed as in the proof of the previous lemma. We suppose $v \in C_V(A)^\times$ is centralized by a nontrivial torus R and we take w inverting R. Consider $v_1 = v + v^w$. Then as in the proof of the previous lemma $C_G(v_1)$ must be a torus. In particular $\mathrm{rk}\, C_G(v_1) = f$ and thus v_1^G is also generic in V. But this contradicts the result of the previous paragraph.

Let T be a maximal torus in $N_G(A)$. For $v \in C_V^\circ(A)^\times$ as $C_G^\circ(v) = A$, the orbit v^T is generic in $C_V^\circ(A)$ and as $C_V^\circ(A)$ is connected, $C_V^\circ(A)^\times$ is a single orbit under T. But if $A_1 \neq A$ is a conjugate of A normalized by T then $V = C_V(A) \oplus C_V(A_1)$ as a T-module and thus $\bar{V}^\times = (V/C_V(A))^\times$ is also a single orbit under T. Since $C_{\bar{V}}(A) \neq 1$, it follows that $C_{\bar{V}}(A) = \bar{V}$, or in other words $[V,A] \leq C_V(A)$, and $[V,A,A] = 0$. □

The following corollary is an analog of a result given in [27] and [28].

Corollary 2.41 ([28], (2.1) of [27]) *Let G be a group of finite Morley rank which is isomorphic to $\mathrm{SL}_2(K)$ with K an algebraically closed field of characteristic 2. Let V be a faithful $\mathbb{F}_2G$-module. Let S be a Sylow 2-subgroup of G. Assume that $T \leq S$ is definable and nontrivial, and:*

(i) $[V, T, T] = 1$,

(ii) $\mathrm{rk}\,(V/C_V(T)) \leq \mathrm{rk}\,(T)$.

Then the following hold:

(a) $\mathrm{rk}\,(T) = \mathrm{rk}\,(V/C_V(T))$,

(b) $T = S$,

(c) $V/C_V{}^\circ(G)$ *is a natural* $\mathbb{F}_2$*-module for* G,

(d) $C_V(S) = [V, S]C_V(G)$.

Proof. Point (d) is a special case of (c). We have proved $(a - c)$ under the assumption that $C_V(G) = 0$. All that we need to prove now is that $C_{V/C_V{}^\circ(G)}(G) = 0$.

Let $V_0/C_V{}^\circ(G) = C_{V/C_V{}^\circ(G)}(G)$. Then $[V_0, G, G] = 1$ so by the Three Subgroups Lemma $[V_0, G] = 1$, as claimed. □

2.10 Balance and components

We remind the reader that for a group of finite Morley rank G, $F(G)$ stands for the Fitting subgroup, the subgroup of G generated by all its normal nilpotent subgroups. It is definable and nilpotent (Theorem 7.3 of [11]).

Fact 2.42 *Let H be a K-group of finite Morley rank. Then $C_H{}^\circ(F^\circ(H)) = Z^\circ(F^\circ(H)) * E^\circ(H)$. In particular, if H is solvable then $C_H{}^\circ(F^\circ(H)) = Z^\circ(F^\circ(H))$.*

Proof. Let $K = C_H{}^\circ(F^\circ(H))$. Then $Z^\circ(F^\circ(H)) \leq Z^\circ(K)$. If $\sigma(K)/Z^\circ(F^\circ(H))$ is infinite, then as the quotient has a definable characteristic abelian subgroup, K contains a characteristic connected nilpotent group N properly containing $Z^\circ(F^\circ(H))$. Then $N \leq (F^\circ(H) \cap K)^\circ = Z^\circ(F^\circ(H))$, a contradiction. So $\sigma(K)/Z^\circ(F^\circ(H))$ is finite.

In particular, as K is connected, $[K, \sigma(K)] \leq Z(K)$. Thus $[K, [K, \sigma(K)]] = (1)$ and by a standard application of the three subgroups lemma, $[K^{(\infty)}, \sigma(K)] = (1)$. By Fact 2.20 $K = K^{(\infty)}\sigma(K)$, and as K is connected this yields $K = K^{(\infty)}\sigma^\circ(K) = K^{(\infty)}Z^\circ(F^\circ(H))$. Let $E = K^{(\infty)}$. As $[E, \sigma(K)] = (1)$, we have $\sigma(E) = Z(E)$ and by Fact 2.20, $E/Z(E)$ is semisimple. Hence $E \leq E(K) \leq E^\circ(H)$. Thus $K \leq Z^\circ(F^\circ(H)) * E^\circ(H)$ and the reverse inclusion is immediate. □

Fact 2.43 ([3]) *Let H be a connected solvable group of finite Morley rank and S a Sylow 2-subgroup of H. Assume S is unipotent. Then $S \leq F(H)$, and therefore S is a characteristic subgroup of H.*

The following result is known as the Thompson $A \times B$-Lemma.

Fact 2.44 ([3], 10.4 (i)) *Let G be a group of finite Morley rank whose definable p-subgroups are nilpotent-by-finite, and let A and P be definable p-subgroups of G with A normalizing P.*

If B is a definable subgroup of G containing no element of order p, which normalizes P and centralizes both A and $C_P(A)$, then B centralizes P.

The next result is referred to as the *L-balance* property:

Fact 2.45 *Let H be a K-group of finite Morley rank of even type, and U a 2-subgroup of H. Then $E^\circ(C(U)) \leq E^\circ(H)$.*

Proof. Let T be a torus contained in a component of $E^\circ(C(U))$. Let P be $O_2(H)$. Now $C_P(U) \leq O_2(C(U))$, so T commutes with $C_P(U)$. By the Thompson $A \times B$-lemma, with $B = T$, T commutes with $O_2(H)$. As such tori generate $E^\circ(C(U))$, $E^\circ(C(U))$ centralizes $O_2(H)$. On the other hand $E^\circ(C(U))$ also centralizes $O(H)$ since $E^\circ(C(U))$ is generated by unipotent 2-subgroups (Corollary to Fact 2.43). Thus $E^\circ(C(U))$ centralizes $F^\circ(H) = O_2{}^\circ(H) * O(F(H))$. But the connected component of the centralizer of $F^\circ(H)$ in H is $Z^\circ(F^\circ(H)) * E^\circ(H)$ (Fact 2.42), so $E^\circ(C(U)) \leq E^\circ(H)$. □

Fact 2.46 *Let H be a connected K-group of finite Morley rank, of even type, and let U be a 2-subgroup of H. Then $E^\circ(C(U)) \lhd E(H)$.*

Proof. By Fact 2.22 $Z(E(H))$ is finite, and $E(H)$ is a central product of quasisimple algebraic groups. As H is connected, it acts by inner automorphisms on $E(H)$. Hence so does U.

By Fact 2.45, $E^\circ(C(U)) \leq E(H)$, so $E^\circ(C(U)) = E(C_{E(H)}{}^\circ(U))$. As U acts by inner automorphisms, $C_{E(H)}(U)$ is the central product of $C_L(U)$ as L varies over the factors of $E(H)$, and $E^\circ(C(U))$ is correspondingly the central product of the groups $E^\circ(C_L(U))$.

For any factor L of $E(H)$, U acts on L as a 2-subgroup $\bar{U}$ of L. If this group is trivial then $E(C_L(U)) = L$, and otherwise $E(C_L(U)) = E(C_L(\bar{U})) = 1$; this last result is a consequence of the result of Borel and Tits on the relation between unipotent subgroups and parabolic subgroups given above as Fact 2.27, as explained for example in [17, 13-4] or [18, §3] . □

3 Baumann's Theorem, Pushing up, and $C(G, T)$

The following analog of a theorem of Stellmacher in the finite case will be assumed in the present section. A proof will be given in the appendix, following closely on the proof as given in [27] and using the information about $\mathrm{SL}_2(K)$ collected in the previous section.

Theorem 3.1 ([27]) *Let $\mathcal{G}$ be a group of finite Morley rank of even type. Let M be a definable connected subgroup of $\mathcal{G}$ such that $\overline{M} = M/O_2(M) \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2, and $F^*(M) = O_2(M)$. Assume that for S a Sylow 2-subgroup of M:*

(P) *no nontrivial definable connected subgroup of S is normalized by both M and $N_{\mathcal{G}}(S)$*

Set $Q = O_2(M)$, $L_0 = O^2(M)$, $V = [Q, L_0]$, and $D = C_Q{}^\circ(L_0)$.
Then the following hold:

1. *V is an elementary abelian 2-group central in Q.*
2. *$V/V \cap Z(M)$ is a natural $\mathbb{F}_2(\overline{M})$-module.*
3. *$Q = DV$, a central product.*
4. *$S/\Omega_1{}^\circ(Z(S))$ is an elementary abelian 2-group.*
5. *$Z^\circ(Q)$ is an elementary abelian 2-subgroup.*

Here $O^2(M)$ is the smallest definable normal subgroup H of M such that M/H is a 2-group; since $\bar{M} = M/O_2(M)$ is simple, this is the smallest definable normal subgroup of M covering $\bar{M}$, and coincides with $M^{(\infty)}$. As in finite group theory $F^*(G)$ denotes the "generalized Fitting subgroup" of a group G and for G of finite Morley rank the definition follows that in finite group theory: $F^*(G) = F(G)E(G)$.

3.1 The Baumann pushing up Theorem

In this subsection we obtain analogues in our context of results from [8].

Theorem 3.2 *Let G be a group of finite Morley rank of even type. Let M be a definable connected subgroup of G such that $\overline{M} = M/O_2(M) \cong \mathrm{SL}_2(K)$ with K an algebraically closed field of characteristic 2. Assume that $F^*(M) = O_2(M)$. If S is a Sylow 2-subgroup of M then the following hold:*

1. *S contains a nontrivial definable connected subgroup which is normalized by both M and $N_G{}^\circ(S)$.*
2. *If in addition*

   (P') *no nontrivial definable subgroup of S is normalized by both M and $N_G(S)$*

   then there exists an automorphism α definable in G such that $S = Z(O_2(M))^\alpha O_2(M)$.

Proof. 1. We use the notation of Theorem 3.1 as well as the structural information provided there concerning the groups Q, V, and D. Note that when one assumes that S contains no nontrivial definable connected subgroup which is normalized by both M and $N_G{}^\circ(S)$, all the assumptions of Theorem 3.1 including (P) are fulfilled (G replaces $\mathcal{G}$). We will show eventually that $\hat{V} = \langle V^{N_G{}^\circ(S)} \rangle$ is normal in M. This will suffice to prove 1 since it is obvious that $N_G^\circ(S)$ normalizes $\hat{V}$.

Note that Q is connected by Fact 2.26, thus S is connected. As L_0 covers $\bar{M}$ and $V \leq L_0$ we have $S = Q(S \cap L_0) = D(S \cap L_0)$. As $F^*(M) = Q$ and $\bar{M}$ is simple, $Z(S) \leq Q$. We note further that $Q \cap (L_0 Z(S)) = (Q \cap L_0)Z(S) = (D \cap L_0)VZ(S) = VZ(S)$

We set $W = VZ^\circ(S)$. As $F^*(M) = Q$, we have $W \le Q$, and by Theorem 3.1 (1) and (5), W is an elementary abelian subgroup central in Q. Let f denote the rank of the field K over which $M/O_2(M)$ is defined. Note that $\mathrm{rk}\,(S/Q) = f$.

A few remarks on the structure of W are in order. We have $W/Z^\circ(S) \simeq V/V \cap Z^\circ(S)$. By Theorem 3.1 (4) we have $[S, V] \le Z(S)$ and then by Theorem 3.1 (2) we find $V \cap Z(S) = [S, V](V \cap Z(M))$, and $\mathrm{rk}\,(W/W \cap Z(S)) = f$. If $i \in W \setminus Z(S)$ then as $Q \le C_S(i) \le S$, we have $C_S(i) = Q$ by Theorem 3.1 (2). Expressed in terms of the co-rank, namely $\text{co-rk}_S\, C_S(i) = \mathrm{rk}\, S - \mathrm{rk}\, C_S(i)$, this becomes $\text{co-rk}_S\, C_S(i) = f$.

On the other hand, suppose $i \in S \setminus Q$ is any involution with $\text{co-rk}_S\, C_S(i) = f$. We will show that $i \in (S \cap L_0)Z(S)$. As $S = D(S \cap L_0)$ we may write $i = i_1 i_2$ for elements $i_1 \in D$, $i_2 \in (S \cap L_0)$. Now $i_2 \in (S \cap L_0) \setminus Q$, so $\text{co-rk}_V\, C_V(i_2) \ge f$ because i_2 acts nontrivially on the natural module $V/V \cap Z(M)$. On the other hand i_1 commutes with V and hence $C_V(i) = C_V(i_2)$. Thus $\text{co-rk}_V\, C_V(i) = f$.

We claim that $C_Q(i) = C_D(i)C_V(i)$. Suppose $d \in D$, $v \in V$, and $[dv, i] = 1$. Then $[v, i] = [d, i] \in D \cap V \le C(L_0)$. Thus in view of the action of S on V, $v \in Z(S)$. Thus $v, d \in C(i)$ and our claim is proved. As $D \cap V \le C(i)$ we may work in $\bar{Q} = Q/D \cap V$. Then $\bar{Q}/\overline{C(i)} = \bar{D}/\overline{C_D(i)} \times \bar{V}/\overline{C_V(i)}$. As $f \ge \text{co-rk}_{\bar{Q}}\, \overline{C_Q(i)} = \text{co-rk}_{\bar{D}}\, \overline{C_D(i)} + \text{co-rk}_{\bar{V}}\, \overline{C_V(i)} = \text{co-rk}_{\bar{D}}\, \overline{C_D(i)} + f$, we find $D \le C(i)$. On the other hand i_2 commutes with D, so $D \le C(i_1)$. Thus i_1 commutes with DVL_0 and hence $i_1 \in Z(S)$. Thus $i \in (S \cap L_0)Z(S)$ as claimed.

Now suppose that α is any definable automorphism of S for which $W^\alpha \not\le Q$. Then $W^\alpha \setminus Q \subseteq (S \cap L_0)Z^\circ(S)$, and since $W^\alpha \setminus Q$ generates W^α, also $W^\alpha \le (S \cap L_0)Z^\circ(S)$. Furthermore we claim that $W^\alpha \cap Q \le Z(S)$ in this case. Let $j \in (W^\alpha \cap Q)$. Then $j \in Q \cap (L_0 Z(S)) = VZ(S)$. However looking again at the natural module, as j commutes with an involution in $W^\alpha \cap (S \setminus Q)$, this forces $j \in Z(S)$ as claimed.

Thus if $W^\alpha \not\le Q$ then $W^\alpha Q/Q \simeq W^\alpha / W^\alpha \cap Q$ has rank f since $Z^\circ(S) \le W^\alpha \cap Q \le Z(S)$, and hence $S = W^\alpha Q$.

Suppose now β is another automorphism of S for which $W^\beta \not\le Q$. Then $S = W^\alpha Q = W^\beta Q$. Take $i \in W^\alpha \setminus Q$, and choose $j \in W^\beta$ representing the same nontrivial element of S/Q. Then $ij \in Q = VD$. Let $ij = vd$ with $v \in V$, $d \in D$. Then $(vd)^i = (vd)^{-1} = vd^{-1}$ so $v^i v \in V \cap D \le Z(M)$ and i acts trivially on v in $V/(V \cap Z(M))$. As $\bar{V} = V/V \cap Z(M)$ is a natural module, $i \in S \setminus Q$, and $Z(S)$ covers $[i, \bar{V}]$, we find $v \in Z(S)$ and hence $ij = vd \in C(i)$. However $ij \in Q \cap (L_0 Z(S)) = VZ(S)$ and $C_{VZ(S)}(i) = Z(S)$, so $ij \in Z(S)$, and $i \in W^\beta Z(S)$. Thus $W^\alpha \le (W^\beta Z(S))^\circ = W^\beta$.

We claim now that for X any connected group of automorphisms of S which is interpretable in G, we have $W^\alpha \le Q$ for $\alpha \in X$. Suppose this fails. Then there is a unique element W^α ($\alpha \in X$) in the orbit of W under X such that $[W, W^\alpha] \ne 1$, namely the one for which $W^\alpha \not\le Q$. Evidently the same condition applies to W^β for any $\beta \in X$:

$$\text{If } \beta \in X \text{ then there is a unique } \alpha \in X \text{ such that } [W^\alpha, W^\beta] \ne 1.$$

Let $\hat{W} = \langle W^X \rangle$. Then $C(\hat{W}) = C(\langle W^{X_0} \rangle)$ for some finite $X_0 \subseteq X$. Then $X/N(W) = X_0/N(W)$: if $\beta \in X$, and α is chosen so that $[W^\alpha, W^\beta] \ne 1$, then there is some $\beta_0 \in X_0$ so that $[W^\alpha, W^{\beta_0}] \ne 1$ and as we have seen this forces $W^\beta = W^{\beta_0}$. Accordingly $X/N_X(W)$ is finite and as X is connected, X normalizes W, a contradiction.

In particular $W^{N_G{}^\circ(S)} \le Q$ and thus $\hat{V} = \langle V^{N_G{}^\circ(S)} \rangle \le Q$. Thus $[\hat{V}, L_0] \le [Q, L_0] = V$ and L_0 normalizes $\hat{V}$. Furthermore S normalizes $\hat{V}$ and as $M = L_0 S$, M normalizes $\hat{V}$. This proves 1.

2. This is mostly a variation over the final argument of part 1. The proof of 1 shows that if α is an automorphism of S definable in G such that $W^\alpha \not\le Q$, then $S = W^\alpha Q$. Therefore it suffices to argue that such a definable automorphism exists. If not then $W^{N_G(S)} \le Q$ and thus $\hat{V} = \langle V^{N_G(S)} \rangle \le Q$. Thus $[\hat{V}, L_0] \le [Q, L_0] = V$ and L_0 normalizes $\hat{V}$. Furthermore S normalizes $\hat{V}$ and as $M = L_0 S$, M normalizes $\hat{V}$. But $\hat{V}$ is definable by Fact 2.1 (ii), and as a result the assumption (P') is violated. □

Remark 3.3 The proof of the first part of the foregoing theorem also shows that starting with the configuration delivered by the Stellmacher theorem, we can find a normal subgroup of M which is *continuously characteristic* in S relative to G, namely the group generated by all V^α, where α varies over all automorphisms of S belonging to any connected group of automorphisms of S interpretable in G. However for this to be of any potential use one would also need to strengthen the Stellmacher theorem correspondingly.

3.2 Pushing up to a parabolic subgroup

Theorem 3.4 (Pushing Up) *Let G be a simple K^*-group of finite Morley rank and of even type, Q a unipotent 2-subgroup of G such that $Q = O_2(N^\circ(Q))$, with $B(N^\circ(Q)/Q) \simeq \mathrm{SL}_2(K)$ for some algebraically closed field of characteristic 2. Then $N^\circ(Q)$ contains a Sylow° 2-subgroup of G.*

Proof. By Fact 2.26, Q is connected. We may suppose that Q is nontrivial.

Let $M = B(N^\circ(Q))$. Let S be a Sylow° 2-subgroup of M, and extend S to T a Sylow° 2-subgroup of G. It suffices to show that $N^\circ{}_T(S) = S$.

We make a case division according as the conclusion of Theorem 3.2 1 does or does not hold for M, namely:

(BT) There is a nontrivial connected definable subgroup X of S which is normalized by M and by $N^\circ(S)$.

Suppose first the condition (BT) holds, and fix $X \le S$ accordingly. As $M \le N^\circ(X)$, by Lemma 2.28 S is a Sylow° 2-subgroup of $N^\circ(X)$. As $N_T{}^\circ(S) \le N^\circ(X)$, we have $N_T{}^\circ(S) = S$, as required.

Now we deal with the case in which condition (BT) fails. We first consider the structure of M. Then by Theorem 3.2 1 $M = F^*(M) = L \times Q$. Furthermore Q is elementary abelian, as otherwise we set $X = \Phi(S) = \Phi(Q)$, and X is connected (Fact 2.19) and normalized by both M and $N(S)$, and is nontrivial.

In this situation, we claim that Q is a Sylow° 2-subgroup of $C(L)$. Let U be a Sylow° 2-subgroup of $C(L)$ containing Q. Then $N_U{}^\circ(Q) \le M = LQ$ and $N_U{}^\circ(Q) \cap L = 1$ so $N_U{}^\circ(Q) = Q$ and by the normalizer condition $Q = U$.

Our final goal is to show that L is a standard component in G; then Fact 2.34 shows that S is a Sylow° 2-subgroup of G. As $U \le C(L)$, $C(L)$ certainly contains involutions.

Let i be an involution in $C(L)$. We must show that L is a component of $C^\circ(i)$. Suppose first that $i \in Q$. Then $S \le C^\circ(i)$ and S is a Sylow° 2-subgroup of $C^\circ(i)$ by Fact 2.28. Thus by Fact 2.25 L is a component of $C^\circ(i)$.

Now let i be any involution in $C(L)$. As Q is a Sylow° 2-subgroup of $C(L)$, we may assume that i normalizes Q. In particular, $C_Q{}^\circ(i) \ne 1$. Let j be an involution in $C_Q{}^\circ(i)$. Then L is a component of $C^\circ(j)$ and hence of $C_{C^\circ(i)}{}^\circ(j)$. By Fact 2.46, L is a component of $C^\circ(i)$. □

3.3 The $C(G,T)$ Theorem

In this subsection we prove a "global" $C(G,T)$ theorem in the context of simple K^*-groups of even type. In finite group theory the "local" $C(G,T)$ theorem was proven by Aschbacher in [6, 7]. Later Gorenstein and Lyons gave a proof for K-groups in [16]. [18] (pages 96-98) contains an outline of "Theorem $\mathcal{M}(S)$" which is a "variation of the global $C(G,T)$ theorem" whose proof will be given in later volumes of the same series on the revision of the classification of the finite simple groups.

We refer to Subsection 2.5 for the precise definition of $C(G,T)$ as used here, and a comparison with related notions.

Theorem 3.5 (C(G,T)) *Let G be a simple K^*-group of finite Morley rank of even type with T a Sylow° 2-subgroup. If $C(G,T) < G$ then $G \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2.*

Proof. With G fixed, we define $M(T) = C(G, T)$. With $\Gamma^*(G)$ as in the statement of Corollary 2.13, we will prove:

$(*)$ $\quad M(S) = M(T)$ when (S, T) is an edge in $\Gamma^*(G)$.

From this it follows that the graph $\Gamma^*(G)$ is disconnected, as otherwise $C(G, T)$ would be independent of the choice of T and hence normal in G. Then Corollary 2.13 applies and the theorem follows. Thus it suffices to verify $(*)$.

Suppose toward a contradiction that $(*)$ fails, and take a counterexample (S, T) with $\mathrm{rk}\,(S \cap T)$ maximal. Let $Q = (S \cap T)^\circ$ and $H = N^\circ(Q)$. Certainly Q is a proper subgroup of S and T. We will show in due course that Theorem 3.4 applies to H.

We claim that there is a Sylow° 2-subgroup S_1 of G such that $M(S) = M(S_1)$, $S_1 \cap H \geq (S \cap H)^\circ$, and $S_1 \cap H$ contains a Sylow° 2-subgroup of H. Indeed, let S_1 be any Sylow° 2-subgroup of G which contains a Sylow° 2-subgroup of H containing $(S \cap H)^\circ$. Then $\mathrm{rk}\,(S \cap S_1) \geq \mathrm{rk}\,(S \cap H) > \mathrm{rk}\, Q$ since $N_S{}^\circ(Q) > Q$. Hence $M(S) = M(S_1)$, as claimed, since otherwise the choice of the pair (S, T) is contradicted.

Similarly we may take T_1 a Sylow° 2-subgroup of G such that $M(T_1) = M(T)$ and $(T_1 \cap H)^\circ$ is a Sylow° 2-subgroup of H containing $(T \cap H)^\circ$. As $(S \cap T)^\circ \leq (S \cap H)^\circ \cap (T \cap H)^\circ$, $(S \cap T)^\circ \leq S_1 \cap T_1$. Thus $M(S_1) \neq M(T_1)$ and $\mathrm{rk}\,(S_1 \cap T_1) \geq \mathrm{rk}\,(S \cap T)$; so S and T may be replaced by S_1 and T_1, which means that we may now suppose that $S \cap H$ and $T \cap H$ contain Sylow° 2-subgroups of H.

With this choice of S and T, we have $O_2{}^\circ(H) \leq (S \cap T)^\circ = Q$, so $O_2{}^\circ(H) = Q$, and by Fact 2.26:

$$O_2(H) = Q$$

Let $\bar{H} = H/Q$. It is easy to see that $\Gamma^*(\bar{H})$ is disconnected, and specifically that $u = \overline{(S \cap H)^\circ}$ and $v = \overline{(T \cap H)^\circ}$ lie in different components. Indeed if (Q_i) is a sequence of Sylow° 2-subgroups of H such that $\bar{Q}_i$ links u to v in $\Gamma^*(\bar{H})$, then the meaning of this in H is that the Q_i are Sylow° 2-subgroups of H such that the rank of any consecutive intersection $Q_i \cap Q_{i+1}$ is greater than $\mathrm{rk}\,(S \cap T)$. If therefore Q_i is extended to a Sylow° 2-subgroup R_i of H, then by the choice of the counterexample (S, T) we find $M(R_i) = M(R_{i+1})$ along the path. Thus $M(S) = M(T)$, a contradiction.

Since $\bar{H}$ is a K-group and $\Gamma^*(\bar{H})$ is disconnected, by Corollary 2.13 we have:

$$B(\bar{H}) \simeq \mathrm{SL}_2(K) \text{ with } K \text{ algebraically closed of characteristic 2.}$$

Thus Theorem 3.4 applies to H, and H contains a Sylow° 2-subgroup of G. But we also arranged to have S, T meet H in Sylow° 2-subgroups of H, so now we have $S, T \leq H$.

As in the proof of Theorem 3.4, we must again consider whether the conclusion of Theorem 3.2 1 applies to $B(H)$, or not. If it does, then S has a nontrivial definable connected subgroup X which is normalized by $N^\circ(S)$ and by $B(H)$, so in particular by T. Hence $T \leq M(S)$. But then S and T are conjugate in $M(S)$, say by g. This gives $M(T) = M(S^g) = M(S)^g = M(S)$, a contradiction to the choice of S and T.

If on the other hand the conclusion of Theorem 3.2 1 does *not* apply to H, then as before we find that $B(H) = Q \times L$ with $L \simeq \mathrm{SL}_2(K)$, K algebraically closed of characteristic 2, and Q elementary abelian. In particular the Sylow° 2-subgroups of H are elementary abelian. As H contains a Sylow° 2-subgroup of G, the same applies to G, so Fact 2.32 applies to G and $G \simeq \mathrm{SL}_2(K)$ with K algebraically closed of characteristic 2, as required. □

Appendix: Stellmacher's Theorem

In this section we carry over the results in [27] to the context of groups of finite Morley rank of even type, following closely the notation and the arguments of [27]. There are some deviations from the arguments as given there: on the one hand the representation theory of $\mathrm{SL}_2(K)$ over the field of 2 elements now involves infinite dimensional representations, and in particular ranks are used rather than dimensions to compare sizes; in addition rather than passing to a free product with amalgamation (or in graph theoretical terms, a tree) we remain in the context of an ambient group of finite Morley rank. The latter point avoids issues of definability.

An important deviation is in the statement of the theorem itself. Our condition (P) is weaker than the most natural analog of Stellmacher's condition, and it is essential for applications that the proof goes through with this assumption.

The entire section will be devoted to the proof of the following theorem, parallel to the main result of [27] in the finite case:

Theorem 3.1

Let $\mathcal{G}$ be a group of finite Morley rank of even type. Let M be a definable connected subgroup of $\mathcal{G}$ such that $\overline{M} = M/O_2(M) \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2, and $F^*(M) = O_2(M)$. Assume that for S a Sylow 2-subgroup of M:

(*P*) *no nontrivial definable connected subgroup of S is normalized by both M and $N_{\mathcal{G}}(S)$*

Set $Q = O_2(M)$, $L_0 = O^2(M)$, $V = [Q, L_0]$, and $D = C_Q{}^\circ(L_0)$.

Then the following hold:

1. *V is an elementary abelian 2-group central in Q.*
2. *$V/V \cap Z(M)$ is a natural $\mathbb{F}_2(\overline{M})$-module.*
3. *$Q = DV$.*
4. *$S/\Omega_1{}^\circ(Z(S))$ is an elementary abelian 2-group.*
5. *$Z^\circ(Q)$ is an elementary abelian 2-subgroup.*

The finite version of this theorem is proven in [27] using the *amalgam method.* We adapt this to our present context. Fix M, S, $\mathcal{G}$ as in the statement of the theorem. By Fact 2.26 S is a connected group. We let $H = N_{\mathcal{G}}(S)$ and $G = \langle M, H\rangle$. By Corollary 2.2 G is definable in $\mathcal{G}$, and we may replace $\mathcal{G}$ by G. Set $B = M \cap H$. Note that B is a Borel subgroup of M.

A The associated graph

We will consider the bipartite coset graph Γ of G corresponding to the pair of subgroups M and H. The two types of vertices will be the cosets of M and H in G. In particular we will refer to a coset of M as a vertex of *type* M. The edges are the cosets of B in G. An edge Bx has as its vertices the cosets Mx and Hx. The natural action of G on Γ is definable. The following properties given in [14] apply here.

Lemma A.1 (1.1, [27]) *(a) Γ is connected and bipartite.*

(b) G is edge but not vertex transitive on Γ.

(c) The vertex stabilizers in G are conjugate to M or H.

(d) The edge stabilizers in G are conjugate to $M \cap H = B$.

(e) For $\lambda \in \Gamma$, the vertex-stabilizer G_λ is transitive on the set of vertices adjacent to λ.

Lemma A.2 (1.2, [27]) *No nontrivial definable connected subgroup of G is normal in the stabilizer of two adjacent vertices. The kernel of the action of G on Γ is a finite subgroup of $O_2(Z(G))$.*

Proof. If K is a definable subgroup of G which is normal in the stabilizers of two adjacent vertices, then by edge transitivity we may suppose that these vertices are M and H. Then $K \lhd M$ and $K \leq B$, so $K \leq O_2(M)$ and condition (P) applies. Hence K cannot be nontrivial and connected.

In particular if K is the kernel of the action of G on Γ then $K \leq O_2(M)$ and $K^\circ = 1$. As G is connected, $K \leq Z(G)$ as well. □

Since we prefer to work with a faithful action, we will factor out the kernel of the action of G on Γ, which will not affect our hypotheses. (We will also have to check the validity of our conclusions in the original context, at some point.) Thus we will generally suppose:

(*) G acts faithfully on Γ

B The module Z_α

Notation B.1 *Let α, α' be vertices of Γ.*

1. *$d(\alpha, \alpha')$ will denote their distance in Γ.*
2. *$G_\alpha^{(1)}$ is the intersection of the groups G_β for which $d(\alpha, \beta) \leq 1$.*
3. *$Q_\alpha = O_2(G_\alpha)$*
4. *$Z_\alpha = \langle \Omega_1{}^\circ(Z(T)) : T \in \mathrm{Syl}_2(G_\alpha) \rangle$.*
5. *$b_\alpha = \min\{d(\alpha, \beta) : \beta \in \Gamma, Z_\alpha \not\leq G_\beta^{(1)}\}$. Let $b = b_\delta$ with δ of type M.*
6. *(α, α') is a* critical pair *for Γ if α is of type M, $d(\alpha, \alpha') = b$ and $Z_\alpha \not\leq G_{\alpha'}^{(1)}$.*

Remark B.2
1. *Q_α and Z_α are of interest only when α is of type M; otherwise, Q_α is the unique Sylow 2-subgroup of G_α, and Z_α is $\Omega_1{}^\circ(Z(Q_\alpha))$.*
2. *For α of type M, Z_α is the critical object of study. We will see momentarily that this is an elementary abelian 2-group which affords a nontrivial representation of $G_\alpha/Q_\alpha \simeq SL_2(K)$, which will essentially be the natural representation.*
3. *The parameter b_α is well-defined (finite) since Z_α is nontrivial, Γ is connected, and the action of G on Γ is faithful. Furthermore b_α evidently depends only on the type of α, so b is also well-defined. Large values of b lead quickly to implausible (and contradictory) configurations; our main concern will be with the possibilities $b = 2$ and $b = 4$.*
4. *The definition of a critical pair implies that $Z_\alpha \leq G_{\alpha'}$.*

Lemma B.3 (1.3, 3.1, [27]) *Let $\alpha \in \Gamma$ be of type M. Then:*

1. *$Q_\alpha = O_2(G_\alpha^{(1)})$ is a Sylow 2-subgroup of $G_\alpha^{(1)}$.*
2. *For T a Sylow 2-subgroup of G_α, $Z_\alpha > \Omega_1{}^\circ(Z(T))$.*
3. *$Z_\alpha \leq \Omega_1{}^\circ(Z(Q_\alpha))$ and $C_{G_\alpha}(Z_\alpha) = Q_\alpha$.*
4. *$b \geq 2$ is even.*

In particular, G_α/Q_α acts on Z_α, and the action is nontrivial.

Proof. *Ad* 1. We may suppose that $\alpha = M$. For S a Sylow 2-subgroup of M, the vertex $\beta = N^\circ(S)$ is a neighbor of α and hence $G_\alpha^{(1)} \le N^\circ(S)$. Hence a Sylow 2-subgroup of $G_\alpha^{(1)}$ is contained in $O_2(M) = Q_\alpha$. On the other hand M acts transitively on its neighbors, by edge transitivity, so they are of the form $N^\circ(S)$ with S a Sylow 2-subgroup of M. Thus $Q_\alpha \le G_\alpha^{(1)}$ is a Sylow 2-subgroup of $G_\alpha^{(1)}$.

Ad 2. If $Z_\alpha = \Omega_1{}^\circ(Z(T))$ we contradict Lemma A.2.

Ad 3. Again we suppose $\alpha = M$. Let S be a Sylow 2-subgroup of M. Then $Z^\circ(S) \le C_M{}^\circ(Q_\alpha) \le Q_\alpha$ as $F^*(M) = O_2(M)$, so $Z^\circ(S) \le Z(Q_\alpha)$. Hence $Z_\alpha \le \Omega_1{}^\circ(Z(Q_\alpha))$ and $C_{G_\alpha}(Z_\alpha) \ge Q_\alpha$. But G_α/Q_α is simple so by point (2), $C_{G_\alpha}(Z_\alpha) = Q_\alpha$.

Ad 4. As $Z_\alpha \le Q_\alpha \le G_\alpha^{(1)}$ we have $b \ge 1$. It suffices now to check that b is even, or in other words, taking (α, α') to be a critical pair, we claim that α' is of type M. If this is not the case then $O_2(G_{\alpha'})$, which is the Sylow 2-subgroup of $G_{\alpha'}$, is contained in $G_{\alpha'}^{(1)}$. Since $Z_\alpha \le G_{\alpha'}$ by the definition of a critical pair (Remark B.2 4), we have $Z_\alpha \le O_2(G_{\alpha'} \le G_{\alpha'}^{(1)}$, a contradiction.
□

Lemma B.4 (1.4, [27]) *Let (α, α') be a critical pair. Then:*

1. $1 \ne [Z_\alpha, Z_{\alpha'}] \le Z_\alpha \cap Z_{\alpha'}$.
2. $[Z_\alpha, Z_{\alpha'}, Z_{\alpha'}] = 1 = [Z_{\alpha'}, Z_\alpha, Z_\alpha]$.
3. *(α', α) is a critical pair.*

Proof. By the minimality of b we have $Z_\alpha \le G_{\alpha'}$ and thus Z_α normalizes $Z_{\alpha'}$. As this is a critical pair however, $Z_\alpha \not\le Q_{\alpha'}$ and thus $[Z_\alpha, Z'_\alpha] \ne 1$ (Lemma B.3 Ad. 3, Ad. 4). In particular $Z_{\alpha'} \not\le Q_\alpha$ and thus the pair (α', α) is also critical. So $(1,3)$ both follow. Lemma B.3 and (1) imply (2). □

Lemma B.5 (2.2, [27]) *Let (α, α') be a critical pair for Γ and set $\overline{G}_\alpha = G_\alpha/Q_\alpha$. Then:*

1. *$Z_\alpha/Z_\alpha \cap Z(G_\alpha)$ is a natural SL_2-module for $\overline{G}_\alpha$.*
2. *$Z_{\alpha'}Q_\alpha$ is a Sylow 2-subgroup of G_α.*
3. *Setting $S = Z_{\alpha'}Q_\alpha$, $\Omega_1{}^\circ(Z(S)) = [Z_\alpha, Z_{\alpha'}](Z_\alpha \cap Z(G_\alpha))^\circ$*

Proof. As both (α, α'), and (α', α) are critical pairs, we will first suppose that for the pair under consideration we have:

$$\mathrm{rk}\,(Z_{\alpha'}/Z_{\alpha'} \cap Q_\alpha) \ge \mathrm{rk}\,(Z_\alpha/Z_\alpha \cap Q_{\alpha'})$$

We may also assume $G_\alpha = M$.

We apply Corollary 2.41 to $\bar{G}_\alpha$ and its subgroup $T = \bar{Z}_{\alpha'}$, acting on the module $V = Z_\alpha$. With this notation, the hypotheses of the corollary are that Z_α is a faithful module (Lemma B.3), that $[Z_\alpha, Z_{\alpha'}, Z_{\alpha'}] = 1$ (Lemma B.4), and that:

$$\mathrm{rk}\,(Z_\alpha/C_{Z_\alpha}(\bar{Z}_{\alpha'})) \le \mathrm{rk}\,(\bar{Z}_{\alpha'})$$

which decodes to the condition assumed at the outset.

Corollary 2.41 then yields the following four conditions:

1. $\mathrm{rk}\,(\bar{Z}_{\alpha'}) = \mathrm{rk}\,(Z_\alpha/C_{Z_\alpha}(\bar{Z}_{\alpha'}))$, and thus our results apply equally to (α, α') or (α', α);
2. $\bar{Z}_{\alpha'}$ is a Sylow 2-subgroup of $\bar{G}_\alpha$, which was our second point;
3. $Z_\alpha/C_{Z_\alpha}(\bar{G}_\alpha)$ is indeed a natural module;
4. $C_{Z_\alpha}(Z_{\alpha'}) = [Z_\alpha, Z_{\alpha'}]C_{Z_\alpha}(G_\alpha)$; this is our final claim, taking into account: $C^\circ_{Z_\alpha}(Z_{\alpha'}) = \Omega_1{}^\circ(Z(Z_{\alpha'}Q_\alpha))$.

□

C The case $b = 2$

We know that $b \geq 2$ is even. In this section we show that the case $b = 2$ leads to the configuration described in the theorem. Subsequently we will show that the case $b > 2$ leads to a contradiction.

Recall that at an early stage we modified G to ensure that the action on Γ is faithful, and that we are presently engaged in verifying the theorem in that case. Since the present case does not lead to a contradiction, but rather to conclusions about the structure of G, it will also be necessary to argue that these conclusions pass over to the general case.

We recall the notation involved in analyzing the structure of M:

$$\begin{array}{ll} Q = O_2(M) & L_0 = O^2(M) \\ V = [Q, L_0] & D = C_Q{}^\circ(L_0) \end{array}$$

The following lemma will be useful in this subsection as well as in the following.

Lemma C.1 *If α, β are vertices of type M in Γ with $d(\alpha, \beta) = 2$, then $G_\alpha \cap G_\beta$ contains a unique Sylow 2-subgroup of G_α and G_β.*

Proof. There is a vertex γ of the form $N^\circ(T)$ adjacent to both α and β, with T a Sylow 2-subgroup of G_α and G_β. If the intersection contained another Sylow 2-subgroup of G_α then by Lemma 2.39 the two together would generate G_α. $\square$

Proposition C.2 (3.2, [27]) *Assume that $b = 2$ and that the action of G on Γ is faithful. Then the following hold:*

1. *$Q = DV$, and V is an elementary abelian 2-group central in Q.*
2. *For S a Sylow 2-subgroup of M, $S/\Omega_1{}^\circ(Z(S))$ is an elementary abelian group.*
3. *$Z_\alpha = Z^\circ(Q_\alpha)$. In particular, $Z^\circ(Q)$ is an elementary abelian group.*

Proof. Let (α, α') be a critical pair for Γ with $\alpha = M$. Then the subgroups Q, L_0, D, V lie in G_α and in particular $Q = Q_\alpha$.

As $b = 2$, Lemma C.1 implies that $G_\alpha \cap G_{\alpha'}$ contains a unique Sylow 2-subgroup S of G_α.

(1) $S = Z_\alpha Q_{\alpha'}$

By Lemma B.5 $Z_\alpha Q_{\alpha'}$ is a Sylow 2-subgroup of $G_{\alpha'}$. Since it is contained in G_α as well, it coincides with S. The same applies to $Z_{\alpha'} Q_\alpha$.

(2) $Q_\alpha = Z_\alpha(Q_\alpha \cap Q_{\alpha'})$

$Z_\alpha \leq Q_\alpha \leq S = Z_\alpha Q_{\alpha'}$. Thus (2) holds.

Now we introduce some additional notation. We fix $g \in G_\alpha$ so that $G_\alpha = \langle Z_{\alpha'}, Z_{\alpha'}^g \rangle Q_\alpha$, which is possible since $Z_{\alpha'}$ covers a Sylow 2-subgroup of $G_\alpha/Q_\alpha \simeq \mathrm{SL}_2(K)$. Set $F = \langle Z_{\alpha'}, Z_{\alpha'}^g \rangle$.

(3) $Z_\alpha \leq [S, Z_\alpha][S^g, Z_\alpha] Z(G_\alpha)$

We work in the natural module $\bar{Z}_\alpha = Z_\alpha / C_{Z_\alpha}(G_\alpha)$. Then $[S, \bar{Z}_\alpha]$ is a 1-dimensional subspace of $\bar{Z}_\alpha$, as is $[S^g, \bar{Z}_\alpha]$. On the other hand $[S, Z_\alpha] = [Z_{\alpha'}, Z_\alpha] \leq Z_{\alpha'} \cap Z_\alpha \leq C_{Z_\alpha}(Z_{\alpha'})$, so $[S, \bar{Z}_\alpha] \leq C_{\bar{Z}_\alpha}(Z_{\alpha'})$ and $[S, \bar{Z}_\alpha] \cap [S, \bar{Z}_{\alpha'}] \leq C_{\bar{Z}_\alpha}(F) = C_{\bar{Z}_\alpha}(G_\alpha) = 1$.

Thus $\bar{Z}_\alpha = [S, \bar{Z}_\alpha] \oplus [S^g, Z_\alpha]$ and (3) follows.

(4) $Q_\alpha = Z_\alpha(Q_\alpha \cap Q_{\alpha'} \cap Q_{\alpha'}^g)$

By (1) $[S, Z_\alpha] \leq Q_{\alpha'}$ and $[S^g, Z_\alpha] \leq Q_{\alpha'}^g$, so by (3) $Z_\alpha \leq (Z_\alpha \cap Q_\alpha \cap Q_{\alpha'}) Q_{\alpha'}^g$. Now $Q_\alpha \cap Q_{\alpha'} \leq S^g = Z_\alpha Q_{\alpha'}^g \leq (Z_\alpha \cap Q_\alpha \cap Q_{\alpha'}) Q_{\alpha'}^g$ so $Q_\alpha \cap Q_{\alpha'} \leq (Z_\alpha \cap Q_\alpha \cap Q_{\alpha'})(Q_\alpha \cap Q'_\alpha \cap Q_{\alpha'}^g) \leq Z_\alpha(Q_\alpha \cap Q'_\alpha \cap Q_{\alpha'}^g)$, and this combines with (2) to give (4).

(5) $Q_\alpha = C_{Q_\alpha}{}^\circ(F)Z_\alpha$

Evidently $Q_\alpha \cap Q_{\alpha'} \cap Q^g_{\alpha'} \leq C_{Q_\alpha}(F)$ and thus (5) follows from (4).

(6) $FZ_\alpha \lhd G_\alpha$

We have $G_\alpha = FQ_\alpha$. Now $[Q_\alpha, FZ_\alpha] = [C_{Q_\alpha}(F)Z_\alpha, FZ_\alpha] \leq Z_\alpha$, and $[F, FZ_\alpha] \leq FZ_\alpha$, so $[G_\alpha, FZ_\alpha] \leq FZ_\alpha$.

(7) $Z_\alpha \leq F$; in particular $F \lhd G_\alpha$ and $L_0 \leq F$.

By (3) $Z_\alpha = [Z_{\alpha'}, Z_\alpha][Z^g_{\alpha'}, Z_\alpha]C_{Z_\alpha}(G_\alpha) = [F, Z_\alpha]\Omega_1{}^\circ(Z(G_\alpha))$. Consider the factors. We have $\Omega_1{}^\circ(Z(G_\alpha)) \leq Z_{\alpha'} \leq F$. Also $[Z_{\alpha'}, Z_\alpha] \leq Z_{\alpha'} \leq F$ and $[Z^g_{\alpha'}, Z_\alpha] \leq Z^g_{\alpha'} \leq F$. Thus $[F, Z_\alpha] \leq F$.

Thus $Z_\alpha \leq F$ and $F = FZ_\alpha \lhd G_\alpha$. As $G_\alpha = FQ_\alpha$, the quotient G_α/F is a 2-group and $L_0 \leq F$.

(8) $Q_\alpha = DV$

We apply (5). $C^\circ_{Q_\alpha}(F) \leq D$ by (7). As $Z_\alpha/C_{Z_\alpha}(G_\alpha) = [G_\alpha/Q_\alpha, Z_\alpha/C_{Z_\alpha}(G_\alpha)]$, we have $Z_\alpha \leq [L_0, Z_\alpha]C_{Z_\alpha}(G_\alpha) \leq VD$. Thus (8) follows.

(9) $\Phi(S) \leq D\Omega_1(Z(S))$.

As $[Z_\alpha, Z_{\alpha'}]$ centralizes Q_α and $Z_{\alpha'}$, and $S = Z_{\alpha'}Q_\alpha$, we find $[Z_\alpha, Z_{\alpha'}] \leq \Omega_1(Z(S))$. Now $S = Z_{\alpha'}Q_\alpha = Z_{\alpha'}DV$. As $V = [Q_\alpha, L_0] \leq [Q_\alpha, F] = [Z_\alpha, F] \leq Z_\alpha$, we find $S = DZ_\alpha Z_{\alpha'}$.

Let $\hat{S} = S/D\Omega_1(Z(S))$. Then $\hat{S} = \langle \hat{Z}_\alpha, \hat{Z}_{\alpha'} \rangle$. Furthermore $[Z_\alpha, Z_{\alpha'}] \leq \Omega_1(Z(S))$ and thus $[\hat{Z}_\alpha, \hat{Z}_{\alpha'}] = 1$. Hence $\hat{S}$ is elementary abelian and (9) follows.

(10) $S/\Omega_1{}^\circ(Z(S))$ is elementary abelian.

The groups $[\Phi(S), S]$ and $\mho^1(\Phi(S))$ are contained in D by (9), and are normal in S Hence they are normalized by $L_0S = G_\alpha$. But as they are characteristic in S, they are normal in $N^\circ(S)$ as well. As these groups are also connected, by our basic assumption (P), this forces them to be trivial. Thus $\Phi(S) \leq \Omega_1(Z(S))$, and as $\Phi(S)$ is connected, (10) follows.

(11) $Z^\circ(S)$ is elementary abelian.

$\mho^1(Z^\circ(S))$ is connected, definable, and characteristic in S, and is contained in $C_{Z(Q_\alpha)}(F)$ which is contained in D. Thus $\mho^1(Z^\circ(S))$ is normalized by $L_0S = G_\alpha$ and by $N^\circ(S)$, which by our main assumption (P) implies (13).

(12) $Z_\alpha = Z^\circ(Q_\alpha)$.

By Lemma B.3(2), $Z_\alpha \leq Z(Q_\alpha)$. As $Q_\alpha = C_{Q_\alpha}(F)Z_\alpha$, we have $Z(Q_\alpha) = C_{Z(Q_\alpha)}(F)Z_\alpha$. We have $C_{Z(Q_\alpha)}(F) \leq Z(S)$ so $Z^\circ(Q_\alpha) \leq Z^\circ(S)Z_\alpha = Z_\alpha$ by (13).

This proves all parts of the theorem. □

Corollary C.3 *Assume that $b = 2$. Then the following hold:*

1. *$Q = DV$, and V is an elementary abelian 2-group central in Q.*
2. *For S a Sylow 2-subgroup of M, $S/\Omega_1{}^\circ(Z(S))$ is an elementary abelian group.*
3. *$Z_\alpha = Z^\circ(Q)$. In particular, $Z^\circ(Q)$ is an elementary abelian group.*

Proof. This is the same statement as the previous without the proviso that G act faithfully on Γ. So let K be the kernel of the action of G on Γ, a finite central 2-group, and let G_1, M_1, S_1 be the quotients of G, M, S by K. Set:

$$\begin{array}{ll} Q_1 = O_2(M_1) & L_1 = O^2(M_1) \\ V_1 = [Q_1, L_1] & D_1 = C_{Q_1}{}^\circ(L_1) \end{array}$$

By the previous proposition our three claims hold for these groups. Note that $Q_1 = Q/K$ and $L_1 = L_0K/K$. Thus $V_1 = VK/K$. We will check also that $D_1 = DK/K$. Certainly $DK/K \leq D_1$. Conversely, let $\hat{D}$ be the preimage of D_1 in G. Then $[\hat{D}, L_0] \leq K$ so by Fact 2.1 $[\hat{D}, L_0] = 1$ and $\hat{D}^\circ \leq D$. As $\hat{D}^\circ$ covers D, $\hat{D} \leq \hat{D}^\circ K \leq DK$.

Ad 1. From $Q_1 = D_1 V_1$ it follows that $Q \leq DVK$. Since Q is connected we conclude that $Q = DV$.

Ad 2. Let S_0 be the preimage of $\mho^{1\circ}(S_1)$ in S. Then $[S, S_0] \leq K$. As S is connected and K is finite, by Fact 2.1 we find $S_0 \leq Z(S)$. Further S_0/K is elementary abelian and $\Phi(S_0{}^\circ)$ is connected, so $S_0{}^\circ$ is elementary abelian. Thus $S_0{}^\circ \leq \Omega_1(Z(S))$. Now $\Phi(S) \leq S_0$ and $\Phi(S)$ is connected so $\Phi(S) \leq S_0{}^\circ \leq \Omega_1{}^\circ(Z(S))$.

Ad 3. Let $Z_{1\alpha}$ be Z_α computed in G_1. It suffices to check that Z_α covers $Z_{1\alpha}$ and that $Z^\circ(Q)$ covers $Z^\circ(Q_1)$. Let A be the preimage in G of $Z^\circ(Q_1)$. Then $[A, Q] \leq K$. As Q is connected, $A \leq Z(Q)$. Thus $Z^\circ(Q)$ covers $Z^\circ(Q_1)$. The argument for Z_α is similar. □

D The case $b > 2$

In this final section we eliminate the case $b > 2$. As b is even, we have $b \geq 4$. The case $b \geq 6$ leads more quickly to a contradiction, while the case $b = 4$ takes a closer analysis.

Notation D.1 *Let (α, α') be a critical pair in Γ. A path of length b from α to α' is fixed, and its vertices are denoted by $(\alpha, \alpha+1, \ldots, \alpha+b)$ or, counting from the other end, $(\alpha'-b, \ldots, \alpha'-1, \alpha')$.*

In the next Lemma we discuss the prolongation of a path linking a critical pair "to the left" in a natural way.

Lemma D.2 (2.3, [27]) *Let (α, α') be a critical pair in Γ. Then there is a vertex β such that $d(\alpha, \beta) = 2$ and:*

(a) $Z_\beta \not\leq G_{\alpha'}$,

With such a choice of β we have:

(b) $\langle O_2(G_\beta \cap G_\alpha), Z_{\alpha'} \rangle = G_\alpha$.

(c) $(\beta, \alpha' - 2)$ *is a critical pair,*

(d) If $b > 2$ then $[Z_\beta, Z_{\alpha'-2}] \leq Z(G_\alpha)$.

Proof. Suppose first that β has been chosen satisfying (a) with $d(\alpha, \beta) = 2$. Note that $d(\beta, \alpha') = b + 2$ as a consequence of condition (a). Let λ be adjacent to α, β, and let $S = O_2(G_\lambda) = O_2(G_\alpha \cap G_\beta)$ by Lemma C.1. As $\lambda \neq \alpha + 1$, S is distinct from $O_2(G_{\alpha+1}) = Z_{\alpha'} Q_\alpha$. Thus $\langle S, Z_{\alpha'} \rangle$ covers G_α / Q_α and hence $\langle S, Z_{\alpha'} \rangle = G_\alpha$. This is condition (b). For (c), note that $d(\beta, \alpha' - 2) \leq b$ while $Z_\beta \not\leq G^{(1)}_{\alpha'-2}$ as otherwise we would find $Z_\beta \leq O_2(G_{\alpha'-1}) \leq G_{\alpha'}$. Thus $(\beta, \alpha' - 2)$ is a critical pair. Thus (b) and (c) both hold.

If $b > 2$ then $[Z_{\alpha'-2}, Z_{\alpha'}] = 1$. As $(\beta, \alpha' - 2)$ is a critical pair $[Z_\beta, Z_{\alpha'-2}] \leq Z_{\alpha'-2} \cap Z_\beta$. Thus the group $[Z_\beta, Z_{\alpha'-2}]$ is centralized by $Z_{\alpha'}$ and also by S as $S = Z_{\alpha'-2} Q_\beta$. Now (b) implies (d).

Accordingly we turn our attention to condition (a). Let $\lambda \neq \alpha + 1$ be any other neighbor of α. Then as seen above, while checking (b), we have $\langle O_2(G_\lambda), Z_{\alpha'} \rangle = G_\alpha$. We will find β adjacent to λ so that $Z_\beta \not\leq G_{\alpha'}$. Then as $\beta \neq \alpha$, we have $d(\alpha, \beta) = 2$.

Suppose toward a contradiction that $Z_\beta \leq G_{\alpha'}$ for every neighbor β of λ, so that in fact $Z_\beta \leq G_{\alpha'} \cap G_{\alpha'-2}$ for each such β. Let $T = O_2(G_{\alpha'} \cap G_{\alpha'-2}) = O_2(G_{\alpha'-1}) = Z_\alpha Q_{\alpha'}$ and set $V_\lambda = \langle Z_\beta : d(\lambda, \beta) = 1\rangle$. Then our hypothesis amounts to: $V_\lambda \leq T$. As $T = Z_\alpha Q_{\alpha'}$ this yields Thus $[V_\lambda, Z_{\alpha'}] \leq [Z_\alpha, Z_{\alpha'}] \leq Z_\alpha \leq V_\lambda$, and hence V_λ is normalized by $Z_{\alpha'}$.

As V_λ is normal in G_λ and $\langle O_2(G_\lambda), Z_{\alpha'}\rangle = G_\alpha$, we find that V_λ is normalized by G_α as well. This contradicts Lemma A.2 □

As a matter of notation, when we apply the foregoing lemma, we will call the vertex β which is selected "$\alpha - 2$". Formally, this has no special meaning, but it serves as an aide-mémoire.

Proposition D.3 (2.4, [27]) $b < 6$.

Proof. Suppose towards a contradiction that $b \geq 6$. Fix a vertex $\beta = \alpha - 2$ as afforded by Lemma D.2, and let a common neighbor of α and $\alpha - 2$ be called $\alpha - 1$. We consider the following groups:

$$V_\alpha = \langle Z_{\alpha-2}^{G_\alpha}\rangle Z_\alpha \qquad V_{\alpha-2} = \langle Z_\alpha^{G_{\alpha-2}}\rangle Z_{\alpha-2}$$

Then $V_\alpha \lhd G_\alpha$ and $V_{\alpha-2} \lhd G_{\alpha-2}$. As $b > 2$ we have $V_\alpha \leq Q_\alpha$ and $V_{\alpha-2} \leq Q_{\alpha-2}$.

(1) $[Q_\alpha, V_\alpha] \leq Z(G_\alpha)$.

It suffices to check that $[Q_\alpha, Z_{\alpha-2}] \leq Z(G_\alpha)$. As $Z_{\alpha'-2}Q_{\alpha-2}$ is a Sylow 2-subgroup of G_α, we have $[Q_\alpha, Z_{\alpha-2}] \leq [Z_{\alpha'-2}Q_{\alpha-2}, Z_{\alpha-2}] = [Z_{\alpha'-2}, Z_{\alpha-2}]$ and condition (d) of Lemma D.2 applies.

The idea now is to "reflect" the "path" $(\alpha - 2, \ldots, \alpha')$ around $\alpha - 2$ and to consider the view from within the resulting long "path".

As $(\alpha - 2, \alpha' - 2)$ is a critical path, $Z_{\alpha'-2}$ covers a Sylow 2-subgroup of $G_{\alpha-2}/Q_{\alpha-2}$ and thus we may choose an element $t \in G_{\alpha-2}$ such that $G_{\alpha-2} = \langle Z_{\alpha'-2}, Z_{\alpha'-2}^t\rangle Q_{\alpha-2}$. We consider the sequence of vertices $((\alpha'-2)^t, (\alpha'-4)^t, \ldots, \alpha^t, \alpha-2, \alpha, \ldots, \alpha'-2)$ in which $\alpha - 2$ is the central point, and only the even terms, as indicated, play any real role.

(2) $V_\alpha \leq G_{(\alpha'-2)^t}$.

We check first that $V_\alpha \leq G_{(\alpha'-6)^t}$. For $g \in G_\alpha$ we have $d(\alpha, (\alpha-2)^g) \leq 2$ and $d(\alpha - 2, (\alpha' - 6)^t) = d(\alpha - 2, \alpha' - 6) \leq b - 4$ and thus $d((\alpha-2)^g, (\alpha'-6)^t) \leq b$. Thus $Z_{(\alpha-2)^g} \leq G_{(\alpha'-6)^t}$ and $V_\alpha \leq G_{(\alpha'-6)^t}$.

Now suppose toward a contradiction that $V_\alpha \not\leq G_{(\alpha'-2)^t}$. Then $V_\alpha \not\leq Q_{(\alpha'-4)^t}$. Thus we may fix i, $i = 4$ or 6, so that $V_\alpha \leq G_{(\alpha'-i)^t}$ while $V_\alpha \not\leq Q_{(\alpha'-i)^t}$. The two possibilities can be analyzed to some extent simultaneously.

We fix $\beta \in (\alpha-2)^{G_\alpha} \cup \{\alpha\}$ such that $Z_\beta \not\leq Q_{(\alpha'-i)^t}$; and we take $\beta = \alpha$ if possible. Set $R = [Z_\beta, Z_{(\alpha'-i)^t}]$.

As $Z_\beta \leq G_{(\alpha'-i)^t}$, we have $R \leq Z_{(\alpha'-i)^t}$. As $d((\alpha'-i)^t, (\alpha'-2)^t) \leq 4 < b$ we have $[R, Z_{(\alpha'-2)^t}] \leq [Z_{(\alpha'-i)^t}, Z_{(\alpha'-2)^t}] = 1$. Thus R centralizes $Z_{(\alpha'-2)^t}$.

Now $d((\alpha'-i)^t, \alpha) \leq (b - i) + 4 \leq b$ so $Z_{(\alpha'-i)^t} \leq G_\alpha$ and thus $Z_{(\alpha'-i)^t} \leq O_2(G_{\alpha-1})$. In particular $R \leq O_2(G_{\alpha-1})$.

We now consider two cases separately:

(*Case 1*) $\quad Z_{(\alpha'-i)^t} \leq Q_\alpha$.

Then $R = [Z_\beta, Z_{(\alpha'-i)^t}] \leq [V_\alpha, Q_\alpha] \leq Z(G_\alpha)$ by (1). By the choice of t, $G_{\alpha-2} = \langle G_{\alpha-2} \cap G_\alpha, Z_{(\alpha'-2)^t}\rangle$ and thus $R \leq Z(G_{\alpha-2})$ as well. As $\beta \in (\alpha-2)^{G_\alpha} \cup \{\alpha\}$, we have $R \leq Z(G_\beta)$.

On the other hand we have $Z_{(\alpha'-i)^t} \leq Q_\alpha \leq G_\beta$ acting nontrivially on Z_β. As $\bar{Z}_\beta = Z_\beta / C_{Z_\beta}(G_\beta)$ is a natural module for $\bar{G}_\beta = G_\beta/Q_\beta$, the commutator R is nontrivial in $\bar{Z}_\beta$, and thus $R \not\leq Z(G_\beta)$, a contradiction.

Now suppose:

(*Case 2*) $\quad Z_{(\alpha'-i)^t} \not\leq Q_\alpha$.

As $d((\alpha'-i)^t,\alpha) \le (b-i)+4$ we conclude that $i=4$ and that $(\alpha,(\alpha'-i)^t)$ is a critical pair. Hence $\beta=\alpha$.

Now $R=[Z_\alpha, Z_{(\alpha'-i)^t}] \le [V_{\alpha-2}, Q_{\alpha-2}] \le Z(G_{\alpha-2})$ by (1).

We have $G_\alpha \le \langle O_2(G_{\alpha-1}), Z_{\alpha'}\rangle$ and hence $G_{\alpha^t} \le \langle O_2(G_{\alpha-1})^t, Z_{(\alpha')^t}\rangle$ But R centralizes $G_{\alpha-2}$, hence $O_2(G_{\alpha-1})^t$, and $d((\alpha'-4)^t, \alpha'^t) = 4 < b$, so $R \le Z_{(\alpha'-4)^t} \le Q_{\alpha'^t}$ and $[R, Z_{\alpha'^t}] = 1$. Thus R centralizes G_{α^t} and as t centralizes R, we have $R \le Z(G_\alpha)$ as well. But $\bar{Z}_\alpha = Z_\alpha/C_{Z_\alpha}(G_\alpha)$ is a natural module for G_α/Q_α, and $\bar{R} = [\bar{Z}_\alpha, Z_{(\alpha'-i)^t}]$ with $Z_{(\alpha'-i)^t}$ acting non-trivially, a contradiction.

(3) V_α, $Z_{\alpha-2}Z_\alpha$, and $Q_\alpha \cap Q_{\alpha-2}$ are normal in $G_{\alpha-2}$.

$Z_{\alpha-2}Q_{(\alpha'-2)^t}$ is a Sylow 2-subgroup of $G_{(\alpha'-2)^t}$ as $(\alpha-2,\alpha'-2)$ is a critical pair; but this is a subgroup of $V_\alpha Q_{(\alpha'-2)^t}$ which is a 2-group by point (2). Hence $V_\alpha \le Z_{\alpha-2}Q_{(\alpha'-2)^t}$.

$G_{\alpha-2}$ is generated by $G_{\alpha-2}\cap G_\alpha$ and $Z_{(\alpha'-2)^t}$.

Now G_α normalizes V_α and $[V_\alpha, Z_{(\alpha'-2)^t}] \le [Z_{\alpha-2}Q_{(\alpha'-2)^t}, Z_{(\alpha'-2)^t}] \le Z_{\alpha-2} \le V_\alpha$. Thus V_α is normal in $G_{\alpha-2}$.

Again, $G_{\alpha-2}\cap G_\alpha$ normalizes $Z_{\alpha-2}Z_\alpha$ and by the calculation of the previous paragraph $[Z_{(\alpha'-2)^t}, Z_\alpha Z_{\alpha-2}] \le [Z_{(\alpha'-2)^t}, V_\alpha] \le Z_{\alpha-2}$ so $Z_{(\alpha'-2)^t}$ also normalizes $Z_{\alpha-2}Z_\alpha$. Thus $Z_{\alpha-2}Z_\alpha$ is normal in $G_{\alpha-2}$.

Finally, $Q_{\alpha-2}\cap Q_\alpha = C_{G_{\alpha-2}}(Z_\alpha Z_{\alpha-2})$.

(4) $Q_\alpha \cap Q_{\alpha-2} \lhd G_\alpha$.

Let X be the normal closure of $Q_{\alpha-2}\cap Q_\alpha$ in G_α. Then $X \le Q_\alpha$ and our claim is that $X \le Q_{\alpha-2}$.

Let $Y=[V_\alpha, Q_\alpha\cap Q_{\alpha-2}]$. By (1) Y is central in G_α and thus $Y=[V_\alpha, X]$ as well.

Since Y is central in G_α it centralizes a Sylow 2-subgroup of $G_{\alpha-2}$. But Y is normal in $G_{\alpha-2}$ by (3), so Y is central in $G_{\alpha-2}$. Thus $[Z_{\alpha-2}, X] \le [V_\alpha, X] \le Z(G_{\alpha-2})$. As $\bar{Z}_{\alpha=2} = Z_{\alpha-2}/C_{Z_{\alpha-2}}(G_{\alpha-2})$ is a natural module and $[\bar{Z}_{\alpha-2}, X]=0$, we find $X \le Q_{\alpha-2}$ as claimed.

The final contradiction is derived as follows. As $\alpha-1$ is conjugate under G_α to $\alpha+1$, $(\alpha-2)$ is conjugate under G_α to a neighbor λ of $\alpha+1$. Suppose $\lambda=(\alpha-2)^g$ with $g\in G_\alpha$. As $d(\lambda,\alpha'-2) < b$, we have $Z_{\alpha'-2} \le Q_\alpha\cap Q_\lambda = (Q_\alpha\cap Q_{\alpha-2})^g = Q_\alpha \cap Q_{\alpha-2}$ by (4). Then $[Z_{\alpha'-2}, Z_{\alpha-2}]=1$, while $(\alpha-2,\alpha'-2)$ is a critical pair, a contradiction. □

Proposition D.4 (3.3, [27]) $b \ne 4$.

Proof. Suppose toward a contradiction that $b=4$. Fix a critical pair (α,α'). Choose $\alpha-2$, and then $\alpha-4$, in accordance with Lemma D.2 so that $d(\alpha,\alpha-2)=d(\alpha-2,\alpha-4)=2$ and $(\alpha-4,\alpha)$ and $(\alpha-2,\alpha+2)$ are critical pairs.

(1) $Z_\alpha = (Z_\alpha\cap Z_{\alpha+2})[Z_\alpha, Z_{\alpha-4}]$

This reflects the fact that the module $\bar{Z}_\alpha = Z_\alpha/C_{Z_\alpha}(G_\alpha)$ is a natural module. $Z_{\alpha-4}$ covers a Sylow 2-subgroup of G_α/Q_α so $[Z_{\alpha-4},\bar{Z}_\alpha]$ is a 1-dimensional subspace of this module, and similarly $[Z_{\alpha'},\bar{Z}_\alpha]$ is a 1-dimensional subspace. As $Z_{\alpha'}$ and $Z_{\alpha-4}$ generate G_α modulo Q_α, by Lemma D.2(b), we find $\bar{Z}_\alpha = [Z_{\alpha-4},\bar{Z}_\alpha]\oplus[Z_{\alpha'},\bar{Z}_\alpha]$.

As $[Z_{\alpha'}, Z_\alpha] \le Z_{\alpha'}\cap Z_\alpha$ centralizes $Z_{\alpha'}Q_\alpha = O_2(G_{\alpha+1})$, we have $[Z_{\alpha'}, Z_\alpha] \le Z_\alpha\cap Z_{\alpha+2}$ and (1) follows.

We introduce the following additional notation.

$$U = Z_\alpha Z_{\alpha-2} Z_{\alpha+2}; \qquad \tilde{D} = Q_{\alpha-4}\cap Q_{\alpha-2}\cap Q_\alpha\cap Q_{\alpha+2}\cap Q_{\alpha'}$$

We observe that U is a subgroup with $U' = [Z_{\alpha-2}, Z_{\alpha+2}] \ne 1$, as Z_α centralizes all three factors and $[Z_{\alpha-2}, Z_{\alpha+2}] \le Z_{\alpha-2}\cap Z_{\alpha+2}$, since $b=4$.

(2) $Q_\alpha = \tilde{D} * U$.

This is similar to the proof of point (4) in Proposition C.2. As $Z_{\alpha-2} \le Q_\alpha \le O_2(G_{\alpha+1}) = Z_{\alpha-2}Q_{\alpha+2}$ we find $Q_\alpha \le Z_{\alpha-2}(Q_\alpha \cap Q_{\alpha+2})$. Similarly using successively $Q_\alpha \cap Q_{\alpha+2} \le O_2(G_{\alpha+3}) = Z_\alpha Q_{\alpha'}$ and $Q_\alpha \cap Q_{\alpha+2} \cap Q_{\alpha'} \le O_2(G_{\alpha-1}) = Z_{\alpha+2}Q_{\alpha-2}$ we find $Q_\alpha \le U \cdot (Q_{\alpha-2} \cap Q_\alpha \cap Q_{\alpha+2} \cap Q_{\alpha'})$.

For the final step, $Q_{\alpha-2} \cap Q_\alpha \cap Q_{\alpha+2} \cap Q_{\alpha'} \le O_2(G_{\alpha-3}) = Z_\alpha Q_{\alpha-4} = (Z_\alpha \cap Z_{\alpha+2})Q_{\alpha-4}$, using (1), and as $Z_\alpha \cap Z_{\alpha+2} \le Q_{\alpha-2} \cap Q_\alpha \cap Q_{\alpha+2} \cap Q_{\alpha'}$, we find $Q_\alpha = U \cdot \tilde{D}$, and the two factors evidently commute.

(3) $UZ(G_\alpha) \lhd G_\alpha$.

Set

$$F = \langle Z_{\alpha-4}, Z_{\alpha'} \rangle$$

By Lemma D.2(b), $G_\alpha = FQ_\alpha$. By (2) $[U, Q_\alpha] \le U$ so it remains to be seen that $[F, U] \le UZ(G_\alpha)$.

Let $U_0 = U[U, F]$. Then $U_0 = U(U_0 \cap \tilde{D})$. Now U_0 centralizes $\tilde{D}$ and $\tilde{D}$ centralizes F and U, so $U_0 \cap \tilde{D} \le Z(G_\alpha)$, and (3) holds.

(4) $\bar{U} = UZ^\circ(G_\alpha)/Z_\alpha Z^\circ(G_\alpha)$ *is a nontrivial* G_α/Q_α*-module.*

Point (3) implies that G_α/Q_α acts on $\bar{U}$. It remains to show that this action is nontrivial.

If $(\alpha-1)^g = \alpha+1$ we will show that g acts nontrivially. Let $\lambda = (\alpha-2)^g$. Then $d(\lambda, \alpha+2) \le 2$ and hence $[Z_\lambda, Z_{\alpha+2}] = 1$. As $[Z_{\alpha-2}, Z_{\alpha+2}] \ne 1$ it follows easily that the action of g on $\bar{U}$ is nontrivial.

(5) $Z_{\alpha'}$ acts quadratically on $\bar{U}$.

$\bar{U} = \bar{Z}_{\alpha-2}\bar{Z}_{\alpha+2}$ where the bar refers to factoring out $Z_\alpha Z^\circ(G_\alpha)$. As $Z_{\alpha'}$ centralizes $Z_{\alpha+2}$ it suffices to consider the action on $\bar{Z}_{\alpha-2}$.

Now $[Z_{\alpha-2}, Z_{\alpha'}] \le Q_{\alpha+2}$. As $Q_{\alpha+2} \le O_2(G_{\alpha'-1}) = Z_\alpha Q_{\alpha'}$, we have $[Q_{\alpha+2}, Z_{\alpha'}] \le [Z_\alpha, Z_{\alpha'}] \le Z_\alpha$. Thus $[Z_{\alpha-2}, Z_{\alpha'}, Z_{\alpha'}] \le Z_\alpha$ and (5) follows.

(6) $\bar{U}$ is a natural module for $F/C_F(\bar{U})$.

Here $F = \langle Z_{\alpha-4}, Z_{\alpha'} \rangle$ as in (3). As Q_α acts trivially on $\bar{U}$ and $FQ_\alpha = G_\alpha$, $F/C_F(\bar{U}) \simeq G_\alpha/Q_\alpha$ is of type SL_2.

We apply Corollary 2.41 with $G = F/C_F(\bar{U})$ and $T = Z_{\alpha'}$. In view of point (5), we need only check that $\mathrm{rk}\,(\bar{U}/C_{\bar{U}}(Z_{\alpha'})) \le \mathrm{rk}\, Z_{\alpha'}$, which is clear, to conclude that $\bar{U}/C_{\bar{U}}(F)$ is a natural module. But $\mathrm{rk}\,\bar{U} \le 2f$ where f is the rank of the base field, so $\bar{U}$ must itself be a natural module.

(7) $Z^\circ(G_\alpha) \le Z_\alpha$.

We know $Z^\circ(G_\alpha) \le Q_\alpha$. We need to show that $Z^\circ(G_\alpha)$ is elementary abelian. Let $S = O_2(G_{\alpha+1}) = Z_{\alpha'}Q_\alpha$. It suffices to show that $Z^\circ(S)$ is elementary abelian.

$Z(S) \le Q_\alpha$ and $U \cap \tilde{D} \le Z(G_\alpha)$ so $Z(S) = [Z(S) \cap U] * [Z(S) \cap \tilde{D}]$. By (6) $\mathrm{rk}\,(\bar{U}) = \mathrm{rk}\,(\bar{Z}_{\alpha-2}) + \mathrm{rk}\,(\bar{Z}_{\alpha+2})$, so $[U \cap Z(G_\alpha)]^\circ \le Z_\alpha$. Hence $\Phi(Z^\circ(S)) \le Z(\tilde{D}) \le Z(G_\alpha)$. Our original hypothesis (P) forces $\Phi(Z^\circ(S)) = 1$ and (7) follows.

After these preparations we reach a contradiction as follows. By (7) the action of G_α on $\bar{U}$ is induced by an action on U. By (6) this action is transitive on $(\bar{U})^\times$. If $u \in U \setminus Z_\alpha$ is an involution, then the class uZ_α consists entirely of involutions. By transitivity of the action, U is elementary abelian. But $U' \ne 1$.

This contradiction shows that $b \ne 4$. □

Proof of Theorem 3.1. Lemma B.3 and Propositions C.2, D.3, and D.4 yield the result. □

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