Groups of finite Morley rank and even type with strongly closed abelian subgroups

Tuna Altınel
Institut Girard Desargues
UPRES-A 5028 Mathématiques
Batiment 101 (mathématiques)
Université Claude Bernard Lyon-1
43 blvd du 11 novembre 1918
69622 Villeurbanne CEDEX, France
e-mail: altinel@desargues.univ-lyon1.fr

Alexandre Borovik
Department of Mathematics, University of Manchester
Institute of Science and Technology
Manchester M60 1QD, England
e-mail: borovik@lanczos.ma.umist.ac.uk

Gregory Cherlin*
Department of Mathematics, Rutgers University
Hill Center, Piscataway, New Jersey 08854, U.S.A
e-mail: cherlin@math.rutgers.edu

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1 Introduction

According to a long-standing conjecture in model theory, simple groups of finite Morley rank should be algebraic. The present paper is part of a series aimed ultimately at proving the following:

Conjecture 1 (Even Type Conjecture) *Let G be a simple group of finite Morley rank of even type, with no infinite definable simple section of degenerate type. Then G is algebraic.*

An infinite simple group G of finite Morley rank is said to be of *even type* if its Sylow 2-subgroups are of bounded exponent. It is of *degenerate type* if its Sylow 2-subgroups are finite. If the main conjecture is correct, then there should be no groups of degenerate type. So the flavor of the Even Type Conjecture is that the classification in the even type case reduces to an extended Feit-Thompson Theorem. Those who are skeptical about the main conjecture would expect degenerate type groups to exist. The Even Type Conjecture confirms that this is the heart of the matter.

We believe that it is realistic to aim at a proof of the Even Type Conjecture with existing tools. For the moment we concentrate on a special case, called the "tame" case (see §1 for definitions). Generally speaking, "nontame" proofs are nontrivial deformations of "tame" proofs, involving a closer analysis of the more pathological configurations that arise. The proof of any version of the Even Type Conjecture will be a rather elaborate affair (following the main lines of the characterization of groups of characteristic 2 type – and a bit more – in finite group theory), and for this reason we have pursued the rapid development of the theory in the tame case as opposed to the more systematic development of the general theory. However others, notably Jaligot, are systematically pursuing the issues that arise in elimination of tameness, so we hope to see the Even Type Conjecture proved in full generality in the near future. The only use of the tameness hypothesis which is made in the present paper is via Fact 1.2 below; if this can be proved without use of tameness then we have no further need of that hypothesis here. Note however that that fact is applied repeatedly. We will take pains to mention such uses explicitly.

The main result proved here is a classification theorem which is an analog of a theorem of Goldschmidt ([13]) in the finite case, belonging to a family of characterizations of SL_2 which are very helpful in dealing with fusion analysis.

Theorem 1.1 *Let G be a simple tame K^*-group of finite Morley rank and of even type. Suppose that G contains an infinite definable abelian subgroup A which is contained in the connected component S of a Sylow 2-subgroup of G, and which is strongly closed in S. Then $G \simeq \mathrm{SL}_2(K)$ where K is some algebraically closed field of characteristic 2.*

Here "strongly closed" means the following: if $a \in A$, $g \in G$, and $a^g \in S$, then $a^g \in A$.

This theorem is the key to the treatment of components in groups of even type (which is the "bit more" alluded to above). We expect it to combine with still conjectural forms of Pushing Up and Baumann's theorem to yield a short proof of the analog of Aschbacher's global $C(G,T)$ theorem. A further indication of the usefulness of this theorem in the classification of tame simple K^*-groups of finite Morley rank of even type is a work in preparation by Borovik, Cherlin and Corredor which proves a version of Aschbacher's standard component theorem by reduction to the strongly closed abelian case. Moreover, there is now a detailed plan for completing the proof of the Even Type Conjecture, at least in the tame case, by first using Theorem 1.1 and some further analysis to eliminate certain standard components which would obstruct the use of the amalgam method, and then applying the amalgam method and a classification theorem for BN-pairs (cf. [12] or [20]) to arrive at a point at which the analog of an identification theorem of Niles applies. The present paper, together with ongoing work on Pushing Up and the global $C(G,T)$ theorem, constitutes the last in the series of papers laying the foundations of the analysis of tame groups of even type by providing some general tools which are mostly connected with fusion analysis.

One of the main tools in proving Theorem 1.1, or any of the classification theorems of this type, is the following classificaton theorem:

Fact 1.2 ([3]) *Let G be a simple, tame, K^*-group of finite Morley rank of even type. If G has a weakly embedded subgroup then $G \simeq \mathrm{SL}_2(K)$, where K is an algebraically closed field of characteristic* 2.

Here weak embedding (cf. §2.4) is a natural generalization of strong embedding, far more flexible in practice (see for example the rapid elimination of cores in 2-local subgroups at the end of [3]). Tameness will be discussed in §2.

Tameness does not actually enter into the proof of Theorem 1.1 given here, except insofar as it is required when Fact 1.2 is invoked. Theorem 1.1 can be rephrased more precisely as follows: *under the stated hypotheses, omitting the tameness, G has a weakly embedded subgroup.* After the present work was complete, Jaligot completed a proof of the generalization of Fact 1.2 in which the tameness hypothesis is dropped ([19, 18]). Taking this into account, we see that Theorem 1.1 is also valid without a tameness hypothesis, and as should now be clear, the proof of that form of Theorem 1.1 is in fact given in the present paper, modulo ([19, 18]).

The proof of Theorem 1.1 reduces to the following result of Aschbacher-Seitz type, which is occasionally of independent interest.

Theorem 1.3 *Let G be a simple tame K^*-group of finite Morley rank and of even type. Suppose that G has a standard component L of the form $\mathrm{SL}_2(K)$ for some algebraically closed field of characteristic 2. Let A be a Sylow 2-subgroup of L and U be the connected component of a Sylow 2-subgroup of $C(L)$. If U is nontrivial then AU is a Sylow° 2-subgroup of G.*

As one might expect, considerably more is true (ultimately, standard components can be completely eliminated). However it appears that this statement, as formulated here, covers the critical configuration on which more general analyses depend. Standard components will be defined in §6. As in the case of Theorem 1.1, one may also drop the tameness hypothesis here, via ([19, 18]), using the arguments of the present paper without any further modification.

The paper is organized as follows. The next section contains necessary background material relating to a variety of conventional group theoretic issues, in the forms appropriate to the study of groups of finite Morley rank. In §3 we lay out more specifically the basic facts relating to strongly closed abelian subgroups, and the relevant K-group statement. The main line of argument then goes as follows. After some adjustment of the strongly closed abelian group A to a "minimal" such, we claim that G has a weakly embedded subgroup, and hence can be identified by Fact 1.2. When this argument fails, the obstruction is always a configuration of the form $L \times U$ where L is a subgroup isomorphic to $\mathrm{SL}_2(K)$ for some algebraically closed field of characteristic 2, which furthermore meets A in a Sylow 2-subgroup of L, and where U is an infinite 2-group commuting with L. There are two cases here: the more degenerate case in which $A \cap L < A$, where we speak of "A-special" components L, and the more plausible configuration in which L contains A, which will capture the bulk of our attention.

We eliminate A-special components in §4 using the theory of groups generated by pseudoreflection groups, which is very powerful here. Then in §5 we show how the weak embedding argument works in the absence of the configuration $L \times U$ $(A \leq L)$. Then we devote two sections to the analysis of $L \times U$: some Sylow analysis in §6, then a brief look at the Thompson rank formula (cf. [3]) leading to a concluding contradiction arrived at by two Thompson rank computations which yield inconsistent answers, in §7.

The analysis in §§6-7 amounts to the proof of Theorem 1.3, as is explained in §6.

2 Background

In the present section we review the main facts required for the proof of Theorem 1.1. We use some of the basic facts and notions as given in [9] without explicit reference, but the more substantial points are all given explicitly below.

Definition 2.1

1. *A* section *of a group G is a quotient of the form H/K where H and K are subgroups of G and $K \lhd H$. Such a section is said to be* definable *if H and K are definable.*
2. *A* bad group *is a simple infinite group of finite Morley rank in which every proper definable connected subgroup is nilpotent.*
3. *A* bad field *is a structure of finite Morley rank consisting of an algebraically closed field together with a distinguished proper infinite subgroup of its multiplicative group.*
4. *A* tame group *is a group such that none of its proper sections is a bad group, and which does not interpret a bad field. We note that for groups which do not interpret bad fields, the assumption that there are no bad sections is equivalent to the assumption that there are no sections of degenerate type in the sense of the introduction. (This is not obvious, but follows from results given in [9].)*
5. *A K*-group *is a group G of finite Morley rank such that every infinite definable simple section of G is isomorphic to an algebraic group over an algebraically closed field.*
6. *A K^**-group *is a group G of finite Morley rank such that every infinite proper definable section of G is isomorphic to an algebraic group over an algebraically closed field. Equivalently, G is either a K-group, or a simple group all of whose definable subgroups are K-groups. As we are concerned here with techniques relevant to an inductive proof of the Even Type Conjecture, we confine ourselves in practice to the study of simple K^*-groups of even type.*

2.1 2-Sylow theory

There is a good Sylow theory for the prime 2 in our context:

Fact 2.2 ([10])

1. *The Sylow 2-subgroups of a group of finite Morley rank are conjugate.*
2. *If S is a Sylow 2-subgroup of a group of finite Morley rank then S is nilpotent-by-finite and its connected component is the central product of a definable, connected, nilpotent subgroup of bounded exponent and a divisible, abelian 2-group. Moreover, these two subgroups are uniquely determined.*

This provides a rather good analog to the general structure of the connected component of a Sylow subgroup in an algebraic group, where depending on the characteristic we may be dealing with a maximal unipotent subgroup, or the 2-torsion in a torus (semisimple elements).

Accordingly we adopt the terminology suggested by this case:

Definition 2.3

1. *A* unipotent *subgroup is a connected definable subgroup of bounded exponent (in our context, typically a 2-group and hence nilpotent by Fact 2.2).*
2. *A* torus *is a definable divisible abelian group.*
3. *For any prime p, a p*-torus *is a divisible abelian p-group. (A nontrivial p-torus is not definable; but its definable closure [Definition 2.38] is a torus.)*
4. *A group of finite Morley rank is of* even type *if the connected component of a Sylow 2-subgroup is unipotent and nontrivial.*
5. *A group of finite Morley rank is of* odd type *if the connected component of a Sylow 2-subgroup is a nontrivial 2-torus.*

6. *A group of finite Morley rank is of* mixed type *if the connected component of a Sylow* 2*-subgroup is the central product of a nontrivial unipotent subgroup and a nontrivial* 2*-torus.*

7. *A group of finite Morley rank is of* degenerate type *if the connected component of a Sylow 2-subgroup is trivial (that is, the Sylow 2-subgroups are finite).*

The degenerate case is by far the hardest one to come to terms with. The conjecture of course is that degenerate type and mixed type do not arise. The nonexistence of infinite simple groups of finite Morley rank of degenerate type is a strong form of Feit-Thompson for this context. On the other hand one can dispose of the mixed type case *a priori*, when working in the K^*-context.

Fact 2.4 ([17]) *A simple K^*-group of finite Morley rank cannot be of mixed type.*

This was proved initially under a tameness hypothesis, and the removal of this hypothesis by some further fusion analysis constituted a major step forward. The success of those arguments is one of the ingredients in our optimism regarding the general form of the Even Type Conjecture.

For the sake of brevity we will refer to the connected components of Sylow 2-subgroups as *Sylow*$^\circ$ 2*-subgroups*. (Ron Solomon chauvinistically proposes the reading “SylOhio” for this notation.)

The following useful lifting result is given in [28].

Fact 2.5 *Let G be a group of finite Morley rank, and N a definable normal subgroup. Then the Sylow 2-subgroups of G/N are the images of Sylow 2-subgroups of G.*

2.2 Nilpotent and solvable groups

Definition 2.6 *Let H be a group of finite Morley rank.*

1. *A $2^\perp$-group is a group that does not have involutions.*

2. *$O(H)$ will denote the largest, definable, connected, normal $2^\perp$-subgroup. This is called the* core *of H.*

3. *We write $N^\circ(H)$ for $N(H)^\circ$, $C^\circ(H)$ for $C(H)^\circ$, and so forth.*

4. The solvable radical*, denoted $\sigma(H)$, is the largest normal solvable subgroup of G*

5. *$F(H)$ is the* Fitting subgroup*, the largest normal nilpotent subgroup.*

6. *$O_2(H)$ is the largest normal 2-subgroup of H. In groups of even type, this is definable.*

The solvable radical and Fitting subgroup are definable in G (see e.g. [24]), but not necessarily connected.

Fact 2.7 ([25]) *Let G be a nilpotent group of finite Morley rank. Then $G = D * C$, where D and C are definable characteristic subgroups of G, D is divisible, and C is of bounded exponent.*

Fact 2.8 ([9], Exercise 1, p. 97) *An infinite nilpotent p-group of finite Morley rank and of bounded exponent has infinitely many central elements of order p.*

Proof. See [1]. $\square$

Fact 2.9 ([9], Exercise 5, p. 98) *Any infinite normal subgroup of a nilpotent group of finite Morley rank contains infinitely many central elements.*

Proof. See [1]. $\square$

Fact 2.10 ([23]) *Let G be a connected solvable group of finite Morley rank. Then $G/F^\circ(G)$ (hence, $G/F(G)$) is a divisible abelian group.*

Fact 2.11 ([9], Theorem 9.29) *Let G be a connected solvable group of finite Morley rank. Then the Sylow p-subgroups of G are connected for any prime p.*

Fact 2.12 ([8], Theorem 2 and [7], Proposition C) *Let G be a solvable group of finite Morley rank and H a normal Hall π-subgroup of G of bounded exponent. Then any subgroup K of G with $K \cap H = 1$ is contained in a complement to H in G, and the complements of H in G are definable, and are conjugate to each other.*

Fact 2.13 ([3]) *Let H be a connected solvable group of finite Morley rank and S a Sylow 2-subgroup of H. Assume S is unipotent. Then $S \leq F(H)$, and therefore S is a characteristic subgroup of H.*

2.3 Automorphisms

Fact 2.14 ([10]) *Let T be a p-torus in a group of finite Morley rank. Then $[N_G(T) : C_G(T)] < \infty$. Moreover, there exists $c \in \mathbb{N}$ such that $[N_G(T) : C_G(T)] \leq c$ for any torus in G.*

Fact 2.15 ([9], Theorem 8.4) *Let $\mathcal{G} = G \rtimes H$ be a group of finite Morley rank where G and H are definable, G is an infinite simple algebraic group over an algebraically closed field, and $C_H(G) = 1$. Then, viewing H as a subgroup of $Aut(G)$, we have $H \leq Inn(G)\Gamma$, where $Inn(G)$ is the group of inner automorphisms of G and Γ is the group of graph automorphisms of G.*

Fact 2.16 ([3], Proposition 3.4) *Let G be a connected K-group of even type and T a connected $2^\perp$-group acting on G. Then T leaves invariant a Sylow$^\circ$ 2-subgroup of G.*

Corollary 2.17 *With the notation of the previous fact, if U is a T-invariant unipotent 2-subgroup of G then U is contained in a T-invariant Sylow$^\circ$ 2-subgroup of G.*

Proof. Apply the fact to $N^\circ(U)$ with U maximal T-invariant and unipotent. $\square$

Fact 2.18 ([7]) *Let G be a group of finite Morley rank and H a definable normal subgroup of G. If x is an element of G such that $\overline{x}$ is a p-element of $\overline{G} = G/H$ then the coset xH contains a p-element.*

Note in particular that if G is torsion-free, then G/H is torsion-free.

Fact 2.19 ([22]) *Let α be a definable involutive automorphism of a group of finite Morley rank G. If α has no nontrivial fixed points then G is abelian and inverted by α.*

Fact 2.20 ([9], Theorem 9.7) *Let $A \rtimes G$ be a group of finite Morley rank such that A is abelian and $C_G(A) = 1$. Let $H \lhd G_1 \lhd G$ be definable subgroups with G_1 connected and H infinite abelian. Assume also that A is G_1-minimal. Then*

$$K = \mathbb{Z}[Z(G)^\circ]/ann_{\mathbb{Z}[Z(G)^\circ]}(A)$$

is a definable algebraically closed field, A is a finite dimensional vector space over K, G acts on A as vector space automorphisms and H acts by scalars. In particular, $G \leq \mathrm{GL}_n(K)$ for some n, $H \leq Z(G)$ and $C_A(G) = 1$.

Fact 2.21 ([9], Theorem 9.8) *Let $A \rtimes G$ be a solvable group of finite Morley rank with A abelian and definable, G definable and connected. Let $B \leq A$ be either G'-minimal or G-minimal. Then G' centralizes B.*

Fact 2.22 ([1]) *Let Q and E be subgroups of a group of finite Morley rank such that Q is normal, $2^\perp$, connected, solvable, and definable, and E is a definable connected 2-group of bounded exponent. Then $[Q,E]$=1.*

The following fact was stated in [3] with the assumption that Q is a unipotent group. The definition of a unipotent group involves connectedness although the proof of the fact needs only Q to be of bounded exponent. So we state the fact in this general form. It is worth noting that we will use it in this more general form in Lemma 7.18.

Fact 2.23 ([3]) *Let Q and X be definable subgroups of a group of finite Morley rank with Q a 2-group of bounded exponent, X a $2^\perp$-group, and X acting on Q, and suppose that X acts trivially on the factors Q_i/Q_{i-1} of a definable normal series for Q. Then X acts trivially on Q.*

2.4 Weak embedding

Definition 2.24 *Let G be a group of finite Morley rank. A proper definable subgroup M of G is said to be* weakly embedded *if it satisfies the following conditions:*

(i) Any Sylow 2-subgroup of M is infinite.

(ii) For any $g \in G \setminus M$, $M \cap M^g$ has finite Sylow 2-subgroups.

Fact 2.25 ([3]) *Let G be a group of finite Morley rank of even type. A proper definable subgroup M of G is a* weakly embedded subgroup *if and only if the following hold:*

(i) M has infinite Sylow 2-subgroups.

(ii) For any unipotent 2-group U of M, $N_G(U) \subseteq M$.

Corollary 2.26 ([3]) *Let G be a group of finite Morley rank of even type, and M a proper definable subgroup of G containing a Sylow 2-subgroup S of G. Then M is a* weakly embedded subgroup *if and only if:* for any unipotent 2-group U of S, $N_G(U) \subseteq M$.

We will need a rather precise formulation of a criterion for the existence of a weakly embedded subgroup, which was proved in [3] but stated somewhat less explicitly.

Fact 2.27 (cf. [3]) *Let G be a simple group of finite Morley rank of even type and H a proper definable subgroup with infinite Sylow 2-subgroups, which contains the connected component of the normalizer of any nontrivial unipotent 2-subgroup of H. Let S be a Sylow° 2-subgroup of H. Then there is a subgroup H_1 of H containing S, for which $N(H_1)$ is weakly embedded in G.*

Proof. We recall the line of argument. Let $\mathbb{U}$ be the graph whose vertices are the nontrivial unipotent 2-subgroups of G, and whose edges are the pairs (U_1, U_2) with $[U_1, U_2] = 1$. Let C be the connected component of $\mathbb{U}$ which contains S. Let $H_1 = \langle \bigcup \mathrm{C} \rangle$. Then $N(H_1)$ is the setwise stabilizer of C with respect to the natural action of G on $\mathbb{U}$, and is weakly embedded in G. □

The relevant classification theorem was given in the introduction as Fact 1.2. Here we wish to record some further information regarding what is known in the absence of a tameness hypothesis. This will allow us to somewhat reduce the number of occasions on which we invoke the full classification theorem.

Fact 2.28 ([1, 3]) *If G is a simple K^*-group of even type of finite Morley rank with a weakly embedded subgroup M, then M° is solvable. In particular, $N_G^\circ(U)$ is solvable for any unipotent 2-subgroup U of G (Fact 2.25).*

This is proved for strongly embedded subgroups in ([1], Theorem 1.5) in a slightly sharper form, and for weakly embedded subgroups which are not strongly embedded in ([3], Theorem 5.1).

2.5 Structure of K-groups

Fact 2.29 ([1]) *Let G be a connected nonsolvable K-group of finite Morley rank. Then $G/\sigma(G)$ is isomorphic to a direct sum of simple algebraic groups over algebraically closed fields. In particular the definable connected $2^{\perp}$-sections of G are solvable.*

Fact 2.30 ([4]) *Let G be a perfect group of finite Morley rank such that $G/Z(G)$ is a simple algebraic group. Then G is an algebraic group. In particular, $Z(G)$ is finite ([16] Section 27.5).*

Our general reference for central extensions of algebraic groups (especially those that are algebraic) is [29] (especially chapters 3 and 7).

Notation 2.31 *Let G be a group of finite Morley rank.*

1. *We write $E(G)$ for the product of the subnormal connected quasisimple subgroups of G. So as in the finite case, $E(G)$ is a central product of quasisimple groups. In the K-group case these are algebraic.*
2. *$B(G)$ will denote the subgroup generated by the unipotent 2-subgroups of G. $B(G)$ is a definable connected subgroup of G*

Fact 2.32 *Let H be a connected K-group of finite Morley rank and of even type with abelian Sylow$^\circ$ 2-subgroups. Then $H = L_1 \times \cdots \times L_n \times \sigma(H)$ is a direct product, with $L_i \simeq \mathrm{SL}_2(K_i)$ for suitable algebraically closed fields K_i of characteristic 2.*

Proof. By Fact 2.22, we have $[B(H), O(H)] = 1$. Thus $[B(H), \sigma^\circ(H)] = 1$. It follows that $[B(H)^{(\infty)}, \sigma(H)] = 1$, using the connectivity of $B(H)^{(\infty)}$ and the three-subgroups lemma, and thus by Facts 2.29 and 2.30 (together with the information provided by [29]), $B(H)^{(\infty)}$ is a central product $L_1 * \cdots * L_n$ with $L_i/Z(L_i) \simeq \mathrm{SL}_2(K_i)$. As K_i has characteristic 2, this is a direct product of simple groups. So it suffices now to check that $H = B(H)^{(\infty)}\sigma(H)$ to conclude; this is essentially Fact 2.29, but we must check that if $U \leq H$ and $U/\sigma(H)$ is unipotent, then $U \leq B(H)$. This follows from the lifting of Sylow 2-subgroups, Fact 2.5. $\square$

Fact 2.33 *Let H be a connected K-group of finite Morley rank of even type such that $O_2(H) = 1$. Then $H = O(H) * E(H)$.*

Proof. The argument is similar to the proof of Fact 2.32. As $O_2(H) = 1$ and H is of even type, Fact 2.13 implies that $\sigma(H)^\circ = O(H)$. Thus we get $[B(H), \sigma^\circ(H)] = 1$, and eventually $[B(H)^{(\infty)}, \sigma^\circ(H)] = 1$. By Facts 2.29 and 2.30, $B(H)^{(\infty)} = L_1 * \cdots * L_n$ where the L_i are quasisimple algebraic groups over algebraically closed fields of characteristic 2. Using Facts 2.29 and 2.5 we get $H = B(H)^{(\infty)}\sigma^\circ(H) = B(H)^{(\infty)}O(H)$. As $B(H)^{(\infty)} = E(H)$, we are done. $\square$

Fact 2.34 *Let L be a K-group of even type with $L = L_1 \times \ldots \times L_t$, where the L_i are simple algebraic groups. If K is a definable simple subgroup of L normalized by a Sylow 2-subgroup of L then $K = L_i$ for some i.*

Proof. Let U be a Sylow 2-subgroup of L which normalizes K. Then $U = U_1 \times \ldots \times U_t$ where each U_i is a Sylow 2-subgroup of L_i. For some i we have $1 \neq [K, U_i] \leq K \cap L_i \lhd K$. Therefore $K \leq L_i$ and we may assume $L = L_i$ is simple.

We claim that $U \leq K$. In any case as U normalizes K, $U \cap K$ is a Sylow 2-subgroup of K. Since $UK = C_{UK}(K) \times K$ by Fact 2.15, we have $U = (U \cap K) \times C_U(K)$. Let $V = C_U(K)$. It suffices to show that $V = 1$. Consider the subgroup $H = N_L(V)$. By a theorem of Borel and Tits ([5], [16] §30.3), as explained in [14] 13-4, $F^*(H) = O_2(H)$, or equivalently $C_H(O_2(H)) \leq O_2(H)$. But $V \leq O_2(H) \leq U$ so $O_2(H) = V \times (O_2(H) \cap K) = V$. Thus $K \leq C(O_2(H)) = O_2(H)$, a contradiction. This shows $V = 1$.

Thus $U \leq K \leq L$. Let $w \in K$ be an involution with $U \cap U^w = 1$. Then $\langle U, U^w \rangle = L$ and thus $K = L$. $\square$

Finally, we rely on both the additivity and the definability of rank, which are not general properties of Morley rank in general structures of finite Morley rank, but do hold in the specific context of groups. Definability may be phrased as follows:

Fact 2.35 ([9], Corollary 4.24) *Let G be a group of finite Morley rank and $\phi(\overline{x}, \overline{y})$ be a first-order formula in the language of groups. For each r, the set $\{\overline{a} \in M^k : \mathrm{rk}(M^n, \overline{a}) = r\}$ is definable.*

We note also that these sets are nonempty for only finitely many values of r; in other words, the parameter space is decomposed definably into finitely many sets on which the rank function is constant. This plays a role in some applications of the Thompson rank formula (for which, see §7).

2.6 2-local subgroups

Definition 2.36 *A* 2*-local subgroup of a group of finite Morley rank is the normalizer of a nontrivial definable* 2*-subgroup.*

The following result is stated in [3] under the hypothesis that the group in question is tame. For our present use we require the more explicit form which records what is actually proved:

Fact 2.37 ([3]) *Let G be a simple K^*-group of finite Morley rank of even type and H a 2-local subgroup of G with $O(H) \neq 1$. Then G has a weakly embedded subgroup.*

2.7 Miscellaneous

Definition 2.38 *If G is a group of finite Morley rank and $X \subseteq G$, then the* definable closure *$d(X)$ of X is the intersection of all the definable subgroups of G that contain X. The* descending chain condition *on definable subgroups in groups of finite Morley rank implies that $d(X)$ is a definable subgroup.*

Fact 2.39 ([9], Exercise 2, p. 92) *Let G be a group of finite Morley rank. Assume $X \subseteq G$. Then $C_G(X) = C_G(d(X))$.*

Fact 2.40 ([30], cf. [26, 31]) *Let K be a field of finite Morley rank and T a definable subgroup of the multiplicative group $K^\times$ containing the multiplicative group of an infinite subfield of K. Then $T = K^\times$.*

As a historical aside, we note that the foregoing is one of the key ingredients in the proof of Fact 2.30 (in the absence of a tameness hypothesis). Unfortunately when [4] was written the authors were not aware of the history of this result.

The following fact is a slight generalization of a lemma in [11].

Fact 2.41 ([3]) *Let E be a unipotent 2-group of exponent at most 4. Assume that*

$$0 \to Z \to E \to E/Z \to 0$$

is an exact sequence, where Z is central and both Z and E/Z are isomorphic to K_+, K a field of characteristic 2 which is closed under taking square roots. Assume also that $T \cong K^*$ acts on E inducing the natural action on both Z and E/Z. Then E is abelian and it is either homocyclic or else is elementary abelian of the form $E = E_1 \oplus E_2$, splitting as a T-module. In the case where E is homocyclic one can obtain the multiplication table of E by fixing x_0 and x_1 in E such that $x_0^2 = x_1$. In fact, any element of E can be written as $x_0^a x_1^b$, with $a, b \in T$, and the product of two distinct elements is given by

$$x_0^{a_1} x_1^{b_1} x_0^{a_2} x_1^{b_2} = x_0^{a_1+a_2} x_1^{b_1+b_2+\sqrt{a_1 a_2}}$$

Zil'ber's Indecomposability Theorem is one of the basic tools in many arguments. Here we give the three corollaries that are used throughout the paper.

Fact 2.42 ([9], Corollaries 5.28 and 5.29, p. 86) *Let G be a group of finite Morley rank.*

1. *The subgroup generated by a set of definable, connected subgroups of G is definable, connected, and it is the setwise product of finitely many of them.*
2. *If H and K are definable subgroups of G with H connected then $[H, K]$ is definable and connected.*
3. *The subgroup of G generated by any family of definable connected subgroups is again definable and connected, and is generated by finitely many of them.*

We also need Clifford theory; most of this is purely module-theoretic.

Fact 2.43 *Let H, G be groups with $H \triangleleft G$ and let V be an irreducible G-module. Then:*

1. V is completely reducible as an H-module and its irreducible H-submodules are G-conjugate.

2. If $V \rtimes G$ has finite Morley rank and H° acts nontrivially on V then V is the direct sum of finitely many irreducible H-submodules.

Proof. The first part is clear as the sum of all G-conjugates of any H-irreducible submodule is G-invariant. For the second part, note that an H-irreducible submodule is either finite or connected. If it is connected, then the submodule generated by its conjugates is the sum of finitely many of them, by Fact 2.42 (2). If on the other hand the submodule is finite, then it is centralized by H°, and thus H° acts trivially on V. $\square$

The degenerate situation not accounted for above is represented by a vector space on which a linear group G acts, with H a finite group of scalars, in finite characteristic.

We note one more purely group-theoretic fact which is quite useful. It follows directly from basic commutator laws.

Fact 2.44 ([15]) *Let H, K be subgroups of the group G. Then H and K normalize $[H, K]$.*

We make occasional use of Frattini subgroups.

Notation 2.45 *Let P be a nilpotent p-group of bounded exponent. Then the* Frattini subgroup *$\Phi(P)$ is the subgroup generated by P' and $\{x^p : p \in P\}$. In the context of groups of finite Morley rank, if P is definable then $\Phi(P)$ is definable since on the one hand P' is definable, and on the other hand $\Phi(P)/P'$ is clearly definable in the quotient.*

Our group theoretic notation and terminology not explicitly explained above is standard. Standard notions which require some adaptation in passing from abstract groups (or algebraic groups) to groups of finite Morley rank are explained further in [9]. The present paper is a sequel to [1, 3] (and, ideally, also [18]), but familiarity with those papers is not essential beyond the points recalled above. It should be noted that the study of the "even type" groups as an isolated case is justified by [2, 17].

3 Strong closure

In this section we will prove some basic properties of strongly closed abelian subgroups, including a K-group fact which will be frequently used in the sequel. We use the following terminology, some of which was mentioned in the introduction.

Notation 3.1 *Let A, G be groups.*

1. *If $A \leq H \leq G$ then A is* strongly closed in H *relative to G if whenever $a \in A$, $g \in G$, and $a^g \in H$, we have $a^g \in A$.*
2. *When G is of finite Morley rank, we say that A is a* strongly closed 2-subgroup *of G if A is a 2-subgroup, is contained in a Sylow$^\circ$ 2-subgroup of G, and is strongly closed in at least one such Sylow$^\circ$ 2-subgroup.*

This terminology must be used with care. It would simplify matters slightly in the 2-group situation if A were taken connected, and as we shall see momentarily, this is harmless for our purposes (the case in which A is finite is also of some interest but was already handled in [3]). We now list a number of elementary properties which are not only quite useful in themselves, but have the general effect of rendering the terminology more robust; notably, it does not matter which particular Sylow$^\circ$ 2-subgroup is considered, in applying the strong closure property, as long as it contains the specified group A. Except for (v), these properties were stated and proved in [3] for abelian groups that are strongly closed in a Sylow 2-subgroup containing them. We restate them under the assumption that the strong closure is only relative to a Sylow$^\circ$2-subgroup. This weakening of the strong closure condition (that is, this strengthening of the final result) is of considerable practical significance for applications.

Lemma 3.2 *Let G be a group of finite Morley rank and of even type, and let A be a definable abelian 2-subgroup of G such that A is strongly closed in a Sylow$^\circ$ 2-subgroup S of G.*

(i) *A is strongly closed in any Sylow$^\circ$ 2-subgroup that contains A.*

(ii) *A is a normal subgroup of any Sylow$^\circ$ 2-subgroup that contains it.*

(iii) *$N(A^\circ)$ controls fusion in A.*

(iv) *Any definable $N(A^\circ)$-invariant subgroup of A is also strongly closed.*

(v) *If N is a definable normal subgroup of G then AN/N is strongly closed in G/N.*

Proof.

For $(i-ii)$ argue as in [3]. For (iii) we proceed as follows. Let $a \in A$ and $g \in G$ be such that $a^g \in A$ as well. Let S_1 be a Sylow$^\circ$ 2-subgroup of $C(a^g)$ containing A°. After conjugation by an element of $C(a^g)$ we may assume that A° and $A^{\circ g}$ are in S_1. Let S be Sylow$^\circ$ 2-subgroup of G in which A is strongly closed. There exists $h \in G$ such that $S_1^h \leq S$. In particular, $A^{\circ h}, A^{\circ gh} \leq S$. By strong closure of A in S we have $A^{\circ h} = A^\circ = A^{\circ gh}$. Hence g normalizes A°. This proves (iii). Now (iv) follows from (iii).

For (v), we argue as in [13]. Let S be a Sylow$^\circ$ 2-subgroup containing A. By Fact 2.5 SN/N is a Sylow$^\circ$ 2-subgroup of G/N containing AN/N. Let $a \in A$ and $g \in G$ such that $\overline{a}^{\overline{g}} \in \overline{S}$. This implies $a^g \in SN$. By conjugacy of Sylow 2-subgroups there exists $n \in N$ such that $a^{gn} \in S$. But A is strongly closed, hence $a^{gn} \in A$, which implies that $\overline{a}^{\overline{g}} \in \overline{A}$. $\square$

Note that (iv) implies that an infinite definable strongly closed abelian 2-group can be taken to be a connected elementary abelian 2-group.

We now prove an important K-group fact:

Lemma 3.3 *Let H be a connected K-group of even type with an infinite strongly closed definable connected abelian 2-subgroup A. If A is not normal in H then H has a normal subgroup of the form $L \simeq \mathrm{SL}_2(K)$, with K an algebraically closed field of characteristic 2, and $L \cap A$ is a Sylow 2-subgroup of L; thus $H = L \times C_H(L)$.*

Proof. We proceed by induction on the rank of H. Let H_1 be the subgroup of H generated by the conjugates of A. This is definable and connected by Fact 2.42. We may suppose $H = H_1$, as otherwise we conclude rapidly by induction.

If $Z(H)$ is infinite, then induction applies to $\bar{H} = H/Z(H)$. Note that the image $\bar{A}$ is again strongly closed abelian, by Lemma 3.2 (v), and is connected and not normal in $\bar{H}$. In fact $N_{\bar{H}}(\bar{A}) = \overline{N_H(A)}$: if $A^g \leq A \cdot Z(H)$, then $A^g \leq AA^g$, a 2-group, and therefore $A^g = A$ by strong closure. Thus we get $\bar{L} \triangleleft \bar{H}$, $\bar{L} \simeq \mathrm{SL}_2(K)$, with $\bar{L} \cap \bar{A}$ a Sylow 2-subgroup of $\bar{L}$. Then taking L as the full preimage of $\bar{L}$, and $L_1 = L^{(\infty)}$, we find that L_1 is a perfect central extension of $\mathrm{SL}_2(K)$, thus $L_1 \simeq \mathrm{SL}_2(K)$ (Fact 2.30 and [29]) and $L_1 \triangleleft H$. Furthermore L_1 meets $A \cdot Z(H)$ in a Sylow 2-subgroup A_1 of L_1. Let T be a maximal torus in $N_{L_1}(A_1)$. Then $A_1 = [T, A_1] \leq [T, AZ(H)] = [T, A] \leq A$. The last inequality follows from Corollary 2.17 and the fact that A is strongly closed.

If $Z(H)$ is finite, then we may factor it out, and the quotient is centerless; we may return from $H/Z(H)$ to H as in the preceding paragraph. So we may suppose $Z(H) = 1$.

Now suppose $O_2{}^\circ(H) \neq 1$. By strong closure of A in $O_2{}^\circ(H) \cdot A$, we have $O_2{}^\circ(H) \leq N(A)$ and hence $[O_2{}^\circ(H), A] \leq A \cap O_2{}^\circ(H)$. If $B = O_2{}^\circ(H) \cap A$ is not trivial, then $B \triangleleft H$ by strong closure, so B is central in $H_1 = H$, a contradiction. Thus $O_2{}^\circ(H) \cap A = 1$ and hence $[O_2{}^\circ(H), A] = 1$. But then $O_2{}^\circ(H)$ is central in $H_1 = H$, giving the same contradiction. Thus $O_2{}^\circ(H) = 1$. Similarly since A centralizes $O(H)$ by Fact 2.22, we find $O(H) = 1$. Thus $\sigma(H)$ is finite, $\sigma(H) \leq Z(H)$, and $\sigma(H) = 1$.

Since $\sigma(H) = 1$, H is a product $L_1 \times \cdots \times L_n$ of simple algebraic groups (Fact 2.29) over fields of characteristic 2. Let S be a Sylow 2-subgroup of H containing A (given the structure of H, this is also a Sylow° 2-subgroup). Then A is strongly closed in S and S is invariant under the action of some maximal torus T of H. Thus A is T-invariant, and as $C(T) = T$ we find that A is the product of the subgroups $A \cap L_i$. In particular for some i, $A \cap L_i$ is infinite. With such an i fixed, we will write L for L_i and B for $A \cap L$. From the structure of simple algebraic groups we find $L \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2; as B is T-invariant it is a product of root subgroups, and as B is strongly closed in L and the root system of L is indecomposable, B must be a Sylow 2-subgroup of L. Also, $H = L \cdot C_H(L)$. Thus our claim holds in this case. □

Note that, in the notation of Fact 3.3, Fact 2.15 implies that $A = (L \cap A) \times C_A(L)$. In the sequel, whether $C_A(L)$ is trivial or not will play an important role. Therefore we make the following definition:

Definition 3.4 *Let G be group of finite Morley rank of even type with a strongly closed abelian 2-subgroup A. Suppose G has a definable subgroup $L \cong \mathrm{SL}_2(K)$ normalized by A such that $L \cap A$ is a Sylow 2-subgroup of L. In particular, $A = (L \cap A) \times C_A(L)$. If $C_A(L) \neq 1$ then L is called an A-*special component.

4 No A-special components

In the present section we will prove the following theorem:

Theorem 4.1 *Let G be a simple K^*-group of finite Morley rank, of even type, and suppose that G contains a nontrivial connected abelian 2-subgroup A which is strongly closed in a Sylow° 2-subgroup, and which is a minimal infinite $N(A)$-invariant group. Then G has no special component with respect to A.*

We will make use of a fact concerning K-groups. The tori occurring in A-special components will be pseudoreflection groups in the sense of the following definition, and this will lead us to consider K-groups generated by pseudoreflection groups.

Definition 4.2 *If A is an elementary abelian 2-group then a nontrivial torus T acting on A is called a* group of pseudoreflections *on A if $A = C_A(T) \times [A, T]$ and T acts faithfully on the second factor, and transitively on its nonzero elements.*

Remark 4.3 *If G is a K^*-group of finite Morley rank of even type with a nontrivial definable connected strongly closed abelian subgroup A, and L is an A-special component, then L contains a group of pseudoreflections on A.*

Theorem 4.4 *Let $A \rtimes H$ be a connected K-group of finite Morley rank and of even type, in which A is an elementary abelian definable 2-subgroup and H acts irreducibly and faithfully on A. Assume that H contains a group T of pseudoreflections on A. Then A can be given a vector space structure over an algebraically closed field K in such a way that $H \simeq \mathrm{GL}(A)$ acting naturally.*

Proof. Observe that A is infinite and connected. Furthermore $O_2(H) = 1$, since $O_2(H)$ centralizes a nontrivial subgroup B of A, and by irreducibility we have $B = A$.

$H = E(H) {*} O(H)$ by Fact 2.33. It follows from Fact 2.21 that $O(H)$ is abelian: indeed, if $B \leq A$ is minimal nontrivial $O(H)$-invariant, then $O(H)'$ centralizes B and hence by irreducibility $C_A(O(H)') = A$, and $O(H)' = 1$. Thus $O(H) = Z^\circ(H)$.

Assume first that:

$$(*) \qquad Z^\circ(H) \neq 1$$

Then by Fact 2.20 A has a natural vector space structure over an algebraically closed field K, with $Z^\circ(H)$ acting via scalars and H acting linearly. We assume $\dim A > 1$.

Now T has some eigenspace $L \leq A$ on which T does not act trivially (Fact 2.23), and as T is a group of pseudoreflections, T must act transitively on $L \setminus (0)$, and hence L is 1-dimensional, and $Z^\circ(H)$ induces all scalars. Thus the elements of T are pseudoreflections also from a linear point of view.

Let H_1 be the subgroup of H generated by pseudoreflection subgroups. As H acts irreducibly and $H_1 \lhd H$, the action of H_1 on A is completely reducible (Fact 2.43). Write $A = A_1 \oplus \cdots \oplus A_n$ as a sum of irreducible H_1-submodules. Each pseudoreflection subgroup acts nontrivially on exactly one factor A_i. Hence H_1 is the direct product of subgroups $H_1^{(i)}$, where $H_1^{(i)}$ acts trivially on all factors A_j for $j \neq i$; A_i is an irreducible $H_1^{(i)}$-module. In particular the A_i are *all* the irreducible H_1-submodules of A, and these factors are therefore permuted by H, which is connected and irreducible. Accordingly there is only one such factor, and A is irreducible as an H_1-module.

In particular there are two pseudoreflection subgroups T_1, T_2 of H which do not commute. The group $\langle T_1, T_2 \rangle$ fixes a subspace of codimension 2 and acts on a complementary space as a subgroup of $\mathrm{GL}_2(K)$. It follows by inspection that this group contains a subgroup of root type in the sense of [21].

Let H_0 be the subgroup of H generated by subgroups of root type. Consider an irreducible H_0-submodule B of A. Note that $\dim B > 1$, as otherwise H_0 acts trivially on B and hence on A. By McLaughlin's theorem [21] H_0 induces $\mathrm{SL}(B)$ or $\mathrm{Sp}(B)$ on B. If T is a pseudoreflection subgroup of H then T fixes a subspace of codimension 1, and hence fixes a nonzero vector in B. So B is H_1-invariant and thus $A = B$ is H_0-irreducible. Now $\mathrm{SL}(A)$ or $\mathrm{Sp}(A)$ is normal in H. In the former case we have $H = \mathrm{GL}(A)$ as claimed, and in the latter case H is an extension of $\mathrm{Sp}(A)$ by the scalars, which does not in fact contain a pseudoreflection group except in dimension 2, where in any case $\mathrm{Sp}(A) = \mathrm{SL}(A)$.

Now suppose that

$$(\neg *) \qquad Z^{\circ}(H) = 1$$

In other words, $H = E(H)$. In this case we will arrive eventually at a contradiction. Note that we can no longer view A as a finite dimensional vector space.

We show first that H is simple. Let T be a pseudoreflection subgroup of H and let H_1 be a simple factor of H not commuting with T. Using Fact 2.44, we have $[T, H_1] = H_1$. Then T normalizes H_1 and acts by inner automorphisms, so T normalizes opposite Sylow 2-subgroups (maximal unipotent subgroups) S^+, S^- of H_1. Set $A^{\pm} = C_A(S^{\pm})$. Then $A^+ \cap A^- = 0$, in additive notation, since H_1 is generated by $S^+ \cup S^-$ and has no fixed points on A. As the groups $A^{\pm}$ are T-invariant, T acts trivially on at least one of them, say A^+. Let $B \leq A$ be H_1-irreducible. Then B meets A^+ and thus T fixes $B \cap A^+$. As B is H_1-irreducible and T normalizes H_1, T stabilizes B. If $B = A$ then $H = H_1$ because otherwise H will contain another component, say H_2 and the centralizer in A of a Sylow 2-subgroup of H_2 will be a proper H_1-invariant subgroup of A. Suppose $B < A$. As A is completely reducible as an H_1-module, it is the direct sum of H-conjugates of B (Fact 2.43). The argument used to show that B is T-invariant proves that T stabilizes each of these conjugates. Since T acts as a pseudoreflection group, the action of T on at least one of the conjugates of B, which we may suppose to be B, is trivial. Then $H_1 = [T, H_1]$ acts trivially on B, a contradiction.

Thus H is simple. Let P be a maximal parabolic subgroup corresponding to deletion of a terminal node in (a component of) the Dynkin diagram, and L the associated Levi factor. Now L contains a maximal torus of H and hence contains a pseudoreflection group T.

Suppose that V is a composition factor for A as an L-module, and that T acts trivially on V. As L' is simple, either L' acts trivially on V, or $[T, L'] = 1$, in which case $[T, L] = 1$.

We may exclude the case $T \leq Z(L)$ as follows. If $T \leq Z(L)$ then T is a root torus in L. Hence T and some conjugate T^g generate a subgroup $L_1 \simeq \mathrm{SL}_2$ in H such that $A/C_A(L_1)$ has rank $2t$ where $t = \mathrm{rk}\, T$. We consider the action of L_1 on $A_1 = A/C_A(L_1)$. T is a torus of L_1 and normalizes two "opposite" Sylow 2-subgroups S^+, S^- in L_1, each of which centralizes a nontrivial T-invariant subgroup of A_1; and the two subgroups involved are disjoint as S^+, S^- generate L_1. Now the Weyl group stabilizes $C_{A_1}(T)$ and interchanges the centralizers of S^+ and S^-, so T acts nontrivially on each of these two subgroups. However as T is a pseudoreflection group this is not possible.

Our conclusion is that L' acts trivially on any composition factor on which T acts trivially. However L' cannot act trivially on all the factors of a composition series for A, as the $2^{\perp}$ elements of L' would then act trivially on A itself (Fact 2.23). Accordingly, let V be a composition factor of A on which L' acts nontrivially. By the above, T also acts nontrivially on V, and therefore acts as a pseudoreflection group on V. By induction on $\mathrm{rk} H$ we may suppose therefore that $L \simeq \mathrm{GL}(V)$ acts naturally on V. In particular $Z(L)$ acts as an algebraically closed field K on V.

We may suppose that $V = A_1/A_0$ where L normalizes A_0 and A_1, and L' acts trivially on A_0. As A_0 is T-invariant and T acts nontrivially on V, T acts trivially on A_0. Thus L acts trivially on A_0. Let $T_1 = Z(L)$ and let $a \in T_1^{\times}$. Then commutation with a induces an isomorphism $\gamma : V \to [a, A_1]$ which is an isomorphism of L-modules. Thus we may suppose that V is a subgroup of A. Furthermore T acts trivially on every composition factor of A/V and hence by the above L acts trivially on every such composition factor, forcing L to act trivially on A/V since it is generated by $2^{\perp}$-elements.

In particular if $\hat{T}$ is a maximal torus of H contained in L, then $V = [\hat{T}, A]$ and thus the Weyl group W of H acts on V. Let $w \in W$ invert T_1: then for $v \in V^{\times}$ and α a scalar we find $(\alpha v)^w = \alpha^{-1} v^w$ and on considering $((\alpha+\beta)v)^w$ this yields $(\alpha+\beta)^{-1} = \alpha^{-1} + \beta^{-1}$, a contradiction. □

Lemma 4.5 *Let H be a connected K-group of finite Morley rank and of even type, and suppose that H contains a nontrivial definable connected abelian subgroup A which is strongly closed in a Sylow$^\circ$ 2-subgroup of H. Suppose that $L \le H$ is isomorphic to $\mathrm{SL}_2(K)$ for some field K and meets A in a Sylow 2-subgroup of L. Then L is contained in the product of the normal subgroups L^* of H with the same properties: $L^* \simeq \mathrm{SL}_2(K)$ for some field (depending on L^*) and L^* meets A in a Sylow 2-subgroup of L^*.*

Proof. Let H_0 be the product of the normal subgroups L^* of H isomorphic to $\mathrm{SL}_2(K)$ (for various fields K) and meeting A in a Sylow 2-subgroup. Then H normalizes the factors of H_0 and acts on each by inner automorphisms by Fact 2.15, so $H = H_0 \times C_H(H_0)$. Let $H_1 = C_H(H_0)$ and let $\bar{L}$ be the projection of L into H_1. If this is trivial then we have our claim, and otherwise $L \simeq \bar{L}$. Furthermore $A = (A \cap H_0) \times C_A(H_0)$ since A also acts by inner automorphisms on H_0 and centralizes $A \cap H_0$. Now $\bar{A}$ is strongly closed abelian in H_1 by Lemma 3.2 (v), and H_1 is again connected. Therefore by Lemma 3.3 and the definition of H_0 it follows that $\bar{A}$ is normal in H_1. But $\bar{L}$ does not normalize $\bar{A}$, so we have a contradiction. □

Recall that a connected definable subgroup L of G is called "special" with respect to A, or A-special, if it is isomorphic to $\mathrm{SL}_2(K)$ for some field K, has $L \cap A$ as a Sylow 2-subgroup, is normalized by A, and commutes with an involution i belonging to A.

Corollary 4.6 *Let G be a K^*-group of finite Morley rank of even type with a nontrivial definable connected strongly closed abelian subgroup A. If L is an A-special component and $L \le H < G$ with H definable and connected, and $A \le H$, then $L \lhd H$.*

Proof. $L \le \prod_i L_i$ with $L_i \lhd H$ and L_i meets A in a Sylow 2-subgroup. As $[L, A \cap L_i] \lhd L$ (Fact 2.44), and $[L, A] \ne 1$, it follows that $L = [L, L_i \cap A] \le L_i$ for some i, hence $L = L_i$. □

Proof of Theorem 4.1.

A is strongly closed in a Sylow$^\circ$ 2-subgroup, and is a minimal infinite $N(A)$-invariant group. Assume toward a contradiction that there is at least one A-special component L and let $a \in A$ be an involution centralizing L. By the preceding corollary L is normal in $C^\circ(a)$. Then the maximal torus of L normalizing $L \cap A$ acts as a group of pseudoreflections on A.

Let $H = N(A)/C(A)$. Then we have shown that H (hence also H°) contains a pseudoreflection group. As H acts irreducibly and faithfully on A, $O_2(H) = 1$. Furthermore by Fact 2.43 A is the direct sum of finitely many H°-irreducible factors A_i, which are conjugate under the action of H/H°.

By Theorem 4.4, each factor A_i carries a natural structure of K_i-vector space for an appropriate field K_i, and H° acts on each factor A_i like $\mathrm{GL}(A_i)$. In particular the tori in A-special components stabilize the A_i and hence for any A-special component L, $L \cap A \subseteq A_i$ for some i.

Suppose:

(Case 1) The number of factors A_i is at least 2

and consider an A-special component L for A with $L \cap A \le A_1$. Let $T \le N_L(A)$ be a maximal torus of L. Now $A = (L \cap A) \cdot C_A(L)$ with $C_A(L)$ T-invariant. In particular $C_A(L)$ contains A_j for $j > 1$. Hence $L \lhd C^\circ(A_j)$ (Corollary 4.6). The connected group H° acts on $C(A_j)$ and hence normalizes L. Thus $L \cap A$ is H°-invariant, forcing $L \cap A = A_1$. Thus there is a unique A-special component meeting A in A_1, and the same applies to any A_i. Let L_i be the A-special component with $L_i \cap A = A_i$. For $i \ne j$ as $L_i \le C(A_j)$, it follows that L_i normalizes L_j. Accordingly the group K generated by the L_i is their product. We claim that $N(K)$ satisfies the criterion of Fact 2.27: for any nontrivial unipotent $U \le N(K)$ we have $N^\circ(U) \le N(K)$.

Observe first that by construction $N(A) \le N(K)$. Furthermore, for any component L_i we claim that $N^\circ(L_i) = N^\circ(K)$. Evidently $N^\circ(K) \le N^\circ(L_i)$. We show the converse. For any j we have $L_j \leq N^\circ(L_i)$, and as L_j is A-special, Corollary 4.6 gives $L_j \triangleleft N^\circ(L_i)$. Thus $N^\circ(L_i) \leq N^\circ(K)$.

Now take a Sylow° 2-subgroup S of $N^\circ(K)$ containing A, and a unipotent 2-subgroup U of S. We claim that $N^\circ(U) \leq N^\circ(K)$. As S normalizes A and K, and A is a Sylow 2-subgroup of K, we have $S = A \cdot C_S(K)$. In particular $S \leq C(A)$ and thus $U \leq C(A)$, so $A \leq N^\circ(U)$. Now if $A \triangleleft N^\circ(U)$ then $N^\circ(U) \leq N(A) \leq N(K)$, as desired. On the other hand if A is not normal in $N^\circ(U)$ then there is a component $X \triangleleft N^\circ(U)$ of the form $\mathrm{SL}_2(K)$ for some algebraically closed field K, such that X meets A in a Sylow 2-subgroup. In particular X is normalized by A and is therefore either A-special, or contains A. If X is A-special then X is one of the L_i and $N^\circ(U) \leq N^\circ(X) = N^\circ(K)$.

Suppose finally that $A \leq X \triangleleft N^\circ(U)$. Then there is a torus T of $N_X(A)$ acting transitively on A. But then $T \leq H^\circ$ and this contradicts our case assumption.

Thus in this case $N(K)$ satisfies the criterion of Fact 2.27. Thus there is a weakly embedded subgroup M of G, which we may suppose contains a Sylow 2-subgroup of K. In view of the structure of K, it then follows that $K \leq M$. This contradicts Fact 2.28.

Now suppose:

(Case 2) $\qquad$ H° is irreducible on A

Then H acts like $\mathrm{GL}(A)$ with respect to some K-structure on A, for K a suitable field. This case will lead directly to a contradiction.

Again consider a Sylow 2-subgroup S of $N^\circ(A)$. Then S must centralize a nonzero element $a_0 \in A$. As the elements of A are conjugate under the action of H°, we may suppose that $a_0 = a$. In particular S acts on L via inner automorphisms and again $S = A \cdot C_S(L)$. But $C_S(L)$ commutes with the torus of L, which lies in H°, and $C_S(L)$ covers the Sylow 2-subgroup of $\mathrm{GL}(A)$, a contradiction unless A is 1-dimensional and H° is the multiplicative group of K. Since $N_L(A)$ contains a torus acting faithfully on A, and having a nontrivial fixed point, this is impossible. □

5 The main configuration

In this section, as in Section 4, A will be an infinite definable strongly closed abelian 2-subgroup minimal and invariant under the action of $N(A)$ (Lemma 3.2 (iv)). G is the simple K^*-group of finite Morley rank and even type which is under analysis. We will prove:

Theorem 5.1 *Under the stated hypotheses, either G has a weakly embedded subgroup or there is a definable subgroup L of G isomorphic to $\mathrm{SL}_2(K)$, with K an algebraically closed field of characteristic 2, such that A is a Sylow 2-subgroup of L and $C(L)$ contains a nontrivial unipotent 2-subgroup.*

To apply the classification of groups with weakly embedded subgroups in the present state of knowledge, we need to assume that G is tame.

Corollary 5.2 *If in addition G is tame, then either $G \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2, or there is a definable subgroup L of G isomorphic to $\mathrm{SL}_2(K)$, with K an algebraically closed field of characteristic 2, such that A is a Sylow 2-subgroup of L and $C(L)$ contains a nontrivial unipotent 2-subgroup.*

The second possibility will provide the main configuration which must be analyzed until a contradiction is reached in succeeding sections. This contradiction is an analog, in a very special case, of a theorem of Aschbacher and Seitz on centralizers of standard components.

More specifically, we study the subgroup $N^\circ(A)$. We show that either this subgroup satisfies the criterion of Fact 2.27, that is that $N^\circ(U)$ is contained in $N^\circ(A)$ for all unipotent $U \leq N^\circ(A)$, or that a component L of the desired type appears.

Lemma 5.3 *Let X be a group of finite Morley rank of even type with an infinite definable connected strongly closed abelian 2-subgroup A. Let H be a definable subgroup of X which contains A, with $A \lhd H$, and assume that for every nontrivial unipotent 2-subgroup $U \leq C_H(A)$, $N^\circ(U) \leq H$. Assume that K is a definable subgroup of X such that $K \cap A$ is infinite. Then $(K \cap A)^\circ$ is a strongly closed abelian 2-subgroup in K.*

Proof. Let $B = K \cap A$. Let S_1 be a Sylow° 2-subgroup of $H \cap K$ and S_2 a Sylow° 2-subgroup of K such that $S_1 \leq S_2$. We will show that $S_2 = S_1$. As $B \lhd H \cap K$, we have $B^\circ \leq S_1$. Also note that $(S_2 \cap H)^\circ = S_1$. Let $x \in N_{S_2}(S_1)$ and $b \in B^\circ$. Then $b^x \in S_1$ and as A is strongly closed in X (in particular in AS_1), we conclude $b^x \in A$. Thus $b^x \in A \cap S_1 \leq B$ and we conclude $B^{\circ x} \leq B$. So we have $N_{S_2}(S_1) \leq N(B^\circ)$. But $B^\circ \leq C(A)$ and by assumption this implies $N(B^\circ) \leq H$. Hence, $N_{S_2}(S_1) \leq H$. This implies $S_1 = S_2$ as claimed. Therefore it will be sufficient to check that B° is strongly closed in S_1.

Now let $k \in K$ and $b \in A \cap S_1$. Assume that $b^k \in S_1$. As S_1 is connected, it is contained in a Sylow° 2-subgroup of H which necessarily contains A as well. Therefore $b^k \in A$, and we have $b^k \in A \cap S_1 \leq K \cap A$. Hence $A \cap S_1$ is strongly closed in S_1 relative to K. In particular, so is $(A \cap S_1)^\circ = (A \cap K)^\circ = B^\circ =$. □

We now embark directly on the proof of Theorem 5.1, more specifically on the study of $N^\circ(A)$. As a matter of notation, set:

$$H = N^\circ(A)$$

Lemma 5.4 *If U is a nontrivial unipotent 2-subgroup of $C^\circ(A)$ then either $N^\circ(U) \leq H$, or G contains a subgroup L of type $\mathrm{SL}_2(K)$ in characteristic 2, which commutes with U, and has A as a Sylow 2-subgroup.*

Proof. As $U \leq C(A)$, we have $A \leq N^\circ(U)$. If A is not normal in $N^\circ(U)$, then using Lemma 3.3 we get $N^\circ(U) = L \times C_{N^\circ(U)}(L)$, where $L \cong \mathrm{SL}_2(K)$ and $L \cap A$ is a Sylow 2-subgroup of L. Theorem 4.1 implies that $A \leq L$; in particular, A is a Sylow 2-subgroup of L. Since U and L normalize each other and L is simple, we conclude $[U, L] = 1$. □

In view of Lemma 5.4 we make the following assumption:

$(*)$ If U is a nontrivial unipotent 2-subgroup of H which commutes with A then $N^\circ(U) \leq H$.

Lemma 5.5 *With the notation and hypotheses as above, suppose that U is a nontrivial unipotent 2-subgroup of H such that $N^\circ(U)$ is not contained in H. Then*

1. *$N^\circ(U)$ is of the form $L \cdot C_{N^\circ(U)}(L)$ with L definable and of the form $\mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2. Furthermore, with this notation:*
2. *$C_{N^\circ(U)}(L) \leq H$, $L \not\leq H$, and H contains a Borel subgroup of L.*
3. *(a) $N_A(U) = L \cap A = N_A(L)$.*
   Furthermore for any $u \in U^\times$:
   (b) these groups also coincide with $C_A(u)$; and $L \triangleleft C^\circ(u)$.
4. *U is an elementary abelian group.*

Proof.
Ad 1. Let $B = N_A(U)$. As UA is a nontrivial unipotent 2-subgroup containing A as a normal subgroup, we find that $A \cap Z(UA)$ is infinite by Facts 2.8 and 2.9, and thus B is infinite. By our assumption $(*)$, we have $N^\circ(B^\circ) \leq H$. Hence $N^\circ(U) \not\leq N^\circ(B^\circ)$. On the other hand B° is strongly closed in $N^\circ(U)$ by Lemma 5.3. Hence after applying Lemma 3.3 we have $N^\circ(U) = L \times C_{N^\circ(U)}{}^\circ(L)$ with L of the form $\mathrm{SL}_2(K)$, and $L \cap B$ a Sylow 2-subgroup of L. In particular (1) holds.

Ad 2. $C_{N^\circ(U)}{}^\circ(L) \leq N^\circ(L \cap B) \leq H$ by assumption $(*)$. $L \not\leq H$ since L does not normalize $A \cap L$. H contains a Borel subgroup of L by $(*)$, applied to $L \cap B$. This proves (2).

Ad 3 (a). By definition $B = N_A(U)$. We first show that $B \leq N_A(L) \leq L$. $L \triangleleft N^\circ(U)$ and $B \leq N(U)$ so B permutes the components of $N^\circ(U)$. As B centralizes $L \cap B$, B normalizes L. Thus $B \leq N_A(L)$.

Now let $B_1 = N_A(L)$. Then $B_1 = (L \cap B_1) \cdot C_{B_1}(L)$. If $C_{B_1}(L) \neq 1$ we contradict Theorem 4.1 as follows. Let i be an involution in $C_{B_1}(L)$. $A \leq C^\circ(i)$, thus $C^\circ(i)$ satisfies the hypotheses of Lemma 4.5. We therefore conclude that $C^\circ(i)$ contains a normal subgroup L_1 isomorphic to SL_2 and which intersects A in a Sylow 2-subgroup. This means L_1 is A-special, which contradicts Theorem 4.1. Thus $B_1 \leq L$, as claimed.

In particular $B \leq L$ and thus B is a Sylow 2-subgroup of L. In particular B is connected, and $B = N_A(U) = L \cap A = N_A(L)$.

Ad 4. Notice first that L and U are normal in $N^\circ(U)$, with L simple, and thus $[L, U] = 1$. Now let $V = \Phi(U) = \Phi(UB)$ where Φ denotes the Frattini subgroup. Our claim is that $V = 1$. Assuming the contrary, we may replace U by V (which is also connected, as it coincides with $U'\langle x^2 : x \in U\rangle$). As $N^\circ(U) \leq N^\circ(V) < G$, we find $N^\circ(V) = L_1 \cdot C_{N^\circ(V)}(L_1)$ with L_1 again a component of type SL_2. Here $L_1 \geq B$ and thus $L \cap L_1 \neq 1$. But $L \cap L_1 \triangleleft N^\circ(U)$ and thus $L \leq L_1$. On the other hand U acts on L_1 by inner automorphisms and centralizes L, hence centralizes L_1. So $L_1 \leq N^\circ(U)$ and $L_1 = L$. Now:

$$UB < N^\circ{}_{UA}(UB) \leq N^\circ{}_{UA}(\Phi(UB)) = N^\circ{}_{UA}(\Phi(U)) \leq N^\circ(V) \leq N(L)$$

Thus $UB < N_{UA}(L)$ and $B < N_A(L)$, a contradiction. Thus $V = 1$, proving (4).

Ad 3 (b). Let $B_1 = C_A{}^\circ(u)$ with $u \in U^\times$. By Lemma 5.3, B_1 is strongly closed abelian in $C_u{}^\circ$. As $L \leq C_u{}^\circ$, our assumption $(*)$ implies that no infinite subgroup of B_1 is normal in $C_u{}^\circ$. Thus Lemma 3.3 implies that $C_u{}^\circ$ is of the form $L_1 \cdot C_{C_u{}^\circ}(L_1)$ with L_1 a component of type SL_2. As in the argument above, we find $L \leq L_1$, then U centralizes L_1, and finally $L_1 = L$. Thus $L \triangleleft C_u{}^\circ$ and we have $C_u^\circ = L \times C_{C_u^\circ}(L)$.

Now let $x \in C_A(u)$. Then $L \cap L^x \geq B \neq 1$ and since $L^x \leq C_u^\circ$ as well $L \cap L^x \lhd L^x$. This implies $L = L^x$, on other words $x \in N_A(L)$. This last subgroup was shown in 3 (a) to be equal to B. Thus $C_A(u) \leq B$. As $B = L \cap A$ by 3 (a), B centralizes U, and we conclude that $C_A(u) = B$. This proves 3 (b). $\square$

For the remainder of the analysis we fix the following

Notation 5.6 *1. U is a unipotent 2-subgroup of H;*

2. $L \lhd N^\circ(U)$ is of type SL_2, and $L \not\leq H$

3. $B = L \cap A$, and T is a torus in L normalizing B, so that $B \rtimes T$ is a Borel subgroup of L contained in H;

4. We take U to be a maximal *unipotent 2-subgroup of $C(L)$.*

We elaborate somewhat on the configuration identified in the previous lemma.

Lemma 5.7 *With the hypotheses and notation as above, we have*

1. *If $t \in T^\times$ then $C_A(t) = 1$. In particular $A \cap U = 1$.*
2. *AU is a Sylow$^\circ$ 2-subgroup of H.*
3. *$B = Z(AU) = (AU)' = [A, u]$ for $u \in U^\times$.*
4. *$N(AU) \leq N(A) \cap N(UB)$.*

Proof.

Ad 1. As $C_A(t) \lhd C_{N(A)}(t)$ and $U \leq C_{N(A)}(t)$, U normalizes $C_A(t)$. As $C_B(t) = 1$, the assumption $C_A(t) \neq 1$ forces U to centralize elements of $A \setminus B$. But $C_A(U) = B$.

Ad 3. We know $B \leq Z(AU)$. That $Z(AU) \leq B$ follows easily from the fact that $C_A(u) = B$ for $u \in U^\times$.

Now consider a commutator $\gamma = [u, a]$ with $u \in U^\times$, $a \in A^\times$. By Lemma 5.5 (4), we have $1 = [u^2, a] = \gamma^u \gamma$, so $\gamma \in C_A(u) = B$. Thus $[u, A] \leq B$ and as T acts on both of these groups, with the action transitive on $B^\times$, we find $[u, A] = B$ for any $u \in U^\times$.

Ad 4. As $C_A(u) = B$ for $u \in U^\times$, it follows that A and BU are maximal elementary abelian subgroups of AU. We will show that $A \cup BU$ contains all involutions of AU. This implies that A and BU are the only maximal elementary abelian 2-subgroups of AU. Furthermore A is strongly closed in AU and hence is normalized by $N(AU)$; hence BU is also normalized by $N(AU)$.

So consider an involution $au \in AU$. As a, u, and au are involutions, they commute and thus if $u \neq 1$ we find $a \in B$. Thus $I(AU) = A^\times \cup (BU)^\times$, as claimed, and (4) follows.

Ad 2. We consider a Sylow$^\circ$ 2-subgroup S containing AU and an element $s \in N_S{}^\circ(AU)$. Then s acts on UB/B and thus there is some $u \in U^\times$, $b \in B$ with $u^s = ub$. But $[u, A] = B$ so after replacing s by sa for a suitable $a \in A$, we find $u^s = u$. Now $L \lhd C^\circ{}_u$ and s normalizes B, so s normalizes L and after a further adjustment by an element of B, we have $s \in C(L)$. All of this shows that $N^\circ{}_S(AU) \leq AC(L)$. However we have chosen U to be maximal unipotent in $C(L)$ and hence this implies $N^\circ{}_S(AU) = AU$, forcing $S = AU$. $\square$

Lemma 5.8 *U is a Sylow$^\circ$ 2-subgroup of $C(T)$.*

Proof. Let V be a Sylow$^\circ$ 2-subgroup of $N_{C(T)}(U)$. Being in $N^\circ(U)$, V normalizes L. V centralizes L since V centralizes T and its action on L is by inner automorphisms. In particular $V \leq C(B)$. Then the assumption $(*)$ implies $V \leq H$, and by the maximal choice of U (Notation 5.6), we have $U = V$. $\square$

Lemma 5.9 $[E(C_H{}^\circ(T)), B] = 1$.

Proof. Let $H_1 = C_H{}^\circ(T)$, and $\hat{H}_1 = C^\circ(T)$. By Lemmas 5.5 and 5.8, and Fact 2.32, we have $H_1 = E(H_1) \times \sigma(H_1)$ and $\hat{H}_1 = E(\hat{H}_1) \times \sigma(\hat{H}_1)$, where both $E(H_1)$ and $E(\hat{H}_1)$ are products of components of type SL_2 over an algebraically closed field of characteristic 2.

As U is a Sylow° 2-subgroup of both H_1 and $\hat{H}_1$, and $E(H_1) \leq E(\hat{H}_1)$ is normalized by U, we find that each component of $E(H_1)$ is a component of $E(\hat{H}_1)$, using Fact 2.34, so $E(H_1) \triangleleft \hat{H}_1$.

Take an involution $w \in L$ inverting T. Then w acts on $\hat{H}_1$ and permutes the components of $E(\hat{H}_1)$ while centralizing U, so w normalizes each component of $E(\hat{H}_1)$, and hence normalizes $E(H_1)$.

Let T_1 be a maximal torus of $E(H_1)$ normalizing U. Then T_1 acts on $C(U)$ and hence normalizes L. Therefore $[w, T_1] \leq L \cap E(H_1) \leq C_L(T) = T \leq \sigma H_1$, so $[w, T_1] \leq E(H_1) \cap \sigma(H_1) = 1$. As the torus T_1 acts on L and commutes with w, we find $[T_1, L] = 1$.

Consider $H_2 = C^\circ(T_1)$. By Lemma 5.3, $(H_2 \cap A)^\circ$ is a strongly closed abelian 2-subgroup in H_2, and by Lemma 4.5 L is contained in the product of the normal subgroups L^* of H_2 with the same structure: type SL_2, with $L^* \cap (H_2 \cap A)^\circ$ a Sylow 2-subgroup of L^*. Applying the hypothesis $(*)$, since $L \not\leq H$, we find that there can be at most one such factor L^*, and thus $L \leq L^* \triangleleft H_2$.

We consider the Weyl group of $E(H_1)$ relative to T_1. If w_1 is any involution of $E(H_1)$ normalizing T_1, then w_1 commutes with T and permutes the components of H_2, so w_1 normalizes L^*. Now as w_1 is an involution acting on L^* and centralizing T, we find $[w_1, L^*] = 1$ and in particular $[w_1, B] = 1$. As $E(H_1)$ is generated by $T_1 \cdot (U \cap E(H_1))$ together with such involutions, we find $[E(H_1), B] = 1$. □

Lemma 5.10 $B \triangleleft H$.

Proof. We have $A \leq O_2^\circ(H) \leq AU$. If $O_2^\circ(H) > A$ then $(O_2^\circ(H))' = B$ by Lemma 5.7, and hence $B \triangleleft H$, as claimed. Therefore we will suppose that $O_2^\circ(H) = A$, and in particular, by Fact 2.13, H is not solvable.

$\bar{H} = H/\sigma^\circ(H)$ is a central product of quasisimple algebraic groups in characteristic 2, whose Sylow 2-subgroup is covered by U. As the torus T commutes with U, it follows that $T \leq \sigma(H)$.

Let $\bar{R}$ be a Borel subgroup of $\bar{H}$ containing $\bar{U}$, with full preimage R. By Schur-Zassenhaus (Fact 2.12) R splits as $AU \rtimes T_0$ for some $2^\perp$-group T_0 containing T. We claim that $AU = F(R)$. If we assume the contrary, then $O(N(AU)) \neq 1$, and then by (Fact 2.37), G has a weakly embedded subgroup M. Then M° is solvable (Fact 2.28), but a conjugate of M contains A, and hence also H, contradicting our hypothesis above.

As $AU = F(R)$, it follows that T_0 is a torus by Fact 2.10. Now $\sigma^\circ H$ splits as $A \rtimes T_2$ with $T \leq T_2 \leq T_0$. Hence $H \leq AN_H^\circ(T_2) \leq AC_H^\circ(T)$ by Fact 2.14, since T is the definable closure of its torsion subgroup, by Fact 2.40. So it will suffice to show that $C_H{}^\circ(T)$ normalizes B. Note also that as $H \leq AC_H{}^\circ(T)$, we know that $C_H{}^\circ(T)$ is not solvable.

Now $E(C_H{}^\circ(T))$ centralizes B by the preceding lemma, and $C_H{}^\circ(T) = E(C_H{}^\circ(T)) \times \sigma(C_H{}^\circ(T))$, as noted earlier. Let $V = U \cap E(C_H{}^\circ(T))$. Then $\sigma(C_H{}^\circ(T))$ normalizes $C_A(V)$ and $C_A(V) = B$ (Lemma 5.5). □

We now reach a contradiction as follows. By a Frattini argument $N(A) = N^\circ(A)N(AU) \leq N(B)$ and thus B is $N(A)$-invariant, which violates the minimal choice of A. This completes the proof of Theorem 5.1.

6 2-Sylow structure

We have shown that in a simple K^*-group G of finite Morley rank of even type with an infinite definable abelian subgroup A which is strongly closed in a Sylow° 2-subgroup, if G has no weakly embedded subgroup then after taking A minimal $N(A)$-invariant, we arrive at a situation in which A is the Sylow 2-subgroup of some proper subgroup L of the form $\mathrm{SL}_2(K)$, where the Sylow° 2-subgroup of $C^\circ(L)$ is nontrivial. This is the point of Theorem 5.1. In the present section we show that this allows us to get a detailed description of the Sylow° 2-subgroups of G. In the main case, they will resemble Sylow subgroups of SL_3 in characteristic 2. We will also show that $C^\circ(A)$ is solvable.

Lemma 6.1 *Let G be a K^*-group of finite Morley rank of even type containing a definable subgroup $A \rtimes T$ isomorphic to a Borel subgroup of $\mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2, with A the Sylow 2-subgroup and T a maximal torus normalizing A. Then $N^\circ(A) = C^\circ(A) \cdot T$.*

Proof.

Let $\bar{T}$ be the image in $N(A)/C(A)$ of T. By Fact 2.16 there is a Sylow° 2-subgroup $\bar{S}$ of $[N(A)/C(A)]^\circ$ which is $\bar{T}$-invariant. As the 2-group $\bar{S}$ acts on A, $C_A(\bar{S})$ is nontrivial and $\bar{T}$-invariant. Hence $\bar{S}$ centralizes A, and as the action is faithful, $\bar{S} = 1$.

As $[N(A)/C(A)]^\circ$ is a connected K-group with trivial Sylow° 2-subgroup, it is solvable (Fact 2.33). As A is $N^\circ(A)$-minimal (consider T), by Fact 2.21 the induced action of $N^\circ(A)$ on A is abelian. As $\bar{T}$ acts transitively on A, we find $[N(A)/C(A)]^\circ = \bar{T}$, which yields the claim. □

We now find it convenient to introduce the notion of a *standard component*, which in groups of even type is defined as follows.

Definition 6.2 *Let G be a group of finite Morley rank and even type, and L a definable connected quasisimple subgroup of G. Then L is called a* standard component *for G if $C(L)$ contains an involution, and L is normal in $C^\circ(i)$ for all such involutions.*

This definition does not include the important condition which is required in the finite case:

(∗) L does not commute with any of its conjugates.

It turns out that this condition can eventually be proved on the basis of the definition as we have given it, but this result depends on the strongly closed abelian type classification given here, so it is not available at this point.

Lemma 6.3 *Let G be a simple K^*-group of even type with an infinite definable strongly closed abelian subgroup A. Suppose that L is a definable subgroup of G with $L \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2, that A is a Sylow 2-subgroup of L, and that L commutes with some involution. Then L is a standard component for G.*

Proof. Let i be an involution commuting with L. Since A is strongly closed abelian and lies in $C^\circ(i)$, by Lemma 3.3 there is a normal subgroup L_1 of $C^\circ(i)$ of the form $\mathrm{SL}_2(K_1)$, which meets A in a Sylow 2-subgroup; then L normalizes L_1 and hence the normal closure of $L_1 \cap A$ in L is contained in L_1. Thus $L = L_1$ is normal in $C^\circ(i)$ and L is a standard component for G. □

Lemma 6.4 *Let G be a simple K^*-group of even type with an infinite definable strongly closed abelian subgroup A. Suppose that L is a definable subgroup of G with $L \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2, that A is a Sylow 2-subgroup of L, and that a Sylow° 2-subgroup U of $C(L)$ is nontrivial. Then AU is not a Sylow° 2-subgroup of $C(A)$.*

Proof. Suppose on the contrary that AU is a Sylow$^\circ$ 2-subgroup of $C(A)$, and hence also of $N(A)$ by Lemma 6.1. Let $H = N^\circ(L)$. We claim:

(1) For $V \le AU$ nontrivial unipotent, we have $N^\circ(V) \le H$

We will first show that this produces a contradiction. If (1) holds then Fact 2.27 applies and yields a weakly embedded subgroup M of G which we may suppose contains AU, and hence also L. This violates Fact 2.28.

We first verify (1) in the special case $V = A$:

(1A) $$N^\circ(A) \le N(L)$$

As $N^\circ(A)$ is generated by its Borel subgroups, let B be one such. We may suppose that U is chosen to be a subgroup of B. As AU is a Sylow$^\circ$ 2-subgroup of B, B splits as $(AU) \rtimes T_1$ with T_1 a $2^\perp$-group.

We show now that T_1 is abelian. Let $K = C^\circ(AU)$. Then as $K \le C^\circ(U)$, K normalizes L. As $K \le C(A)$, $K = A \cdot C_K{}^\circ(L)$. By Fact 2.12, $C_K{}^\circ(L)$ splits as $U \times H_1$ with $H_1 = O(K)$. If $O(K) \ne 1$ then by Fact 2.37 there is a weakly embedded subgroup of G, which as we have observed produces a contradiction. Thus $O(K) = 1$ and $C^\circ(AU) = AU$. In particular $F^\circ(B) = AU$, and hence T_1 is abelian, by Fact 2.10.

Let T be a torus in $N_L(A)$. We claim $T \le \sigma(N(A))$, because $T \le C(AU \bmod A)$, and AU/A is a Sylow$^\circ$ 2-subgroup of $N(A)/A$ (Fact 2.29). Applying Fact 2.29 to $N^\circ(A)$, our claim follows. In particular $T \le B$ and we may suppose that $T \le T_1$. Let $T_2 = T_1 \cap C^\circ(A)$. Then $T_1 = T \cdot T_2$ as T is transitive on A and T_1 is abelian.

For $t \in T_2$, we have $[t, AU] = [t, U] \le C_{AU}(T) = U$. Thus $W =: [T_2, AU] \le U$, $W \ne 1$, and $T_2 \le N^\circ(W) \le N(L)$ as L is a component of $C(W)$. So $B = AUTT_2 \le N(L)$, as required, and (1A) follows.

We now deal with the general case of (1). We have $A \le N^\circ(V)$. If $A \lhd N^\circ(V)$, then we have $N^\circ(V) \le N^\circ(A) \le N(L)$. Assume A is not normal in $N^\circ(V)$. Then by Lemma 3.3, we have $N^\circ(V) = L_1 \times C_{N^\circ(V)}{}^\circ(L_1)$, with $L_1 \simeq \mathrm{SL}_2(K_1)$ for some algebraically closed field K_1 of characteristic 2, and $A \cap L_1 \in \mathrm{Syl}_2(L_1)$. As $L_1, V \lhd N^\circ(V)$, we have $[V, L_1] = 1$.

Let T_1 be a maximal torus in $N_{L_1}(A)$. Now $V, T_1 \le N^\circ(A) \le N^\circ(L)$ by (1A). As T_1 acts regularly on A, V centralizes A and T_1, and both act on L, we find $[V, L] = 1$. As L is a standard component for G, we have $L \lhd N^\circ(V)$, as required. □

Our goal now is the following instance of Theorem 1.3.

Proposition 6.5 *Suppose that:*

1. *G is a simple K^*-group of finite Morley rank, and of even type.*
2. *L is a standard component of type SL_2 in G, A is a Sylow 2-subgroup of L.*
3. *U is a Sylow$^\circ$ 2-subgroup of $C(L)$, and is nontrivial.*

Then AU is a Sylow$^\circ$ 2-subgroup of $C(A)$.

This in conjunction with the previous lemma provides the contradiction that completes the proof of Theorem 1.1. We will now show that it also proves Theorem 1.3:

Proof of Theorem 1.3. : The hypotheses are as above, and we now assume in addition that AU is a Sylow$^\circ$ 2-subgroup of $C(A)$. We must show that AU is a Sylow$^\circ$ 2-subgroup of G.

Let T be a maximal torus in $N_L(A)$ and let S be a T-invariant Sylow$^\circ$ 2-subgroup of $N_G(AU)$ (Corollary 2.17). It suffices to show that $S = AU$. Now $Z(S) \cap AU$ is infinite and T-invariant, hence in view of the action of T on AU, either A is central in S or $Z(S) \cap AU \le U$. But if $Z(S)$ meets U nontrivially, and u is an involution in $Z(S) \cap U$, then as L is a standard component, and $L, S \le C_u$, we find that S normalizes L, and hence S normalizes $L \cap AU = A$; thus again $Z(S)$ meets A, and hence contains A.

We conclude that in any case $A \le Z(S)$, so $S \le C(A)$. Hence $S = AU$ as claimed. □

For the remainder of the paper we devote our attention to the proof of Proposition 6.5. We fix the following additional hypotheses and notation, whose numbering continues that of the proposition. In view of hypothesis (5) following, we will seek a contradiction.

Notation 6.6 *4. $N_L(A) = A \rtimes T$ with T a maximal torus of L. S is a Sylow 2-subgroup of $C(A)$, chosen so that S° is T-invariant (corollary to Fact 2.16).*

5. $S^\circ > AU$.

6. $V = N_S(L)$ and $\hat{U} = C_V(L)$. Thus $\hat{U}^\circ = U$ and $V = A \times \hat{U}$.

7. $W = [\Omega_1{}^\circ Z](N_{S^\circ}(V^\circ) \bmod V^\circ)$, or in other words: the pullback to S of $\Omega_1{}^\circ Z(N_{S^\circ}(V^\circ)/V^\circ)$.

Our goal at this point is to work out the structure of W first, and then to show that $W = S^\circ$.

We insert a useful general remark which has already made an appearance above.

Lemma 6.7 *G has no weakly embedded subgroup. In particular, 2-local subgroups are core-free.*

Proof. If G has a weakly embedded subgroup M, then we may suppose that M contains U. By weak embedding, M contains L, and this contradicts Fact 2.28. The last statement is Fact 2.37. □

Lemma 6.8 *V is elementary abelian.*

Proof. Supposing the contrary, we have $\Phi(V) = \Phi(\hat{U}) \neq 1$ with Φ denoting the Frattini subgroup. Thus $N_S(V) \leq N_S(\Phi(V)) = N_S(\Phi(\hat{U})) \leq N_S(L)$ as L is a component of $C(\Phi(\hat{U}))$ and S normalizes A. Hence $N_S(V) = V$ and $S = V$, $S^\circ = AU$, a contradiction. □

Thus W has exponent at most 4.

Lemma 6.9 *For $v \in V \setminus A$:*

1. *$C_S(v) = V$;*
2. *$[W, v] = A$.*

Proof. We have $v = ua$ with $u \in \hat{U}^\times$ and $a \in A$. Hence $C_S(v) = C_S(u)$ and $[W, v] = [W, u]$, so we may take $v = u \in \hat{U}^\times$.

Ad 1. As L is a component of $C(v)$ and S normalizes A, we have $C_S(v) \leq N_S(L) = V$.

Ad 2. We show that $[W, v] \leq A$. As $[W, v]$ is nontrivial (6.6 (5) and part (1)) and T-invariant, our claim follows. So let $\gamma = [w, v]$ with $w \in W$. We may suppose $w \in W \setminus V$.

Suppose first that $\gamma \in V$. As $w^2 \in V^\circ$, we have $1 = [w^2, v] = \gamma^w \gamma$, so $w \in C(\gamma)$; if $\gamma \notin A$ we find $w \in V$ by part (1), a contradiction. Accordingly, if $\gamma \in V$ then $\gamma \in A$. This applies in particular if $v \in V^\circ$ since $\gamma \in V^\circ$ in this case.

To complete the analysis we show that $\gamma \in V$. Let $v_0 \in V^\circ \setminus A$. Then $\gamma^{v_0} = [w^{v_0}, v] = \gamma$ as $w^{v_0} \in wA$. Hence $\gamma \in C_S(v_0) = V$, as required. □

Corollary 6.10 *W/A is elementary abelian.*

Proof. For $w \in W \setminus V$, as $w \in C(w^2)$ it follows that $w^2 \in A$. □

This gives adequate control on W. We aim next at showing $W = S^\circ$. To this end, we introduce the following additional notation.

Notation 6.11

1. *$\hat{W} = [\Omega_1{}^\circ Z](N_S(W) \bmod W)$.*
2. *$W_1 = [Z(\hat{W} \bmod A) \cap W]^\circ$.*
3. *$V_1 = [Z(\hat{W} \bmod A) \cap V]$.*

We will also use the notion of a *continuously characteristic subgroup*. Our usual notion of characteristic subgroup is actually "definably characteristic" – invariant under definable automorphisms of the ambient group. The condition for "continuously characteristic" is weaker: invariance under all definable *connected* groups of automorphisms of the ambient group.

Lemma 6.12 $V_1 > A$.

Proof.
Assume $V_1 = A$. Then $W_1 \cap V^\circ = A$. As W is normal in $\hat{W}$, we have $W_1 > A$. For $v \in V^\circ \setminus A$ we find $[W_1, v] \leq A$ is nontrivial and T-invariant, so $[W_1, v] = A$ and $\mathrm{rk} W_1/A = \mathrm{rk} A$. Similarly $\mathrm{rk} W/V^\circ = \mathrm{rk} A$ and hence $W = W_1 \cdot V^\circ$.

In addition we have $1 \to A \to W_1 \to W_1/A \to 1$, with T acting, and the action of T on both A and W_1/A is standard (field multiplication), as $W_1/A \simeq [W_1, v]$ as a T-module for any $v \in V^\circ \setminus A$. It follows that W_1 is abelian by Fact 2.41.

To conclude, we will show that V° is continuously characteristic in W. This implies $V^\circ \triangleleft \hat{W}$, and hence $V_1 > A$.

Suppose first that W_1 is elementary abelian. Then $I(W) = V^{\circ\times} \cup W_1^\times$ and in particular V° and W_1 are the only maximal elementary abelian subgroups of W. Thus V° is continuously characteristic in W.

Now suppose that W_1 is homocyclic of exponent 4. Then $I(W_1) = A^\times$. If $I(W) = V^{\circ\times}$ then again V° is continuously characteristic in W, so suppose that there is an involution of the form wv with $w \in W_1 \setminus A$ and $v \in V^\circ$. Then we can take $v \in U$. Then every coset of V° in $W \setminus V^\circ$ contains a T-conjugate of wv: $(wv)^t = w^t v$. It is then easy to see that the involutions of $W \setminus V^\circ$ are of the form $w^t va$ with w, v fixed and t, a varying respectively over T and A. If two such involutions $w_1 va_1$ and $w_2 va_2$ commute, we find $1 = [w_1 v, w_2 v] = [w_1, v][w_2, v]$, forcing $w_1 = w_2$. It follows that V° is the unique maximal connected elementary abelian subgroup of W, and is characteristic in W. □

Corollary 6.13 $S^\circ = W$.

Proof. Take $v \in V_1 \setminus A$. Then $[\hat{W}, v] = A = [W, v]$ and hence $\hat{W} = W \cdot C_{\hat{W}}(v) = W$. Thus $S^\circ = W$. □

Lemma 6.14 $C^\circ(A)$ *is solvable.*

Proof. Let $H = C^\circ(A)$ and $\bar{H} = H/A$. As $\bar{H}$ is a connected K-group of even type with abelian Sylow° 2-subgroups, by Fact 2.32 $\bar{H} = \bar{L}_1 \times \cdots \times \bar{L}_n \times \sigma(\bar{H})$, with $\bar{L}_i \simeq \mathrm{SL}_2(K_i)$, where each K_i is an algebraically closed field of characteristic 2. By Fact 2.30 (and [29]), $\bar{L}_i$ is covered by $L_i \simeq \mathrm{SL}_2(K_i)$, and then $H = L_1 \times \cdots \times L_n \times \sigma(H)$.

However the structure of S° does not allow this for $n \geq 1$: for $v \in V \setminus A$, we see that $V = C_S(v)$ meets L_1 in a nontrivial central subgroup of S°. As $Z(S^\circ) = Z(W) = A$ by Lemma 6.9, we have $A \cap L_1 \neq 1$ – but A is central in H. □

7 Final analysis

For our concluding argument we need a form of the *Thompson rank formula*, which is an analog of the Thompson order formula for finite groups, in the context of groups of finite Morley rank of even type. This has turned out to be a very useful tool ([3]), but the version of the rank formula given in [3] applies only to groups containing a finite number of conjugacy classes of involutions. Here we give a more general form which does not require this hypothesis.

We need a variant of the definability lemma given in [3]:

Lemma 7.1 (Variant of Lemma 6.2 of [3]) *Let G be a group of finite Morley rank of even type.*

(i) If i, j are involutions then there is at most one involution in $d(\langle ij\rangle)$.

(ii) If i and j are nonconjugate involutions then $d(\langle ij\rangle)$ contains an involution.

(iii) The function $\theta(i, j)$ which associates to each pair (i, j) of involutions the unique involution in $d(\langle ij\rangle)$, when there is one, is definable.

Proof. The first two points are standard (cf. [3]). We prove (*iii*). For any a, define $Z_a = Z(C(a))$. As Z_a is an abelian group of finite Morley rank and of even type, it has the form $B_a \oplus D_a$ with B_a a 2-group of bounded exponent and D_a a $2^\perp$-group. As $Z_a \leq G$, the exponent of B_a is uniformly bounded. Accordingly $D = D_a$ is definable, uniformly, from a. As G is of even type, D_a contains no involutions. As a is of uniformly bounded order modulo D_a, the group $\langle a\rangle D_a$ is also definable uniformly in the parameter a.

So to conclude it suffices to check:

$$I(d(\langle a\rangle)) = I(\langle a\rangle D_a)$$

Of course $d(\langle a\rangle) \leq \langle a\rangle D_a$, so it suffice to check that any involution i in $\langle a\rangle D_a$ lies in $d(\langle a\rangle)$. In fact, one may check that $\langle a\rangle D_a$ contains at most one involution, and that if it does contain an involution, then so does $d(\langle a\rangle)$, because a is then of even order modulo $d(\langle a\rangle)^\circ$. $\square$

Fact 7.2 ([9], Exercise 14 p. 65) *Suppose G is a group of finite Morley rank and A and B are two definable subsets of G. If f is a definable function from A onto B, $B = \sqcup_i B_i$ is a finite partition of B into definable sets, and for $b \in B_i$ $\mathrm{rk} f^{-1}(b) = r_i$ is constant, then* $\mathrm{rk}A = \max_i(r_i + \mathrm{rk}B_i)$.

Let G be a group of finite Morley rank and suppose there exists $n > 1$ such that $\sqcup_1^n X_i$ is a definable partition of $I(G)$, where each X_i is a union of conjugacy classes of involutions whose centralizers have constant rank c_i; thus each conjugacy class contained in X_i has rank $g - c_i$, with $g = \mathrm{rk}G$.

Take two distinct X_i, say X_1 and X_2, and define the map θ of Lemma 7.1 using these two sets. Thus $\theta : X_1 \times X_2 \to I(G)$ is definable. For each i and $f \leq \mathrm{rk}(X_1 \times X_2)$, let $X_{if} = \{x \in X_i : \mathrm{rk}(\theta^{-1}(x)) = f\}$. Then $X_i = \sqcup X_{if}$, where f varies over the set of fiber ranks for which X_{if} is nonempty. This is a finite partition of X_i into definable sets X_{if} which are again unions of conjugacy classes.

As the restriction of θ to X_{if} has fibers of constant rank, by Fact 7.2, $\mathrm{rk}(X_1) + \mathrm{rk}(X_2) = \mathrm{rk}(X_{if}) + f \leq \mathrm{rk}(X_i) + f$ for some i and some fiber rank f.

On each set X_i, conjugacy induces a definable equivalence relation $\sim$. If C_i is a single conjugacy class contained in X_i, then $\mathrm{rk}(X_i) = \mathrm{rk}(X_i/\sim) + \mathrm{rk}(C_i) = \mathrm{rk}(X_i/\sim) + g - c_i$. We will denote $c_i - \mathrm{rk}(X_i/\sim)$ by $\tilde{c}_i$, so that $\mathrm{rk}X_i = g - \tilde{c}_i$. Thus the inequality $\mathrm{rk}(X_1) + \mathrm{rk}(X_2) \leq \mathrm{rk}X_i + f$ becomes $g - \tilde{c}_1 + g - \tilde{c}_2 \leq g - \tilde{c}_i + f$, that is: $g \leq \tilde{c}_1 + \tilde{c}_2 - \tilde{c}_i + f$. We will write $\tilde{c}_3$ for the relevant value of $\tilde{c}_i$, but one should bear in mind that the subscripts $1, 2, 3$ stand for three indices i_1, i_2, i_3, some of which may coincide. This formula has the same form as the one given in [3], except that c_i is replaced by $\tilde{c}_i$.

We resume the analysis from the point reached in §6. The notation and hypotheses were fixed in 6.5 and 6.6. The main notations were as follows.

Notation 7.3

1. *G is a simple K^*-group of finite Morley rank, and of even type.*
2. *L is a standard component of type* SL_2 *in G, A is a Sylow 2-subgroup of L, and $N_L(A) = A \rtimes T$.*
3. *U is a Sylow° 2-subgroup of $C(L)$, and is nontrivial.*
4. *S is a Sylow 2-subgroup of $C(A)$ containing AU, and $W = S^\circ$ is T-invariant.*
5. *$V = N_S(L)$ and $\hat{U} = C_V(L)$. Thus $\hat{U}^\circ = U$ and $V = A \times \hat{U}$.*

Furthermore, we recall the following essential properties.

Fact 7.4

1. *$N^\circ(A) = C^\circ(A) \rtimes T$ is solvable.*
2. *V is elementary abelian.*
3. *$W > AU$*
4. *$A = Z(W)$ and W/A is elementary abelian.*
5. *For $v \in V \setminus A$, $C_S(v) = V$ and $[W, v] = A$.*

Lemma 7.5 *For $u \in U^\times$, we have $C^\circ(u) = U \times L$.*

Proof. Set $H = C^\circ(u)$. Then $L \lhd H$ as L is a standard component. Thus $H = L \cdot C_H{}^\circ(L)$. $C_H{}^\circ(L)$ is solvable with 2-Sylow° U (Fact 7.4, (1,2)), and hence splits as $U \rtimes T_1$ (Fact 2.12). T_1 normalizes V°.

For $t \in T_1$ and $w \in W$ we have $[u, w] = [u, w]^t = [u, w^t]$ and hence $w^t \in wV$ (Fact 7.4 (4,5)). Hence for $v \in V^\circ$ we find $[w, v] = [w, v]^t = [w^t, v^t] = [w, v^t]$ (Fact 7.4 (2)) and $[V^\circ, T_1] \le ZW = A$. Thus T_1 acts trivially on V°/A and on A, hence also on V°, by Fact 2.23. So $T_1 \le O(C^\circ(u))$. By Lemma 6.7, $O(C^\circ(u)) = 1$. □

Now we work with the following sets of involutions.

Notation 7.6

1. *I_1 is the set of involutions which are conjugate to an involution in A.*
2. *I_2 is the set of involutions which are conjugate to an involution in U.*

Remark 7.7 The sets I_1 and I_2 are disjoint, as a Sylow° 2-subgroup of the centralizer of one of these involutions is nonabelian in the first case, and abelian in the second (Fact 7.4 (4), Lemma 7.5). Furthermore, by the preceding lemma, the ranks of conjugacy classes are constant over I_2 (this is not an issue for I_1, as it consists of a single class).

Lemma 7.8 *W is a Sylow° 2-subgroup of G.*

Proof. Let W_0 be a Sylow° 2-subgroup of $N_G(W)$. Then $W_0 \le N^\circ(A)$ and as $N^\circ(A)/C^\circ(A)$ is a torus, we have $W_0 \le C^\circ(A)$. Hence $W_0 = W$. The claim follows. □

Lemma 7.9 *For $a \in A^\times$ we have $C^\circ(a) = C^\circ(A)$.*

Proof. Let B be a Borel subgroup of $C^\circ(a)$ containing W. Then B splits as $W \rtimes T_0$ with T_0 a connected $2^\perp$-group. By Fact 7.4 (4), $T_0 \leq N^\circ(A)$. Moreover $N^\circ(A) = C^\circ(A) \rtimes T$ and $[T_0, a] = 1$, so $T_0 \leq C^\circ(A)$. As $C^\circ(a)/\sigma(C^\circ(a))$ is a product of simple algebraic groups in characteristic 2 and A commutes with T_0, which covers a maximal torus in the quotient, we find $A \leq \sigma(C^\circ(a))$. As $\sigma^\circ(C^\circ(a)) \leq WT_0$, we find $A \leq Z\sigma^\circ(C^\circ(a))$.

Let $B = A \cap Z(C^\circ(a))$, and $T_1 = \{t \in T : a^t \in B\}$. For $t \in T_1$, we have $C^\circ(a) \leq C^\circ(a^t)$, and hence $C^\circ(a) = C^\circ(a^t)$, and therefore $B = B^t$. Thus $T_1 = N_T(B)$. Now T_1 acts transitively on $B^\times$, so the pair (B, T_1) represents a subfield of the field represented by the pair (A, T). By finiteness of rank, either $B = A$ as claimed, or else B is finite (cf. [27], Lemme 3.2). Assume that B is finite.

Let $r \in C^\circ(a)$ act modulo $\sigma(C^\circ(a))$ as an involution normalizing $T_0\sigma C^\circ(a)$ and exchanging T_0W with an opposite Borel subgroup. Adjusting r by an element of $\sigma(C^\circ(a))$, we may suppose that r normalizes T_0. Thus r acts on $C_{\sigma(C^\circ(a))}(T_0)$ and in particular $A^r \leq C_{\sigma(C^\circ(a))}(T_0)$. Let $B_1 = A \cap A^r$. Then $C(B_1)$ contains $\sigma^\circ(C^\circ(a))$, W, and W^r; so $B_1 \leq Z(C^\circ(a))$. Thus B_1 is finite.

Now by Lemma 7.8 $\Phi(O_2(C^\circ(a))) \leq A$ and $\Phi(O_2C^\circ(a))$ is r-invariant, so $\Phi(O_2(C^\circ(a))) \leq B_1$ is finite. Thus $O_2{}^\circ(C^\circ(a))$ is elementary abelian. Furthermore $\mathrm{rk}(O_2{}^\circ(C^\circ(a))) \geq \mathrm{rk}(AA^r) = 2\mathrm{rk}(A)$ and by the structure of W this forces $O_2{}^\circ(C^\circ(a)) = AA^r$. Hence $[T_0, O_2{}^\circ(C^\circ(a))] = 1$.

Let H be the subgroup generated by all conjugates of T_0. Then $[H, O_2{}^\circ(C^\circ(a))] = 1$. Since $C^\circ(a) = H\sigma^\circ(C^\circ(a))$ and A commutes with both factors, our claim follows. □

Lemma 7.10 *For $a \in A^\times$ we have $C(a) = C(A)$.*

Proof.
It follows from the preceding lemma, together with $A = Z(W)$, that $A = O_2{}^\circ(Z(C^\circ(a)))$. Hence $A \lhd C(a)$. By Lemma 6.1 we have $TC(A)/C(A) \lhd N(A)/C(A)$. Note also that A is $TC(A)/C(A)$-minimal. By Fact 2.21:

$$TC(A)/C(A) \leq Z(N(A)/C(A)),$$

so $[T, C(a)] \leq C(A)$. Thus for $t \in T$, $a \in A^\times$ and $c \in C(a)$, we have $a^{tc} = a^{[t^{-1},c^{-1}]ct} = a^{ct} = a^t$. Since T acts transitively on A we conclude that $C(a) = C(A)$. □

Lemma 7.11 *Let $\theta : I_1 \times I_2 \to I(G)$ be the Thompson map. If A is strongly closed in W, then the image of θ is contained in I_2.*

Proof. Let $i = \theta(a, u)$ lie in the image. We may suppose $u \in U$. As $i \in C(u)$, we have $i \in N(L)$ (Lemma 7.5), and hence i centralizes a conjugate of A in L. Without loss of generality i centralizes A.

Now a belongs to a conjugate A_1 of A and $i \in C(a) = C(A_1)$, by Lemma 7.10. If A, A_1 generate a 2-subgroup of $C^\circ(i)$ then since A is strongly closed in W we find $A_1 = A$ and $a \in A$, $i = ua \in I_2$. Otherwise, applying strong closure again in connection with Lemma 3.3 and Theorem 4.1 we get components $L_1, L_2 \lhd C^\circ(i)$ containing A_1, A respectively, of the form SL_2. As A and A_1 do not commute, these components coincide.

Now $u \in C(i)$ and $[u, A] = 1$, so $u \in N(L_1)$, and $i \in d(\langle (ua) \rangle) \leq \langle u \rangle \cdot L_1$. Since $L_1 \leq C(i)$, $i \notin L_1$, so $i \in u \cdot L_1$. As $u \in C(A)$, u acts on L_1 like an element a_1 of A. So $iua_1 \in L_1 \cap C(L_1) = 1$, and $i \in uA \leq V^\circ \setminus A \subseteq I_2$. □

The next few lemmas are only relevant when A is not strongly closed in W. As we have indicated, the added generality will be useful in practice.

Lemma 7.12 *Suppose that A_1 is a conjugate of A such that $A_1 \neq A$ and $A_1 \cap W \neq 1$. Then:*

1. *$A_1 \leq W$ and $A_1 \cap A = 1$. Set $B = A \cdot A_1$. Then:*
2. *$B \cap V = A$ and $W = BV^\circ$.*

3. $I(W) = I(V) \cup I(B)$.

4. $B = \langle A^g : g \in G, [A, A^g] = 1 \rangle = \langle A^{N(B)} \rangle$.

Proof. *Ad* 1. Take $a \in (A_1 \cap W)^\times$. Then $A \leq C^\circ(a) = C^\circ(A_1)$, so $A_1 \leq C^\circ(A)$, which is solvable. So $A_1 \leq O_2{}^\circ(C^\circ(A)) = W$.

If $a \in (A_1 \cap A)^\times$ then $C^\circ(a) = C^\circ(A) = C^\circ(A_1)$ and hence $A_1 \leq Z(W) = A$, a contradiction. Thus $A_1 \cap A = 1$. Accordingly $B = AA_1$ is an elementary abelian subgroup of W.

Ad 2. B is elementary abelian and $\mathrm{rk}B = 2\mathrm{rk}A$. If $v \in (B \cap V) \setminus A$, then by Fact 7.4 (5) we have $B \leq V$. But $V \setminus A \subseteq I_2$ by Fact 7.4 (5) and $B \setminus A$ meets I_1, a contradiction. Thus $B \cap V = A$.

For $v \in V^\circ \setminus A$, by Fact 7.4 (5) and rank considerations, we have $[v, B] = A$, so $W = BC^\circ{}_W(v) = BV^\circ$.

Ad 3. If $bv \in I(W)$ with $b \in B$ and $v \in V^\circ$, then b and v commute as $b^2 = v^2 = (bv)^2 = 1$. Now Fact 7.4 (5) implies that either $b \in V^\circ$ or $v \in B$. This proves (3).

Ad 4. If $A_2 \neq A$ is a conjugate of A and $[A, A_2] = 1$, then $A_2 \leq W$, so $A_2^\times \subseteq I(W)$, while $A_2 \cap V = 1$. So $A_2 \leq B$. This proves the first equality. For the second, take $A_2 = A^g \leq B$. We will show that $g \in N(B)$. We have $B^g = \langle A_2^h : h \in G, [A_2, A_2^h] = 1 \rangle$, so $B \leq B^g$. Thus $B = B^g$ and $g \in N(B)$. $\square$

Lemma 7.13 *Suppose A is not strongly closed in W, and let $B = \langle A^g : g \in G, [A, A^g] = 1 \rangle$. Then:*

1. $N(B)$ *acts transitively on* $B^\times$.

2. $\mathrm{rk}U = \mathrm{rk}A$.

Proof. Let $\bar{H} = (N(B)/C(B))^\circ$. Note that $N^\circ(A) \leq N^\circ(B)$. We claim:

$$(*) \qquad B \text{ is } \bar{H}\text{-irreducible}$$

Take $B_1 \leq B$ $\bar{H}$-irreducible. As $[U, B_1] \leq B_1$, we find $B_1 \cap A \neq 1$; in view of the action of T, $A \leq B_1$. In particular, B contains a unique $\bar{H}$-irreducible submodule, so $B_1^{N(B)} = B_1$. As $A \leq B_1$, point (3) of the preceding lemma implies $B = B_1$.

Now $O_2(\bar{H}) = 1$, as otherwise $C_B(O_2(\bar{H})) < B$ is $\bar{H}$-invariant. In view of Fact 2.32:

$$\bar{H} = E(\bar{H}) \times O(\bar{H})$$

with $E(\bar{H})$ a product of groups $L_i \simeq \mathrm{SL}_2(K_i)$ for suitable fields K_i of characteristic 2.

$O(\bar{H})$ stabilizes $C_B(\bar{U}) = A$. Furthermore $\bar{T} \leq O(\bar{H})$ since $[T, U] = 1$ and $W = BU$. By Fact 2.21, $O(\bar{H})'$ centralizes A, hence B by irreducibility, and $O(\bar{H})$ is abelian. It follows that $O(\bar{H}) = \bar{T}$.

By Fact 2.20 $\bar{T}$ gives B a K-structure for some field K, and in view of the action on A, T may be identified with $K^\times$. Furthermore the action of $E(\bar{H})$ is K-linear. As $\mathrm{rk}B = 2\mathrm{rk}A$, B has dimension 2. One expects $E(\bar{H})$ to reduce to $\mathrm{SL}_2(K)$ with the same field and the natural representation, and we will now check this.

Let L_0 be a component of $E(\bar{H})$. Then B is L_0-irreducible and hence $L_0 = E(\bar{H})$. Let $\bar{T}_0$ be a maximal torus of L_0 normalizing $\bar{U}$, and let $\bar{w}$ be an involution in L_0 inverting $\bar{T}_0$. Let $v_1 \in A^\times$, $v_2 = v_1^{\bar{w}}$, and consider the representation of L_0 over K with respect to this basis. Then $\bar{U}$ is represented by strictly lower triangular matrices $M(u)$ and (in view of the action of $\bar{w}$ on $\bar{T}_0$) $\bar{T}_0$ is represented by diagonal matrices $D(t) \in SL_2(K)$; let $m(u)$ and $d(t)$ be the corresponding elements of K, that is $m(u) = M_{21}(u)$ and $d(t) = D_{11}(t)$. Let K_a and K_m be, respectively, the image of m on $\bar{U}$ and the image of d on $\bar{T}_0$. Then K_a is an additive subgroup of K, K_m is a multiplicative subgroup of $K^\times$, and K_m acts on K_a by $(t, u) \mapsto t^{-2}u$. As $\bar{T}_0$ acts transitively on $\bar{U}$, it follows that $K_m \cup \{0\}$ is also closed under addition, and thus is a subfield of K. By finiteness of rank, $K_m \cup \{0\} = K$. Thus the base field K_0 of $\bar{L}_0$ can be identified with K, and the action is as expected.

In particular $N(B)$ acts transitively on $B^\times$ and we can compare the ranks: $\mathrm{rk}U = \mathrm{rk}K_0 = \mathrm{rk}K = \mathrm{rk}A$. $\square$

Lemma 7.14 *If A is not strongly closed in W, then $I(G) = I_1 \cup I_2$.*

Proof. We show first that $I(W) \subseteq I_1 \cup I_2$ in this case. By Lemma 7.12 (3), we have $I(W) = I(B) \cup I(V)$, and we know $I(V) \subseteq I_1 \cup I_2$. By the preceding lemma, $I(B) \subseteq I_1$. Thus $I(W) \subseteq I_1 \cup I_2$, and to conclude it will now suffice to show that W is a Sylow 2-subgroup of G, and not just a Sylow° 2-subgroup.

Suppose that $s \in N(W)$ and $s^2 \in W$. We must show that $s \in W$.

Let $B = \langle A^g : g \in G, [A, A^g] = 1\rangle$. Let $a \in A^\times$, $[s, a] = 1$. Then $s \in C(a) = C(A)$, so $[s, A] = 1$.

Note that B and V° are normal in $N(W)$. In particular $[s, V^\circ] \leq V^\circ$. Take $u \in U^\times$ with $[s, u] \in A$. By Fact 7.4 (5) after replacing s by a suitable sb with $b \in B$, we may suppose that $[s, u] = 1$. Then s acts on $C^\circ(u) = LU$ (Lemma 7.5) and normalizes W. In particular, s acts on L like an element of A, so after a second adjustment we may suppose that $[s, L] = 1$. Then $s^2 \in C_W(L) = U$. Choose $b \in B \setminus A$ so that $[s, b] \in A$. Then $1 = [s^2, b]$ and $s^2 \in U$, so $s^2 \in A$ by Fact 7.4 (5). But $A \cap U = 1$, so $s^2 = 1$.

As $\mathrm{rk}U = \mathrm{rk}A$, it follows as in Lemma 6.9 (2) that $[U, b] = A$, and hence after a third adjustment, we may take $[s, b] = 1$. Possibly we have now obtained $s = 1$, in which case we are finished. Assume not, and we will reach a contradiction.

As $L \leq C_s$ is a standard component and s is an involution, we have $L \triangleleft C_s{}^\circ$. Now b acts on $C_s{}^\circ$ and it is easy to see that L is the only component of $C_s{}^\circ$, so b normalizes L and hence also $C^\circ(L) \cap W = U$. But this is not so. We have reached a contradiction. □

We now undertake some Thompson rank computations. We use the following notation throughout:

$$f = \mathrm{rk}A; u = \mathrm{rk}U; g = \mathrm{rk}G; c_A = \mathrm{rk}C(A)$$

Lemma 7.15

$$\mathrm{rk}\, I_2 \geq 4f + u$$

Proof. We consider the following partial Thompson map:

$$\theta_0 : I_2 \times I_2 \to I_2$$

which is defined for pairs u_1, u_2 distinct in I_2 for which $d\langle (u_1 u_2)\rangle$ contains an element of I_2.

The map θ_0 has large fibers. The set $\{(a, b) \in I(L) \times I(L) : ab \text{ is a } 2^\perp\text{-element of } L\}$ is a generic subset of $I(L) \times I(L)$. It follows that for $u_1, u_2 \in U$, we have $\theta_0(u_1 a, u_2 b) = u_1 u_2$ over a generic subset of $I(L) \times I(L)$. Thus, as $\mathrm{rk}(I(L)) = 2f$, the fiber rank of θ_0 is at least $4f + u$. On the other hand, over a generic subset of I_2 the fiber rank is at most $\mathrm{rk}(I_2 \times I_2) - \mathrm{rk}\, I_2 = \mathrm{rk}\, I_2$, so we find:

$$\mathrm{rk}\, I_2 \geq 4f + u$$

□

Lemma 7.16 *Generically, the Thompson map $\theta : I_1 \times I_2 \to I(G)$ maps into I_2.*

Proof. If A is strongly closed in W, then this map literally maps into I_2. If A is not strongly closed in W, then $I(G) = I_1 \cup I_2$ and thus it suffices to show that the set $D = \{x \in I_1 \times I_2 : \theta(x) \in I_1\}$ has rank less than $\mathrm{rk}I_1 + \mathrm{rk}I_2$. Suppose therefore:

$$\mathrm{rk}D = \mathrm{rk}I_1 + \mathrm{rk}I_2$$

Then the fiber ranks for θ over points of I_1 will be $\mathrm{rk}I_2$.

But we may compute this fiber rank exactly. We are assuming that A is not strongly closed in W, and hence that W is a Sylow subgroup of G. Let $B = \langle A^g : g \in G, [A, A^g] = 1\rangle$. Let $a \in I_1$, say $a \in A^\times$, and let $a = \theta(a_1, v)$. Then $a_1, v \in C(a) = C(A)$. It follows that $a_1, v \in W$ and hence that $a_1 \in B$, $v \in V \setminus A$. Thus $a = [a_1, v]$, and the rank of $\theta^{-1}(a)$ is $\mathrm{rk}B + \mathrm{rk}V - \mathrm{rk}A = 3f$. This is considerably less than our lower bound for $\mathrm{rk}I_2$, a contradiction. □

Lemma 7.17 $g = c_A + 4f$.

Proof. We consider the Thompson map $\theta : I_1 \times I_2 \to I(G)$ or more exactly its restriction θ_0 to the preimage of I_2, which is generically defined in $I_1 \times I_2$, and quite possibly total. We claim that the rank of the fibers of θ_0 above I_2 is constant, and equal to $4f$. Granted this, we have $4f = \mathrm{rk}(dom\,\theta_0) - \mathrm{rk}(im\,\theta_0) = \mathrm{rk}I_1$, or $g - c_A = 4f$, as claimed.

So we now carry out the fiber rank computation. Fix $u \in U$. For $a, b \in I(L)$ we have, generically, that ab is semisimple and that $a \cdot ub$ corresponds to u under the Thompson map (that is, $d(\langle uab\rangle) = \langle u\rangle d(\langle ab\rangle)$, with the second factor a torus). Thus $r \geq 4f$.

Conversely if $\theta(a, v) = u \in U^\times$ then $u \in C(a) = C(A_1)$ for some conjugate A_1 of A, and thus $a \in C^\circ(u) = U \times L$, forcing $a \in L$. Also $v \in N(L)$ and as u is the image of (a, v) under the Thompson map, we have $u \in vL$, equivalently $v \in uL$ and so $v \in u \cdot I(L)$. This shows that $r \leq 4f$. $\square$

Lemma 7.18 *$C^\circ(A) = W \rtimes T_1$ for some torus T_1 (meaning T_1 is definable, abelian, and divisible) with $\mathrm{rk}N_{T_1}(V^\circ) \leq \mathrm{rk}U$ and $\mathrm{rk}T_1/N_{T_1}(V^\circ) \leq f - \mathrm{rk}U$. In particular $\mathrm{rk}T_1 \leq f$.*

Proof. $C^\circ(A)$ is solvable with Sylow° 2-subgroup W. Thus it splits definably as $W \rtimes T_1$ for some $2^\perp$ group T_1. Now $C^\circ(A)$ is core-free, by Lemma 6.7, so the Fitting subgroup is W and the quotient T_1 is divisible abelian (Fact 2.10).

Now we estimate $\mathrm{rk}T_1/N_{T_1}(V^\circ)$. We first estimate $\mathrm{rk}I(W)$. For each coset C of V° in W, other than V° itself, the rank of the set of involutions in C is f: if $w \in W \setminus V^\circ$ and w, wv are both involutions, with $v \in V^\circ$, then w and v commute and hence $v \in A$. Thus the rank of $I(W)$ is at most $2f$. On the other hand the rank of $I(V^\circ) \setminus A^\times$ is $f + \mathrm{rk}U$. Thus our estimate will follow if we show simply that distinct conjugates of V° under T_1 meet only in A. This is clear: if $v \in [(V^\circ)^{t_1} \cap (V^\circ)^{t_2}] \setminus A$ then $(V^\circ)^{t_1} = {C^\circ}_W(v) = (V^\circ)^{t_2}$.

We claim lastly that $\mathrm{rk}N_{T_1}(V^\circ) \leq \mathrm{rk}U$. Let $V_1 \leq V^\circ$ be minimal subject to: $V_1 > A$ and $V_1 = C_{V^\circ}(X)$ for some $X \leq N_{T_1}(V^\circ)$. (Most probably $V_1 = V^\circ$ and $X = 1$.) Fix $v \in V_1 \setminus A$. The structure of $C^\circ(v)$ is given by Lemma 7.5 since v is conjugate in W to an element of U, and this implies that $T_0 = C_{T_1}(V_1)$ is finite. As T_1 is abelian, $N_{T_1}(V^\circ)/T_0$ acts on V_1. If $t \in N_{T_1}(V^\circ) \setminus T_0$ fixes a point of $(V_1/A)^\times$, then t fixes a point of $V \setminus A$ (Fact 2.23) and hence we contradict the minimality of V_1 by considering $C_V(T_0 \cup \{t\})$. Thus $N_{T_1}(V^\circ)/T_0$ acts semiregularly on V_1/A and $\mathrm{rk}N_{T_1}(V^\circ) = \mathrm{rk}(N_{T_1}(V^\circ)/T_0) \leq \mathrm{rk}(V_1/A) \leq \mathrm{rk}U$. $\square$

We will use the notation $u = \mathrm{rk}U$ and $t_1 = \mathrm{rk}T_1$ in conjunction with the notation of the preceding lemma. In particular, as $t_1 \leq f$ by Lemma 7.18, we have $c_A = 2f + u + t_1 \leq 3f + u$.

Now we can show, finally, that: *the configuration we have obtained is inconsistent.*

Proof.

We have $\mathrm{rk}\,I_2 = g - c_V + \mathrm{rk}\,(I_2/\sim)$ where $c_V = u + 3f$ is the rank of $C(v)$ for $v \in I_2$ and $\sim$ is the equivalence relation of conjugacy in G. We apply Lemma 7.15, taking $g = c_A + 4f$, and then evaluate c_A and c_V:

$$u \leq \mathrm{rk}I_2 - 4f = (c_A - c_V) + \mathrm{rk}(I_2/\sim) = (t_1 - f) + \mathrm{rk}(I_2/\sim)$$

or $\mathrm{rk}(I_2/\sim) \geq u + f - t_1 \geq u$.

Now representatives for $I_2/\sim$ are found in U, so $\mathrm{rk}(I_2/\sim) \leq u$. Accordingly

$$\mathrm{rk}(I_2/\sim) = u \text{ and } f = t_1$$

Then in view of the last lemma, $\mathrm{rk}\,N_{T_1}(V^\circ) = u$. As $N_{T_1}(V^\circ)/T_0$ acts semiregularly on V_1/A (in the proof of that lemma) we have $\mathrm{rk}\,V_1/A = u$, and $V_1 = V^\circ$. Now looking at the action of $N_{T_1}(V^\circ)/T_0$ on V_1, and the ranks, we conclude that $V_1^\times$ is a single conjugacy class, and as $V_1 = V^\circ$ we find $\mathrm{rk}\,(I_2/\sim) = 0$, a contradiction. $\square$

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