HOMOGENEOUS FINITE GROUPS

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ABSTRACT

The paper gives an explicit classification of the finite groups which are homogeneous in the sense of Fraïssé.

1. *Introduction*

A group G (or for that matter, any type of algebraic or combinatorial structure) is said to be *homogeneous* if any isomorphism between two finitely generated subgroups (respectively, substructures) is induced by some automorphism. The study of homogeneous structures has long engaged the attention of model theorists; the *combinatorial* case (where there are relations but no functions, so 'finitely generated substructure' simply means 'finite induced substructure') is discussed in [**6**, **17**, **18**] for example. The classification of homogeneous rings and groups, not necessarily finite, has also been investigated in detail, but there are strong indications that in the nilpotent case the homogeneity condition is not sufficiently strong to allow a full classification; see for example [**8**, **23**, **24**].

We examined homogeneous solvable groups in [**7**], showing that apart from the difficulties associated with homogeneous nilpotent groups of class 2, one can pin down the structure of these groups quite precisely, even in the infinite case. As one would expect, in the context of finite groups the homogeneity condition is so restrictive that one may make an explicit and relatively short list of all the possibilities – at least, under the assumption that the classification of the finite simple groups is correct. In our argument we will make some attempt to limit our use of this assumption. For example, we will prove the special case of the Feit–Thompson theorem that we actually require, since this can be done rather directly, and in fact all we will really need from classification theory is the contents of [**21**], together with the earlier results on which that depends, such as [**2**]. Possibly more direct arguments can be found. The present article was conceived as a companion to [**7**], but has been delayed by the search for arguments of a comparably elementary character.

Our result is as follows.

MAIN THEOREM. *A finite group G is homogeneous if and only if G is of the form $H \times K$ with H, K homogeneous finite groups of relatively prime order satisfying*

(1) *H is solvable;*

(2) *K is either trivial or is of the form* $\mathrm{SL}(2,5)$ *or* $\mathrm{PSL}(2,p)$ *with $p = 5$ or 7.*

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As the homogeneous finite solvable groups are known explicitly, this gives an explicit classification of the homogeneous finite groups. The finite solvable case was first completed by P. Neumann.

FACT 1 [**7**]. A finite solvable group H is homogeneous if and only if it is of the form $A \times H_0$ with A, H_0 homogeneous finite groups of relatively prime orders, satisfying the following:

(1) A is abelian.

(2) If H_0 is nontrivial then it is of the form $V \rtimes T$, where the possibilities for the pair (V, T) and the action of T on V are as follows:

(a) V is homogeneous abelian of odd order and T is a cyclic 2-group generated by an element which inverts V.

(b) V is an elementary abelian 3-group of order 9, and T is the quaternion group, acting faithfully as a subgroup of $\mathrm{SL}(V)$.

(c) V is an elementary abelian 2-group of order 4, and T is cyclic of order 3, acting faithfully on V.

(d) V is a homogeneous finite nonabelian 2-group and T is either trivial, or is cyclic of order 3, acting faithfully on V.

Evidently a finite abelian group A is homogeneous if and only if it is a product of homocyclic groups of relatively prime orders.

FACT 2 [**7**]. There are up to isomorphism exactly two homogeneous finite nonabelian 2-groups, and each of them admits an automorphism of order 3. These groups are the quaternion group Q of order 8 and a group Q^* of order 64, class 2, and exponent 4, with $Z(Q^*) = \Omega_1(Q)$ elementary abelian of order 4.

The group Q^* and its automorphism group are discussed in detail in [**7**].

We will only need the classification theorem for finite simple groups in our determination of the homogeneous quasisimple groups (that is, perfect central extensions of simple groups), but this remark may be misleading, as the latter classification is needed also to get the more general structural information in the main theorem. More precisely, we prove the following two results.

THEOREM 1. *Let K be a quasisimple homogeneous finite group Then*

(1) *if K is not simple then $K \simeq \mathrm{SL}(2, 5)$;*

(2) *if K is simple then $K \simeq \mathrm{PSL}(2, p)$ with $p = 5$ or 7.*

We have stated the two points separately as only the second one requires substantial background information. Note however that we will invoke [**12**] to complete the proof of Theorem 1(1).

THEOREM 2. *Let G be a finite homogeneous group. Then $G = H \times K$ with H, K homogeneous and of relatively prime orders, satisfying*

(1) *H is solvable;*

(2) *K is quasisimple or trivial.*

The proof of Theorem 2 does not require any substantial information about finite simple groups, apart from Theorem 1(2). On the other hand we do need Theorem 1(2) to control the automorphism group of a quasisimple homogeneous group.

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Our notation is standard. We use the notation $\Omega_i(G)$ only when G is a p-group for some prime p; then $\Omega_i(G)$ is the subgroup of G generated by elements of order at most p^i.

2. *Preliminaries*

Our first lemma shows that one of the implications in the main theorem is a triviality, and hence the main theorem will follow from Theorems 1 and 2 once one checks the homogeneity of the quasisimple groups listed.

LEMMA 1. *Let G be a finite group.*

(1) *If G is homogeneous and K is characteristic in G, then K is homogeneous. Furthermore in this case if K contains an element of order n, then every element of order n in G belongs to K.*

(2) *If $G = H \times K$ with H characteristic in G then G is homogeneous if and only if H and K are homogeneous and of relatively prime orders.*

This is straightforward. The following principle is also very useful.

LEMMA 2. *Let G be a group and H be any subgroup such that every element of $\operatorname{Aut} H$ lifts to an automorphism of G. Then $N(H)$ induces a normal subgroup of $\operatorname{Aut} H$ on H.*

Note that Lemma 2 applies in particular when G is homogeneous, and H is any finite subgroup.

LEMMA 3. *Let G be a homogeneous finite p-group. Then either G is homocyclic or $p = 2$, in which case the possibilities are those listed in Fact* 2(2).

This information is contained in Facts 1 and 2 and is also a special case of other group-theoretic results. For p odd it is contained in a result of Shult [**25**]: if the elements of order p in G lie in a single orbit under $\operatorname{Aut} G$, and p is odd, this already implies that G is abelian. Similarly for $p = 2$ the result can be read off from the classification of Suzuki 2-groups [**14**].

The point of this is that we may combine Lemmas 1 and 3 to conclude the following.

COROLLARY 1. *Let G be a homogeneous finite group. Then either $O_p(G)$ is homocyclic, or $p = 2$ and $O_2 G$ is Q or Q^*, as in Fact* 2.

In this connection it is useful to know the following.

LEMMA 4. *Let G be Q or Q^*. Then $\operatorname{Aut} G$ is solvable.*

For $\operatorname{Aut} Q^*$, see [**7**].

We will need to analyze the group of automorphisms induced by a finite homogeneous group on a characteristic subgroup. The following result will be useful.

LEMMA 5. *Let G be a homogeneous finite group and H be a characteristic subgroup. Then for any natural number n there is a number n^* such that every element of G/H of*

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order n is represented by some element of order n^*. In particular any two elements of G/H of order n will lie in the same orbit under the action of $\mathrm{Aut}(G/H)$.

Proof. The second claim follows from the first by homogeneity in G.

For the first claim, let π be the set of prime divisors of n. We claim that if an element $\bar{g}$ of order n in G/H is lifted to an element g whose order n^* is a π-number, then the value of n^* is independent of all choices made.

We deal first with the case in which n is a power of a prime p. If g, g' are p-elements of G both of order n modulo H, say with $o(g') \leqslant o(g)$, then $o(g') = o(g^k)$ for some k and hence by homogeneity g' and g^k have the same order modulo H. Hence $(p, k) = 1$ and g, g' have the same order, as claimed.

The general case follows by factoring a lifting g of $\bar{g}$ as $\prod_p g_p$ where the g_p are commuting p-elements. □

3. *The proof of Theorem* 2

In this section we assume the validity of Theorem 1 and we prove Theorem 2. We will make use of the generalized Fitting subgroup $F^*(G) = F(G) \cdot E(G)$, in Bender's notation: that is, $F(G)$ is the Fitting subgroup and $E(G)$ is the central product of the subnormal quasisimple subgroups of G. The proof will depend on the following results.

PROPOSITION 1. *Let G be a finite homogeneous group with $E(G) = 1$. Then G is solvable.*

PROPOSITION 2. *Let G be a finite homogeneous group with $E(G)$ nontrivial. Then $E(G)$ is quasisimple.*

This last result relies on the Feit–Thompson theorem, but the following simple special case suffices.

PROPOSITION 3. *There is no quasisimple homogeneous finite group of odd order.*

Assuming these ingredients, we prove Theorem 2 as follows. Let G be a homogeneous finite group, $K = E(G)$, and $H_0 = C(K)$. By Lemma 1, H_0 and K are homogeneous and so by Propositions 1 and 2, H_0 is solvable and K is trivial or quasisimple. We may suppose that K is nontrivial. Then by Theorem 1, K is $\mathrm{SL}(2,5)$ or $\mathrm{PSL}(2,p)$ for $p = 5$ or 7.

We show that $G = H_0 K$. If this fails, then some element $g \in G$ induces an outer automorphism of K. We may take g to be a 2-element. Note that $g^2 \in H_0$ and $g \notin K$; in particular g is not an involution. Some power h of g is an involution in $H_0 \cap K$ and this forces $K = \mathrm{SL}(2,5)$. Then some power of g is an element of order 4 in $H_0 \cap K$ and this is a contradiction.

Thus $G = H_0 K$. If $H_0 \cap K = 1$ then we have the desired conclusion. The alternative is that $K = \mathrm{SL}(2,5)$ and $H \cap K = Z(K)$. In this case $Z(K)$ is a Sylow 2-subgroup of H_0 by Lemma 1(1) and hence $H_0 = Z(K) \times O(H_0)$, $G = O(H_0) \times K$, which is of the desired form. This completes the proof.

For the proof of Proposition 1, as we assume that $E(G) = 1$ we have $C(F(G)) = Z(F(G))$ and thus for the solvability of G it suffices to check the solvability of

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$G/C(F(G))$, which embeds into $\prod_p \mathrm{Aut}(O_p(G))$; so it suffices to check the solvability of $G/C(O_p(G))$ for each p. This will reduce to the case $p = 5$, as we see from the next lemma.

LEMMA 6. *Let G be a homogeneous finite group, and p be a prime. If $G/C(O_p(G))$ is nonsolvable, then $p = 5$, $O_p(G)$ is an elementary abelian group of order 25, and $G/C(O_p(G)) \simeq \mathrm{SL}(2,5)$.*

Proof. Let A_p be $G/C(O_p(G))$ viewed as a subgroup of $\mathrm{Aut}\, O_p(G)$, and write $V_p = \Omega_1(O_p(G))$. Suppose that A_p is not solvable. The kernel of the natural map $A_p \longrightarrow \mathrm{Aut}\, V_p$ acts trivially on each quotient $\Omega_i(O_p(G))/\Omega_{i-1}(O_p(G))$ and is therefore solvable, so the subgroup $\bar{A}_p$ of $\mathrm{Aut}\, V_p$ induced by G is nonsolvable.

By Lemma 2, $\bar{A}_p \lhd \mathrm{GL}(V_p)$. If $\bar{A}_p$ is not solvable we conclude that $\mathrm{SL}(V_p) \leqslant A_p$. Furthermore by Lemma 5 applied to $G/C(V_p)$, any two elements of $\bar{A}_p$ of equal order are conjugate in $\mathrm{GL}(V_p)$. This applies in particular to $\mathrm{SL}(V_p)$ and since $\dim V_p > 1$ in the nonsolvable case, we find easily that $\dim V = 2$ and $p \leqslant 5$ by considering elements of order p, multiplication maps by elements of norm 1 (identifying V_p with the additive group of a finite field) or diagonal matrices, according to whether $\dim V_p > 3$, $\dim V_p = 3$, or $\dim V_p = 2$ with $p > 5$. As we assume $\bar{A}_p$ is nonsolvable, we have $\dim V_p = 2$ and $p = 5$.

We also have $O_5(G) = V_5$, since otherwise using the fact that $O_5(G)$ is homocylic with $\Omega_1\, O_5(G) = V_5$, and that $\mathrm{SL}(V_5) \leqslant \bar{A}_5$ with A_5 normal in $\mathrm{Aut}(O_5(G))$, we can easily find elements of order 5 in A_5 which are nonconjugate in $\mathrm{Aut}\, O_5(G)$, one acting trivially on V_5, the other nontrivially. Similarly, $G/C_G(V_5) \simeq \mathrm{SL}(2,5)$ as otherwise this group, viewed inside $\mathrm{GL}(V_5)$, contains nonconjugate diagonal matrices of equal order. □

We will reduce the proof of Proposition 1 to the following special case.

LEMMA 7. *Let G be a homogeneous finite group with $C(O_5(G)) \leqslant O_5(G)\cdot Z(G)$. Then G is solvable.*

Proof. It suffices to show that $G/C(O_5(G))$ is solvable, so by Lemma 6 we may suppose toward a contradiction that $O_5(G)$ is elementary abelian of order 25, with $G/C(O_5(G)) \simeq \mathrm{SL}(2,5)$. Write V for $O_5(G)$. Let $a \in G$ be a 2-element acting on V by inversion. Then $a^2 \in Z(G)$. We claim that $G = V \times C(a)$.

For $g \in G$ a $2'$-element we have $[a,g] \in O_5(G)\cdot Z(G)$ and $1 = [a^2, g]$, so a inverts $[a,g]$ and hence $[a,g] \in V$. Then for a suitable $v \in V$ we have $[a, gv] = 1$ and hence $gv \in C(a)$. Thus $G = V \times C(a)$. Therefore $C(a)$ contains an element of order 5 which contradicts Lemma 1(1) as V is characteristic in G. □

Proof of Proposition 1. We suppose that G is finite and homogeneous with $E(G) = 1$. We will show that G is solvable. Let $K = C(O_5(G))$. Then $E(K) = 1$ and $O_5(K) \leqslant Z(K)$ so by Lemma 6, K induces a solvable group of automorphisms on $O_p(K)$ for all p, and hence K is solvable. Hence by Fact 1, $K = O_5(G) \times H$ with H a $5'$-group. As $O_5(H) = 1$ it also follows from Lemma 6 that $G/C(F(H))$ is solvable. By Fact 1, $\mathrm{Aut}(H/F(H))$ is solvable, so $C(F(H))/C(H)$ is solvable. Thus it remains only to show that $C(H)$ is solvable. Let $H^* = C(H)$. This is a homogeneous finite group with $O_5(H^*) = O_5(G)$ and

$$C_{H^*}(O_5(H^*)) = K \cap C(H) = O_5(H^*) \times Z(H) \leqslant O_5(H^*) \cdot Z(H^*).$$

Thus Lemma 7 applies and H^* is solvable. □

Proof of Proposition 2. Let G be a homogeneous finite group with $E(G)$ nontrivial. We wish to show that $E(G)$ is quasisimple. We may suppose that $G = E(G)$. Thus G is a central product of quasisimple components.

Let K be one of these components, p be a prime dividing $|K|$, and p^n be the maximal order of a p-element of $Z(K)$. As K is quasisimple (and in particular perfect) there are elements of order p in $K/Z(K)$, and these are represented by elements of order p^{n+1} in K, by homogeneity. Let $a \in K$ have order p^{n+1}. Then the normal closure of a in G is K; so for any element of this order, its normal closure is a component of G, and $a^p \in Z(G)$.

From these considerations it follows easily that the orders of the components of G are relatively prime. An application of the Feit–Thompson theorem shows there is only one component; alternatively, as these components are characteristic and hence homogeneous, we may use Proposition 3 instead. □

The rest of this section is devoted to a short proof of Proposition 3, which is of course a special case of the Feit–Thompson theorem.

LEMMA 8. *Let G be a homogeneous finite group of odd order, p be a prime, and P be a p-subgroup of G. Then $\Omega_1 P \leqslant Z(P)$.*

Proof. Assuming the contrary, let i be minimal such that $Z_i(P) \backslash Z(P)$ contains an element a of order p, and set $A = \langle a, \Omega_1 Z(P) \rangle$. Thus A is elementary abelian of rank at least 2 and $N_P(A)$ induces a nontrivial automorphism of A, since if j is minimal with $[a, Z_j(P)] \neq 1$ one finds $[a, Z_j(P)] \leqslant \Omega_1 Z_{i-1}(P) \leqslant A$.

By Lemma 2, $N_G(A)$ induces a normal subgroup of $\mathrm{GL}(A)$ and as this subgroup contains a p-element, it contains $\mathrm{SL}(A)$; but that group contains involutions, a contradiction. □

Proof of Proposition 3. Let G be a homogeneous finite group of odd order, and p be the least prime divisor of the order of G. We show that G is solvable, proceeding by induction on $|G|$.

We will show that for any p-subgroup H of $G, N(H)/C(H)$ is a p-group. This is a standard criterion for the existence of a normal p-complement [**11**, 7.3.5]. Hence $G' < G$, and induction applies to conclude.

Accordingly we fix a p-subgroup H of G and consider $N(H)/C(H)$. Let $A = \Omega_1 H$, and P be a Sylow p-subgroup containing A. Then $A \leqslant Z(P)$ by Lemma 8, and $N(A)/C(A)$ is a p'-group, normal in $\mathrm{GL}(A)$. As $N(A)/C(A)$ has odd order and $p > 2$, it follows that $N(A)/C(A)$ is trivial (if A is cyclic, we use the minimality of p at this point).

Thus $N(H)$ acts trivially on $\Omega_1 H$. As p is odd, it follows that $N(H)/C(H)$ is a p-group [**11**, 5.3.10].

4. *Homogeneity of* SL(2, 5) *and* PSL(2, p) *for* $p = 5$ *or* 7

One checks the homogeneity of these three groups by a computation which we summarize here.

Let G be a finite group. In order to check the homogeneity of G, in practice one checks the following two conditions.

(H1) Any two isomorphic subgroups of G lie in the same orbit under the action of $\mathrm{Aut}\, G$.

(H2) $\mathrm{Aut}\, G$ induces $\mathrm{Aut}\, H$ on any subgroup of G.

It is convenient to identify PSL(2, 5) with Alt5 and PSL(2, 7) with GL(3, 2). Note that an outer automorphism of GL(3, 2) fuses two orbits of elements of order 7.

For cyclic subgroups, conditions (H1) and (H2) mean that elements of equal order are in the same orbit under $\mathrm{Aut}\, G$. In the cases at hand one verifies this directly for elements of prime order or order 4, and conditions (H1) and (H2) then follow for subgroups of order $2p$, with p an odd prime, as they will be either the centralizer or the normalizer of a Sylow p-subgroup in each of our three cases.

For Sylow subgroups one must check (H2). For example in GL(3, 2) we get an outer automorphism of the group of upper triangular unipotent matrices by conjugating first by a permutation matrix corresponding to the transposition (1, 3), followed by the transpose-inverse automorphism.

Leaving these three cases aside, there remain:

(1) an elementary abelian 2-group of order 4 in PSL(2, 7);

(2) Sym4, Alt4 inside PSL(2, 7);

(3) a group of order 21 inside PSL(2, 7);

(4) normalizers of Sylow 2-subgroups and 3-subgroups in SL(2, 5).

For the first case one may work inside D_4. In case (2) property (H2) is not an issue and as these groups normalize groups occurring in case (1), property (H1) is also clear. In case (3) we have the normalizer of a Sylow 7-subgroup so property (H1) is not an issue. Furthermore if P is a Sylow 7-subgroup, since there is an outer automorphism of G carrying an element of P to a nonconjugate element of P, (H2) is clear as well.

Finally in case (4) only (H2) is an issue. The groups in question are $Q \rtimes C_3$ and $C_3 \rtimes C_4$ with nontrivial actions. For the first of these, once the claim has been verified for Q it follows in this case. For the second, one makes an additional computation.

5. *Theorem* 1(1)

This result can be read off rather tediously from the classification of the finite simple groups and computations of Schur multipliers, but one may easily give more elementary proofs, based on the following.

LEMMA 9. *Let K be a homogeneous finite group whose center has order divisible by an odd prime p. Then K has a normal p-complement.*

Proof. By homogeneity any element of K of order p belongs to the center. Then for any p-subgroup P of K, $N(P)$ induces a p-group of automorphisms on P, by Huppert's theorem [**11**, 5.3.10], and hence K has a normal p-complement [**11**, 7.3.5]. □

Proof of Theorem 1(1). G is homogeneous, finite, quasisimple, and not simple. By Lemma 9, $Z(G)$ is a 2-group, and by homogeneity every involution in G is central. This is the situation considered by Griess in [**13**] (more generally, for not necessarily quasisimple groups). In the quasisimple case his result states that G is isomorphic either to $\mathrm{SL}(2, q)$ or $\hat{A}_7$, the covering group of Alt_7. This result makes rather heavy use of the theory of finite simple groups, and we will indicate below how to argue more directly.

Thus it suffices to check that most of these groups are not homogeneous. For $q \geqslant 7$ there are elements of $\mathrm{SL}(2, q)$ of order $q+1$, corresponding to multiplication maps in the field $\mathbb{F}_{q^2}$, whose characteristic polynomials are distinct. Such elements are not in the same orbit under $\mathrm{Aut}(\mathrm{SL}(2, q))$. In $\hat{A}_7$ there are elements of order 3 which are not conjugate by an automorphism, as their centralizers are of distinct orders. □

Griess' result depends in turn on the classification of groups in which all 2-local subgroups are solvable and two results of Goldschmidt on strongly closed subgroups of Sylow 2-subgroups, as well as further information. This argument can be eliminated using homogeneity as follows. We begin with a quasisimple homogeneous finite group K whose center is a nontrivial 2-group. Let $A = \Omega_1(Z(K))$ and fix a Sylow 2-subgroup S of K. By homogeneity and control of fusion, $\mathrm{Aut}\, S$ induces $\mathrm{GL}(A)$ on A, and in particular S is a Suzuki 2-group. Hence by [**14**] either S is generalized quaternion or S/A is elementary abelian. To get rid of the second case one can either look more closely at the Suzuki groups or one can examine the simple groups with elementary abelian subgroups: $\mathrm{PSL}(2, q)$ with q a power of 2 or congruent to ± 3 modulo 8; Janko's first group J_1; and groups of Ree type. However J_1 and groups of Ree type have trivial Schur multiplier; in the case of groups of Ree type, the computation that eliminates the 2-part of the Schur multiplier involves only the structure of the normalizer of a Sylow 2-subgroup (see [**1**]).

Thus in the proof of Lemma 1 we need only consider the case in which $K/Z(K)$ has dihedral Sylow 2-subgroups [**12**]. Hence $K/Z(K)$ is isomorphic to Alt_7 or $\mathrm{PSL}(2, q)$ with $q \geqslant 5$ odd, and as $Z(K)$ is a 2-group, $K \simeq \hat{A}_7$ or $\mathrm{SL}(2, q)$ respectively. Of course [**12**] is a major result of classification theory; perhaps a more direct analysis can be used to reduce to $\mathrm{SL}(2, 5)$ in this case.

6. *Theorem* 1(2)

This is the most troublesome point. It can be read off rather tediously from the classification of the finite simple groups. A more direct approach makes use of the following fact (which incorporates the classification of groups of 2-rank at most 2 [**2**]).

FACT 3 [**21**, special case of Theorem A]. Let G be a finite simple group of 2-rank r. Suppose that G has an elementary abelian 2-subgroup E of rank r such that E is a Sylow 2-subgroup of $C(E)$, and $N_G(E)$ induces $\mathrm{GL}(E)$ on E. Then G is of one of the following forms:

(1) $\mathrm{PSL}(2, q)$ with $q \geqslant 5$ odd; $\mathrm{PSL}(3, q)$, $\mathrm{PSU}(3, q)$, $G_2(q)$, or ${}^2D_4(q)$ with q odd;
(2) Alt(7), M_{11}, Co_3, or F_3.

Actually [**21**, Theorem A] does not require E to be a 2-Sylow of $C(E)$, thereby allowing the additional case of ON, and covers the more general situation in which

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$N_G(E)$ acts transitively on the flags of E, allowing the following additional cases: PSU(3, 4); M_{23}; Mc; Ly; J_1; and groups of Ree type in characteristic 3.

To bring these results to bear on Theorem 1, it suffices to prove the following.

PROPOSITION 4. *Let G be a homogeneous finite group with $O_2(G) = 1$, having 2-rank $r \geqslant 3$, and let E be an elementary abelian 2-subgroup E of rank r. Then $N(E)/C(E) \simeq \mathrm{GL}(E)$ and E is a Sylow 2-subgroup of $C(E)$.*

Then applying Fact 3 we get a relatively short list of possibilities for a homogeneous quasisimple finite group. We may then eliminate the nonhomogeneous ones by inspection.

For example, the groups Alt7, PSL$(3, q)$ (q odd), $G_2(q)$ ($q = p^m, p \geqslant 5$), ${}^2D_4(q)$ (q odd), Co_3, and F_3 have pairs of elements of orders 3, $q-1$, $2p$, 4, 2, and 3 respectively which lie in distinct orbits under the automorphism group, as their centralizers are of distinct orders; cf. [**5**, **9**, **10**, **22**]. Similarly in M_{11} there are nonconjugate elements of order 11, and as all automorphisms are inner this is a contradiction [**3**, **4**; Table 1]; and in $G_2(3^m)$ there are nonconjugate elements of order 3, as well as conjugate pairs of such elements which generate an elementary abelian subgroup of rank 2, violating the following.

LEMMA 10. *If G is a homogeneous finite group containing two conjugate elements of prime order p which generate an elementary abelian subgroup of order p^2, then all elements of order p are conjugate in G.*

The proof is immediate, applying homogeneity to elementary abelian p-subgroups of rank at most 2.

For the case of PSU$(3, q)$ with q odd we may use the following evident principle.

LEMMA 11. *Let G be a homogeneous group and H be a subgroup such that for every automorphism α of G, we have either $H^\alpha = H$ or $H^\alpha \cap H = 1$. Then H is homogeneous.*

Thus if PSU$(3, p^m)$ were homogeneous, its Sylow subgroups would be homogeneous and hence, for p odd, would be abelian, a contradiction. Note however that the Sylow 2-subgroup of PSU(3, 4) is in fact the homogeneous group Q^*.

We are left with the case of PSL$(2, q)$ for $q \geqslant 9$ odd to consider, which resembles the case of SL$(2, q)$ mentioned earlier. In this case there are diagonal matrices of order $(q-1)/2$, or matrices of order $(q+1)/2$, nonconjugate in Aut(PSL$(2, q)$). (It suffices to take whichever of these two numbers is odd.)

Thus to complete our arguments it remains only to prove Proposition 4.

7. *Proposition* 4

LEMMA 12. *Let G be a finite homogeneous group.*

(1) *If $O_2(G) = 1$ then G has one conjugacy class of involutions.*

(2) *If G has one conjugacy class of involutions then the rank of the center of a Sylow 2-subgroup of G is at most* 2.

Proof. (1) This follows easily from Glauberman's Z^* theorem and Lemma 10. We may also argue as follows. In view of Lemma 10, if there are nonconjugate involutions in G then any commuting pair of involutions are nonconjugate.

Conversely, it is easy to see that any nonconjugate pair of involutions i, j will commute; otherwise they will generate a dihedral group D_n of order $2n$ with $n > 2$ even, and then D_n contains a conjugate pair of commuting involutions. Now let i, j be commuting involutions and let I, J, K be the conjugacy classes of i, j, and ij respectively. Then $\langle K \rangle \leqslant \langle I, J \rangle$ and $[\langle K \rangle, \langle I, J \rangle] = 1$, so $\langle K \rangle$ is commutative. Thus $\langle K \rangle \leqslant O_2(G)$, a contradiction.

(2) Let S be a Sylow 2-subgroup of G, and $V = \Omega_1(Z(S))$. As $N(S)$ controls fusion in $Z(S)$, $N(V)/C(V)$ is nontrivial, and is normal in $\mathrm{GL}(V)$ by Lemma 2. On the other hand $N(V)/C(V)$ has odd order, so this forces the rank of V to be at most 2. □

Proof of Proposition 4. Let G be a homogeneous finite group with $O_2 G = 1$, of 2-rank $r \geqslant 3$, and let E be an elementary abelian 2-subgroup of rank r. Let S be a 2-Sylow subgroup of G. We must prove that

(1) $N(E)/C(E) \simeq \mathrm{GL}(E)$;

(2) E is a Sylow 2-subgroup of $C(E)$.

(1) Let $A = C_S(E) < S$ (by Lemma 12). Then $E = \Omega_1(Z(A))$ and $N_S(A) > A$ so $N(E)$ acts nontrivially on E and induces a normal subgroup of $\mathrm{GL}(E)$ by Lemma 2. This proves (1).

(2) Let Q be a Sylow 2-subgroup of $C(E)$. By the Frattini argument $N(E) = N(Q) \cdot C(E)$. Thus $N(Q)$ induces $\mathrm{GL}(E)$ on E. It follows that Q is homocyclic (cf. [**21**, §3]). Let $A = N(Q)/C(Q)$. We have the following properties:

(i) Q is homocyclic;

(ii) A is a normal subgroup of $\mathrm{Aut}(Q)$ inducing $\mathrm{GL}(\Omega_1 Q)$ on $\Omega_1 Q$;

(iii) every element of order 2 in A acts nontrivially on $\Omega_1 Q$.

This forces $Q = \Omega_1 Q$, that is, $Q = E$. □

Thus the main point in the classification of homogeneous finite groups is the analysis given in [**21**], which depends on a number of specific identification theorems as well as some key cohomological information regarding groups of the form $E \cdot \mathrm{GL}(E)$.

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