

On Groups of Finite Morley Rank with Weakly Embedded Subgroups

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1. INTRODUCTION

The present paper is a contribution to the classification of tame groups of finite Morley rank. A group of finite Morley rank is called *tame* if it involves no bad groups and no bad fields in the sense of [7]; see also Section 2. It is conjectured that an infinite simple tame group of finite Morley rank is isomorphic to an algebraic group over an algebraically closed field. It is known [2] that a minimal counterexample is either of even type, meaning that the Sylow 2-subgroups are of bounded exponent, or of odd type, meaning that the Sylow 2-subgroups are finite extensions of divisible abelian groups. We deal here with tame simple groups of even type having a “weakly embedded” subgroup, a notion related to the notion of a strongly embedded subgroup in the finite case, but having no exact analog in that case.



In [1] the classification of simple tame groups with a strongly embedded subgroup was derived, in an inductive framework, as follows.

Fact 1.1 [1]. Let G be an infinite, simple, K^* -group of finite Morley rank of even type with a strongly embedded subgroup M . Then M is solvable. If, in addition, G is tame, then it is isomorphic to $\mathrm{PSL}_2(K)$, where K is an algebraically closed field of characteristic 2.

A K^* -group is a group whose proper infinite definable simple sections are isomorphic to algebraic groups over algebraically closed fields. A minimal counterexample to the conjectured classification is necessarily a simple K^* -group. We remark that it is known that in the case of simple K^* -groups, the hypothesis of tameness is equivalent to the noninvolvement of bad fields and the presence of at least one involution (cf. [11], Theorems 13.3 (iv) and B.3).

In the context of infinite groups the following notion is a natural generalization of strong embedding and is considerably easier to achieve in practice.

DEFINITION 1.2. Let G be a group of finite Morley rank. A proper definable subgroup M of G is said to be weakly embedded if it satisfies the following conditions:

- (i) Any Sylow 2-subgroup of M is infinite.
- (ii) For any $g \in G \setminus M$, $M \cap M^g$ has finite Sylow 2-subgroups.

In the present paper we prove the following theorem, which generalizes Fact 1.1:

THEOREM 1.3. *Let G be a simple, tame, K^* -group of finite Morley rank of even type. If G has a weakly embedded subgroup, then $G \cong \mathrm{PSL}_2(K)$, where K is an algebraically closed field of characteristic 2.*

An equivalent statement, in view of the structure of $\mathrm{PSL}_2(K)$ and Fact 1.1, which, moreover, corresponds to the form that the proof actually takes, reads as follows.

THEOREM 1.4. *If G is a simple tame K^* -group of finite Morley rank of even type and M is a weakly embedded subgroup of G , then M is strongly embedded in G .*

For standard terminology regarding groups of finite Morley rank we refer to [11]. The more specialized terminology relating to tame groups will be reviewed below.

In the final section we give a number of applications of the main theorem: the classification of simple tame K^* -groups of even type with all 2-local subgroups solvable; the elimination of connected cores in 2-local

subgroups within simple tame K^* -groups of even type; and a generalization of the Glauberman Z^* -theorem to the finite strongly closed abelian case. These relatively direct arguments show the flexibility of the weak embedding theorem as a tool for further analysis. Less straightforward uses of the theorem should serve to treat the infinite strongly closed abelian case, which then combines with the method of weak embedding to give a version of the Aschbacher standard component theorem. This is the subject of work in progress by various combinations of the present authors and Corredor and is part of a larger plan to completely classify the simple tame groups of finite Morley rank of even type.

Recently Jaligot [16] generalized the result of [2], concerning the nonexistence of simple tame K^* -groups of mixed type, eliminating the use of tameness. Given the restrictive nature of this assumption, one will want to investigate further the possibility of removing this hypothesis from at least part of the classification project for simple K^* -groups of finite Morley rank (notably the case of groups of even type). In the context of the present paper, tameness amounts to the assumption that no bad field is involved, since we deal throughout with simple K^* -groups that contain involutions. The arguments given here make no use of the noninvolvement of bad fields until Section 9, so our intermediate results will be formulated without invoking that hypothesis.

2. BACKGROUND

In this section we review some general results that will be used in the sequel, most of which are standard.

DEFINITION 2.1. 0. A *section* of a group G is a quotient of the form H/K , where H and K are subgroups of G and $K \triangleleft H$. Such a section is said to be definable if H and K are definable.

1. A *bad group* is a simple infinite group of finite Morley rank in which every proper definable connected subgroup is nilpotent.

2. A *bad field* is a structure of finite Morley rank consisting of an algebraically closed field together with a distinguished proper infinite subgroup of its multiplicative group.

3. A *tame group* is a group such that none of its proper sections is a bad group, and which does not interpret a bad field.

4. A *K -group* is a group G of finite Morley rank such that every infinite definable simple section of G is isomorphic to an algebraic group over an algebraically closed field.

5. A K^* -group is a group G of finite Morley rank such that every infinite proper definable simple section of G is isomorphic to an algebraic group over an algebraically closed field.

6. If G is a group of finite Morley rank and $X \subseteq G$, then the definable closure $d(X)$ of X is the intersection of all of the definable subgroups of G that contain X . The *descending chain condition* on definable subgroups in groups of finite Morley rank imply that $d(X)$ is a definable subgroup.

DEFINITION 2.2. 1. A group G is of even type if its Sylow 2-subgroups are of bounded exponent; in this case they are definable.

2. A subgroup of a group G of finite Morley rank is *unipotent* if it is definable, connected, and of bounded exponent.

3. The (definable) subgroup generated by the unipotent 2-subgroups of a group G is denoted $B(G)$. If $G = B(G)$, then G is said to be a B -type group.

4. $O_2(G)$ is the largest connected definable normal 2-subgroup of G , and $O(G)$ is the largest connected definable normal 2^\perp -subgroup of G . (A 2^\perp -group is a group containing no involution.)

Some of the facts quoted below are given as exercises in [11]. In cases where proofs of these facts are not given, the reader is referred to [1] or [2].

Fact 2.3 [11], *Exercise 10*, p. 78]. Let G be a group of finite Morley rank. G° contains all connected definable subgroups of G .

Fact 2.4 [25]. Let G be a group of finite Morley rank. The subgroup generated by a set of definable connected subgroups of G is definable and connected, and it is the setwise product of finitely many of them.

Fact 2.5 [25]. Let G be a group of finite Morley rank. Let $H \leq G$ be a definable connected subgroup. Let $X \subseteq G$ be any subset. Then the subgroup $[H, X]$ is definable and connected.

Fact 2.6 [23]. Let G be a group. If H and N are subgroups of the group G , with N normal in G , and if the set of commutators $\{[h, n] : h \in H, n \in N\}$ is finite, then so is the group $[H, N]$.

COROLLARY 2.7. *Let G be a group of finite Morley rank with two definable subgroups H and N such that H normalizes N . Then $[H, N]$ is a definable subgroup of G .*

Proof. Repeat the argument used in [23] to prove the same fact for algebraic groups, replacing closed subgroups with definable subgroups. ■

COROLLARY 2.8. *Let G be a group of finite Morley rank such that $G = HN$, where H and N are two definable subgroups with $N \triangleleft G$. The smallest normal subgroup of G that contains H , namely $[H, N]H$, is definable.*

Fact 2.9 [11, Proposition 10.2]. Let G be a group of finite Morley rank and $i, j \in I(G)$. Then either i and j are $d(\langle ij \rangle)$ -conjugate or they commute with a third involution of $d(\langle ij \rangle)$.

Fact 2.10 [11, Lemma 5.35 (i)]. Let G be a group of finite Morley rank and X be a subset of G such that the elements of X commute. Then the definable closure $d(X)$ of X is an abelian subgroup.

Fact 2.11 [18]. Let A be an abelian group of finite Morley rank. Then $A = DB$ (central product), with D and B 0-definable, D divisible, and B of bounded exponent.

Fact 2.12 [11, Exercise 10, p. 93]. The definable closure of a cyclic subgroup of a group of finite Morley rank is the direct product of a divisible group and a finite cyclic group.

Proof. Let G be a group of finite Morley rank, and let $C = \langle x \rangle$ be a cyclic subgroup of G . Then $d(C)$ is an abelian subgroup by Fact 2.10. By Fact 2.11, $d(C) = DB$, with D and B definable, D divisible, and B of bounded exponent. Since $d(C)/D$ is of bounded exponent, $x^n \in D$ for some n . Choose the smallest such n . Since D is divisible, there is $z \in D$ such that $x^n = z^n$, and thus $(xz^{-1})^n = 1$. As $x \in D\langle xz^{-1} \rangle$, $d(C) = D\langle xz^{-1} \rangle = D \times \langle xz^{-1} \rangle$. ■

Fact 2.13 [8]. Let G be a group of finite Morley rank and H be a definable normal subgroup of G . If x is an element of G such that \bar{x} is a p -element of $\bar{G} = G/H$, then the coset xH contains a p -element.

Fact 2.14 [20]. Let α be a definable involutive automorphism of a group of finite Morley rank G . If α has finitely many fixed points, then G has a definable normal subgroup of finite index that is abelian and inverted by α .

Fact 2.15 [11, Exercise 14, p. 73]. Let G be a 2^\perp -group of finite Morley rank. Assume that σ is a definable involutive automorphism of G . Then $G = C_G(\sigma)G^-$, where $G^- = \{g \in G : \sigma(g) = g^{-1}\}$. There is a bijection between $C_G(\sigma) \times G^-$ and G given by $(c, x) \mapsto cx$. In particular, $C_G(\sigma)$ is a connected group if G is connected.

Proof. The reader is referred to the hints in [11]. ■

Fact 2.16 [11, Lemma 6.2]. If G is an infinite nilpotent group of finite Morley rank, then $Z(G)$ is infinite.

Fact 2.17 [11, Lemma 6.3]. Let G be an infinite nilpotent group of finite Morley rank. If $H < G$ is a definable group of infinite index, then $N_G(H)/H$ is infinite.

Fact 2.18 [11, Lemma 5.41]. Let G be a group of finite Morley rank. If H is a nilpotent-by-finite (resp. solvable-by-finite) subgroup of G , then H° , i.e. $H \cap d(H)^\circ$, is a nilpotent (resp. solvable) group.

Fact 2.19 [11, Exercise 5, p. 98]. Any infinite normal subgroup of an infinite nilpotent group of finite Morley rank contains infinitely many central elements.

Fact 2.20 [11, Exercise 1, p. 97]. An infinite nilpotent p -group of finite Morley rank and of bounded exponent has infinitely many central elements of order p .

Fact 2.21 [11, Exercise 2, p. 175]. Let Q and E be subgroups of a group of finite Morley rank such that Q is normal, connected, solvable, and definable and does not contain involutions and E is a definable connected 2-group of bounded exponent. Then $[Q, E] = 1$.

Fact 2.22 [3]. Let G be a perfect group of finite Morley rank such that $G/Z(G)$ is a quasi-simple algebraic group. Then G is an algebraic group. In particular, $Z(G)$ is finite ([15], Sect. 27.5).

Fact 2.23 [11, Theorem 8.4]. Let $\mathcal{G} = G \rtimes H$ be a group of finite Morley rank where G and H are definable, G is an infinite simple algebraic group over an algebraically closed field, and $C_H(G) = 1$. Then, viewing H as a subgroup of $\text{Aut}(G)$, we have $H \leq \text{Inn}(G)\Gamma$, where $\text{Inn}(G)$ is the group of inner automorphisms of G and Γ is the group of graph automorphisms of G .

For any group G of finite Morley rank, we write $\sigma(G)$ for the solvable radical of G . This is the largest solvable normal subgroup of G , and it is definable in G but not necessarily connected, even if G is connected ([11, Theorem 7.3]).

Fact 2.24 [1]. Let G be a connected nonsolvable K -group of finite Morley rank. Then $G/\sigma(G)$ is isomorphic to a direct sum of simple algebraic groups over algebraically closed fields.

We turn now to the basic properties of strongly and weakly embedded (Definition 1.2) subgroups.

DEFINITION 2.25. A proper definable subgroup M of a group G of finite Morley rank is said to be strongly embedded in G if it satisfies the

following conditions:

- (i) M contains involutions.
- (ii) For every $g \in G \setminus M$, $M \cap M^g$ does not contain involutions.

Fact 2.26 [11, Theorem 10.19], [24, (6.4.3)]. Let G be a group of finite Morley rank with a proper definable subgroup M . Then the following are equivalent:

- (i) M is a strongly embedded subgroup.
- (ii) $I(M) \neq \emptyset$, $C_G(i) \leq M$ for any $i \in I(M)$, and $N_G(S) \leq M$ for any Sylow 2-subgroup S of M .
- (iii) $I(M) \neq \emptyset$, and $N_G(S) \leq M$ for any nontrivial 2-subgroup S of M .

Fact 2.27 [2]. Let G be a group of finite Morley rank with a weakly embedded subgroup M . Then the following hold:

- (i) For any Sylow 2-subgroup S of M , $N_G(S) \leq M$.
- (ii) If S is a Sylow 2-subgroup of M , then S is a Sylow 2-subgroup of G .

The next result is given more generally in [2]; we cite it in the form appropriate for groups of even type.

Fact 2.28 [2]. Let G be a group of finite Morley rank of even type. A proper definable subgroup M of G is a weakly embedded subgroup if and only if the following hold:

- (i) M has infinite Sylow 2-subgroups.
- (ii) For any unipotent 2-group U of M , $N_G(U) \subseteq M$.

COROLLARY 2.29. *Let G be a group of finite Morley rank of even type, and M a proper definable subgroup of G containing a Sylow 2-subgroup S of G . Then M is a weakly embedded subgroup if and only if the following hold:*

- (i) S is infinite.
- (ii) For any unipotent two-group U of S , $N_G(U) \subseteq M$.

Proof. If U is any unipotent 2-subgroup of M and $U^g \leq S$, with $g \in M$, then $N_G(U) = N_G(U^g)^{g^{-1}} \leq M^{g^{-1}} = M$. ■

This version makes the following evident.

COROLLARY 2.30. *Let M be a weakly embedded subgroup of the group G of even type. Any definable proper subgroup H of G containing M is also weakly embedded.*

A related notion is the k -generated core.

DEFINITION 2.31. Let S be a Sylow 2-subgroup of G . The k -generated core, relative to S , is the definable closure of the group generated by all subgroups $N_G(R)$, where $R \leq S$ contains an elementary abelian subgroup of rank at least k . It is denoted $\Gamma_k(S)$.

COROLLARY 2.32. Let G be a group of even type with an infinite Sylow two-subgroup. If for some k $\Gamma_k(S) < G$, then $\Gamma_k(S)$ is a weakly embedded subgroup of G .

The following proposition gives a characterization of weak embedding in simple groups of finite Morley rank of even type, which will be useful in the last section of the paper.

PROPOSITION 2.33. Let G be a simple group of finite Morley rank of even type and H a proper definable subgroup with infinite Sylow 2-subgroups, which contains the connected component of the normalizer of any connected definable 2-subgroup of H . Then G has a weakly embedded subgroup.

Proof. We consider the graph \mathbb{U} , whose vertices are the unipotent 2-subgroups of G with edges between any two groups whose intersection is infinite.

We show first that this graph is disconnected. H cannot contain all of the unipotent subgroups of G , as G is simple. We claim that if U is a unipotent 2-subgroup not contained in H , then the intersection $U \cap H$ is finite. If, on the contrary, $V = (U \cap H)^\circ$ is nontrivial, then as $N_G(V)^\circ \leq H$, we find $N_U(V)^\circ = V$, so $U = V \leq H$, a contradiction. Thus \mathbb{C} is disconnected.

Now let \mathbb{C} be a connected component of the graph \mathbb{U} . Note that G acts on the graph \mathbb{U} by conjugation. Furthermore, as every component of \mathbb{U} contains the connected component of a Sylow 2-subgroup, G conjugates the connected components of \mathbb{U} transitively. Let $M = N_G(\langle \cup \mathbb{C} \rangle)$, and let $S \in \mathbb{C}$ be the connected component of a Sylow 2-subgroup. Observe that M contains the stabilizer of \mathbb{C} in G . In fact, M coincides with this stabilizer, since by the Frattini argument $M \leq \langle \cup \mathbb{C} \rangle N(S)$, which evidently stabilizes \mathbb{C} . In particular, $M < G$.

For $U \leq S$ unipotent, we have $U \in \mathbb{C}$, and hence $N(U)$ stabilizes \mathbb{C} , so $N(U) \leq M$. Thus by Corollary 2.29 M is weakly embedded in G . ■

DEFINITION 2.34. Let $G = XY$ be a group of finite Morley rank where X and Y are definable subgroups of G and $X \triangleleft G$. A subgroup A of X is said to be Y -minimal if it is infinite, definable, normalized by Y , and minimal with respect to these properties. In particular, A is connected.

Fact 2.35 [26]. Let $G = A \rtimes H$ be a group of finite Morley rank where A and H are infinite definable abelian subgroups and A is H -minimal. Assume $C_H(A) = 1$. Then the following hold:

(i) The subring $K = \mathbb{Z}[H]/\text{ann}_{\mathbb{Z}[H]}(A)$ of $\text{End}(A)$ is a definable algebraically closed field; in fact, there is an integer l such that every element of K can be represented as the endomorphism $\sum_{i=1}^l h_i$, where $h_i \in H$.

(ii) $A \cong K^+$, H is isomorphic to a subgroup T of K^\times , and H acts on A by multiplication.

(iii) In particular, H acts freely on A , $K = T + \dots + T$ (l times), and with additive notation $A = \{\sum_{i=1}^l h_i a : h_i \in H\}$ for any $a \in A^*$.

Fact 2.36 [11, Theorem 9.7]. Let $A \rtimes G$ be a group of finite Morley rank such that A is abelian and $C_G(A) = 1$. Let $H \triangleleft G_1 \triangleleft G$ be definable subgroups with G_1 connected and H infinite abelian. Assume also that A is G_1 -minimal. Then

$$K = \mathbb{Z}[Z(G)^\circ]/\text{ann}_{\mathbb{Z}[Z(G)^\circ]}(A)$$

is a definable algebraically closed field, A is a finite-dimensional vector space over K , G acts on A as vector space automorphisms, and H acts scalarly. In particular, $G \leq GL_n(K)$ for some n , $H \leq Z(G)$, and $C_A(G) = 1$.

Fact 2.37 [11, Corollary 9.6]. Let $A \rtimes G$ be a group of finite Morley rank where G is definable and connected and $C_G(A) = 1$. Assume G has a normal, infinite, abelian, and definable subgroup H and there is an H -minimal subgroup $B \leq A$ such that A is generated by the conjugates B^g for $g \in G$. Then there is an interpretable algebraically closed field K and a finite-dimensional vector space structure on A , say $A \cong K^n$, such that G acts on A as K -vector space automorphisms (i.e., $G \leq GL_n(K)$) and H by scalar multiplication (i.e., $H \leq Z(G) \leq Z(GL_n(K))$).

Fact 2.38 [11, Theorem 9.8]. Let $A \rtimes G$ be a solvable group of finite Morley rank with A abelian and definable, and G definable and connected. Let $B \leq A$ be either a G' or G -minimal subgroup. Then G' centralizes B .

Fact 2.39 [9]. Let G be a solvable group of finite Morley rank and H a normal Hall π -subgroup of G . Then H has a complement in G . If H has a bounded exponent, then the complements of H in G are definable and conjugate to each other.

Fact 2.40 [8]. Let G be a connected solvable group of finite Morley rank. Then the Hall π -subgroups of G are connected.

Fact 2.41 [4]. Let G be a solvable group of finite Morley rank, $N \triangleleft G$, and let H be a Hall π -subgroup of G for some set π of primes. Then:

(i) $H \cap N$ is a Hall π -subgroup of N , and all Hall π -subgroups of N are of this form.

(ii) If N is definable, then HN/N is a Hall π -subgroup of G/N , and all Hall π -subgroups of G/N are of this form.

Fact 2.42 [2]. Let Y be a connected solvable group of finite Morley rank and S a Sylow 2-subgroup of Y . Assume S (which is connected by Fact 2.40) is unipotent. Then $S \leq F(Y)$, and therefore S is a characteristic subgroup of Y .

PROPOSITION 2.43. *Let $G = H \rtimes Q$ be a group of finite Morley rank. Let $H_1 \triangleleft H$ be a solvable Q -invariant, definable π -subgroup of bounded exponent in G . Assume that Q is a solvable, definable π^\perp -subgroup. Then*

$$C_H(Q)H_1/H_1 = C_{H/H_1}(Q).$$

Proof. It is enough to show that $C_{H/H_1}(Q) \leq C_H(Q)H_1/H_1$. Let $L/H_1 = C_{H/H_1}(Q)$. $[L, Q] \leq H_1$. We have $H_1Q \leq LQ \leq HQ$. We will apply Fact 2.39 to H_1Q . For $x \in L$, $Q^x \leq H_1Q$. Therefore, by Fact 2.39, $Q^x = Q^h$ for some $h \in H_1$. This implies that $xh^{-1} \in N_L(Q)$ and, therefore, $x \in H_1N_L(Q) = H_1C_L(Q) \leq H_1C_H(Q)$. ■

COROLLARY 2.44. *Let $G = H \rtimes Q$ be a solvable group of finite Morley rank, with H and Q definable. Assume H is a π -group of bounded exponent and Q is a π^\perp -group. Then $H = [H, Q]C_H(Q)$.*

Proof. In the above proposition, let $H_1 = [H, Q]$. Then $L = H$. ■

COROLLARY 2.45. *Let Q and X be definable subgroups of a group of finite Morley rank with Q a unipotent 2-group, X a 2^\perp -group, and X acting on Q , and suppose that X acts trivially on the factors Q_i/Q_{i-1} of a definable normal series for Q . Then X acts trivially on Q .*

Fact 2.46 [11, Exercise 4, p. 294]. If G is a group of finite Morley rank whose definable connected 2^\perp -sections are nilpotent, then either G° does not have involutions or G has infinite Sylow 2-subgroups.

This last fact reflects a tame group phenomenon, since by Theorem B.1 of [11] the connected definable sections of tame groups without involutions are nilpotent. In K^* -groups, the structure of simple algebraic groups leads

to the following conclusion:

PROPOSITION 2.47. *Let G be a K^* -group of finite Morley rank. Then every proper, definable, connected section of G that does not contain an involution is solvable.*

The following two facts were proved in [2], under the tameness assumption. Using Fact 2.22 in place of [2, Theorem 4.1] and some minor changes in the arrangement of the proofs (cf. [16]), they hold generally for K -groups.

Fact 2.48 [2, Lemma 5.20]. *If a nonsolvable connected K -group H has a weakly embedded subgroup, then $H/O(H)$ is isomorphic to $(P)SL_2(K)$, where K is an algebraically closed field.*

DEFINITION 2.49. $\mathcal{Z}(G)$ denotes the graph whose vertices are the nontrivial unipotent 2-subgroups of G and whose edges are the pairs of unipotent 2-subgroups which commute.

Fact 2.50 [2, Proposition 5.21]. *If X is a B -type K -group and $\mathcal{Z}(X)$ is not connected, then $X \cong PSL_2(K)$, where K is an algebraically closed field of characteristic 2.*

The following definability result is occasionally useful.

Fact 2.51 [22, Corollaire 4.16]. *In a simple algebraic group over an algebraically closed field, definability from the field and definability from the pure group coincide.*

It was shown in [2] that this fact also holds for quasi-simple algebraic groups.

We write C_i for the centralizer of the involution i in an ambient group when the latter is understood.

3. PRELIMINARY ANALYSIS

Until Section 9, with the exception of the more general context dealt with in Section 6, G will denote a connected simple K^* -group of even type, and M will be a weakly embedded subgroup of G . In Section 9 we will make use of the noninterpretability of bad fields in G . We assume toward a contradiction that M is not strongly embedded. We will be concerned with the analysis of the resulting configurations for the remainder of the paper.

The argument goes as follows. As M is weakly but not strongly embedded in G , it follows from Facts 2.26 and 2.27 that there is an involution $\alpha \in M$ whose centralizer is not contained in M . K -group information reviewed above allows us to determine this centralizer with some precision:

its connected component will be the product of a solvable group without involutions and a copy L of $\mathrm{PSL}_2(K)$ for some algebraically closed field K of characteristic 2, the latter meeting M in a Borel subgroup B of L . We can then show that there is a Sylow 2-subgroup of M containing α , whose connected component is normalized by a torus in B , and exploiting this information we determine the structure of the connected component S of a Sylow 2-subgroup of M , along the general lines of [17] and [19]. Using this information, we first prove that M° is solvable and then use a variant of the Thompson order formula to eliminate most of the possibilities for the structure of S . There will remain the possibility that S is homocyclic and inverted by α , in which case we rework the arguments used in [1], themselves a reworking of arguments in [13].

Fix an involution $\alpha \in M$ such that $C_\alpha \not\leq M$. We set

$$H = C_\alpha^\circ \quad \text{and} \quad L = B(H). \quad (1)$$

We will show below that $H = L \times O(H)$ and $L \cong \mathrm{PSL}_2(K)$ with K a field of characteristic 2. We will then argue that $B = L \cap M$ is a Borel subgroup of L , and that a torus T in B normalizes the connected component of some Sylow 2-subgroup of M .

LEMMA 3.1. *$H \cap M$ is weakly embedded in H and $L \cap M$ is weakly embedded in L .*

Proof. Since M is weakly embedded in G , it suffices to show that the Sylow 2-subgroups of $H \cap M$ are infinite, and that L is not contained in M .

Let S be a Sylow 2-subgroup of M containing α . Then α acts on S° and $C_\alpha \cap S$ is infinite, as otherwise by Fact 2.14 α would invert S° , and S contains infinitely many involutions. Thus the Sylow 2-subgroups of $C_\alpha \cap M$ are infinite, and hence the same applies to $H \cap M$.

It remains to be seen that L is not contained in M . Suppose, on the contrary, that $L \leq M$. Let S be a Sylow 2-subgroup of H . By the Frattini argument $C_\alpha \leq L \cdot N(S) \leq L \cdot N(S^\circ) \leq LM = M$, as M is weakly embedded. This contradicts our choice of α . ■

PROPOSITION 3.2. *$H = L \times O(H)$, and $L \cong \mathrm{PSL}_2(K)$, where K is an algebraically closed field of characteristic 2. $M \cap L$ is a Borel subgroup of L and $O(H) \leq M$.*

Proof. If H is solvable then by Fact 2.42 H has a unique, normal, Sylow 2-subgroup, which contradicts Lemma 3.1: any weakly embedded subgroup of H would contain this Sylow subgroup and its normalizer.

Now Fact 2.48 and Lemma 3.1 imply that $H/O(H) \cong \mathrm{PSL}_2(K)$, where K is an algebraically closed field of characteristic 2. Therefore $H = LO(H)$,

where $L = B(H)$. As $L \cap M$ is a weakly embedded subgroup of L , $\mathcal{Z}(L)$ (Definition 2.49) is not connected. Therefore, by Fact 2.50, $L \cong \text{PSL}_2(K)$, where K is an algebraically closed field of characteristic 2. From this it follows that $L \cap O(H) = 1$, and $H = L \times O(H)$.

Now as $M \cap H$ is weakly embedded in H , it follows that $M \cap H = B \times O(H)$, with B a Borel subgroup of L . ■

COROLLARY 3.3. *The involution α lies outside H .*

We will write

$$M \cap L = A \rtimes T, \quad (2)$$

with A unipotent and T a torus in H . $A \rtimes T$ can be identified with $K_+ \rtimes K^\times$ with the multiplicative group K^\times acting by multiplication on the additive group.

PROPOSITION 3.4. *Let X be a connected K -group of even type and Y a connected 2^\perp -group acting definably on X . Then Y leaves invariant the connected component of a Sylow 2-subgroup of X .*

Proof. It suffices to find a solvable Y -invariant subgroup containing a Sylow 2-subgroup of X , since then by Fact 2.42 its connected component has a unique maximal connected 2-subgroup, which is then Y -invariant. Therefore we may replace X by $X/\sigma(X)$, and assume $\sigma(X) = 1$. By Fact 2.24, X is then a product of simple algebraic groups over algebraically closed fields of characteristic 2. As Y is connected, it normalizes each factor, and thus it suffices to deal with the case in which X is a simple algebraic group over an algebraically closed field of characteristic 2. By Fact 2.23, Y acts by inner automorphisms, and since a definable quotient of a 2^\perp -group of finite Morley rank is again a 2^\perp -group (Fact 2.13), Y acts on X as (part of) a torus Y_1 . As Y_1 is contained in a Borel subgroup, and a Borel subgroup contains a Sylow 2-subgroup of X , we have found the desired Y -invariant connected solvable subgroup. ■

PROPOSITION 3.5. *There is a Sylow 2-subgroup of M containing A whose connected component is normalized by $\langle \alpha \rangle \times T$.*

Proof. Let Q be a maximal connected 2-subgroup of M normalized by $\langle \alpha \rangle \times T$. We claim that Q is the connected component of a Sylow 2-subgroup of M .

Supposing the contrary, let S be a Sylow 2-subgroup of M containing Q and α and let $R \leq S$ be the preimage in S of $Z(N_S(Q)/Q)$. Then $R^\circ > Q$ (Facts 2.17 and 2.16) and $[\alpha, R^\circ] \leq Q$. Let K/Q be the centralizer of α in $N(Q)/Q$. Then any subgroup of K containing Q is α -invariant, and the connected component of a Sylow 2-subgroup of K properly contains Q .

Thus to get a contradiction it suffices to find a Sylow 2-subgroup of K/Q whose connected component is T -invariant, and for this the previous proposition suffices. ■

For the remainder of the paper, we fix an $(\langle \alpha \rangle \times T)$ -invariant maximal connected 2-subgroup of M , denoted S . The following is a consequence of the preceding discussion.

COROLLARY 3.6. $C_S(\alpha)^\circ = A$.

4. SYLOW 2-SUBGROUPS

In this section we analyze the structure of the connected component of a Sylow 2-subgroup of G and obtain a complete classification of the different possibilities. This classification forms the basis of the arguments in the remaining sections. The result is suggested by the finite analog in [17], although the proof uses some different ideas.

This part of the argument does not involve weakly embedded subgroups, and we anticipate that it will be useful in some other situations in which one gets an action of a torus commuting with an involution in the precise manner of the preceding section. Accordingly, we now give the result in its general form.

THEOREM 4.1. *Let $H = S \rtimes T$ be a group of finite Morley rank, where S is a definable, connected 2-group of bounded exponent, and T is also definable. Assume that S has a definable subgroup A such that $A \rtimes T \cong K_+ \rtimes K^\times$ for some algebraically closed field K of characteristic 2, with the multiplicative group acting naturally on the additive group. Assume also that α is a definable involutory automorphism of H such that $C_H(\alpha)^\circ = A \rtimes T$. Under these assumptions S is isomorphic to one of the following groups:*

(i) *If S is abelian, then either S is homocyclic with $I(S) = A^\times$, or $S = E \oplus E^\alpha$, where E is an elementary abelian group isomorphic to K_+ . In the latter case, $A = \{xx^\alpha : x \in E\}$.*

(ii) *If S is nonabelian, then S is an algebraic group over K whose underlying set is $K \times K \times K$, and the group multiplication is as follows: For $a_1, b_1, c_1, a_2, b_2, c_2 \in K$,*

$$\begin{aligned} & (a_1, b_1, c_1)(a_2, b_2, c_2) \\ &= (a_1 + a_2, b_1 + b_2, c_1 + c_2 + \epsilon\sqrt{a_1a_2} + \sqrt{b_1b_2} + \sqrt{b_1a_2}), \end{aligned}$$

where ϵ is either 0 or 1.

In this case α acts by $(a, b, c)^\alpha = (a, a + b, a + b + c + \sqrt{ab})$ and $[\alpha, S] = \{(0, b, c) : b, c \in K\}$. In particular, if S is nonabelian, then S has exponent four.

LEMMA 4.2. $C_S(t)$ is finite for every $t \in T^\times$.

Proof. Fix $t \in T^\times$ and let $Q = C_S(t)$. We assume toward a contradiction that Q is infinite. As both $\langle \alpha \rangle$ and T centralize T and normalize S , Q is $\langle \alpha \rangle \times T$ -invariant. Since Q is infinite, $C_Q(\alpha)$ is infinite and $C_Q(\alpha)^\circ$ is a nontrivial connected 2-subgroup of $C_H(\alpha)^\circ$. This implies that $C_Q(\alpha)^\circ \leq A$. This contradicts the action of t on A . Therefore, Q is finite. ■

COROLLARY 4.3. If S and T are as in the statement of Theorem 4.1, then $C_S(t) = C_S(T)$ for every $t \in T^\times$.

Proof. $C_S(t)$ is a T -invariant finite group and T is connected. ■

LEMMA 4.4. Let S, T , and α be as in the statement of Theorem 4.1. Let R be a nontrivial definable connected $\langle \alpha, T \rangle$ -invariant subgroup of S . Let R_1 be a maximal proper definable connected normal $\langle \alpha, T \rangle$ -invariant subgroup of R that contains R' . Then the following hold:

- (i) R/R_1 is an elementary abelian group.
- (ii) $C_{R/R_1}(\alpha) = R/R_1$, or in other words, $[\alpha, R] \leq R_1$.
- (iii) $\text{rk}(R) = \text{rk}(R_1) + \text{rk}(A)$.

Proof.

(i) As $\Omega_1(R/R_1)^\circ$ is a nontrivial definable connected subgroup of R/R_1 , R/R_1 is an elementary abelian 2-group.

(ii) By Fact 2.14, $C_{R/R_1}(\alpha)^\circ$ is a nontrivial definable connected subgroup of R/R_1 . Thus, $C_{R/R_1}(\alpha) = R/R_1$.

(iii) We will use $\bar{}$ -notation to denote quotients by R_1 . By Lemma 4.2 we can find $x \in R \setminus C_R(T)R_1$. By Proposition 2.43 and Corollary 4.3, $C_T(\bar{x}) = 1$, so $\text{rk}(\bar{x}^T) = \text{rk}(T) = \text{rk}(A)$. We have $\bar{x}^T \subseteq R/R_1$, and thus $\text{rk}(\bar{x}^T) \leq \text{rk}(R) - \text{rk}(R_1)$, so $\text{rk}(R) \geq \text{rk}(R_1) + \text{rk}(A)$. Since we also have $\text{rk}(R) = \text{rk}C_R(\alpha) + \text{rk}(\alpha^R)$ and $\alpha^R \subseteq \alpha R_1$ by (ii), we conclude $\text{rk}(R) \leq \text{rk}(A) + \text{rk}(R_1)$. ■

PROPOSITION 4.5. Let S and T be as in the statement of Theorem 4.1. Then for every $t \in T^\times$, $C_S(t) = 1$.

Proof. Let $Q = C_S(t)$. By Lemma 4.2 and Corollary 4.3, $Q = C_S(T)$ is finite. We suppose toward a contradiction that $Q \neq 1$. Let R be a minimal definable connected $\langle \alpha, T \rangle$ -invariant subgroup of S that contains Q . Let R_1 be a maximal proper definable connected normal $\langle \alpha, T \rangle$ -invariant

subgroup of R that contains R' . By Lemma 4.4 (i) and (iii), R/R_1 is a connected elementary abelian 2-group of rank $\text{rk}(T)$. We therefore have the following short exact sequence:

$$0 \rightarrow M_0 \rightarrow M \rightarrow M_1 \rightarrow 0,$$

where $M = R/R_1$, M_1 is the natural T -module K_+ (by Fact 2.35, T acts on M_1 by scalar multiplication), and $M_0 = QR_1/R_1$ is the kernel of the action of T on M . By choice of R and R_1 , $M_0 \neq (0)$.

We will show that this is a split exact sequence. We will use additive notation for the groups M_0 , M , and M_1 . We fix $x \in M \setminus M_0$ and write $x^{g+h} = x^g + x^h + a(g, h)$ (taking $x^0 = 0$). Then a is a definable function form $(T \cup \{0\}) \times (T \cup \{0\})$ into M_0 , which satisfies the following cocycle condition for every $g, h, k \in T \cup \{0\}$:

$$a(g, h) + a(g + h, k) = a(g, h + k) + a(h, k).$$

Since M_0 is finite, for fixed g the function $a(g, _)$ generically takes on a value a_g . This defines a definable function from T into M_0 , which associates with every $g \in T$ this generic value a_g . Applying the cocycle condition with an arbitrary pair g, h of elements from T and $k \in T$ independent from g and h , one obtains $a(g, h) + a_{g+h} = a_g + a_h$. This means that $a(g, h)$ is a definable coboundary, and hence the extension splits definably. This splitting contradicts the connectedness of R , and we conclude that $M_0 = (0)$ and thus $Q = 1$. ■

COROLLARY 4.6. *Let S and T be as in the statement of Theorem 4.1. If X is a definable normal T -invariant subgroup of S , then for any element t of T^\times , $C_{S/X}(t) = 1$.*

Proof. Let T_1 be the definable closure of $\langle t \rangle$. Then T_1 is a definable 2^\perp -group and $C_{S/X}(t) = C_{S/X}(T_1) = C_S(T_1)X/X$ be Proposition 2.43. By Proposition 4.5 this is trivial. ■

COROLLARY 4.7. *Let S , T , and α be as in the statement of Theorem 4.1. Any definable normal T -invariant subgroup X of S is connected. In particular, $C_S(\alpha)$ is connected, and thus $C_S(\alpha) = A$.*

Proof. As T is connected, it centralizes X/X° . By the preceding corollary, we get $X = X^\circ$. ■

We will use work of Davis and Nésin on *Suzuki 2-groups* [12] to handle certain minimal cases.

DEFINITION 4.8. A *Suzuki 2-group* is a pair (S, T) where S is a nilpotent 2-group of bounded exponent and T is an abelian group that acts on S by group automorphisms and which is transitive on the involutions

of S . A Suzuki 2-group is said to be a free Suzuki 2-group if T acts on S freely, i.e., for any $g \in S$ and $t \in T$, $g^t = g$ implies either $g = 1$ or $t = 1$. A Suzuki 2-group (S, T) is said to be abelian if S is abelian.

In Theorem 4.1, if S is abelian and homocyclic with $\Omega_1(S) = A$, then $S \rtimes T$ is a free Suzuki 2-group of finite Morley rank. In [12], Davis and Nesin prove the following:

Fact 4.9 [12]. A free Suzuki 2-group of finite Morley rank is abelian.

The following proposition is a slight generalization of a lemma in [12] with essentially the same proof:

PROPOSITION 4.10. *Let E be a unipotent 2-group of exponent at most 4. Assume that*

$$0 \rightarrow Z \rightarrow E \rightarrow E/Z \rightarrow 0$$

is an exact sequence, where Z is central and both Z and E/Z are isomorphic to K_+ , where K is a field of characteristic 2 that is closed under taking square roots. Assume also that $T \cong K^*$ acts on E , inducing the natural action on both Z and E/Z . Then E is abelian, and it is either homocyclic or else is elementary abelian of the form $E = E_1 \oplus E_2$, splitting as a T -module. In the case where E is homocyclic, one can obtain the multiplication table of E by fixing x_0 and x_1 in E such that $x_0^2 = x_1$. Then any element of E can be written as $x_0^a x_1^b$, with $a, b \in T \cup \{0\}$, and the product of two distinct elements is given by

$$x_0^{a_1} x_1^{b_1} x_0^{a_2} x_1^{b_2} = x_0^{a_1+a_2} x_1^{b_1+b_2+\sqrt{a_1 a_2}}$$

Proof. We choose two distinct elements x_0 and x_1 of E with $x_0 \in E - Z$ and $x_1 \in Z^*$; furthermore, we take $x_1 = x_0^2$ unless E is elementary abelian. Using these two elements, we coordinatize E by $(T \cup \{0\})^2$ (which may also be thought of as K_+^2) by associating the pair (a, b) in $(T \cup \{0\})^2$ with the element $x_0^a x_1^b$. Every element of E is associated with a unique pair (a, b) in this way, with the convention $x_i^0 = 1$.

For a_1, b_1, a_2, b_2 in $T \cup \{0\}$ we have

$$x_0^{a_1} x_1^{b_1} x_0^{a_2} x_1^{b_2} = x_0^{a_1+a_2} x_1^{b_1+b_2+f(a_1, a_2)},$$

with $f: (T \cup \{0\})^2 \rightarrow T \cup \{0\}$. In coordinates this gives

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 + a_2, b_1 + b_2 + f(a_1, a_2)).$$

In some of what follows we ignore the possibility that an exponent vanishes; such cases are important, but we leave the reader to check that they conform sufficiently well to the general case.

As

$$(x_0^a x_0^b) = (x_0 x_0^{a^{-1}b})^a,$$

the group law on E is determined by the behavior of the function $g(x) = f(1, x)$; that is, $f(a, b) = ag(a^{-1}b)$. Note that $g(0) = 0$.

The associativity of the group law in E may be expressed in terms of g as follows:

$$(a + b)g((a + b)^{-1}c) + ag(a^{-1}b) = ag(a^{-1}(b + c)) + bg(b^{-1}c),$$

where $a, b, c \in T \cup \{0\}$ and all terms are defined.

Setting $a = 1$, $b = x$ and $c = y$, we get

$$(1 + x)g((1 + x)^{-1}y) + g(x) = g(x + y) + xg(x^{-1}y), \quad (3)$$

or equivalently,

$$(1 + x)g((1 + x)^{-1}y) + xg(x^{-1}y) = g(x + y) + g(x).$$

As the left side is invariant under the substitution of $x + 1$ for x , the same applies to the right side:

$$g(1 + x) + g(1 + x + y) = g(x) + g(x + y).$$

Taking $y = x$, we get an additive law $g(1 + x) = g(1) + g(x)$.

Take $y = x(x + 1)$ in Eq. 3. This produces

$$xg(x) = g(x^2) + xg(x + 1),$$

and applying our additive law, this simplifies to $g(x^2) = xg(1)$, or in other words, $g(x) = g(1)\sqrt{x}$, and $f(a, b) = g(1)\sqrt{ab}$. Thus E is commutative.

Note that $x_0^2 = x_1^{g(1)}$, and hence by our choice of x_1 , $g(1)$ is either 0 or 1. In the former case E is elementary abelian and splits as a T -module. In the latter case E is homocyclic with the indicated multiplication. ■

LEMMA 4.11. *If $A < S$, then $\text{rk}(C_{S/A}(\alpha)) = \text{rk}(A)$ and $C_{S/A}(\alpha)$ is an elementary abelian group. Furthermore, $C_{S/A}(\alpha)$ is isomorphic as a T -module with A .*

Proof. Let $X/A = C_{S/A}(\alpha)$, which is nontrivial by Fact 2.14. Commutation with α induces an isomorphism of X/A with A . It is surjective because the image is nontrivial and T -invariant. ■

The next proposition classifies the abelian 2-groups that satisfy the conditions of Theorem 4.1:

PROPOSITION 4.12. *Let S be an abelian 2-group satisfying the conditions of Theorem 4.1. Then either S is homocyclic with $I(S) = A^\times$, or $S = E \oplus E^\alpha$, where E is a T -invariant elementary abelian group. In the latter case, $A = \{xx^\alpha : x \in E\}$ and both E and E^α are T -modules.*

Proof. Note that the assumption that S is abelian implies that for $x \in S$, xx^α is centralized by α , and thus $xx^\alpha \in A$. As a result, α inverts S/A .

Suppose $I(S) = A^*$. By [14, Theorem 17.2], S is a direct sum of cyclic groups. Since $\text{Aut}(S)$ is transitive on $I(S)$, it follows that S is homocyclic. We therefore assume that $I(S \setminus A) \neq \emptyset$.

Let $E = \Omega_1(S)$, an $(\langle \alpha \rangle \times T)$ -invariant definable subgroup of S . By Corollary 4.7, E is connected. As S/A is inverted by α , $E/A \leq C_{S/A}(\alpha)$. Lemma 4.11 and the connectedness of definable normal T -invariant subgroups of S imply that $E/A = C_{S/A}(\alpha)$. In particular, E and T satisfy the conditions of Proposition 4.10. Thus, E is an elementary abelian subgroup that is the direct sum of two T -modules E_1 and E_2 . As the actions of α and T commute, we may assume that $E_2 = E_1^\alpha$.

We claim that $S = E$. Let $R = \Omega_2(S)$. For $x \in R$, as $xx^\alpha \in A$, we have $x^2(x^2)^\alpha = (xx^\alpha)^2 = 1$, which implies that x^2 is an involution inverted by α ; hence $x^2 \in A$. Therefore R is a T -invariant subgroup of S such that $R/A \leq C_{S/A}(\alpha)$. But from the previous paragraph we know that $C_{S/A}(\alpha) = E/A$. Thus $R = E$ and hence $S = E$. ■

For the remainder of the proof of Theorem 4.1 we assume that S is nonabelian unless it is mentioned otherwise. We choose S_1 to be a maximal definable proper normal $(\langle \alpha \rangle \times T)$ -invariant subgroup of S containing S' . Note that, by Corollary 4.7, S_1 is connected. Lemma 4.4 applies to S and S_1 . We note the resulting conclusions, which will be used in the sequel:

- (i) S/S_1 is an elementary abelian group.
- (ii) $C_{S/S_1}(\alpha) = S/S_1$, or in other words, $[\alpha, S] \leq S_1$.
- (iii) $\text{rk}(S) = \text{rk}(S_1) + \text{rk}(A)$.

We next show that $S_1 = [\alpha, S]$, which will imply that this group is uniquely determined.

PROPOSITION 4.13. *$S_1 = [\alpha, S]$ is abelian, and α inverts S_1 .*

Proof. Let $X = \{[\alpha, x] : x \in S\}$, a subset of S_1 inverted by α , of rank $\text{rk}S - \text{rk}A = \text{rk}S_1$. Thus X is generic in S_1 . As S_1 is connected, we find $S_1 = \langle X \rangle$, and furthermore, $X \cap gX$ is generic in S_1 for any $g \in S_1$. If

$g, h, gh \in X$, then α inverts all three elements and hence $[g, h] = 1$. Thus for $g \in X$, $C_{S_1}(g)$ contains the generic subset $X \cap g^{-1}X$, and hence $C_{S_1}(g) = S_1$, $X \subseteq Z(S_1)$. As X is generic, we conclude that S_1 is abelian and then $S_1 = [\alpha, S]$.

As S_1 is abelian, the subset of S_1 inverted by α is a subgroup, and as this set contains the generic set X , S_1 must be inverted by α . ■

In particular, α centralizes the involutions of S_1 ; hence

COROLLARY 4.14. $\Omega_1(S_1) = A$, and thus $A \triangleleft S$.

COROLLARY 4.15. $A \leq Z(S)$.

Proof. As A is normal in S , $A \cap Z(S) \neq 1$. But $A \cap Z(S)$ is T -invariant and T acts on A transitively. Therefore, $A \leq Z(S)$. ■

Next we prove a special case of Theorem 4.1. The proof makes use of computations very similar to those used in [12] to prove a related result.

THEOREM 4.16. *Let α, S, A , and T be as in Theorem 4.1. If in addition S is of exponent 4, then the conclusions of Theorem 4.1 hold.*

Proof. By Proposition 4.12 we may assume that S is nonabelian. If $S_1 = A$, then S, T , and the actions of T on A and S/A are as described in Proposition 4.10, which forces S to be abelian. Thus $S_1 > A$. As $\Omega_1(S_1) = A$, S_1 is of exponent 4. Moreover, the action of T on A and S_1/A is as described in the assumptions of Proposition 4.10; note that commutation with α (i.e., squaring) gives a T -module isomorphism of S_1/A with A . Hence S_1 is homocyclic of exponent 4.

It follows that for $x \in S$, commutation with x gives an endomorphism h_x of S_1 . Since $A = \Omega_1(S_1)$ lies in the kernel of h_x , the image is elementary abelian. In other words, $[S, S_1] \leq A$.

We will now see that the map

$$\begin{aligned} ad_\alpha: S/S_1 &\rightarrow S_1/A \\ xS_1 &\mapsto [\alpha, x]A \end{aligned}$$

is a well-defined T -module isomorphism.

As $[S, S_1] \leq A$, the map from S to S_1/A induced by commutation with α is a homomorphism: $[\alpha, xy] = [\alpha, y][\alpha, x]^y \equiv [\alpha, x][\alpha, y]$ modulo A . The kernel of this map contains S_1 , so we have an induced homomorphism ad_α , which is surjective by Lemma 4.13. As S/S_1 and S_1/A have the same rank as A , the kernel of ad_α is finite; since it is also T -invariant, it is trivial by Corollary 4.7. As α commutes with T , it also respects the T -module structure.

As S/S_1 is elementary abelian and S is of exponent 4, for any $x \in S$ we have $x^2 \in \Omega_1(S_1) = A$. Thus S/A is elementary abelian. Combining the T -module isomorphism given by ad_α with Proposition 4.10, we find that S/A splits as a T -module: $S/A = S_0/A \oplus S_1/A$.

We can now completely coordinatize S in terms of the base field K . Fix $x_0 \in S_0 \setminus A$, and set $x_1 = [\alpha, x_0]$, $x_2 = x_1^2$. Now $A \rtimes T \cong K_+ \rtimes K^\times$ for some algebraically closed field K of characteristic 2. We identify K_+ with $T \cup \{0\}$, and then we identify S as a set with $K_+ \times K_+ \times K_+$: (a, b, c) corresponds to $x_0^a x_1^b x_2^c$, where elements of T act by conjugation and $x_i^0 = 1$.

For $a_1, b_1, c_1, a_2, b_2, c_2 \in K$ we have

$$x_0^{a_1} x_1^{b_1} x_2^{c_1} x_0^{a_2} x_1^{b_2} x_2^{c_2} = (x_0^{a_1} x_0^{a_2})(x_1^{b_1} x_1^{b_2})([x_1^{b_1}, x_0^{a_2}] x_2^{c_1} x_2^{c_2}).$$

If we let $[x_1, x_0^t] = x_2^{g(t)}$ and apply Proposition 4.10 to S_0 and S_1 , we get the following formula:

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + \epsilon\sqrt{a_1 a_2} + \sqrt{b_1 b_2} + b_1 g(b_1^{-1} a_2)),$$

where ϵ is either 0 or 1, depending on whether S_0 is elementary abelian or homocyclic, respectively. Note that g is an additive map.

The associativity of the group law implies

$$(b + c)g((b + c)^{-1} a) = cg(c^{-1} a) + bg(b^{-1} a).$$

Letting $a = (b + c)x$ implies

$$bg(b^{-1} cx) = cg(c^{-1} bx);$$

hence,

$$g(yx) = yg(y^{-1} x).$$

In particular, if $x = y$, then $g(x^2) = xg(1)$. By taking square roots we conclude

$$g(x) = \sqrt{x}g(1).$$

We will show finally that $g(1) = 1$, in other words that $[x_0, x_1] = x_2$. We have

$$x_0^2 = (x_0^{-2})^\alpha = [(x_0^{-1})^\alpha]^2 = (x_1 x_0^{-1})^2,$$

or as S has exponent 4: $1 = x_1 x_0^{-1} x_1 x_0 = x_1^2 [x_0, x_1]^{-1}$, and our claim follows.

This shows that the structure of S is determined by the structure of S_0 and finishes the proof of the theorem apart from the calculation of α , which may be done directly, using $x_0^\alpha = x_0[x_0, \alpha] = x_0x_1^{-1}$:

$$\begin{aligned} (x_0^a x_1^b x_2^c)^\alpha &= (x_0^\alpha)^a (x_1^\alpha)^b (x_2^\alpha)^c \\ &= (x_0 x_1^{-1})^a (x_1^{-1})^b x_2^c \\ &= (x_0)^a (x_1 x_2)^a (x_1 x_2)^b x_2^c \\ &= (a, a, a)(0, b, b + c) \\ &= (a, a + b, a + b + c + \sqrt{ab}) \end{aligned}$$

■

Having handled the minimal nonabelian case, we assume that S is a counterexample of minimal rank to the statement of Theorem 4.1.

PROPOSITION 4.17. $2S_1 \leq Z(S)$; equivalently, $[S, S_1] \leq A$.

Proof. The equivalence of the two conditions is straightforward, as $A = \Omega_1(S_1)$ and S_1 is abelian.

As S is a counterexample of minimal rank, it follows from Lemma 4.11 that α , T , and S/A satisfy the assumptions of Theorem 4.1. By induction, we have the following three possibilities for S/A :

(i) S/A is abelian: In this case certainly $[S, S_1] \leq A$.

(ii) S/A is nonabelian and in part (ii) of Theorem 4.1, $\epsilon = 1$: The analysis in Theorem 4.16 shows that $S/A = S_0/A \cdot S_1/A$, where S_0/A is homocyclic of exponent 4, $2S_0/A = 2S_1/A$.

Fix $s_1 \in S_1$ and choose $s_0 \in S_0$ such that $s_0^2 = s_1^2$. Then $[S_1, s_1^2] = 1$ and $[S_0, s_1^2] = [S_0, s_0^2]$. As S_0/A is abelian, $[S_0, s_0] \leq A$ and $[S_0, s_0^2] = 1$; thus s_1^2 commutes with $S_0 S_1 = S$.

(iii) S/A is nonabelian and in part (ii) of Theorem 4.1, $\epsilon = 0$: In this case we will obtain a contradiction. The analysis in Theorem 4.16 shows that $S/A = S_0 A \cdot S_1/A$, where S_0/A is elementary abelian. Let $B/A = Z(S/A)$. B is a homocyclic group of exponent 4. We will show that every element of $S_0 \setminus A$ inverts B , which yields a contradiction by considering a triple x, y, xy of elements in $S_0 \setminus A$.

The image \bar{x} of x in S/A acts on S_1 . Let $X = \{[\bar{x}, s] : s \in S_1\}$. Then $XA/A = B/A$, by inspection in S/A , and the involution \bar{x} inverts the elements of X , as well as the elements of A . Thus \bar{x} inverts $XA = B$, as claimed. ■

COROLLARY 4.18. S/A is abelian.

Proof. As in the preceding proof, if S/A is not abelian, then by induction we have $S/A = S_0/A \cdot S_1/A$, where S_0/A and S_1/A are abelian. Furthermore, by the preceding proposition, these two factors commute. ■

The next proposition is a special case of a result given in [12], but for the reader's convenience we give the proof.

PROPOSITION 4.19. If S is not abelian, then $S \setminus S_1$ contains an involution.

Proof. If S has exponent 4, then our claim follows from Theorem 4.16. Assume that S has exponent greater than 4 and $I(S) = A^\times$. We will show that this implies S is abelian.

$S/Z(S)$ is an elementary abelian 2-group: if x, y are in S , then as S/A is abelian (Corollary 4.18), we compute $[x, y^2] = [x, y][x, y]^y = [x, y]^2 = 1$.

We claim that $S_1 = Z(S)$. If S_1 has exponent 4, then as S has exponent greater than 4, there is an element of S whose square lies in $S_1 - A$. Hence $S_1 \setminus A$ meets $Z(S)$, and as $(S_1/A)^\times$ is a single T -orbit, and $A \leq Z(S)$, we have $S_1 = Z(S)$ in this case. Now suppose the exponent of S_1 is greater than 4 and $Z(S) < S_1$; hence as $Z(S)$ is T -invariant, $Z(S) = 2S_1$. For $x \in S \setminus S_1$ we can solve $x^2 = s^2$ with $s \in S_1$; hence xs^{-1} has order at most 4 and lies in $S \setminus S_1$. Then $(xs^{-1})^2 = s_1^2$ with $s_1 \in 2S_1$, so $xs^{-1}s_1^{-1}$ is an involution, which is a contradiction. Thus in all cases we get $S_1 = Z(S)$.

As $(S/S_1)^\times$ is a single T -orbit, it will suffice to show now that for $x \in S \setminus S_1$ the conjugates of x under T commute with each other.

Fix $x \in S \setminus Z(S)$ and $i \in A^\times$. Define $g: K \rightarrow K$ by $[x, x^t] = i^{g(t)}$. Note that g depends only on xS_1 , and therefore g is additive (the action of T on S/S_1 is by multiplication on K_+). Furthermore, $g(1) = 0$. Working modulo S_1 , the equations $[\bar{x}, \bar{x}^t] = [\bar{x}^t, \bar{x}] = [\bar{x}, \bar{x}^{t^{-1}}]^t = i^{g(t^{-1})t}$ imply $g(t^{-1}) = t^{-1}g(t)$. Replacing t by $(t(t+1))^{-1}$ and using $(t(t+1))^{-1} = t^{-1} + (t+1)^{-1}$, we get (after simplification) $g(t^2) = g(1)t = 0$ and hence $g(t) = 0$ for any t in T . This implies that S is an abelian group after all, which is a contradiction. ■

PROPOSITION 4.20. For every involution $x \in S \setminus S_1$, $[x, x^\alpha] \neq 1$.

Proof. Let $x^\alpha = xs$ with $s = [x, \alpha] \in S_1$. If x and x^α commute, then s is an involution, hence $s \in A$. Accordingly, there is $s_1 \in S_1$ with $[s_1, \alpha] = s$ and thus $[xs_1, \alpha] = s^2 = 1$, $xs_1 \in A$, $x \in S_1$, a contradiction. ■

PROPOSITION 4.21. The exponent of $Z(S)$ is 2.

Proof. Suppose toward a contradiction that the exponent of $Z(S)$ is at least 4. As $Z(S)$ is a T -invariant subgroup of S_1 , it is homocyclic.

Let y be an involution in $S \setminus S_1$. Then $y^\alpha = yx$, where $x \in S_1$. As $[S, S_1] \leq A$, $[y, y^\alpha] = [y, x] \in A$. Let $s \in Z(S)$ be such that $s^2 = [y, x]$.

Now $1 = (y^\alpha)^2 = (yx)^2 = y^2x^2[x, y] = x^2s^{-2} = (xs^{-1})^2$, so $xs^{-1} \in A$ and $x \in Z(S)$. Thus $[y, y^\alpha] = 1$, and this contradicts the preceding proposition. ■

COROLLARY 4.22. *The exponent of S is 4.*

Proof. We know $S' \leq A$, so $S^2 \leq Z(S)$. ■

Proof of Theorem 4.1. Proposition 4.12 proves the theorem if S is abelian. The nonabelian case is handled by Corollary 4.22 and Theorem 4.16. ■

It is also useful to have the formula for commutation in terms of coordinates. This does not depend on the value of ϵ .

COROLLARY 4.23. *If S is nonabelian, then in the notation of Theorem 4.1 we have the following commutation formula:*

$$[(a, b, c), (a', b', c')] = (0, 0, \sqrt{ab' + a'b}).$$

5. SOLVABILITY OF M°

In this section we will prove

THEOREM 5.1. *If G is a simple K^* -group of even type of finite Morley rank with a weakly embedded subgroup M , then M° is solvable.*

We recall the notation established above: α is an involution such that $C_\alpha^\circ = H$ is of the form $L \times O(H)$ with $L \cong \text{PSL}_2(K)$, $H \cap M = (AT) \times O(H)$ with $AT \cong K_+ \rtimes K^\times$, a Borel subgroup of L . S is the connected component of a Sylow 2-subgroup of M and is $(\langle \alpha \rangle \times T)$ -invariant. $S_1 = [\alpha, S]$. We have $A = \Omega_1(S_1) = O_2(C_M(\alpha))$, except when $S = A$ and therefore $S_1 = 1$. In addition we set $M_1 = B(M)^{(\infty)}$.

LEMMA 5.2. $M^\circ = M_1\sigma(M^\circ)$.

Proof. $M^\circ/\sigma(M^\circ)$ is a product of simple algebraic groups of characteristic 2. By Fact 2.41, the Sylow 2-subgroups of M° cover the Sylow 2-subgroups of $M^\circ/\sigma(M^\circ)$. Hence, M_1 covers this quotient. ■

LEMMA 5.3. *Assume $O_2(M) = 1$. Then M° is solvable.*

Proof. As $O_2(M) = 1$, M° has no definable connected normal 2-subgroup. Thus $\sigma(M^\circ) = \sigma(M^\circ)^\circ$ is a 2^\perp -group (Fact 2.40), and by Fact 2.21 we have $[B(M), \sigma(M^\circ)^\circ] = 1$. Thus $M_1/Z(M_1)$ is a product of simple

algebraic groups, and by the theory of central extensions, Fact 2.22, M_1 is a central product of quasi-simple algebraic groups. As $M^\circ = M_1 O(M^\circ)$, we have $S \subseteq M_1$.

As $C_{M_1}(\alpha)^\circ$ is solvable, α must normalize each quasi-simple factor of M_1 , and hence $A = O_2(C_M(\alpha))$ meets each factor of M_1 . On the other hand, the connected group T normalizes each such factor and acts transitively on A^* , so if M_1 is nontrivial it consists of a single quasi-simple algebraic group. As $C_{M_1}(\alpha)^\circ$ is solvable, α is inner and hence acts on S like an element a of S . Thus $[a, S] = S_1$, which, however, contradicts all of the possibilities for the structure of S , apart from $S = A$. In this case $M_1 \cong \text{PSL}_2(K')$ for some algebraically closed field K' of characteristic 2, and $\langle \alpha \rangle \times T$ acts faithfully on M_1 via inner automorphisms; this can be seen by considering the action on A and bearing in mind that the action of α on M_1 is nontrivial. On the other hand M_1 contains no such subgroup, so we have a contradiction. ■

LEMMA 5.4. *If $O_2(M) \neq 1$ then $A \leq Z(B(M))$.*

Proof. The connected component of the centralizer of α in $Z(O_2(M))$ is a nontrivial subgroup of A . As $Z(O_2(M))$ is T -invariant, it contains A .

Let $B = \Omega_1(Z(O_2(M)))$. Then $[\alpha, B] \leq B \cap S_1 = A$, so $B \leq S_1$ in view of the structure of S and α . Thus $B = A$. It follows that A is normal in M° and hence is central in each conjugate of S and hence also in $B(M)$. ■

LEMMA 5.5. $[O_2(M), M_1] = 1$.

Proof. We may assume $O_2(M) \neq 1$. If $O_2(M) = S$ then $B(M)$ is solvable and $M_1 = 1$. As $O_2(M)$ is $(\langle \alpha \rangle \times T)$ -invariant the remaining possibility is that $A \leq O_2(M) \leq S_1$ and $O_2(M)$ is homocyclic with $A = \Omega_1(O_2(M))$.

It is easy to see that M_1 is generated by its definable connected 2^\perp -subgroups, since this holds modulo the solvable radical, and solvable connected groups split over their O_2 . So it will suffice now to show that each definable connected 2^\perp -subgroup X of $B(M)$ centralizes $O_2(M)$.

X centralizes $A = \Omega_1(O_2(M))$ and hence acts trivially on each section of the form $\Omega_i(O_2(M))/\Omega_{i-1}(O_2(M))$; hence by Corollary 2.45 X centralizes $O_2(M)$. ■

Proof of Theorem 5.1. We may now complete the proof of Theorem 5.1. We may assume $O_2(M) \neq 1$. As $[M_1, O_2(M_1)] = 1$, $\sigma(M_1)^\circ = O_2(M_1) \times O(M_1)$ (Facts 2.39 and 2.42) and $M_1/O(M_1)$ is a perfect central extension of a product of quasi-simple algebraic groups and hence has finite center by the theory of central extensions, Fact 2.22. Thus $O_2(M_1) = 1$, and in particular, $A \cap M_1 = 1$. On the other hand, α normalizes M_1 and therefore centralizes a nontrivial 2-subgroup of M_1 if $M_1 \neq 1$. This forces $M_1 = 1$. ■

We close this section with some useful consequences of the solvability of M° .

PROPOSITION 5.6. *If $i \in I(M^\circ)$ then $C_i \leq M$.*

Proof. As M° is solvable and connected, its unique Sylow 2-subgroup is also connected and hence coincides with S . We noted above that α cannot belong to S , so $\alpha \notin M^\circ$. Since α is an arbitrary involution of M whose centralizer does not lie in M , our claim follows. ■

PROPOSITION 5.7. *Let $g \in G \setminus M$. Then $M^g \cap I(M^\circ) = \emptyset$.*

Proof. Let $g \in G \setminus M$ and suppose $i \in M^g \cap I(M^\circ)$. Then i normalizes S^g . As S^g is of bounded exponent, Fact 2.14 implies that i centralizes an infinite subgroup of S^g . By Proposition 5.6, we conclude that $M \cap M^g$ contains an infinite 2-group, which contradicts weak embedding. ■

COROLLARY 5.8. *$I(G) = I_1 \sqcup I_2$, where I_1 is the set of involutions in G conjugate to an involution in M° and I_2 is the set of involutions in G conjugate to an involution in $M \setminus M^\circ$.*

Proof. Since M contains a Sylow 2-subgroup of G , $I(G) = I_1 \cup I_2$. Thus our claim is simply that no involution in M° is conjugate to an element of $M \setminus M^\circ$, which follows from Proposition 5.7. ■

COROLLARY 5.9. *M controls fusion of involutions in M° .*

6. THE THOMPSON RANK FORMULA

In finite group theory the Thompson order formula gives a useful computation of the order of a group having at least two conjugacy classes of involutions in terms of data that can be computed locally. In the study of groups of finite Morley rank of even type, an analogous computation gives the rank, rather than the order, and seems even more useful than in the finite case. We will refer to this as the Thompson rank formula. In particular, experience to date suggests that situations calling in the finite case for use of the Thompson transfer lemma can be handled in our case by using the Thompson rank formula. Since there is no analog of transfer in our context, this is extremely fortunate.

There is also a version of the Thompson rank formula for groups of odd type, but it is somewhat more technical, and it is less clear how broadly useful it will be. Here we will restrict ourselves to a description of the formula in groups of even type.

In this section we deal with groups of finite Morley rank of even type with no special hypotheses. In particular, we make no assumption regard-

ing the existence of a weakly embedded subgroup. We will make use of the following general principle.

LEMMA 6.1 [11, Exercise 14, p. 65]. *Suppose G is a group of finite Morley rank, and A and B are two definable subsets of G . If f is a definable function from A onto B , $B = \sqcup_i B_i$ is a finite partition of B into definable sets, and for $b \in B_i$ $\text{rk } f^{-1}(b) = r_i$ is constant, then $\text{rk } A = \max_i (r_i + \text{rk } B_i)$.*

Proof. Let $A_i = f^{-1}[B_i]$. Then $A = \sqcup_i A_i$, $\text{rk } A = \max_i \text{rk } A_i$, and $\text{rk } A_i = r_i + \text{rk } B_i$. ■

LEMMA 6.2. *Let G be a group of even type of finite Morley rank.*

(i) *If i, j are nonconjugate involutions then there is a unique involution in $d(\langle ij \rangle)$.*

(ii) *The function $f(i, j)$, which associates with each pair (i, j) of nonconjugate involutions the unique involution of $d(\langle ij \rangle)$, is definable.*

Proof. (i) By Fact 2.12 and the assumption that G is of even type, $d(\langle xy \rangle)$ contains at most one involution. By Fact 2.9 $d(\langle xy \rangle)$ contains at least one involution.

(ii) Let Φ be the collection of all formulas $\phi(x, y)$ in two variables. Let T_0 be the complete theory of G . Adjoin two constants i, j and let T be T_0 together with the axioms:

i. i, j are conjugate involutions.

ii. It is not the case that the set defined by $\phi(x, ij)$ is a group containing ij and a unique involution.

This theory is inconsistent, since in a model G^* , i and j would represent nonconjugate involutions, and $d(\langle ij \rangle)$ would be an ij -definable group containing ij and a unique involution. By the compactness theorem, there is a finite subset Φ_0 of Φ such that the corresponding fragment of T is inconsistent. We can associate with any element g of G the group H_g , defined as the intersection of all of the groups containing g , which are defined by a formula of the form $\phi(x, g)$ with $\phi \in \Phi_0$, with the empty intersection construed as G . Then for i, j nonconjugate involutions in G , H_{ij} contains ij and contains at most one involution; as $d(\langle ij \rangle) \leq H_{ij}$, H_{ij} contains the involution $f(i, j)$. The function associating (i, j) with H_{ij} is definable, and hence so is the map f . ■

PROPOSITION 6.3. *Let G be a group of finite Morley rank g and of even type. Let C_1, C_2 be two distinct conjugacy classes of involutions in G . Let $\theta: C_1 \times C_2 \rightarrow I(G)$ be the map defined by $\theta(x, y) \in I(d(\langle xy \rangle))$. For $i \in I(G)$ let $\rho(i) = \text{rk } \theta^{-1}(i)$. Let $c_l = \text{rk } C(x)$ for $x \in C_l$, $l = 1, 2$. If X_1, \dots, X_k is a*

definable partition of $\theta(C_1 \times C_2)$ into sets such that ρ is constant on each X_j , with value r_j , then

$$2g = c_1 + c_2 + \max_{1 \leq j \leq k} (r_j + \text{rk} X_j).$$

In particular, if G has finitely many conjugacy classes of involutions, then we may take the X_j to be the classes that are contained in $\theta(C_1 \times C_2)$, and writing c_j for $\text{rk} C(x)$, where $x \in X_j$, we get

$$g = c_1 + c_2 + \max_{1 \leq j \leq k} (r_j - c_j).$$

Proof. Here we use the formula $\text{rk} C_l = g - c_l$, and we apply the preceding remarks. Observe that if X_j is a conjugacy class, then r_j is indeed well defined. ■

This equation is very useful because it severely restricts the size of the group G in which these computations are being made. In some cases these restrictions yield rapid contradictions, and in other cases they will serve to “pin down” the structure of G . The main point is that the parameter r_j can be estimated in practice. Examples will be found later, beginning with the end of the next section.

7. THE NONABELIAN CASE

Our standing hypothesis in this section is as follows:

$$S \text{ is nonabelian with } [\alpha, S] = S.$$

We will first analyze the conjugacy classes of involutions and then use the information obtained to apply the Thompson rank formula to eliminate the possibility of having S nonabelian.

7.1. Involutions

We first study involutions in $M - M^\circ$ and then look at fusion in S . Recall that as M° is solvable and connected of even type, it has a unique Sylow 2-subgroup S , which is also connected, and is the connected component of any Sylow 2-subgroup of M . We will show that if S is nonabelian, then G has finitely many conjugacy classes of involutions (two represented in S , and one represented in $M \setminus M^\circ$), so that the Thompson rank formula applies in its simplest form.

We remind the reader of the notation introduced in Corollary 5.8:

Notation 7.1. I_1 is the set of involutions in G conjugate to an involution in M° . I_2 is the set of involutions in G conjugate to an involution in $M \setminus M^\circ$.

LEMMA 7.2. *For $s \in S$, if αs is an involution, then α is conjugate to αs under the action of S .*

Proof. We claim that $s \in S_1 = \{[\alpha, x] : x \in S\}$, which yields our claim. This requires a computation. Relative to our coordinatization of S we have $(a, b, c)^\alpha = (a, a + b, a + b + c + \sqrt{ab})$. From this it follows that the only elements of S inverted by α are the elements of S_1 . ■

We now fix a Sylow 2-subgroup R of M containing S and α , so $S = R^\circ$. Let $R_1/S = \Omega_1(Z(R/S)) > 1$. Recall that $C_\alpha^\circ = L \times O(C_\alpha)$, with $L = B(C_\alpha) \cong \text{PSL}_2(K)$. Moreover, since the arguments of the previous sections can be applied to any involution having the same “offending properties” as those of α , if β is another involution of $M \setminus M^\circ$ whose centralizer is not in M , then $C_\beta^\circ = L_\beta \times O(C_\beta)$, where $L_\beta = B(C_\beta) \cong \text{PSL}_2(K')$ with K' an algebraically closed field of characteristic 2. We also have $L_\beta \cap M = A_\beta \rtimes T_\beta$, which is a Borel subgroup of L_β with $O_2(M \cap L_\beta) = A_\beta$ and T_β a torus. Since S is the only Sylow 2-subgroup of M° , $A_\beta \leq S$, and Theorem 4.1 implies that under the standing hypothesis in this section on the structure of S , we have $\text{rk}(S) = 3\text{rk}(A_\beta)$. Thus if $L_\alpha \leq L_\beta$, we have equality.

LEMMA 7.3. $R = S \rtimes C_R(L)$.

Proof. Certainly R contains $S \rtimes C_R(L)$. Our claim is that $R \leq S \cdot C_R(L)$.

We show first that if $x \in R$ and $[\alpha, x] \in S$, then $x \in S \cdot C_R(L)$. Let $s = [\alpha, x]$. As $\alpha^x = \alpha s$ is an involution, the preceding lemma shows that $\alpha^x = \alpha^{s'}$ for some $s' \in S$. Adjusting x by $(s')^{-1}$, we may suppose that $[x, \alpha] = 1$. Then x normalizes $B(C_\alpha) = L$, so as the induced automorphism is inner, there is an involution i of L acting on L as x does. Then xi centralizes L , and as M is weakly embedded, it follows that $i \in M$; hence $i \in A \leq S$. Thus $x \in S \cdot C_R(L)$.

Fix $x \in R_1 \setminus S$. As $[x, \alpha] \in S$, we have $x = s\beta$ with $s \in S$, $\beta \in C_R(L)$, and $\beta^2 \in C_S(L) = 1$. Thus β is an involution centralizing L , i.e., $L \leq L_\beta$. By the remarks prior to this lemma, we have $L = L_\beta$. In particular, C_β is not contained in M . We may therefore suppose that $\alpha \in R_1$, and hence $[\alpha, R] \leq S$. Thus by our first claim, $R \leq S \cdot C_R(L)$. ■

We remark that actually $\beta = \alpha$ in the situation arising above, as we will see in the course of the following argument.

LEMMA 7.4. *The involutions of $M \setminus M^\circ$ belong to a single conjugacy class, and in fact the involutions of $R \setminus M^\circ$ are permuted transitively by S .*

Proof. As in the previous argument, we may suppose that $\alpha \in R_1$. Let $s\beta$ be an involution with $s \in S$ and $\beta \in C_R(L)$. Then $\beta^2 \in C_S(L) = 1$, so β is an involution centralizing L . We also have $[\alpha, \beta] \in C_S(L) = 1$, and thus α, β commute. It will suffice to prove that $\beta = \alpha$, as then Lemma 7.2 applies.

The remarks preceding Lemma 7.3 and the fact that $L \leq L_\beta$ imply $L = L_\beta$. In particular, we may use the same torus T in our analysis of α or β .

Recall that $S_1 = [\alpha, S]$ is the unique maximal connected normal $(\langle \alpha \rangle \times T)$ -invariant proper subgroup of S containing A . But $[\beta, S]$ is such a subgroup, so $[\beta, S] \leq S_1$, and by symmetry $[\beta, S] = S_1$. In particular, β inverts S_1 . If $\alpha\beta \neq 1$, then since $\alpha\beta$ centralizes L ; we find also that $\alpha\beta$ inverts S_1 , a contradiction. We conclude that $\alpha = \beta$. ■

So far we have worked mainly in M . We also need a little information about involutions in C_α .

LEMMA 7.5. 1. *The involutions in $C_\alpha - C_\alpha^\circ$ are of the form αi with $i \in I(L)$.*

2. $I_1 \cap C_\alpha = I(C_\alpha^\circ)$.

Proof. 1. If $\beta \in I(C_\alpha) - C_\alpha^\circ$, then β induces an inner automorphism of L . Then for some element $i \in L$ with $i^2 = 1$, βi centralizes L and β centralizes i . As βi centralizes L , βi lies in M . As in the proof of the previous lemma, we find $\beta i = \alpha$, $\beta = \alpha i$.

2. $I(C_\alpha^\circ) = I(L) \subseteq I_1$ since $A \leq M^\circ$. $I(C_\alpha) - C_\alpha^\circ \subseteq I_2$ by part (1). ■

Now we consider fusion in S .

LEMMA 7.6. *The involutions of A and $S - A$ are nonconjugate in G .*

Proof. As M controls fusion of involutions in M° , it suffices to show that A is normal in M . By the proof of Lemma 7.4, for any involution β of $M \setminus M^\circ$, A is the centralizer of β in S . This shows that A is normal in M . ■

LEMMA 7.7. *The involutions in S lie in exactly two conjugacy classes in G : $I(A)$ and $I(S) - S_1$.*

Proof. The involutions of A are permuted transitively by T . Thus in view of the preceding lemma, it will suffice to show that the involutions in $S \setminus A$ are all conjugate in M .

These involutions all lie in $S \setminus S_1$, where $S_1 = [\alpha, S]$. As T operates transitively on $(S/S_1)^\times$, it suffices to check that for at least one involution

i of $S \setminus S_1$, i is conjugate in M to any element of the form is with $s \in S_1$ inverted by i , or in other words, $s = [i, x]$ for some $x \in M$. We may take i to be $(1, b, 0)$, where $b^2 + b = \epsilon$, in terms of the explicit coordinatization given in Section 4. Then the elements inverted by i can be computed as those of the form $(0, b', c)$ with $b' = 0$ or 1 ; the same computation shows that $[i, S_1] = A$, and since $[i, \alpha] = (0, 1, 1 + t)$ and $[i, \alpha x] = [i, \alpha][i, x]$ for $x \in S_1$, our claim follows. ■

7.2. Application of the Thompson Rank Formula

We will now eliminate the possibility of having S nonabelian with $[\alpha, S] = S$. Thus we prove

THEOREM 7.8. *If G is a simple K^* -group of finite Morley rank of even type with a weakly embedded subgroup M that is not strongly embedded, then the connected component S of a Sylow 2-subgroup of G is abelian.*

We retain the notation I_1, I_2 from the previous subsection for the classes of involutions conjugate, respectively, to involutions in M° or $M - M^\circ$. θ will be used to denote the function defined in Proposition 6.3.

The Thompson rank formula will yield $\text{rk}G < \text{rk}M + \text{rk}S$ in this case. We first observe that this will yield a contradiction.

LEMMA 7.9. $\text{rk}G \geq \text{rk}M + \text{rk}S$.

Proof. If $g \in G - M$, then $M \cap S^g$ is finite; hence $\text{rk}G \geq \text{rk}(MS^g) = \text{rk}M + \text{rk}S^g = \text{rk}M + \text{rk}S$. ■

Notation 7.10. Set $k = \text{rk}A$.

Note that $k = \text{rk}K$ (where $L = \text{PSL}_2(K)$) (hence the notation) and $k = \text{rk}T$. We have $\text{rk}S = 3k$ when S is nonabelian.

We make a critical calculation for the application of the Thompson rank formula.

LEMMA 7.11. *Suppose S is nonabelian. Let C_1, C_2, C be three conjugacy classes of involutions with $C_1 \subseteq I_1$ and $C_2 \subseteq I_2$, let $i \in C$, and let $r = \text{rk}\{(x, y) \in C_1 \times C_2 : i \in d(\langle xy \rangle)\}$. If $i \in I_1$ then $r \leq 3k$, and if $i \in I_2$ then $r \leq 4k$.*

Proof. The first point is that x and y invert $d(\langle xy \rangle)$ and hence commute with any involution in that group; so we may restrict our attention to pairs of involutions (x, y) lying in C_i .

We deal first with the case $i \in I_1$, and we may suppose $i \in M^\circ$. Then by Proposition 5.6, $C_i \leq M$. Thus $x \in I(S)$ and $y \in \alpha S_1$. Furthermore, xy is a 2-element, so $d(\langle xy \rangle)$ is just the cyclic group generated by xy . The rank of $I(S)$ is $2k$, by computation in terms of the coordinatization, so the rank of

$I(S) \times \alpha S_1$ is $4k$. As each of the two conjugacy classes of involutions in S has rank at least k , and the rank of the fiber of θ is constant on each conjugacy class, no fiber can have rank more than $3k$.

Now suppose $i \in I_2$; we may take $i = \alpha$. By Lemma 7.5 $\text{rk}(I_2 \cap C_\alpha) = \text{rk}(I_1 \cap C_\alpha) = \text{rk}I(L) = 2k$. Thus $\text{rk}\theta^{-1}(\alpha) \leq 4k$. ■

Now we can apply the Thompson rank formula.

LEMMA 7.12. $\text{rk}G \leq \text{rk}M + 2\text{rk}A$.

Proof. Let C_1 be one of the two conjugacy classes of involutions contained in I_1 , and let C_2 be the conjugacy class of involutions conjugate to an element of $M \setminus M^\circ$. Let c_l ($l = 1, 2$) be the rank of the centralizer of an element of C_l . Choose C_1 to minimize c_1 . The Thompson rank formula gives

$$g = c_1 + c_2 + r - c,$$

where for some involution i we have $c = \text{rk}C_i$ and $r = \text{rk}\{(x, y) \in C_1 \times C_2 : i \in d(\langle xy \rangle)\}$.

By the previous lemma:

$$\text{If } i \in I_1 \text{ then } r \leq 3k; \text{ if } i \in I_2 \text{ then } r \leq 4k. \quad (*)$$

We may now conclude as follows.

If $i \in I_1$, then as $c_1 \leq c$ we have $\text{rk}G \leq c_2 + 3k$. We will show that $c_2 = \text{rk}C_\alpha \leq \text{rk}M - k$. $C_\alpha^\circ = L \times O(C_\alpha)$ with $O(C_\alpha) \leq M$, $\text{rk}L = \text{rk}(L \cap M) + k$, and hence $\text{rk}C_\alpha = \text{rk}(C_\alpha \cap M) + k$. But $\text{rk}(C_\alpha \cap S) = k = \text{rk}S - 2k$, and thus $\text{rk}(C_\alpha \cap M) \leq \text{rk}M - 2k$, $\text{rk}C_\alpha \leq \text{rk}M - k$. Thus $\text{rk}G \leq \text{rk}M + 2k$.

If $i \in I_2$, then $c_2 = c$ and $\text{rk}G = c_1 + r$. Here c_1 is no larger than $\text{rk}C_s$ for s an involution of $S \setminus S_1$. By Proposition 5.6, $C_s \leq M$. Since $C_s \cap (ST) = C_s \cap S$ has rank $2k$ and ST has rank $4k$, we find $\text{rk}C_s \leq \text{rk}M - 2k$ and $\text{rk}G \leq \text{rk}M + 2k$. ■

As pointed out at the outset, this implies $\text{rk}G < \text{rk}M + \text{rk}S$, a contradiction. Thus the hypothesis that S is nonabelian is untenable.

8. THE "DIAGONAL" ELEMENTARY ABELIAN CASE

Having eliminated the possibility of having S nonabelian where S is the connected component of any Sylow 2-subgroup of M , the two abelian possibilities remain. In this section we eliminate one of them, namely the *diagonal* case. This is the case where $S = E \times E^\alpha$ and $A = \{xx^\alpha : x \in E\}$.

The arguments of this section, together with the results of the previous section, will prove the following theorem:

THEOREM 8.1. *If G is a simple K^* -group of finite Morley rank of even type with a weakly embedded subgroup M that is not strongly embedded, then the connected component S of a Sylow 2-subgroup of G is homocyclic with $A = \Omega_1(S)$.*

Until the end of this section our standing hypothesis is that, on the contrary, S is elementary abelian with $S = E \times E^\alpha$ and $A = \{xx^\alpha : x \in E\}$. As in the previous sections, L will denote the copy of $\text{PSL}_2(K)$ in C_α .

LEMMA 8.2. *S has at least two conjugacy classes of involutions.*

Proof. Suppose toward a contradiction that $I(S)$ is a single conjugacy class. As M controls the fusion of involutions in M° (Corollary 5.9), $I(S) = i^M$ for any $i \in I(S)$. Since $\text{rk}(i^{M^\circ}) = \text{rk}(M^\circ) - \text{rk}(C_{M^\circ}(i)) = \text{rk}(M) - \text{rk}(C_M(i)) = \text{rk}(i^M) = \text{rk}(S)$ and S is connected, we have $i^M = i^{M^\circ}$. In particular, M° acts on $S \setminus \{1\}$ transitively. We will consider the action of $M^\circ/C_{M^\circ}(S)$ on S . This action is transitive on $S \setminus \{1\}$; thus S is $M^\circ/C_{M^\circ}(S)$ -minimal. This implies (Fact 2.38) that $M^{\circ'} \leq C_{M^\circ}(S)$. Hence, $TC_{M^\circ}(S)/C_{M^\circ}(S) \triangleleft M^\circ/C_{M^\circ}(S)$. Note also that A is a $TC_{M^\circ}(S)/C_{M^\circ}(S)$ -minimal subgroup of S . By Fact 2.37, there exists $x \in M$ such that $S = A \oplus A^x$ is isomorphic to a two-dimensional vector space over an algebraically closed field K whose additive subgroup is isomorphic to A . Being solvable and connected, $M^\circ/C_{M^\circ}(S)$ is conjugate to a subgroup of

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, c \in K^\times \text{ and } b \in K \right\}$$

(see [15, Theorem 17.6]). But this group has a one-dimensional invariant subspace, which contradicts the transitivity of the action of M° on $I(S)$. ■

Now we can prove Theorem 8.1.

Proof of Theorem 8.1. By Lemma 8.2 we know that S has at least two conjugacy classes of involutions. Let u be an involution conjugate to an element of A , and let v be an involution not conjugate to u , but conjugate to an element of $S \setminus A$, so that u and v do not commute (for example, take $u \in A$ and $v \notin M$). As u and v are not conjugate, they commute with a third involution w . We may suppose $w \in M$. If $C_w \leq M$, then $u, v \in S$ (Corollary 5.8), so they commute, a contradiction. If $C_w \not\leq M$, then we may suppose $w = \alpha$ and thus $v \in C_\alpha$. As v is not conjugate to u , $v \notin L$. In particular, $v \in C_\alpha \setminus C_\alpha^\circ$. v acts on L as an inner automorphism, thus v acts as an involution $i \in L$. Equivalently, $vi \in C(L)$, which forces $vi \in M$. Since v is conjugate to an involution in S , C_v is contained in the conjugate

of M containing v , say M^g (Proposition 5.6). This implies $i \in M^g$. In fact, both i and v are in M^{og} , as they are both conjugate to involutions of S (Corollary 5.8). So $vi \in I(M^{og})$. But $C_{vi} \not\leq M^{og}$, a contradiction to Proposition 5.6. ■

9. THE HOMOCYCLIC CASE

By Theorem 8.1 the connected component S of a Sylow 2-subgroup in a simple K^* -group G of finite Morley rank of even type with a weakly embedded subgroup M that is not strongly embedded is homocyclic, with $I(S) = I(A)$. Furthermore, the involutions of A are conjugate under the action of T . This is a picture somewhat reminiscent of the situation inside $\mathrm{PSL}_2(K)$, and, what is more to the point, very reminiscent of a stage that was reached in the analysis of groups with strongly embedded subgroups prior to their identification as $\mathrm{PSL}_2(K)$.

The Thompson rank formula is not very helpful here. If $k = \mathrm{rk} A$ then the Thompson rank formula yields the estimate $\mathrm{rk} G \leq \mathrm{rk} M + 3k$, which is certainly a nontrivial constraint but not tight enough to be really useful. Since at this point our configuration resembles a fairly advanced stage in the analysis of groups with strongly embedded subgroups, we model the rest of the argument on the methods used in [1]. Some variations on the arguments in that paper prove that G is a *split Zassenhaus group of characteristic 2*. Then the following fact yields a contradiction:

Fact 9.1 [10]. Let G be an infinite split Zassenhaus group of finite Morley rank of characteristic 2. Then $G \cong \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic 2.

We recall the terminology.

DEFINITION 9.2. A doubly transitive group G is said to be a *Zassenhaus group* if the stabilizer of any set of three distinct points is trivial. Let G_x denote a one-point stabilizer and $G_{x,y}$ denote a two-point stabilizer. G is said to be a *split Zassenhaus group* if $G_{x,y}$ has a normal complement in G_x . G is said to be (split) of characteristic 2 if this normal complement is a 2-group.

In this section we assume that G is tame. The next lemma and proposition occur in [1], although without a precise statement of their range of generality. We will take pains here to state more explicitly what is actually proved at each step. On the other hand we abbreviate the proofs, which are given in full in [1].

The notation M , α , S , $L \cong \mathrm{PSL}_2(K)$, A , T is to be understood as in previous sections, although for ease of comparison with [1] we will state

the hypotheses used in the next two lemmas more explicitly. On the other hand we abandon the previous convention $H = C_\alpha^\circ$. We will also fix an involution w of L inverting T .

LEMMA 9.3. *Let M be a solvable group of finite Morley rank and A a connected elementary abelian normal p -subgroup such that M° acts transitively on A . Then $M/C(A)$ acts sharply transitively on A .*

Proof. We follow the proof of Lemma 5.1 in [1]. By Facts 2.36 and 2.38 A can be identified with a finite-dimensional vector space over an algebraically closed field of characteristic p in such a way that M° induces scalars and M acts linearly. As M° acts transitively on A , A is one-dimensional and hence M induces scalars. ■

In our particular case this can also be expressed as follows: $M^\circ = C_{M^\circ}(A) \rtimes T$.

PROPOSITION 9.4. *Let H be a connected solvable group of even type and S its Sylow 2-subgroup (which is unique, normal, and connected by Fact 2.42). Suppose that S is homocyclic and H acts transitively on the involutions of S . Then $C_H(A)^\circ = S \times O(H)$, where $A = \Omega_1(S)$.*

Proof. By Fact 2.39, $C_H(A)^\circ = S \rtimes H_1$, where H_1 is a definable connected complement to S . As S is homocyclic abelian, the mapping $x \mapsto x^{2^i}$ is an H_1 -module isomorphism between $\Omega_{i+1}(S)/\Omega_i(S)$ and $\Omega_1(S)$. Therefore, H_1 centralizes $\Omega_{i+1}(S)/\Omega_i(S)$ for each $i \geq 1$. By Corollary 2.45, H_1 centralizes S . ■

We apply the preceding lemma with $H = M^\circ$. Hence $H_1 = O(M)$, and we get $C(A)^\circ = S \times O(M)$.

As in [1], a key objective is now core-killing: $O(M) = 1$. Once we achieve this, we will be able to carry out the remainder of the analysis very much along the lines of [1].

LEMMA 9.5. *M is a maximal proper definable subgroup of G .*

Proof. Suppose $M \leq H < G$ with H definable. Then H is a weakly embedded subgroup of G (Corollary 2.30), and hence as proved in Section 5, H° is solvable. Thus H° normalizes its Sylow 2-subgroup, which is also the Sylow 2-subgroup of M° . By the weak embedding condition on M (Corollary 2.29), $H = M$. ■

COROLLARY 9.6. *If X is a nontrivial definable normal subgroup of M then $N_G(X) = M$.*

Notation 9.7. Set $H = C(T)^\circ$.

LEMMA 9.8. $[T, O(M)] = 1$.

Proof. As $O(M)T$ is a 2^\perp -group, it is nilpotent (here we are using tameness). Therefore, the Sylow p -subgroups of T , which are divisible torsion p -groups, are central in $O(M)T$. But T is the definable closure of any one of these Sylow p -subgroups, by tameness. ■

LEMMA 9.9. $(H \cap M)^\circ = T \cdot O(M)$.

Proof. By Lemma 9.8, $T \cdot O(M) \leq (H \cap M)^\circ$.

Let $s \in (H \cap M)^\circ$. Since T acts transitively on A , for $a \in A$, we have $a^s = a^t$, where $t \in T$. This implies that $(H \cap M)^\circ \leq C(A)T$, and hence $(H \cap M)^\circ \leq (C(A)T)^\circ = C(A)^\circ T$. The conclusion follows from Proposition 9.4. ■

LEMMA 9.10. w and T are not in the same conjugate of M .

Proof. Suppose toward a contradiction that there exists $g \in G \setminus M$ such that $w \in M^g$ and $T \leq M^g$. Note that both w and T are in the connected component of M^g . In fact, T is connected and $w \in I(L) \cap M^g = A^g$. As w inverts T , for any $t \in T$, wt is an involution and is also in A^g . But $w \in I(A^g)$ as well, which contradicts that T does not have involutions. ■

LEMMA 9.11. For any nontrivial definable subgroup U of $O(M)$, $N_G(U)^\circ \leq M$. In particular, the connected component of the centralizer of any nontrivial element of $O(M)$ lies in M .

Proof. If there is a counterexample, take one in which U has maximal rank. As $S \leq N(U)$, the subgroup $M \cap N(U)^\circ$ is weakly embedded in $N(U)^\circ$, so $N(U)^\circ/O(N(U)) \cong \text{PSL}_2(K_1)$ for some algebraically closed field K_1 of characteristic 2 (Fact 2.48). Let $V = N_{O(M)}(U)^\circ$. As $[V, S] = 1$ and V is a 2^\perp -group, $V \leq O(N(U))$. On the other hand, $O(N(U)) \leq C(A) \leq M$, so $O(N(U)) \leq O(M)$ and $V = N_{O(N(U))}(U)$. Thus $N(U)^\circ \leq N(V)$, and by maximality $\text{rk} V = \text{rk} U$. As $O(M)$ is connected nilpotent, $U = O(M)$. But $N(O(M)) = M$. ■

LEMMA 9.12. If $O(M) \neq 1$ then $H = T \times O(M)$.

Proof. By Lemma 9.8, $O(M)$ centralizes T . It remains therefore to prove that if $O(M)$ is nontrivial, then $H \leq M$. We make use of the Weyl group element w inverting T , which acts on H .

If $C_H(w)$ is finite then by Fact 2.14 H is abelian and hence $H \leq N(O(M)) = M$. Suppose therefore that $C_H(w)^\circ$ is nontrivial. Note that w is conjugate to an involution of A (as this is a Sylow 2-subgroup of L). Thus $w \in I(M^{g^s})$ for some $g \in G \setminus M$ and $C_w \leq M^g$. Hence $C_H(w) \leq M^g$. We claim that $C_H(w)^\circ$ is a 2^\perp -group. Suppose toward a contradiction that $i \in I(C_H(w)^\circ)$. As $i \in H$, $T \leq C_i$ and therefore $T \leq M^g$ by Proposition 5.6, a contradiction of Lemma 9.10. Thus $C_H(w)^\circ$ is a 2^\perp -group.

Now $C_H(w)^\circ$ is an infinite, definable, 2^\perp -group contained in $C(A^g)$. It therefore is a definable subgroup of $O(M^g)$ (Proposition 9.4). By Lemma 9.11 $C((C_H(w)^\circ)^\circ) \leq M^g$. In particular $T \leq M^g$, yet another contradiction to Lemma 9.10. ■

PROPOSITION 9.13. $O(M)$ is abelian.

Proof. Suppose not. Then $O(M)' \neq 1$, and by Corollary 9.6, $M = N(O(M)')$. As w normalizes T , it normalizes H . Therefore, w normalizes $H' = O(M)'$ (Lemma 9.12), a contradiction of Corollary 9.6. ■

PROPOSITION 9.14. $O(M) = 1$.

Proof. Suppose $O(M) \neq 1$. We will show that w normalizes $O(M)$, which yields a contradiction of Corollary 9.6.

By Fact 2.15, $H = H^- C_H(w)$, where $H^- = \{t \in H: t^w = t^{-1}\}$. As H is abelian, H^- is a group. Let $X = C_H(w)$. We claim that $X = 1$. If not, then since X is a connected group, X is infinite. As $w \in M^{og}$ for some $g \in G \setminus M$, $C_w \leq M^g$ by Proposition 5.6. Therefore, $X \leq M^{og}$. As X centralizes w , $X \leq C(A^g)$ by Proposition 9.3, and thus $X \leq O(M^g)$. By Lemma 9.11, $C(X)^\circ \leq M^g$. But $T \leq C(X)$ as well, a contradiction of Lemma 9.10. Therefore $X = 1$, and w inverts H . In particular, w inverts $O(M)$. This finishes the proof. ■

COROLLARY 9.15. $Z(M^\circ) = 1$.

The next two lemmas are given in [1] in a slightly different setting as Corollary 4.6 and Lemma 6.1. We first define the *generalized centralizer* of an element of an arbitrary group:

DEFINITION 9.16. Let X be an arbitrary group and $x \in X$. The generalized centralizer of x in X is

$$C_X^*(x) = \{y \in X: x^y = x \text{ or } x^{-1}\}.$$

Note that $C_X^*(x)$ is a subgroup of X and $C_X(z)$ is of index at most 2 in $C_X^*(x)$.

LEMMA 9.17. Let i, j be involutions conjugate to elements of A and let $a \in G$ be centralized by i and inverted by j . Then $a^2 = 1$.

Proof. We may suppose $i \in A$. Then $a \in M$. Moreover, by Lemma 9.3, a centralizes A . We suppose toward a contradiction that $a^2 \neq 1$. Then $j \in C^*(a) \setminus C(a)$. Hence, j and i are not conjugate in $C^*(a)$. Therefore, there exists an involution $k \in C^*(a)$ that commutes with j and i . As k commutes with i , we have $k \in M$ by Proposition 5.6. There are two possibilities for k . If $k \in M^\circ$, then, as k is conjugate to the involutions in

A , Proposition 5.7 implies that $k \in A$. This implies that $j \in M$, and therefore by Proposition 5.7, $j \in A$ as well. But a centralizes A ; hence this case disappears.

It remains to handle the case where $k \in M \setminus M^\circ$. Since k centralizes i and j , which are two conjugate involutions with $i \in A$, k centralizes A and A^h , where A^h is a conjugate of A containing j (Lemma 9.3). Moreover, $k \in C(i^j)$, which implies k centralizes A^j , again by Lemma 9.3. Since $j \notin A$, $A^j \neq A$. Therefore, $A^j \not\leq M$. Therefore, $C_k^\circ = L \times O(C_k)$, where $L \cong \text{PSL}_2(K)$ with $\text{char}(K) = 2$. Since $L = \langle A, A^j \rangle$ and $A^h \leq C_k^\circ$, we have $A^h \leq \langle A, A^j \rangle$. On the other hand, as $A \leq C(a)$ and $j \in C^*(a)$, $A^j \leq C(a)$. Hence $A^h \leq C(a)$. But this forces $a^j = a$. This case vanishes as well. ■

LEMMA 9.18. *If $a \in M^\circ$ is a nontrivial element inverted by an involution $w \in G \setminus M$ that is conjugate to an element of A , then w inverts $C(a)^\circ$.*

Proof. It suffices to show that w centralizes only finitely many elements of $C(a)^\circ$. Suppose, on the contrary, that $U = C(a, w)^\circ$ is nontrivial. By the previous lemma $C(a)$ does not contain involutions that are conjugate to those in A . Hence U , which lies in the connected component of a conjugate of M containing w , is a 2^\perp -group. But $U \leq C(w) = C(A^g)$ (Lemma 9.3) and in fact $U \leq C(A^g)^\circ$, so by Propositions 9.4 and 9.14, $U \leq S^g$, a contradiction. ■

We can now complete the proof of Theorem 1.3 by repeating the arguments in the last three pages of [1]. We have the following information:

$M^\circ = S \rtimes T$ with S a Sylow 2-subgroup of G .

S is homocyclic.

If $A = \Omega_1(S)$, then $A \rtimes T \cong K_+ \rtimes K^\times$ with the natural action.

For $i \in A$, $C(i) \leq M$.

Lemmas 9.17, 9.18

We fix an involution i in A and we make the following definitions, following the lead of [13]:

$$\begin{aligned} T[w] &= \{m \in M^\circ : m^w = m^{-1}\} \\ X_1 &= \{w \in i^G \setminus M : T[w] \cap M^\circ = 1\} \\ X_2 &= \{w \in i^G \setminus M : T[w] \cap M^\circ \neq 1\}. \end{aligned}$$

As the remainder of the argument is identical to that given in [1], up to some terminological variations, we just record the sequence of steps as a

convenience for the reader. These are Lemmas 6.5–6.13 in [1]. We note that the term “involution” in [1] should be replaced by “involution conjugate to an involution in A ” in the present context.

Rank computations depending on the inclusion $C_G(i) \leq M$ yield

LEMMA 9.19. $\text{rk } X_2 = \text{rk } i^G$.

Lemma 9.18 yields

LEMMA 9.20. For $w \in X_2$, $T[w]$ is conjugate to T .

A delicate rank computation then yields

LEMMA 9.21. 1. $\text{rk } X_2 = \text{rk } C(T)S$.

2. $\text{rk } G = \text{rk } C(T) + 2\text{rk } S$.

A direct computation (again using $C(i) \leq M$) yields

LEMMA 9.22. $\text{rk } i^G M^\circ = \text{rk } G$.

The next lemma is proved by arguing that a counterexample would produce two disjoint sets of full rank in G , namely $i^G M^\circ$ and the union of the cosets ucM° , where in the notation of the lemma c varies over $C(a) \setminus M$ and u varies over S .

LEMMA 9.23. For $a \in T^\times$, $C(a) \leq M$.

COROLLARY 9.24. $\text{rk } G = \text{rk } T + 2\text{rk } S$.

One then verifies easily

LEMMA 9.25. 1. If $u_1 g m_1 = u_2 g m_2$ with $u_1, u_2 \in S$ and $m_1, m_2 \in M$, then $u_1 = u_2$ and $m_1 = m_2$.

2. For $g \in G \setminus M$, $\text{rk}(M^\circ g S) = \text{rk } G$.

Proof of Theorem 1.3. To prove Theorem 1.3, all that remains to be done is to eliminate the possibility of S being homocyclic. This is done as in [1] by showing that G is a split Zassenhaus group of characteristic 2 with M° the stabilizer of a point and S the normal complement to a 2-point stabilizer. The previous lemma gives $G = M^\circ \sqcup M^\circ w S$, with w the involution of L inverting T . It remains to decode this, as in the last few lines of [1]. The main point is that in the associated permutation representation, no nontrivial element stabilizes three points. Suppose therefore that $t \in T = M^\circ \cap (M^\circ)^w$, and that t stabilizes a third point, which is of the form uwM° with $u \in S$. Thus $uwM^\circ = tuwM^\circ = u^{t^{-1}}wM^\circ$, and thus by Lemma 9.25, $u = u^{t^{-1}}$. As T acts sharply transitively on A , it follows that either t is trivial as claimed, or u is trivial. ■

10. APPLICATIONS

Theorem 1.3 (equivalently, Theorem 1.4) is expected to be very useful in the classification of simple, tame, K^* -groups of finite Morley rank of even type. In this section, we prove some results that justify these expectations.

The first result that we will obtain using Theorem 1.3 is the classification of simple, tame, K^* -groups of finite Morley rank of even type in which the 2-local subgroups are solvable-by-finite. This classification follows from the following proposition:

PROPOSITION 10.1. *Let G be a K^* -group of finite Morley rank of even type with infinite 2-subgroups such that the 2-local subgroups are solvable-by-finite. If S is a Sylow 2-subgroup of G , then $N_G(S^\circ)$ is a weakly embedded subgroup.*

Proof. Let $M = N_G(S^\circ)$. By Corollary 2.29, it is enough to show that if U is a unipotent 2-subgroup of S (equivalently of S°), then $N_G(U) \leq M$. Let U be a counterexample of maximal rank to this statement. Clearly, $U < S^\circ$. Fact 2.17 implies that $|N_{S^\circ}(U): U| = \infty$. Let S_1 be a Sylow 2-subgroup of $N_G(U)$ and $U_1 = N_{S^\circ}(U)$. Since $N_G(U)^\circ$ is solvable (Fact 2.18), S_1° is the unique Sylow 2-subgroup of $N_G(U)^\circ$ (Fact 2.40). Therefore, $S_1^\circ \geq U_1^\circ$. If $S_1^\circ > U_1^\circ$, then $N_{S_1^\circ}(U_1^\circ) > U_1^\circ$, and thus there exists $g \in G \setminus M$, which normalizes U_1° . This contradicts the choice of U . Hence, $S_1^\circ = U_1^\circ$. Now let $g \in N_G(U) \setminus M$. g normalizes S_1° . But S_1° is a unipotent 2-subgroup of M that properly contains U , a contradiction to the choice of U . ■

COROLLARY 10.2. *If G is a simple, tame, K^* -group of finite Morley rank of even type whose 2-local subgroups are solvable-by-finite, then $G \cong \text{PSL}_2(K)$, where K is an algebraically closed field of characteristic 2.*

The next application of Theorem 1.3 is the elimination of cores of 2-local subgroups in simple, tame, K^* -groups of finite Morley rank of even type. This result requires more preparation. First we prove a version of Thompson's $A \times B$ -lemma for groups of finite Morley rank. This will later be used to prove a K -group statement.

The proof of Thompson's $A \times B$ -lemma that we will give is a translation of the arguments in [24].

LEMMA 10.3 (cf. [24, (1.13), p. 8]). *Let $G = HK$ be a group of finite Morley rank where H is a definable normal π -subgroup and K is a definable π^\perp -subgroup. Then $[H, K, K] = [H, K]$.*

Proof. By Corollary 2.7, $[H, K]$ and $[H, K, K]$ are definable subgroups of G . Let $N = [H, K]K$. Then N is definable by Corollary 2.8. We claim that N is the smallest normal definable subgroup of G such that G/N is a π -group. Let M be a definable normal subgroup of G such that G/M is a π -group. Since $KM/M \cong K/K \cap M$ and KM/M is a π -group while

$K/K \cap M$ is a π^\perp -group (Fact 2.13), these two quotient groups are trivial. Therefore, $K \leq M$. But $M \triangleleft G$. Therefore, $N \leq M$. This proves our claim, which implies that N is a definably characteristic subgroup of G .

Now we carry out the same argument with $[H, K, K]$ instead of $[H, K]$ to show that $N_1 = [H, K, K]K$ is the smallest definable normal subgroup of N such that N/N_1 is a π -group. Since N_1 is definably characteristic in N , which is definably characteristic in G , $N_1 \triangleleft G$. But $K \leq N_1$, and hence $N_1 = N$. Therefore, $[H, K, K] = N_1 \cap [H, K] = N \cap [H, K] = [H, K]$. ■

We state the $A \times B$ -lemma in two equivalent forms:

PROPOSITION 10.4 (cf. [24, (1.15), (1.15)']). *Let G be a group of finite Morley rank. The following are equivalent:*

(i) *Let A be a definable p^\perp -subgroup of G . Let B be a definable p -subgroup of $C_G(A)$. Suppose $A \times B$ normalizes a definable p -subgroup P . If A centralizes $C_p(B)$ then A centralizes P .*

(ii) *Let Q be a definable p -subgroup of G and U be a definable subgroup of Q such that $C_Q(U) \leq U$. Suppose A is a definable p^\perp -subgroup of G that normalizes Q and centralizes U . Then A centralizes Q .*

Proof. We first prove (i) implies (ii). Let U and Q be as in (ii). The group $A \times U$ normalizes Q . By the assumption A centralizes $C_Q(U)$. Therefore we can apply (i) with $B = U$ and $P = Q$ and conclude that A centralizes Q .

Now we prove (ii) implies (i). By considering the semidirect product $P \rtimes (A \times B)$, we may assume that $B \cap P = 1$. In particular, $N_P(B) = C_p(B)$. Let $Q = BP$. Q is a definable p -group. If $U = N_p(B)B$ then $C_Q(U) \leq C_Q(B) \leq N_Q(B) = N_p(B)B = U$. An application of (ii) proves the result. ■

We will prove Proposition 10.4 (ii). First a special case:

LEMMA 10.5 (cf. [24, (1.16)]). *Let G be a group of finite Morley rank. Let X be a definable π -subgroup of G and Y be a normal definable subgroup of X such that $C_X(Y) \leq Y$. Suppose that A is a definable π^\perp -subgroup of G that normalizes X and centralizes Y . Then A centralizes X .*

Proof. As $[A, Y] = 1$ and $Y \triangleleft X$, we have $[X, Y, A] = 1$. Clearly, $[Y, A, X] = 1$. By the 3-subgroup lemma, $[A, X, Y] = 1$. Therefore, $[A, X] \leq C_X(Y) \leq Y$. This implies $[A, X, A] = [X, A, A] \leq [Y, A] = 1$. Lemma 10.3 implies that $[A, X] = [X, A, A] = 1$. ■

Now we prove the $A \times B$ -lemma:

PROPOSITION 10.6 (cf. [24]). *Let G be a group of finite Morley rank whose Sylow p -subgroups are nilpotent-by-finite. Let Q be a definable p -subgroup of*

G and U be a definable subgroup of Q such that $C_Q(U) \leq U$. Suppose A is a definable p^\perp -subgroup of G that normalizes Q and centralizes U . Then A centralizes Q .

Proof. Let $Y = C_Q(A)$. Then $U \leq Y$, and we have $C_Q(Y) \leq C_Q(U) \leq U \leq Y$. Let $X = N_Q(Y)$. By Lemma 10.5, A centralizes X . Therefore, $X \leq Y$. But Q is a nilpotent-by-finite p -group and thus satisfies the normalizer condition. Therefore, $X = Y = Q$. ■

For the elimination of cores in 2-local subgroups we will need to begin with the algebraic case, in which case it is folklore, based on the $A \times B$ -lemma. For the convenience of the reader we go through the algebraic case first.

Fact 10.7 ([6], as exposed in [15, Corollary 30.3 A]). Let G be a reductive algebraic group and let U be a closed unipotent subgroup of G . Then $N_G(U)$ is contained in a parabolic subgroup $\mathcal{P}(U)$ of G such that $U \leq R_U(\mathcal{P}(U))$, where R_U denotes the unipotent radical.

The following fact seems to have a long history. Our reference is [5], where the fact is proved for finite groups, but the argument works for algebraic groups as well.

Fact 10.8 [5, (47.5)]. Let G be a semisimple algebraic group and P be a parabolic subgroup. Then $C(R_U(P)) \leq R_U(P)$.

LEMMA 10.9. Let H be a group of finite Morley rank and P be a definable 2-subgroup of H . Then $O(N_H(P)) = O(C_H(P))$.

Proof. Since $O(N_H(P)) \triangleleft N_H(P)$ and $O(N_H(P)) \cap P = 1$, $O(N_H(P)) \leq O(C_H(P))$. On the other hand, $O(C_H(P))$ is a definably characteristic subgroup of $C_H(P)$, which is a normal subgroup of $N_H(P)$. Therefore, $O(C_H(P)) \leq O(N_H(P))$. ■

LEMMA 10.10. Let G be a simple algebraic group over an algebraically closed field of characteristic 2 and P be a definable 2-subgroup. Then $O(N_G(P)) = 1$.

Proof. For notational simplicity we restrict ourselves to simple algebraic groups that are simple as abstract groups, although the lemma is still true with finite centers. Let G and P be as in the statement of the lemma. Suppose toward a contradiction that $O(N_G(P)) \neq 1$. By Fact 10.7, $N_G(P)$ lies in a parabolic subgroup H of G .

To avoid introducing new notation, until the end of this proof O_2 will be used to denote the largest closed normal (not necessarily connected) subgroup of a given closed group in G . In this notation, $R_U(H) = O_2(H)^\circ$. By Lemma 10.9, $O(N(P)) = O(C(P))$. We consider the action of $P \times$

$O(N(P))$ on $O_2(H)^\circ$. We have $C_{O_2(H)}(P) \leq N_{O_2(H)}(P) \leq O_2(N(P))$. But $[O(N(P)), O_2(N(P))] = 1$. Hence, $O(N(P))$ centralizes $C_{O_2(H)}(P)$. By Thompson's $A \times B$ -lemma, $O(N(P)) \leq C(O_2(H)^\circ)$. But $C(O_2(H)^\circ) \leq O_2(H)^\circ$ by Fact 10.8, a contradiction. ■

PROPOSITION 10.11. *Let H be a K -group of even type with $P \leq H$ a definable 2-subgroup. Then $O(N_H(P)) \leq O(H)$.*

Proof. By Lemma 10.9, $O(N_H(P)) = O(C_H(P))$. Let $O = O(C_H(P))$.

Using Facts 2.40 and 2.47, we may assume H has infinite Sylow 2-subgroups.

We make a case division and first assume H° is solvable. Let S be a Sylow 2-subgroup of H . By Fact 2.42, S° is the only Sylow 2-subgroup of H° . In this case it is enough to prove that $O \leq C(S^\circ)$. In fact, $C(S^\circ)^\circ = Z(S^\circ) \oplus X$, where X is a connected definable 2^\perp -group. Hence $O \leq C(S^\circ)$ implies $O \leq O(C(S^\circ))$, and since $O(C(S^\circ)) \triangleleft H^\circ$, we have $O \leq O(C(S^\circ)) \leq O(H)$.

We prove $O \leq C(S^\circ)$ using the $A \times B$ -lemma. The group $O \times P$ normalizes S° . We are done if we can show that $[O, C_{S^\circ}(P)] = 1$. But O normalizes $C_{S^\circ}(P)$ and $O \triangleleft C_H(P)$. Therefore, $[O, C_{S^\circ}(P)] \leq O \cap C_{S^\circ}(P) = 1$ and the $A \times B$ -lemma implies $O \leq C(S^\circ)$.

Now we assume H° is nonsolvable. By Fact 2.24 $H^\circ/\sigma(H^\circ)^\circ$ is isomorphic to the central product of a finite number of quasi-simple algebraic groups. We assume first that $\sigma(H^\circ)^\circ$ has a nontrivial Sylow 2-subgroup, say U . Then U is connected (Fact 2.40). By Fact 2.42, U is the unique Sylow 2-subgroup of $\sigma(H^\circ)^\circ$. The group $O \times P$ normalizes U . Moreover, $[O, C_U(P)] \leq O \cap C_U(P) = 1$. Therefore, by the $A \times B$ -lemma, O centralizes U .

We claim that $O \leq \sigma(H^\circ)^\circ$. To prove this claim we analyze $\bar{H} = H/U$. Let $H_1/U = C_{\bar{H}}(\bar{P})$. Then $H_1 = \{h \in H : [h, P] \leq U\}$. In particular, $H_1 \leq N_H(PU)$. Therefore, H_1 normalizes $O(N_H(PU))$. Note that this last subgroup is equal to $O(C_H(PU))$ by Lemma 10.9. Since $C_H(PU) \triangleleft C_H(P)$ and O centralizes both P and U , $O = O(C_H(PU))$. Therefore, H_1 normalizes O , which implies that H_1/U normalizes OU/U . As a result, $\bar{O} \leq O(C_{\bar{H}}(\bar{P}))$. By induction on the rank and degree of H , $\bar{O} \leq O(\bar{H})$. $O(\bar{H}) = O_1/U$ for some definable, connected, solvable, normal subgroup of H . Hence, $O \leq \sigma(H^\circ)^\circ$. The arguments of the first part applied to U and O imply that $O \leq O(\sigma(H))$. But $O(\sigma(H)) = O(H)$.

The above paragraphs handle the case in which $\sigma(H^\circ)^\circ$ contains an involution. We can therefore assume that $\sigma(H^\circ)^\circ = O(H)$. This implies that $H^\circ/O(H)$ is the central product of quasi-simple algebraic groups over fields of characteristic 2. If $O \leq O(H)$, then there is nothing to do; thus we assume that $O \not\leq O(H)$ and first show that we can find $t \in I(H)$ such

that $O(C_t) \not\leq O(H)$. Let S be a Sylow 2-subgroup of H that contains P . Let $t \in I(Z(S)^\circ)$. As $Z(S)^\circ$ is a connected, definable, 2-group of bounded exponent in $C_H(P)$, it centralizes O by Fact 2.21. Hence, $O \leq C_t$. Note also that since $Z(S)^\circ$ contains infinitely many involutions (Fact 2.20) and $H^\circ/O(H)$ has finite center, t can be chosen so that $C_t < H$. By induction on the rank and degree of H , $O(C_{C_t}(P)) \leq O(C_t)$. But $O \leq O(C_{C_t}(P))$. Hence, if $O \not\leq O(H)$, then $O(C_{H/O(H)}(tO(H))) \neq 1$. This contradicts Lemma 10.10. ■

COROLLARY 10.12. *Let H be a K^* -group of even type and P, Q two definable 2-subgroups with $P \neq 1$, Q unipotent, and Q centralizing P . Then $O(N_H(P)) \leq O(N_H(Q))$. In particular, if $P, Q \in \mathcal{Z}(H)$ and P, Q lie in the same connected component of $\mathcal{Z}(H)$, then $O(N_H(P)) = O(N_H(Q))$.*

Proof. It suffices to prove the first claim. Let $X = O(N_H(P))$. As Q normalizes P , Q normalizes X and hence centralizes it by Fact 2.21. Furthermore, P centralizes X , so $X \leq C_{C_H(Q)}(P) \leq C_{N_H(Q)}(P)$, and as $X \triangleleft N_H(P)$, we have $X \leq O(C_{N_H(Q)}(P)) \leq O(N_H(Q))$, the last by Proposition 10.11 applied to the K -group $N_H(Q)$. ■

THEOREM 10.13. *Let G be a simple, tame, K^* -group of finite Morley rank of even type. Then $O(N_G(P)) = 1$ for any definable 2-subgroup P of G .*

Proof. Suppose toward a contradiction that there is a definable 2-subgroup P of G such that $O(N(P)) \neq 1$. Then P is nontrivial. Let Q be a nontrivial unipotent subgroup commuting with P . By the previous corollary, $O(N(Q))$ is nontrivial.

Let $X = O(N(Q))$. We show that $N(X)$ is weakly embedded in G . Evidently $N(X) < G$. Let S be the connected component of a Sylow 2-subgroup containing Q . By the preceding corollary, $X = O(N(U))$ for any nontrivial unipotent $U \leq S$. Hence $N(U) \leq N(X)$ and $N(X)$ is weakly embedded.

By Theorem 1.3 G is a simple algebraic group of even type (specifically, $SL(2, K)$), and so we have a contradiction. ■

Our final application of Theorem 1.3 is the classification of simple, tame, K^* -groups of finite Morley rank of even type with *finite strongly closed abelian subgroups*. An important corollary of this will be Glauberman's Z^* -theorem for simple, tame, K^* -groups of finite Morley rank of even type.

DEFINITION 10.14. Let G be a group of finite Morley rank and S be a Sylow 2-subgroup of G . A nontrivial definable subgroup A of S is said to be strongly closed in S if whenever $x^g \in A$ for $x \in S$ and $g \in G$, we have $x \in A$.

We intend to analyze groups of finite Morley rank with infinite strongly closed abelian subgroups in a subsequent paper. In this section we will only consider finite strongly closed abelian subgroups. Nevertheless, we give some general properties of strongly closed abelian subgroups below.

LEMMA 10.15. *Let G be a group of finite Morley rank and S be a Sylow 2-subgroup of G . A definable abelian subgroup A of S has the following properties if it is strongly closed in S :*

- (i) A is strongly closed in any Sylow 2-subgroup that contains it.
- (ii) If S is a Sylow 2-subgroup that contains A , then $A \triangleleft S$.
- (iii) $N(A)$ controls fusion in A .
- (iv) Any nontrivial, definable, $N(A)$ -normal subgroup of A is strongly closed.

Proof. (i) Let S^h be another Sylow 2-subgroup that contains A . A^h is strongly closed in S^h . Since $A = (A^h)^{h^{-1}}$, $A \leq A^h$. Degree and rank arguments imply $A = A^h$.

(ii) This is clear from the definition and from part (i).

(iii) Let $x \in A$ and $g \in G$ such that $x^g \in A$. $C(x^g) \geq \langle A, A^g \rangle$. Let S_1 and S_2 be two Sylow 2-subgroups of $C(x^g)$ such that $S_2 = S_1^c$ with $c \in C(x^g)$ and $A \leq S_1$, $A^g \leq S_2$. $A^c \leq S_2$ as well and A^c is strongly closed in S_2 . If $S_1 \leq T_1$, where T_1 is a Sylow 2-subgroup of G , then $S_2 \leq T_1^c$ and A^c is strongly closed in T_1^c . But by part (i), A^g is strongly closed in T_1^c as well. Hence, $gc^{-1} \in N(A)$ and $x^{gc^{-1}} = x^g$.

(iv) Let B be a nontrivial, definable, $N(A)$ -normal subgroup of A . Let $x \in S$, where S is as in the statement of the lemma such that $x^g \in B$ for some $g \in B$. This implies $x \in A$. But, by part (iii), there exists $h \in N(A)$ such that $x^h = x^g$. Therefore, $x \in B^{h^{-1}} = B$, as B is an $N(A)$ -normal subgroup of A . ■

The analysis of simple, tame, K^* -groups of finite Morley rank of even type with finite strongly closed abelian subgroups is different from the analysis of those that have infinite strongly abelian closed subgroups, although in both cases weakly embedded subgroups are the main tool. To prove our last result we need the following lemma as well as Proposition 2.33.

LEMMA 10.16. *Let H be a tame K -group of finite Morley rank of even type with a finite strongly closed abelian subgroup A . Then A centralizes $B(H)$.*

Proof. If H° is solvable, then $B(H)$ is the only Sylow 2-subgroup of H° (Fact 2.42). Hence, part (ii) of Lemma 10.15 implies that $B(H)$ normalizes A . Since $B(H)$ is connected, $[A, B(H)] = 1$.

Now we assume that H° is nonsolvable. We will use $\bar{}$ -notation to denote quotients by $\sigma(H^\circ)$. By Fact 2.24, $H^\circ/\sigma(H^\circ) = \bigoplus \bar{X}_i$, where the \bar{X}_i are algebraic groups over algebraically closed fields of characteristic 2. A acts on $H^\circ/\sigma(H^\circ)$.

Let S be a Sylow 2-subgroup of H that contains A . By Lemma 10.15 (ii), $[S^\circ, A] = 1$. Using Fact 2.41 (ii), one can show that \bar{S}° is a Sylow 2-subgroup of \bar{H}° . In particular, $\bar{S}^\circ = \bigoplus \bar{S}_i$, where each \bar{S}_i is some Sylow 2-subgroup of \bar{X}_i . Since A centralizes S° , for $a \in A$, $\bar{S}_i \leq \bar{X}_i \cap \bar{X}_i^a$. This implies that A actually normalizes each component in the direct sum of the \bar{X}_i . So we may assume that \bar{H}° is a simple algebraic group.

The fact that A centralizes a Sylow 2-subgroup of \bar{H}° implies that it acts on \bar{H}° as a group of inner automorphisms. The same is true for the action of \bar{A} on \bar{H}° . We will show that this last action is actually trivial. Let $\bar{a} \in \bar{A}^\times$. If the action of \bar{a} on \bar{H}° is not trivial, then there exists \bar{b} , which can be taken to be in $Z(\bar{S}^\circ)$ such that the action of \bar{a} on \bar{H}° is the same as that of \bar{b} . $N_{\bar{H}^\circ}(\bar{S}^\circ)$ is a Borel subgroup of \bar{H}° . Let T be the full preimage of a torus of this Borel subgroup and $\bar{t} \in \bar{T}$. Then the action of \bar{a}^t on \bar{H}° and that of \bar{b}^t are the same. In particular, $\bar{t}^{\bar{a}} = \bar{t}^{\bar{b}}$ and $\bar{a}^t = \bar{a}[\bar{b}, \bar{t}]$. Thus $[\bar{A}, \bar{a}^t] = [\bar{A}, \bar{a}[\bar{b}, \bar{t}]] = 1$ as $[\bar{b}, \bar{t}] \in \bar{S}^\circ$. This argument implies that A and A^t commute modulo $\sigma(H^\circ)$. Hence A^t normalizes $A\sigma(H^\circ)$. Since A^t is in a Sylow 2-subgroup of $A^tA\sigma(H^\circ)$, it normalizes some A^x where $x \in \sigma(H^\circ)$. Thus $A^t = A^x$ and $tx^{-1} \in N(A)$. This implies $t \in N(A)\sigma(H^\circ)$. We conclude $T \leq N(A)\sigma(H^\circ)$. Thus \bar{T} normalizes \bar{A} . But $\bar{T} = \bar{T}^\circ$, and this last group centralizes \bar{A} . \bar{A} is a group of inner automorphisms of \bar{H}° that centralizes a Borel subgroup, which implies that \bar{A} centralizes \bar{H}° .

Since $\bar{H}^\circ = \bar{B}(\bar{H}^\circ)$, the above paragraph implies that $[A, B(H)] \leq \sigma(H^\circ)$. Thus, if S_1 is a Sylow 2-subgroup of H , then A normalizes $S_1^\circ\sigma(H^\circ)$, which is a solvable group. Hence, by the conclusion of the first paragraph, A centralizes $B(S_1^\circ\sigma(H^\circ))$, and in particular, it centralizes S_1° . We conclude that $[A, B(H)] = 1$. ■

COROLLARY 10.17. *Let H be a tame K -group of even type such that $O(H) = 1$. Assume A is a finite strongly closed abelian subgroup of H . Then $[A, H^\circ] = 1$.*

Proof. We first assume that H° is solvable. Since $O(H) = 1$, $B(H) \neq 1$ unless $H^\circ = 1$. By Fact 2.42, $B(H)$ is the only Sylow 2-subgroup of H° . We therefore leave $[C_H(B(H)), H] \leq C_H(B(H))$, and thus $[A, H^\circ] \leq (C_H(B(H)))^\circ = Z(B(H))^\circ(O(H) = 1)$. This implies that A and A^h lie in the same Sylow 2-subgroup for any $h \in H^\circ$, and hence by strong closure H° normalizes and thus centralizes A .

If H° is nonsolvable, then Fact 2.24 and the fact that H is of even type imply that $H^\circ = B(H)\sigma(H^\circ)$. But both of the subgroups in this factorization are centralized by A . ■

THEOREM 10.18. *Let G be a simple, tame, K^* -group of finite Morley rank of even type with a strongly closed abelian subgroup A . Then A is infinite.*

Proof. We suppose toward a contradiction that A is finite. We will find in G a weakly embedded subgroup whose existence will be detected using Proposition 2.33. Let $M = C(A)$. Note that M contains infinite 2-subgroups, since A is centralized by the connected component of any Sylow 2-subgroup that contains it (Lemma 10.15 (ii)) and G has infinite Sylow 2-subgroups (Fact 2.46). Let U be a unipotent 2-subgroup of M and $H = N(U)$. As U centralizes A , $A \leq H$. Theorem 10.13 and Corollary 10.17 imply that A centralizes H° . Thus the connected component of the normalizer of any unipotent 2-subgroup of M is contained in M . Now Proposition 2.33 implies that G has a weakly embedded subgroup. ■

We conclude with the most important special case of Theorem 10.18.

COROLLARY 10.19 (Glauberman's Z^* -Theorem). *Let G be a simple, tame, K^* -group of finite Morley rank of even type. Let x be an involution in a Sylow 2-subgroup S of G . Then $x^G \cap S \neq \{x\}$.*

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REFERENCES

1. T. Altinel, Groups of finite Morley rank with strongly embedded subgroups, *J. Algebra* **180** (1996), 778–807.
2. T. Altinel, A. Borovik, and G. Cherlin, Groups of mixed type, *J. Algebra* **192** (1997), 524–571.
3. T. Altinel and G. Cherlin, On central extensions of algebraic groups, *J. Symbolic Logic*, to appear.
4. T. Altinel, G. Cherlin, L.-J. Corredor, and A. Nesin, A Hall theorem for ω -stable groups, *J. London Math. Soc.*, to appear.
5. M. Aschbacher, “Finite Group Theory,” Cambridge Studies in Advanced Mathematics, no. 10, Cambridge Univ. Press, Cambridge, 1988.
6. A. Borel and J. Tits, Eléments unipotents et sous-groupes paraboliques de groupes réductifs, *Invent. Math.* **12** (1971), 95–104.

7. A. Borovik, Simple locally finite groups of finite Morley rank and odd type, in "Proceedings of NATO ASI on Finite and Locally Finite Groups, Istanbul, Turkey, 1994," pp. 247–284, NATO Adv. Sci. Inst.
8. A. Borovik and A. Nesin, On the Schur–Zassenhaus theorem for groups of finite Morley rank, *J. Symbolic Logic* **57** (1992), 1469–1477.
9. A. Borovik and A. Nesin, Schur–Zassenhaus theorem revisited, *J. Symbolic Logic* **59** (1994), 283–291.
10. A. V. Borovik, M. DeBonis, and A. Nesin, On a class of doubly transitive ω -stable groups, *J. Algebra* **165** (1994), 245–257.
11. A. V. Borovik and A. Nesin, "Groups of Finite Morley Rank," Oxford Univ. Press, Oxford, 1994.
12. M. Davis and A. Nesin, Suzuki 2-groups of finite Morley rank (or over quadratically closed fields), preprint.
13. M. DeBonis and A. Nesin, On CN groups of finite Morley rank, *J. London Math. Soc. (2)* **50** (1994), 532–546.
14. L. Fuchs, "Infinite Abelian Groups," Academic Press, San Diego, 1970.
15. J. E. Humphreys, "Linear Algebraic Groups," 2nd ed., Springer Verlag, Berlin/New York, 1981.
16. E. Jaligot, Groupes de type mixte, *J. Algebra*, to appear.
17. P. Landrock and R. Solomon, "A Characterization of the Sylow 2-Subgroups of $PSU(3, 2^n)$ and $PSL(3, 2^n)$," Technical Report 13, Aarhus Universitet, Matematisk Institut, January 1975.
18. A. Macintyre, On ω_1 -categorical theories of abelian groups, *Fund. Math.* **70** (1971), 253–270.
19. V. D. Mazurov, On centralizers of involutions in simple groups, *Math. USSR Sbornik* **22** (1974), 535–546.
20. A. Nesin, On sharply n -transitive superstable groups, *J. Pure Appl. Algebra* **69** (1990), 73–88.
21. S. Outing, "Stealing" images on the Internet, *E & P Interactive*, Wednesday, November 8, 1995.
22. B. Poizat, "Groupes Stables," Nur Al-Mantiq Wal-Ma'rifah, Villeurbanne, France, 1987.
23. M. Rosenlicht, On a result of Baer, *Proc. Amer. Math. Soc.* **12** (1961), 984–988.
24. M. Suzuki, "Group Theory II," Grundlehren der mathematischen Wissenschaften, no. 248, Springer-Verlag, Berlin/New York, 1986.
25. B. I. Zilber, Groups and rings whose theory is categorical, *Fund. Math.* **55** (1977), 173–188.
26. B. I. Zilber, Some model theory of simple algebraic groups over algebraically closed fields, *Colloq. Math.* **48** (1984), 173–180.