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Minimal Antichains in Well-founded Quasi-orders with an Application to Tournaments

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We investigate the minimal antichains (in what is essentially Nash-Williams' sense) in a well-founded quasi-order. We prove the following finiteness theorem: If Q is a well-founded quasi-order and k a fixed natural number, then there is a finite set Λ_k of minimal antichains of Q with the property that for any ideal I of Q obtained by excluding at most k elements of Q, I is well-quasi-ordered if and only if its intersection with each antichain in Λ_k is finite. When applied in a suitably sharpened form to an algorithmic problem arising in model theory, this yields a strengthening of the main result of [18]. © 2000 Academic Press

1. INTRODUCTION

In connection with the classification of the countable homogeneous directed graphs [5] the following algorithmic problem arises: determine effectively whether the class of finite tournaments that excludes a specified finite set of "forbidden tournaments" is well-quasi-ordered (wqo).

For the case of a *single* forbidden tournament, this problem was solved in a very concrete way in [18], exploiting the structural information in [20, 21]. We give a new approach here that depends on the same structural information but makes use of a further idea that applies quite generally to well-founded quasi-orders. For the applications to tournaments, one deals with the quasi-order of finite tournaments, where $A \leqslant B$ means that A embeds

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isomorphically into B. This approach yields the decidability of the original question for up to two constraints and has also led to the discovery of some new families of minimal antichains, which puts us in a better position to formulate some meaningful structural conjectures applying specifically to antichains of tournaments.

We begin with a finiteness theorem, whose proof is a direct and rather formal application of Nash-Williams' "minimal bad sequence" argument:

THEOREM 1 (Finiteness Theorem). *Let Q be a well-founded quasi-order, and k an integer. Then there is a finite set Λ_k of minimal antichains such that for any ideal $I \subseteq Q$ determined by excluding at most k elements of Q, the following are equivalent:*

1. *I is well-quasi-ordered.*
2. *I meets each antichain in Λ_k in a finite set.*

This will be proved in Section 2.

The proof gives no information concerning the size of Λ_k and is completely noneffective. However if Q is itself given effectively, so that it makes sense to raise issues of decidability (as will be the case whenever Q consists of all finite structures of some specified type), and if the antichains in Λ_k are known sufficiently effectively, then one can get additional information for the case of $k+1$ constraints:

COROLLARY 1.2. *Suppose that Λ_k satisfies the conditions of Theorem* 1.1 *and that for each antichain J in Λ_k there is an algorithm that determines whether an arbitrary element $a \in Q$ lies below some element of J. Then there is an algorithm that determines whether the ideal formed by excluding $k+1$ specified elements of Q is wqo.*

Here the notion of "minimality" is essentially that used by Nash-Williams, and exactly that used by Gustedt [12], and will be reviewed below along with other relevant background material and terminology.

We will show that each of the two antichains originally encountered in the analysis of the case of forbidden tournaments (with $k=1$) is minimal and satisfies the additional effectivity requirement of the corollary. Thus we conclude:

PROPOSITION 1.3. *The problem of determining whether a class of finite tournaments obtained by excluding two forbidden tournaments is wqo is decidable* (*in polynomial time*).

Note, however, that we do not yet know a *specific* algorithm that will suffice in this case.

In the next section we prove the finiteness theorem and its algorithmic corollary, in the context of well-founded quasi-orders. In Section 3 we consider classes of tournaments determined by the exclusion of a single forbidden subtournament, identify Λ_1, building on earlier work, and verify the necessary effectivity criterion. In Section 4 we introduce a convenient topological point of view, in which equivalence classes of minimal anti-chains are the points of a space and the Λ_k are successive approximations to a dense subset. We conjecture that the isolated points are dense in this space in the case of tournaments and that all the isolated points satisfy our additional effectivity condition. In Section 5 we construct infinitely many inequivalent antichains for which the corresponding classes are isolated. The first family exhibited is one that emerged from a preliminary analysis of Λ_2. In the last section we make a suggestion as to the nature of Λ_2 and state what is known and what we think are the requirements for proving that conjecture: namely, a large variety of structure theorems for various classes of tournaments determined by exclusion of specific pairs of constraints.

In the remainder of this introduction we will review some terminology and background information.

DEFINITION 1.4.

1. A *quasi-order* is a set equipped with a reflexive and transitive relation $\leqslant$; we write $a<b$ if $a\leqslant b$ and $b\nleqslant a$.

2. A quasi-order is *well-founded* if it contains no infinite strictly decreasing sequence.

3. An *antichain* in a quasi-order is a set of incomparable elements.

4. A quasi-order is *well-quasi-ordered* if it is well-founded and contains no infinite antichain.

5. An *ideal* in a quasi-order is a set closed downwards with respect to $\leqslant$.

6. If A is a subset of the quasi-order Q, then the ideal Q_A determined by *excluding* A is the set $\{x\in Q : a\nleqslant x \text{ for all } a\in A\}$. We use the same notation when A is given as a sequence of elements of Q rather than a set.

7. If $(Q, \leqslant)$ is a quasi-order and $A\subseteq Q$, set

$$Q^{\leqslant A}=\{x\in Q : x\leqslant a \text{ for some } a\in A\} \tag{1}$$

$$Q^{<A}=\{x\in Q : x<a \text{ for some } a\in A\} \tag{2}$$

$$Q^{\ll A}=\{x\in Q : x<a \text{ for all but finitely many } a\in A\}. \tag{3}$$

Thus $Q^{\ll A}=Q$ if A is finite, and $Q^{\ll A}\subseteq Q^{<A}$ if A is infinite; only the latter case will arise here.

8. An antichain I in a quasi-order Q is *minimal* if $Q^{<I}=Q^{\ll I}$ and $Q^{<I}$ is wqo.

Note the following criterion that applies when Q is well-founded: I is minimal if and only if I is infinite, and for any infinite antichain $J \subseteq Q^{\leqslant I}$, all the elements of J are equivalent to elements of I.

This notion of minimal is suggested by Nash-Williams' notion of "minimal bad sequence" [23]. One can also introduce a partial ordering on infinite antichains defined as follows: $J < I$ if and only if $J \subseteq Q^{\leqslant I}$ and $J \cap Q^{<I} \neq \emptyset$; then our criterion for minimality agrees with minimality with respect to this partial ordering.

We will vary this notation somewhat when we pass to the case of finite tournaments:

DEFINITION 1.5.

1. $\mathcal{Q}$ is the class of all finite tournaments; its elements are typically denoted by capital letters $A, B, C, \ldots$; and $\mathcal{Q}$ is equipped with the quasi-order defined by $A \leqslant B$ if A is isomorphic to a subtournament of B. This is evidently well-founded and is not wqo (e.g., [13, 16]).

2. For $\Gamma \subseteq \mathcal{Q}$, $\mathcal{Q}_\Gamma$ is the ideal determined by excluding Γ; we also refer to the elements of Γ in this case as *forbidden subtournaments*.

3. Antichains in $\mathcal{Q}$ will typically be denoted $\mathcal{I}$, $\mathcal{J}$, and the like.

We call a finite set Γ of finite tournaments *tight* if $\mathcal{Q}_\Gamma$ is wqo and *loose* otherwise. This terminology reflects the fact that the set Γ serves as a constraint.

The problem with which we began may be stated as follows:

Problem 1. Is there an algorithm that will determine whether a given finite set Γ of finite tournaments is loose or tight?

We believe that this problem is best approached in terms of understanding more fully the space of all minimal antichains of tournaments (modulo a natural equivalence relation), and we will elaborate on this in Section 4.

From wqo theory we will need only the following facts:

Fact 1.1. The product of finitely many well-quasi-orders is a well-quasi-order, under the product ordering $\bar{a} \leqslant \bar{b}$ if and only if $a_i \leqslant b_i$ for all i.

Fact 1.2 (Higman [14]). If Σ is a wqo alphabet then the set $\Sigma^\star$ of finite words in the alphabet is a wqo under the relation of monotone word embeddability: $w \leqslant w'$ if and only if $w_i \leqslant w'_{f(i)}$, where the function f is strictly increasing. In this context, $w = (w_1, \ldots, w_n)$ and $w' = (w'_1, \ldots, w'_{n'})$.

Fact 1.1 is applied in the proof of the finiteness theorem. Fact 1.2 (Higman's theorem) is considerably stronger and is not needed for the proof of the finiteness theorem, but plays a role in the analysis of Λ_1 for the case of finite tournaments and in similar analyses aimed at determining Λ_2.

2. THE FINITENESS THEOREM

In the present section we prove our finiteness theorem, Theorem 1.1, and its corollary. The use of minimal antichains is the key in both cases.

LEMMA 2.1. *Every well-founded quasi-order that is not wqo contains a minimal antichain.*

This can be proven using Nash-Williams' minimal bad sequence argument [23] and is given explicitly by Gustedt in [12].

We now prove the finiteness theorem, Theorem 1.1.

Proof. Let Q be a well-founded quasi-order. We prove by induction on $k \geqslant 0$ that there is a finite set Λ_k of minimal antichains such that for any ideal $I \subseteq Q$ determined by excluding at most k elements of Q, the following are equivalent:

1. I is well-quasi-ordered.
2. I meets each antichain in Λ_k in a finite set.

The base case $k=0$ is immediate. We now describe the passage from Λ_k to Λ_{k+1}.

For any sequence $A=(a_1, ..., a_{k+1})$ of elements of Q for which Q_A is not wqo, fix a minimal antichain I_A in Q_A, using Lemma 2.1. For any sequence $\hat{I}=(I_1, ..., I_{k+1})$ of elements of Λ_k (with repetitions allowed) let $L(\hat{I})$ be the set of sequences $A \in \prod_{i \leqslant k+1} Q^{<I_i}$ that are loose (i.e., Q_A is not wqo), and let $\Lambda(\hat{I})$ be $\{I_A\colon A \text{ is a minimal element of } L(\hat{I})\}$. Note that every element of $L(\hat{I})$ lies above a minimal element. As each factor $Q^{<I_i}$ is wqo, by Fact 1.1 $\Lambda(\hat{I})$ is finite. Set

$$\Lambda_{k+1}=\Lambda_k \cup \bigcup \{\Lambda(\hat{I})\colon \hat{I} \in (\Lambda_k)^{k+1}\}.$$

Let $A=(a_1, ..., a_{k+1})$ be any loose sequence in Q. We must check that Q_A meets one of the antichains in Λ_{k+1} in an infinite set. For each $i \leqslant k+1$ let A_i be the sequence obtained from A by deleting a_i. Then A_i is also loose and by the inductive hypothesis there is an antichain $I_i \in \Lambda_k$ meeting Q_{A_i} in an infinite set. If for some i, $a_i \notin Q^{<I_i}$, then the same antichain meets Q_A in an infinite set.

Suppose therefore that $a_i \in Q^{<I_i}$ for each i, or in other words, with $\hat{I}=(I_1, ..., I_{k+1})$: $A \in \prod_{i \leqslant k+1} Q^{<I_i}$. As A is loose, we have $A \in L(\hat{I})$. Then there is a minimal element $A' \in L(\hat{I})$ below A, and Q_A contains $Q_{A'}$, and hence contains $I_{A'}$, which belongs to Λ_{k+1}. ∎

The next definition is motivated by topological considerations developed in Section 4.

Definition 2.2. A set Λ_k of minimal antichains will be called *k-dense* if it has the property specified in the statement of the finiteness theorem: any non-wqo ideal obtained by excluding at most k elements meets one of the antichains in Λ_k in an infinite set.

We turn now to the algorithmic consequences of this result. For this to make sense one requires that the elements of the quasi-order Q are indexed by natural numbers or at least by words over a finite alphabet. We make this assumption tacitly below and in fact identify the elements of Q with their codes.

Corollary 2.3. *Let Q be a well-founded quasi-order. Suppose that Λ_k is a finite k-dense set of minimal antichains of Q and that each antichain in Λ_k satisfies the following effectivity condition*:

$(EC1)$ For each $I \in \Lambda_k$ there is an algorithm that determines whether an arbitrary element of Q belongs to $Q^{<I}$.

Then there is an algorithm that determines, for any $A \subseteq Q$ of size $k+1$, whether Q_A is wqo.

Proof. By the proof of the finiteness theorem, and using its notation, in order to determine whether Q_A is wqo, for $A \in Q^{k+1}$, it suffices to answer the following two questions:

1. Does A lie above some minimum element of some $L(\hat{I})$?
2. Is there an antichain $I \in \Lambda_k$ such that Q_A meets I in an infinite set?

As the first question depends only on the comparison of A with finitely many elements of Q^{k+1}, this part of the algorithm can be "hard-wired."

For the second question, since $Q^{<I} = Q^{\ll I}$ when I is minimal, our hypothesis on Λ_k amounts to the assumption that this can be checked. ∎

Remark 2.4. The condition $(EC1)$ above can usually be replaced by:

$(EC2)$ For each $I \in \Lambda_k$ there is an algorithm that determines whether an arbitrary element of Q belongs to $Q^{\leqslant I}$.

This differs from $(EC1)$ only in the substitution of $\leqslant$ for $<$. When we deal, for example, with classes of finite structures, under any natural encoding the class Q will have the additional property that a representative set of minimal covers of any element $a \in Q$ is finite and is computable from a, in which case $(EC2)$ implies $(EC1)$.

It should be noted that the solution to our original algorithmic problem, with k varying, cannot be reduced to the solution of each special case, with k fixed; some further uniformity would be needed, in connection with $(EC1)$.

We will apply this corollary to the case of tournaments with $k=1$ in the next section.

3. TOURNAMENTS EXCLUDING ONE FORBIDDEN SUBTOURNAMENT

In this section we work in the quasi-order $\mathcal{Q}$ consisting of all finite tournaments, ordered by embeddability (isomorphism with a subtournament).

We will describe two infinite antichains $\mathcal{I}_L$ and $\mathcal{I}_O$, introduced in [18], such that the set $\Lambda_1 = \{\mathcal{I}_L, \mathcal{I}_O\}$ is 1-dense. This is the main content of [18, 20, 21], apart from the minimality of these antichains, which will be checked here. It turns out that these two antichains also satisfy the condition $(EC1)$, and this then yields the decidability of the problem of recognizing constraint pairs for which the corresponding class of tournaments is wqo.

3.1. *An Antichain of Modified Linear Orders*

The first of our two antichains, $\mathcal{I}_L$, is made up of slightly modified transitive tournaments. We will sketch the proof that it is an antichain and then prove that it is minimal and satisfies condition $(EC1)$ of Corollary 2.3.

DEFINITION 3.1. L_n is a transitive tournament on the vertex set $\{1, ..., n\}$ with the arc relation given by $i \to j$ if $i < j$. N_n is the tournament obtained from L_n by reversing the arcs $(i, i+1)$, for $1 \leqslant i \leqslant n-1$. For $n \geqslant 7$, $\hat{N}_n$ is the tournament obtained from N_n by reversing the arcs $(1, 3)$ and $(n-2, n)$. We define $\mathcal{I}_L = \{\hat{N}_n\}_{7 \leqslant n < \omega}$.

A representation of $\hat{N}_9$ is given in Fig. 1 with only the right-to-left arcs shown.

LEMMA 3.2 [18]. *$\mathcal{I}_L$ is an antichain.*

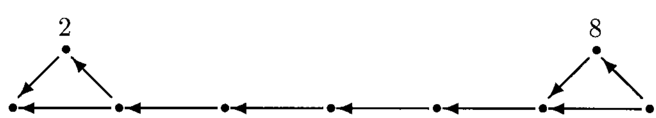

FIG. 1. Some arcs of $\hat{N}_9$.

Proof. An arc (x, y) in a tournament T is said to be a p-arc if (x, y) occurs in at least p distinct copies of C_3 in T. The 2-arcs in $\hat{N}_n$ define a successor relation on the vertices $\{3, ..., n-2\}$ while the 3-arcs occur only at the ends of this path. Thus if $\hat{N}_m$ embeds into $\hat{N}_n$, then the pattern of 3-arcs and 2-arcs in $\hat{N}_m$ must be preserved by the embedding, forcing $m = n$. ∎

We now deal with minimality.

LEMMA 3.3. $\mathscr{Q}^{<\mathscr{I}_L} = \mathscr{Q}^{\ll \mathscr{I}_L}$.

Proof. It suffices to check that any tournament $\hat{N}'_n$ that is obtained from $\hat{N}_n$ by deleting one vertex will embed in $\hat{N}_m$ for all $m \geqslant n$. This is clear by inspection. ∎

In order to show that $\mathscr{Q}^{<\mathscr{I}_L}$ is wqo we apply Fact 1.2 (Higman).

LEMMA 3.4. *$\mathscr{Q}^{<\mathscr{I}_L}$ is wqo.*

Proof. Call a tournament T *decomposable* if its vertex set can be divided into two nonempty sets A, B such that every vertex of A dominates every vertex of B, and *strong* otherwise. Let Σ be the collection of isomorphism types of finite strong tournaments. The isomorphism type of any finite tournament T can be encoded naturally by a word $w = (w_1, ..., w_l)$ where w_i is the isomorphism type of a maximal strong subtournament T_i of T, and w_i precedes w_j if and only if the vertices of T_i dominate the vertices of T_j. Thus by Fact 1.2 (Higman), if the strong tournaments in $\mathscr{Q}^{<\mathscr{I}_L}$ form a wqo class, then $\mathscr{Q}^{<\mathscr{I}_L}$ is also wqo.

That the strong tournaments in $\mathscr{Q}^{<\mathscr{I}_L}$ form a wqo class is clear by inspection: they are isomorphic to tournaments derived from the tournaments N_n by reversing at most one arc: $(1, 3)$ or $(n-2, n)$. ∎

Now we consider condition $(EC1)$ of Corollary 2.3.

LEMMA 3.5. *Let T be a tournament of order $t \geqslant 4$. Then the following are equivalent*:

1. *$T < \hat{N}_n$ for some n.*
2. *$T < \hat{N}_{2t}$.*

Proof. Suppose T embeds into $\hat{N}_n$ and n is minimal. Let A be the complement of the vertex set of the image of T in $\hat{N}_n$. If two consecutive vertices from $\{1, ..., n\}$ are in A then n can be reduced. Thus $|A| \leqslant t$ unless $n = 2t+1$ and T is transitive; in this case, n can still be reduced. Thus in all cases $|A| \leqslant t$ and $n \leqslant 2t$. ∎

COROLLARY 3.6. *The antichain $\mathcal{I}_L$ is minimal and satisfies condition* $(EC1)$ *of Corollary* 2.3.

Proof. Lemmas 3.3, 3.4, and 3.5. ∎

3.2. *An Antichain of Modified Local Orders*

We introduce the second of our two antichains, $\mathcal{I}_O$, which is made up of slightly modified local orders. We will sketch the proof that it is an antichain and then prove that it is minimal and satisfies condition $(EC1)$ of Corollary 2.3.

DEFINITION 3.7. For $n \geqslant 1$, the *local order* S_n is the tournament realized by $\mathbb{Z}/(2n+1)\mathbb{Z}$, with the arc relation

$$a \to b \qquad \text{if and only if} \quad b - a \in \{1, 2, ..., n\}.$$

For $n \geqslant 4$, $\hat{S}_n$ is the tournament obtained from S_n by reversing the arcs $(i, i+n)$ and $(i+n+1, i)$, for $0 \leqslant i \leqslant n-1$. We define $\mathcal{I}_O = \{\hat{S}_n\}_{4 \leqslant n < \omega}$.

DEFINITION 3.8. Given a tournament T and a vertex a in T, let $'a$ and a' be the induced subtournaments on the vertex sets $\{b \in T : b \to a\}$ and $\{b \in T : a \to b\}$, respectively.

$\hat{S}_n$ is isomorphic to its dual under the isomorphism $\phi(x) = -x$ and the automorphism group of $\hat{S}_n$ is transitive. Let $C_3(x, y, A)$ denote the tournament obtained by replacing the vertex z of the 3-cycle on $\{x, y, z\}$ with the tournament A so that $x \to y \Rightarrow V(A) \Rightarrow x$. We note that $0' = C_3(n+1, 1, \{2, ..., n-1\})$, where the induced subtournament on $\{2, ..., n-1\}$ is isomorphic to L_{n-2}.

LEMMA 3.9 [18]. *$\mathcal{I}_O$ is an antichain.*

Proof. Suppose for some $m \leqslant n$ we have $\hat{S}_m$ embedded in $\hat{S}_n$. We study the mapping of the vertices of $\hat{S}_m$ to the vertices of $\hat{S}_n$. Let $x \in V(\hat{S}_m)$.

We know that $x' = C_3(x+m+1, x+1, \{x+2, ..., x+m-1\})$. So the arc $x + (m+1) \to x+1$ is the unique $(m-2)$-arc in x', since $m \geqslant 4$. If x corresponds to the vertex y in $V(\hat{S}_n)$ then the $(m-2)$-arc in x' maps to the $(n-2)$-arc in y', so necessarily $(x+1) \mapsto (y+1)$. As the embedding respects the successor function, $m = n$. ∎

We now deal with minimality.

LEMMA 3.10. $\mathcal{Q}^{<\mathcal{I}_O} = \mathcal{Q}^{\ll \mathcal{I}_O}$.

Proof. Let $\hat{S}'_n$ be the induced subtournament of $\hat{S}_n$ on $V(\hat{S}_n)-\{0\}$. Since the automorphism group of $\hat{S}_n$ is transitive, any proper subtournament of $\hat{S}_n$ embeds in $\hat{S}'_n$. It suffices to show that $\hat{S}'_n$ embeds in $\hat{S}_m$ for $m \geqslant n$.

The following mapping $f\colon V(\hat{S}'_n) \to V(\hat{S}_m)$ defines a suitable embedding:

$$f(j)=\begin{cases} j & \text{if } \ j \leqslant n \\ m-n+j & \text{if } \ j>n. \end{cases} \quad \blacksquare$$

In order to show that $\mathcal{Q}^{<\mathcal{I}_O}$ is wqo we again apply Fact 1.2 (Higman). We will give an encoding of $\mathcal{Q}^{<\mathcal{I}_O}$ by finite words in a wqo alphabet, which is more complicated than the one used for $\mathcal{I}_L$.

LEMMA 3.11. *$\mathcal{Q}^{<\mathcal{I}_O}$ is wqo.*

Proof. $\mathcal{Q}^{<\mathcal{I}_O}$ consists of the tournaments that embed into $\hat{S}'_n$ for some n. The vertex set of $\hat{S}'_n$ is $\{1, ..., 2n\} \subseteq \mathbb{Z}/(2n+1)\mathbb{Z}$; we now think of this set as a subset of $\mathbb{N}$, equipped with the natural order.

For $1 \leqslant i \leqslant n$ let $V_i=\{i, i+n\} \subseteq V(\hat{S}'_n)$. Observe that the arcs in $\hat{S}'_n$ are determined by the following rules:

- Within V_i: $i+n \to i$
- On $V_i \times V_j$, $j>i+1$: $i \to j \to i+n \to j+n \to i$
- On $V_i \times V_j$, $j=i+1$: As in the previous case, except $i \to j+n$.

The arcs between V_i and V_j are illustrated in Fig. 2 for two cases. On the left we see the arcs when $j=i+1$ and on the right we see the arcs when $j>i+1$.

For $S \leqslant \hat{S}'_n$, we associate a vertex-colored directed graph G_S to S as follows. The vertices of G_S are the sets V_i for $1 \leqslant i \leqslant n$ such that $V_i \cap V(S) \neq \emptyset$, and the color of V_i is L, R, or D (left, right, or double) according as $V_i \cap V(S)$ is $\{i\}$, $\{i+n\}$, or V_i. An arc joins V_i to V_j in G_S if $j=i+1$ and both i and $j+n$ are present in S.

The connected components of G_S are vertex colored directed paths, and the isomorphism type of S can clearly be recovered from G_S together with the induced ordering on its components, using the rules given above to

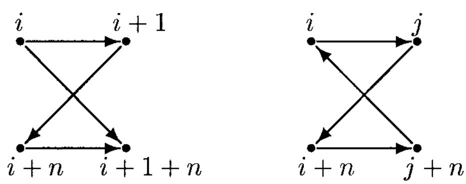

FIG. 2. The arcs between V_i and V_j.

determine the arcs. Observe that in each component of G_S, only the first vertex can be colored L and only the last can be colored R.

Let Π be the set of $\{L, R, D\}$-colored directed paths P in which at most the first vertex of P is colored L and at most the last vertex of P is colored R. We write $P_1 \leqslant P_2$ for $P_1, P_2 \in \Pi$ if there is an embedding of $V(P_1)$ into $V(P_2)$ that respects arcs and that respects colors apart from the possibility that vertices of any color may map to vertices of color D. Notice that Π is wqo under this relation, though not with respect to color-preserving embeddings.

We associate to $S \leqslant \hat{S}'_n$ the word $w = P_1 \cdots P_r \in \Pi^\star$, where $P_1, ..., P_r$ are the components of G_S, in their natural order.

To complete the proof that $\mathcal{Q}^{<\mathcal{I}_O}$ is wqo, it suffices to check that if $S^1 \leqslant \hat{S}'_m$, $S^2 \leqslant \hat{S}'_n$ correspond to $w_1, w_2 \in \Pi^\star$ and $w_1 \leqslant w_2$, then $S^1 \leqslant S^2$. The relation $w_1 \leqslant w_2$ means that each component $P_i{}^1$ of G_{S^1} can be embedded into a component $P_{i'}{}^2$ of G_{S^2} in an order-preserving manner, with L or R-colored vertices, corresponding to one element of $V(S^1)$, possibly mapping to D-colored vertices, corresponding to two elements of $V(S^2)$. If we require that a vertex x of S^1 map to a vertex in the range $\{1, ..., n\}$ in S^2 if and only if x is in the range $\{1, ..., m\}$ in S^1, this induces a map of the vertices of S^1 into those of S^2. This map preserves arcs since the arc relations are determined by the data in G_{S^1}, G_{S^2}, and the corresponding orderings on their components, as noted at the outset. ∎

LEMMA 3.12. *Given a tournament T of order t the following are equivalent*:

1. $T < \hat{S}_n$ *for some* n.
2. $T < \hat{S}_{2t}$.

Proof. Suppose $T < \hat{S}_n$ for some n, i.e., $T \leqslant \hat{S}'_n$. Then T can be encoded by a word w in $\Pi^\star$. The word w encodes a vertex colored directed graph whose components are directed paths. Let V be the vertex set of the directed graph. Let $m = |V|$. Let r be the number of components. Then observe that $m + r \leqslant 2t$.

The vertex colored directed graph encoded by w embeds in the path on $m + r - 1$ vertices each of which is colored D. This path on $m + r - 1$ vertices encodes the tournament $\hat{S}'_{m+r-1}$. Therefore $T \leqslant \hat{S}'_{m+r-1} < \hat{S}'_{2t} < \hat{S}_{2t}$. ∎

COROLLARY 3.13. *The antichain $\mathcal{I}_O$ is minimal and satisfies condition* $(EC1)$ *of Corollary* 2.3.

Proof. Lemmas 3.10, 3.11, and 3.12. ∎

Now Proposition 1.3 follows from Corollaries 3.6 and 3.13, bearing in mind the result of [18] (which depends also on [20, 21]) showing that $A_1 = \{\mathscr{I}_L, \mathscr{I}_O\}$ does indeed meet the conditions of Theorem 1.1 for $k=1$ and $Q=\mathscr{Q}$: the problem of determining whether a class of finite tournaments obtained by excluding a pair of tournaments is wqo, is decidable (in polynomial time).

4. THE SPACE $\mathscr{A}_{\mathscr{Q}}$ OF MINIMAL ANTICHAINS

Before proceeding further it will be useful to take a broader view of the situation. We find a topological point of view convenient. For general notation (such as the terminology for separation properties) we refer to [2].

Definition 4.1. Let Q be a quasi-order.

1. $\mathscr{A}_Q$ is the set of all minimal antichains in Q, equipped with the topology generated by the following basis of open sets: for $X \subseteq Q$ finite, let

$$U_X = \{A \in \mathscr{A}_Q : A \cap Q_X \text{ is infinite}\}.$$

2. $\bar{\mathscr{A}}_Q$ is the quotient of $\mathscr{A}_Q$ modulo the equivalence relation defined by: A and B are equivalent if and only if they belong to exactly the same open sets. Thus $\bar{\mathscr{A}}_Q$ is at least a T_0 space, and one would not expect it to be T_1 in general.

Remark 4.2.

1. Minimal antichains A and B are equivalent in the above sense if and only if $Q^{<A} = Q^{<B}$. (Recall that $Q^{<A} = Q^{\ll A}$ and $Q^{<B} = Q^{\ll B}$.)

2. By abuse of language we will refer to a minimal antichain as *isolated* if its *image* in $\bar{\mathscr{A}}_Q$ is isolated. (There are no isolated points in the topology on $\mathscr{A}_Q$.) Thus isolation means: there is a finite set X such that Q_X contains a unique minimal antichain, up to equivalence, and A is in this equivalence class. In this case we do not necessarily have $A \subseteq Q_X$, but $A \backslash X \subseteq Q_X$ where $A \backslash X$ means the set of $a \in A$ that are not *equivalent* to elements of X.

The notion of k-density introduced in Section 2 is not purely topological: a set is k-dense in $\mathscr{A}_Q$ if it meets each nonempty U_X with $|X| \leqslant k$. However, the union over k of k-dense sets is a dense set and in particular contains the isolated points (that is, contains a representative of each isolated equivalence class in $\mathscr{A}_{\mathscr{Q}}$).

Remark 4.3. Let Q be a well-founded quasi-order. Then the following are equivalent:

1. For each k, there is a finite k-dense set Λ_k whose elements satisfy the effectivity condition $(EC1)$.

2. The set of minimal antichains satisfying $(EC1)$ is dense.

For $2 \Rightarrow 1$, recall the proof of the finiteness theorem.

For the case of tournaments we conjecture:

Conjecture 1.

1. The isolated points are dense in $\bar{\mathscr{A}}_{\mathscr{Q}}$.

2. Each isolated point of $\bar{\mathscr{A}}_{\mathscr{Q}}$ satisfies condition $(EC1)$.

Note that the second condition is unambiguous: condition $(EC1)$ is a property of the equivalence class.

We also conjecture that not all minimal antichains in $\mathscr{Q}$ are isolated. The ones described in the previous section are isolated, because $\mathscr{Q}^{\leqslant \mathscr{I}_L}$ and $\mathscr{Q}^{\leqslant \mathscr{I}_O}$ are each described as $\mathscr{Q}_X$ for appropriate finite sets X, as in [18]. The new antichains given in the next section will also be isolated. These examples will show in particular that $\lim_{k \to \infty} |\Lambda_k| = \infty$, no matter how Λ_k is selected. See also the discussion at the end of [22].

One essential feature of the situation that is not captured at all by the topology is the "bootstrapping" character of the finiteness theorem: knowledge of Λ_k provides meaningful information about Λ_{k+1}. The new examples were found while considering the implications of our knowledge of Λ_1 for the determination of Λ_2. A brief discussion of Λ_2 is given at the end.

5. SOME ISOLATED ANTICHAINS

The basis of our construction lies in the following generalization of the sequences (L_n) (transitive tournaments) and (N_n) (Section 3).

Definition 5.1. For $k, n \geqslant 1$:

1. $L_{k,n}$ is the tournament with vertex set $\{0, ..., n-1\}$ and with arcs determined by the rule: for $i < j$, (i, j) is an arc if and only if $j \equiv i \bmod k$.

2. $N_{k,n}$ is the tournament formed from $L_{k,n}$ by reversing the orientation of each arc connecting successive vertices $i, i+1$. (As a result, $(i, i+1)$ will be an arc in $N_{k,n}$ if and only if $k > 1$.)

We note that $N_{1,n} \simeq N_n$ in this notation. Furthermore $N_{2,2n} \simeq \hat{S}'_n$: taking $\hat{S}'_n$ and $N_{2,2n}$ both to have vertex set $\{0, ..., 2n-1\}$, identified with a subset of $\mathbb{Z}/(2n+1)\mathbb{Z}$, the function $\alpha(v) = (2v+1) \bmod (2n+1)$ induces an isomorphism from $\hat{S}'_n$ to $N_{2,2n}$.

What makes the sequences $(N_{k,n})_{n \in \mathbb{N}}$ suitable for the construction of antichains is their relative rigidity in the following sense.

PROPOSITION 5.2. *Suppose* α: $N_{k,n} \to N_{k',n'}$ *is an isomorphic embedding and one of the following holds*:

1. $k = 1$, $n \geqslant \max(6, 2k'+1)$.
2. $k = 2$, $n \geqslant 6$.
3. $k \geqslant 3$, $n \geqslant 6k+1$.

Then $k = k'$ *and* α *is a constant shift*: $\alpha(i) = i + r$, *with* r *fixed.*

Proof. Each case is treated along the same general lines, with considerable variation in matters of detail.

At this point we require a bit of notation for specific tournaments. If A, B are tournaments (or isomorphism types of tournaments), then AB will stand for a tournament consisting of two disjoint copies A^*, B^* of A and B, with the vertices of A^* dominating those of B^*. It is also convenient to introduce the notation $\Gamma(A)$ for the tournament induced on the set $\{v: a \to v, \text{ all } a \in A\}$, computed in a fixed ambient tournament.

Suppose first that $k = 1$ and $k' > 1$. It will be convenient to write N_n for $N_{1,n}$. The case $k' = 2$ should be treated by inspection separately. For $k' = 3$ we make use of the following: if $C \subseteq N_{k',n'}$ is a 3-cycle and $k' \geqslant 3$, then the set of congruence classes represented by elements of C is disjoint from the set of congruence classes represented by elements of $\Gamma(C)$. In particular, $N_{3,n}$ does not contain a 3-cycle dominating another 3-cycle and hence does not contain a copy of N_7. For $k' > 3$ one may proceed by induction on k', considering a putative embedding of $N_{2k'+1}$ into $N_{k',n'}$, the initial vertex v of the image, and the resulting embedding of $N_{2k'-1}$ into $\Gamma(v)$; as $\Gamma(v)$ has the form LA with L transitive and dominating A, and A embeds into $N_{k'-1,2n'}$, this yields an embedding of $N_{2k'-1}$ into $N_{k'-1,2n'}$, contradicting our induction hypothesis.

Now suppose $k = 2$ and $k' \neq k$. One excludes the case $k' = 1$ by inspection. $N_{2,6}$ consists of two 3-cycles C, C^* such that the arc relation $u \to v$ defines an anti-isomorphism of C^* with C. To rule this out one may make a calculation, beginning with an embedding of the 3-cycle C into $N_{k',n'}$, whose image will either consist of three successive vertices i, $i+1$, $i+2$ or two vertices congruent modulo k' and a third incongruent vertex. In either case one may then trace through the possible extensions of this embedding to $N_{2,6}$ and arrive at a contradiction (more rapidly in the first case than in the second).

If $k = k' = 1$ our claim was checked in Lemma 3.2, and if $k = k' = 2$ it is a slightly simpler version of what was checked there.

This leaves the case $k \geqslant 3$, $n \geqslant 6k + 1$, which is slightly more troublesome. One shows first that for $2k \leqslant i \leqslant (n - 3k)$ and $i < j < i + k$:

$$\alpha(i) \not\equiv \alpha(j) \bmod k'. \tag{1}$$

To see this, let $a_l = \alpha(i - lk)$ and $b_l = \alpha(i + (l + 1)k)$ for $l = 1, 2$. Then $a_1, a_2 \to \alpha(i) \to b_1, b_2 \to \alpha(j) \to a_1, a_2$, and $a_1, a_2 \to b_1, b_2$, and (1) follows by inspection.

Hence for $2k \leqslant i \leqslant (n - 3k)$, the three vertices $\alpha(i)$, $\alpha(i+1)$, $\alpha(i+2)$ are pairwise incongruent vertices of a 3-cycle and hence form a cyclic permutation of some triple $\{i', i'+1, i'+2\}$. In particular if $2k < i \leqslant (n - 3k)$, on considering the images of $i - 1$, i, $i + 1$, and $i + 2$, we find that $\alpha(i+1) = \alpha(i) + 1$. Thus $\alpha(i) = i + r$ with some fixed r over at least the range $2k + 1 \leqslant i \leqslant (n - 3k)$. In particular, as α is an embedding, we find $k' = k$. It then remains to check that α is a constant shift over its whole domain, which may be checked by inspection (most easily by considering the maximal counterexample $i < 2k + 1$ and the minimal counterexample $i > (n - 3k)$). We omit the remaining details. ∎

There are a variety of constructions that transform the sequence $(N_{k,n})$ (k fixed, n variable) into a minimal antichain, all based on the same general principle as in Section 3: "anchoring" the ends. In all cases this anchoring can be thought of as taking place by the addition of one or two additional vertices to a sequence of this general type (though in one case, the generalized Henson antichain, we will describe it slightly differently).

We now describe two anchoring constructions. The second of these will be referred to as the "Henson" construction as it occurs in [13]. Jenkyns and Nash-Williams refer in [16] to a number of independent unpublished constructions of infinite antichains of tournaments, all of which presumably involved a similar anchoring construction. As far as we are aware, only the Henson construction was published.

Definition 5.3.

1. If A is a tournament and v a vertex of A, the tournament obtained by "doubling" the vertex v has vertex set $V(A) \cup \{v^*\}$, with v^* a new vertex, and $u \to v^*$ if and only if $u \to v$ (in particular, $v^* \to v$).

2. $N_{k,n,D}$ is the tournament obtained from $N_{k,n}$ by doubling its initial vertex 0 and then its terminal vertex $n - 1$.

3. The *Henson variant* $N_{k,n,H}$ of $N_{k,n}$ is the tournament obtained from $N_{k,n}$ by reversing the orientation of the arc on $\{0, n-1\}$.

4. $\mathscr{I}_{k,D} = \{N_{k,kn+1,D} : n \geqslant 6\}$; $\mathscr{I}_{k,H} = \{N_{k,kn+1,H} : n \geqslant 6\}$.

Remark 5.4. $\mathscr{I}_{1,D}$ and $\mathscr{I}_{2,H}$ are equivalent to (and contained in) the antichains $\mathscr{I}_L$ and $\mathscr{I}_O$ considered in Section 3, while $\mathscr{I}_{1,H}$ is the earliest published example of an antichain of tournaments [13]. Other examples of anchoring processes will be given subsequently that are relevant to the identification of Λ_2.

PROPOSITION 5.5. *For each* $k \geqslant 1$ *the sets* $\mathscr{I}_{k,H}$ *and* $\mathscr{I}_{k,D}$ *are isolated minimal antichains, all inequivalent.*

Proof. The main verifications are found in the preceding proposition. We will not give much more detail here, though we note that the proof is easier in some cases if n is taken larger, and in any case, we do not claim that our bounds are the best possible.

That the $\mathscr{I}_{k,D}$ and $\mathscr{I}_{k,H}$ are antichains follows easily from the preceding proposition.

The minimality argument is as in Section 3, where it was given twice. The proof in general follows the proof for $\mathscr{I}_O$ exactly. In the analysis there we were more or less forced to rewrite $\hat{S}'_n$ as $N_{2,2n-1}$. With this change in notation, the "components" of a subgraph correspond to intervals in the vertex set.

As before, we leave the details of isolation to the reader, with the comment that our explicit definition of the antichains involved can simply be translated into an appropriate isolating set. ∎

We will give a few more examples of antichains built by anchoring the sequence N_n by the adjunction of a single point. This can also be carried out in general, but we have no full overview of the resulting possibilities, and we do this mainly with an eye to the information it will yield concerning Λ_2.

DEFINITION 5.6. Let $\bar{n}$ be a sequence of positive integers $(n_0, n_1, ..., n_l)$ and $\varepsilon \in \{\pm 1\}$. Set $n = |\bar{n}|$, i.e., $\sum_i n_i$. Then the vertex set of $N_{\varepsilon,\bar{n}}$ is $\{1, ..., n\} \cup \{v\}$ with N_n induced on $\{1, ..., n\}$ and with the following rule for v: if $\varepsilon = 1$, then v dominates the first n_0 vertices of N_n, is dominated by the next n_1, and so on; while if $\varepsilon = -1$, the opposite orientation is taken. We will also write $N_{\bar{n}}$ for $N_{1,\bar{n}}$ as it suffices to deal with one class of examples for most purposes.

Proposition 5.7. *The following collections of codes $\bar{n}$ correspond to sequences $(N_{\bar{n}})$ that form inequivalent isolated minimal antichains*:

1. *With $m \geqslant 1$ fixed and $l \geqslant 3$ varying*: $n_i = m$ *for* $0 < i < l$ *and* $n_0 = n_l = m+1$.
2. *With $m \geqslant 2$ fixed and $l \geqslant 6$ varying*: $n_i = m$ *if* $1 < i < l-1$, $n_i = 1$ *if* $i = 0, 1, l-1$, *or* l.
3. $\bar{n} = (1, n-2, 1)$, $n \geqslant 4$.

There is little to prove here, beyond what we have already indicated in similar cases. The last example is just $\mathscr{I}_{1,H}$ again, included just to illustrate the notation. More generally one can work with sequences of periodic words, slightly modified at the extremes. We cannot be specific here because it is unclear what the possibilities are. Apart from the doubling construction, we do not know what minimal antichains can be created by anchoring by more than one vertex.

In any case, we think it likely that these examples are typical:

Conjecture 2.

1. Let $\mathscr{I}$ be a minimal antichain in $\mathscr{Q}$ such that for some k, we have $N_{k,n} \in \mathscr{I}^<$ for all n. Then for some finite m, every tournament in $\mathscr{I}$ can be obtained from a tournament $N_{k,n}$ by adjoining at most m vertices. (Conceivably one can always take $m = 2$.)
2. Let $\mathscr{I}$ be a minimal antichain in $\mathscr{Q}$. Then there is a fixed bound l such that every tournament in $\mathscr{I}$ can be partitioned into at most l transitive subtournaments. (This says nothing about how these l subtournaments are related.)

6. Λ_2

We have not identified Λ_2. On the other hand, a preliminary investigation of Λ_2 led to the discovery of the role of the tournaments $N_{k,n}$. We have the impression that, in general, all isolated minimal antichains may well have much the same character as the known examples. We think Λ_2 may consist of $\Lambda_1 \cup \{\mathscr{I}_{1,H}, \mathscr{I}_{3,D}\}$ together with the example $(N_{\bar{n}})$ with $\bar{n} = (1, 1, 2, 2, ..., 2, 2, 1, 1)$, mentioned in the preceding section.

We know by the proof of the finiteness theorem and the identification of Λ_1 that the issue is which constraint pairs (A, B) are tight, where $A \in \mathscr{Q}^{<\mathscr{I}_L}$ and $B \in \mathscr{Q}^{<\mathscr{I}_O}$, and neither A nor B is tight. If the conjectured value of Λ_2 above is accurate, there are a large number of such pairs whose tightness remains to be verified. For example:

Conjecture 3. The pairs (N_n, S_n) are tight for all n.

The next result requires some further notation for particular tournaments. We write $A[B]$ for the composition of two tournaments in which the vertices of A are replaced by copies of B, with arcs as in B within the components and as in A between components. We also use the notation AB, introduced in the proof of Proposition 5.2, for the disjoint union of A and B with A dominating B and a *partial composition* operator $C_3(A, B, C)$ in which the three vertices of a 3-cycle are replaced by the tournaments A, B, C, respectively. We write I for the degenerate tournament on one vertex.

PROPOSITION 6.1. *For any transitive tournament L the following triple is tight*:

$$(LC_3(I, I, L)L, L[C_3], C_3[L]).$$

Hence the following pairs are tight:

1. $(C_3[L], LC_3L)$
2. $(L[C_3], C_3(I, I, L))$.

This is proved as in [19], by considering the effect of Ramsey's theorem on a large set of 3-cycles. The cases $(L[C_3], C_3(I, I, L))$ and $(C_3[L], LC_3L)$ are covered explicitly there.

Proposition 6.1 simplifies the analysis of Λ_2 considerably, but as is illustrated by Conjecture 3, there are still many concrete problems to be dealt with before Λ_2 can be pinned down with any precision.

In the past, Kruskal's tree theorem combined with Ramsey's theorem has provided an effective tool for proving structure theorems on which tightness arguments depend (see, for example, the structural analyses in [20, 21] and related references therein). These structure theorems will need to be substantially generalized if anything like our conjectured value of Λ_2 is to hold. This may in turn cast some light on the plausibility of our main conjecture, Conjecture 1.

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