A HALL THEOREM FOR ω-STABLE GROUPS

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Introduction

A theorem of Hall extends the sylow theory to π-subgroups of a finite solvable group for any set of primes π. This theorem has been generalized to solvable periodic linear groups [**21**] and to other similar settings [**19**]. We consider the same problem in a model-theoretic context.

In model theory, the study of ω-stable groups generalizes the classical theory of algebraic groups in the case of a algebraically closed base field, and is closely connected with certain aspects of the general structure theory developed by Shelah, particularly in connection with the more geometric theory of Zilber, Buechler, Hrushovski, Pillay, and others. In this connection, the primary concern is with the structure of simple ω-stable groups of finite rank. One way to approach the study of ω-stable groups is by generalizing some of the methods of finite group theory to this setting. One anticipates that character theory and counting arguments will not be available, though one would hope to recover much of the 'local analysis'. Unfortunately there is as yet no very satisfactory sylow theory for ω-stable groups, except for the prime 2 [**4**]. One of us (Nesin) noticed initially that there is a sylow theorem for connected solvable ω-stable groups of finite Morley rank; this prompted us to look for a generalization along the following lines.

HALL THEOREM. *Let π be a set of primes. Any two maximal π-subgroups of a solvable ω-stable group are conjugate.*

In view of a result in [**6**] reviewed in §4, it would be enough to prove this for just one case, in which π is the set of all primes, so that the π-subgroups are the locally finite subgroups. However the proof we give applies to a general set of primes π, and we do not know a simpler way to get at that particular case.

The critical point in the proof of our Hall theorem seems to be the formulation of a suitable version of the Schur–Zassenhaus Splitting Theorem.

SCHUR–ZASSENHAUS THEOREM. *Let π be a set of primes. Suppose that $A \lhd G$ with A an abelian, π-divisible, $\pi^{\perp}$-group. Suppose that G/A is a locally finite π-group and A satisfies the descending chain condition for centralizers of subsets of G. Then G splits over A, that is, there is a complement to A in G, and any two such complements are conjugate. Furthermore, any group $H \leqslant G$ with $H \cap A = (1)$ is contained in a complement to A.*

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This variant of the Schur–Zassenhaus Theorem can be extracted from two arguments of Brian Hartley, found in [**9**, p. 172] and [**10**, p. 267]. We give the details below. We used a quite similar but less general form of this result originally; the formulation given here was suggested by Borovik, and Professor Hartley pointed out the connection with [**9**, **10**].

We emphasize that the form of the Schur–Zassenhaus Theorem given here is dictated by our approach to the Hall Theorem. As always, the heart of the matter is a small cohomological computation, which is adequately captured by the version of Schur–Zassenhaus given in [**20**], and recorded as Fact 2 below.

We also have a more 'natural' version of the Schur–Zassenhaus Theorem.

Theorem. *Let G be a connected solvable group of finite Morley rank. Let $H \lhd G$ be a π-subgroup such that G/H contains no π-element. Then H has a complement in G.*

This version of Schur–Zassenhaus does not bear directly on the Hall Theorem. The proof makes use of the structure theorem for ω-stable nilpotent groups developed in [**15**], which is derived from a purely algebraic structure theorem on nilpotent poly-π-separated groups. (Following [**14**], a group is π-separated if it is a central product of a π-group of bounded exponent and a π-radical group.) A conjugacy theorem (for definable complements) has been proved by Borovik and Nesin [**3**].

One also has a formation theory in this context. This follows from the Hall Theorem combined with [**6**] and [**8**], but there is one point that needs to be checked; we are grateful to the referee of an earlier version of the paper for pointing this out. The main argument in §4 was found by Altinel. As a special case of this theory, if one defines a Carter subgroup by analogy with the finite case as a locally nilpotent locally finite subgroup containing every element of finite order in its normalizer, then every solvable group of finite Morley rank has a single conjugacy class of Carter subgroups. The treatment given here has benefited significantly from comments by two distinct anonymous referees.

We conclude this Introduction with a rather general remark that seems worth recording. Here we shall use some stability-theoretic notions not used elsewhere in the paper. The rest of the paper does not depend on the next point, but it says something about our context.

Theorem 1. *A locally solvable ω-stable group G is solvable.*

This is based on a theorem of McLain [**13**]: a minimal normal subgroup of a locally solvable group is abelian.

Proof. We may assume that G is connected. Assume toward a contradiction that the theorem fails, and that G is a counterexample of minimal U-rank. Write the U-rank of G as $k \cdot \omega^{\alpha}+\beta$ with $\beta < \omega^{\alpha}$. If H is a proper definable normal subgroup of G of U-rank at least ω^{α}, then by induction H and G/H are solvable, and thus G is solvable. So G satisfies:

$$U(G) \geqslant \omega^{\alpha} \text{ and for } H \lhd G \text{ a proper definable subgroup, } U(H) < \omega^{\alpha}. \qquad (1)$$

By [**2**, Proposition 2.6], G/ZG is simple, and hence abelian by McLain's Theorem. This contradicts the choice of G.

1. *Background*

For a general introduction to ω-stable groups see [**16**]. We use the following standard facts.

Fact 1. *Let G be an ω-stable group.*

1. *The group G satisfies the descending chain condition on centralizers.*
2. *If N is a definable subgroup of G, then N is ω-stable. If in addition N is normal, then G/N is ω-stable.*
3. *If G is abelian then G splits as a direct sum of a group of bounded exponent and a divisible group* [**12**].
4. *For any subgroup H of G, there is a unique minimal definable subgroup containing H. This is called the definable closure of H, and is denoted* $\mathrm{cl}(H)$.
5. *If G has finite Morley rank, and H is a definable connected subgroup of G, then for any set A the subgroup of G generated by the set of commutators $[A, H]$ is definable and connected.*

If π is a set of primes, we use the following terminology. A *π-number* is a number which is a product of primes in π. A *π-group* is a periodic group in which the order of every element is a π-number. A group is *π-radicable* or *π-divisible* (in multiplicative or additive terminology, respectively) if the map $x \mapsto x^n$ is surjective for very π-number n. We call a group a $\pi^\perp$-group if it contains no π-elements. (We avoid the notation π', which would suggest that the group is periodic.) Notice that an ω-stable $\pi^\perp$-group is π-radicable, since every element belongs to a definable abelian subgroup to which Fact 1.3 applies.

Our main inductive method may be expressed as follows. We say that a subgroup A of a group G satisfies the *descending chain condition for centralizers in G* if there is no infinite properly descending chain of groups of the form $C_A(X)$ with $X \subseteq G$. From Facts 1.1–1.4 we may conclude the following.

Corollary. *Let G be an ω-stable group, and π a set of primes. Then there is a series*

$$(1) \lhd G_1 \lhd \ldots \lhd G_n = G$$

with $G_i \lhd G$ such that for each i the section G_i/G_{i-1} is either:

1. *a π-group; or*
2. *a π-divisible $\pi^\perp$-group satisfying the descending chain condition for centralizers in G/G_{i-1}.*

Proof. First one forms a series of definable normal subgroups with abelian quotients. Then one splits each abelian quotient A into its π-component A_π and a complement $A_{\pi^\perp}$. Note that A_π is typically not definable, and $A_{\pi^\perp}$ is typically not normal.

If we insert the terms A_π (more exactly, their preimages in G) between successive terms of the definable normal series, one gets a normal series in which every other term is in general undefinable. What needs to be checked is the following: if A is a definable normal abelian subgroup of an ω-stable group G, and A_π is the π-primary component, then A/A_π has the descending chain condition on centralizers in G/A_π. Indeed, for any subset X of G we have:

$$C_{A/A_\pi}(XA_\pi/A_\pi) = C_A(X)\,A_\pi/A_\pi; \tag{$*$}$$

thus A/A_π will inherit the appropriate chain condition from A.

We check ($*$). We must show that for $a \in A$, if $[a, X] \leqslant A_\pi$ then $a \in C_A(X)\,A_\pi$. Here $C_A(X)$ is a direct sum of a group of bounded exponent and a divisible group, so $A/C_A(X)\,A_\pi$ is a $\pi^\perp$-group. For some π-number n we have:

$$[a^n, X] = [a, X]^n = (1).$$

Thus $a^n \in C_A(X)$, hence $a \in C_A(X)\,A_\pi$.

We also use the version of Schur–Zassenhaus given by Suzuki [**20**, Theorem 8.9].

FACT 2. *Let G be a group with a normal abelian subgroup A, and let L be a subgroup of G such that $A \leqslant L \leqslant G$ and $[G:L] = m < \infty$. Assume that the map $a \mapsto a^m$ is bijective on A, that L splits over A, and that any two complements of A in L are conjugate in L. Then G splits over A, and any two complements to A in G are conjugate. Furthermore, if H is a complement to A in L, then H is contained in a complement to A in G.*

We take note also of a lemma concerning the inductive properties of the notions used here. As we shall not use this lemma we omit the routine proof.

INDUCTION LEMMA. *Let $A \lhd G$, and let $K \leqslant G$ with $K \cap A = (1)$. Set $G_1 = N(K)/K$, $A_1 = C_A(K)\,K/K$.*

1. *If A satisfies the descending chain condition for centralizers in G, then A_1 satisfies the descending chain condition for centralizers in G_1.*
2. *If A is an abelian, π-divisible, $\pi^\perp$-group, then so is A_1.*
3. *G/A is a π-group then G_1/A_1 is a π-group.*

2. *The Hall Theorem*

We begin with a form of the Schur–Zassenhaus Theorem. Another version, in some ways more natural, is given in §3. We shall apply the following result in the case that G/A is a solvable π-group.

THEOREM 2 (Schur–Zassenhaus). *Let π be a set of primes. Suppose that $A \lhd G$ with A an abelian, π-divisible, $\pi^\perp$-group. Suppose that G/A is a locally finite π-group and A satisfies the descending chain condition for centralizers of subsets of G. Then*

1. *G splits over A, that is, there is a complement to A in G;*
2. *any two such complements are conjugate;*
3. *any group $H \leqslant G$ with $H \cap A = (1)$ is contained in a complement to A.*

Proof. We show that Parts 2 and 3 follow directly from Part 1 under our hypotheses, and we prove Part 1 by induction on the cardinality of G/A.

Proof of Parts 2 *and* 3 *assuming that G splits*. We follow [**9**, p. 172, proof of Lemma 4.2]. Let $G = A \rtimes H$, and let H_1 be a subgroup with $H_1 \cap A = (1)$. We shall show that H_1 is conjugate to a subgroup of H.

By the descending chain condition on centralizers in G, there is a finite subset K_0 of H_1 such that $C_A(K_0) = C_A(H_1)$. Let K_1 be a finite subgroup of H_1 containing K_0, and let K_2 be a finite subgroup of H such that $K_1 \subseteq A \rtimes K_2$. By Fact 2, K_1 is conjugate to a subgroup of K_2. In particular K_1 is conjugate to a subgroup of H, and since $G = AH$, there is an element $a \in A$ such that $K_1^a \leqslant H$.

This element a is unique modulo $C_A(H_1)$: if $K_1^b \leqslant H$ and $b \in A$, then $K_0^a, K_0^b \leqslant H$ and $[a^{-1}b, K_0^a] \leqslant A \cap H = (1)$, forcing $a^{-1}b \in C_A(K_0^a) = C_A(K_0) = C_A(H_1)$. In other words, if $K_0^a \leqslant H$ then also $K_1^a \leqslant H$ for all finite groups K_1 with $K_0 \leqslant K_1 \leqslant H_1$, and thus $H_1^a \leqslant H$, as desired.

Proof of Part 1. We follow [**10**, p. 267]. We proceed by induction on the cardinality κ of G/A. If κ is finite, then Fact 2 applies. If κ is infinite, then G can be written as an increasing continuous union of groups G_i (for i $< \kappa$) with $A \leqslant G_i$ and $|G_i/A| < \kappa$. Hence by induction, G_i splits as $A \rtimes H_i$. We can now replace the groups H_i by groups H_i^* which are again complements to A in G_i, and are increasing: $i < j$ implies that $H_i^* \leqslant H_j^*$, proceeding by induction on i. At limit ordinals we take unions, and at successor stages we take H_{i+1}^* to be a suitable conjugate of H_{i+1}, using (ii), (iii).

THEOREM 3. (Hall Theorem). *Let G be solvable and ω-stable. Then any two maximal π-subgroups of G are conjugate.*

Proof. We prove the result for the wider class of solvable groups having a normal series:

$$(1) \lhd G_1 \lhd \ldots \lhd G_n = G$$

such that each quotient G_i/G_{i-1} is abelian, and is either a π-group, or a π-divisible $\pi^{\perp}$-group which satisfies the descending chain condition on centralizers in G/G_{i-1}.

We proceed by induction on the length n of such a series. We may suppose that the result holds for $\bar{G} = g/G_1$. In particular, if G_1 is a π-group the result follows at once for G.

Assume therefore that G_1 is an abelian normal π-divisible $\pi^{\perp}$-subgroup of G with the descending chain condition on centralizers in G. Let H_1, H_2 be maximal π-subgroups of G. By induction on n, there is a $K \leqslant G$ and containing G_1 such that K/G_1 is a maximal π-subgroup of G/G_1 and H_1 and H_2 are conjugate to a subgroup of K. We may assume that H_1 and H_2 are subgroups of K. By Theorem 2, $K = G_1 \rtimes L$ for some L and H_1 and H_2 can be conjugated to a subgroup of L. We may assume that H_1 and H_2 are subgroups of L. Since L is a π-subgroup, we get $H_1 = H_2$.

COROLLARY. *In an ω-stable solvable group, any two maximal locally finite subgroups are conjugate.*

The next logical step after proving a Hall theorem would be to develop a formation theory. We shall show in §4 that the theory given in [**8**] can be applied in our setting with the help of [**6**]. In particular we shall get a satisfactory notion of Carter subgroups.

3. *Another Schur–Zassenhaus Theorem*

Unlike the situation in the context of finite groups, one can image a wide variety of Schur–Zassenhaus theorems for the ω-stable context, depending on the way elements of infinite order and undefinable subgroups are handled. The following somewhat specialized statement is the strongest natural version we know. In this section we make frequent use of Fact 1.5.

THEOREM 4. *Let G be a connected ω-stable solvable group of finite Morley rank. Let $H \lhd G$ be a π-subgroup such that G/H is a $\pi^{\perp}$-group. Then H has a complement in G.*

Our proof is based on the following two facts.

FACT 3 [**14**, **22**]. *Let G be a connected ω-stable solvable group of finite Morley rank. Then G' is nilpotent.*

FACT 4 [**15**]. *Let be an ω-stable nilpotent group. Then $N = BD$ (a central product) with B, D 0-definable, B of bounded exponent, D radicable. The torsion subgroup T of D is central in D, and $D = T \times R$ with R torsion-free and radicable.*

If N has finite Morley rank then T contains finitely many elements of any given order.

We turn to the proof of our second Schur–Zassenhaus theorem.

Proof of Theorem 4. If G contains an infinite definable normal abelian $\pi^{\perp}$-subgroup then one may argue by induction on Morley rank. We assume that G has no such subgroup.

By the facts cited, $G' = BD$ with B, D commuting 0-definable groups, B of bounded exponent and D radicable. Let $D = T \times R$ with T the torsion subgroup of D. As $D' = R'$ is torsion-free and definable, our initial remark implies that $D' = (1)$ and D is abelian, and thus D is central in G'.

We now prove that

$$T \leqslant ZG. \tag{1}$$

Clearly T is a normal subgroup of G. By Fact 4, T has only finitely many elements of each order. As G is connected, the action of G on the set of elements of T of fixed order is trivial. Thus $T \leqslant ZG$.

We now prove that

$$[g, D] \text{ is torsion-free and normal in } G \text{ for } g \in G. \tag{2}$$

Since the subgroup $[g, D]$ is in fact the set of 1-commutators, $[g, D] \simeq D/C_D(g)$ and $C_D(g)$ contains T by (1). So $C_D(g)$ is divisible by Fact 1.3, and hence the quotient is torsion-free. For the normality, let $h \in G$, and compute that $[g, D]^h = [g^h, D] = [g[g, h], D] = [g, D]$ as $D \leqslant ZG'$.

By our initial remark, (2) implies that

$$D \leqslant ZG. \tag{3}$$

Now we claim that

$$G' = B \text{ is a } \pi\text{-group of bounded exponent.} \tag{4}$$

Consider the group G/B of finite Morley rank. We have $(G/B)' = G'/B = BD/B \leqslant Z(G/B)$ by (3). Therefore G/B is nilpotent. Write

$$G/B = B_1/B * D_1/B$$

(a central product), where B_1 and D_1 are definable, characteristic and connected, B_1 has bounded exponent, and D_1/B is divisible. Note that

$$(B_1/B)' \leqslant G'/B = BD/B \simeq D/(B \cap D).$$

So $(B_1/B)'$ has bounded exponent and is contained in the divisible group $(G/B)'$. Thus $(B_1/B)'$ is finite, and as it is connected we have $(B_1/B)' = (1)$ and $(G/B)' = (D_1/B)'$.

Now $(D_1/B)' = D_1' B/B$ is torsion-free; this is seen as in the proof of (1), using the decomposition of D_1/B into its torsion part and a torsion-free part. We have $D_1' B = SB$, where $S = (D_1' B) \cap D_1$ is definable and normal in $D_1' B$. Also $S/(S \cap B)$ is torsion-free, and $(S \cap B)$ is finite and central in S, so $S = (S \cap B) \times S_1$ with S_1 characteristic in S, definable, and torsion-free.

Thus $D_1' B = S_1 \times B$ with $S_1 \lhd G$ and torsion-free, so $S_1 = (1)$ by our first remark. It follows that $D_1' \leqslant B$ and G/B is abelian, that is, $G' = B$.

Finally, the nilpotent group B is a π-group because G has no normal infinite definable $\pi^{\perp}$-subgroup; thus (4) is proved.

If G is nilpotent, then by Fact 4 there is no problem. So assume that G is not nilpotent, and in particular G' is nontrivial. As G' is connected and nilpotent, ZG' is infinite and for some $p \in \pi$ it follows that ZG' contains an infinite definable connected elementary abelian p-group A which is normal in G. Take a minimal such group A. By induction $G/A = H/A \rtimes K/A$ for some subgroup K containing A. It is enough to split K over A.

As $G' \leqslant H$, we have that $K/A \simeq G/H$ is abelian. If A is central in K then K is nilpotent and Fact 4 applies. Assume therefore that $[K, A] \neq (1)$. Fix any $t \in K$ with $[t, A] \neq (1)$. We shall show that $K = A \rtimes C_K(t)$.

We show first that

$$C_A(t), \quad [A, t] \lhd G. \tag{5}$$

We have that $C_A(t)^g = C_A(t^g) = C_A(t[t, g]) = C_A(t)$ and $[t, A]^g = [t^g, A] = [t[t, g], A] = [t, A]$ since $A \leqslant ZG'$.

Now by minimality of A, we have that $C_A(t)$ is finite and $[A, t] = A$. In particular $C_A(t) \leqslant ZG$. Now we claim that

$$C_A(t) = (1). \tag{6}$$

Fix $a \in C_A(t)$, where $a = [b, t]$ with $b \in A$. For $s \in K$ we have $[b, s]^t = [b^t, s^t] = [ba, s^t] = [b, s^t] = [b, s]$, so $[b, K] \subseteq C_A(t)$ is a finite set. As K is connected, we find that $b \in ZK$ and $a = 1$.

Finally we prove that

$$K = A \rtimes C_K(t). \tag{7}$$

We have to show that $K \leqslant AC(t)$. For $s \in K$ we have $[t, s] \in A$, so that $[t, s] = [t, a]$ for some $a \in A$; but $s = (sa^{-1})a$ and $[t, sa^{-1}] = [t, a^{-1}][t, s]^a = 1$, therefore $s \in C(t)A = AC(t)$.

4. *The class* $\mathfrak{U}$ *of* [8]

The Hall Theorem is the beginning of extended sylow theory in solvable groups. One would naturally like to recover as much as possible of the general theory of finite solvable groups, as presented notably in Chapter VI of [11], in the ω-stable case, with appropriate modifications. We shall not undertake this project here in any systematic way, but we will point out some further connections with the literature and some additional developments of this type.

A rather large step in this general direction has already been taken in [8], which deals with sylow theory and formation theory in a broad class of locally finite groups. Accordingly, in the present section we shall review the setting of [8] and its connection with our context.

In [8] a class $\mathfrak{U}$ of locally finite groups is introduced as the largest subgroup-closed collection of locally finite groups such that every group G in $\mathfrak{U}$ satisfies the following two conditions.

$$\begin{array}{c}\text{There is a finite series } (1) = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G \\ \text{with locally nilpotent quotients.}\end{array} \tag{1}$$

$$\text{For each set } \pi \text{ of primes, the maximal } \pi\text{-subgroups of } G \text{ are conjugate.} \tag{2}$$

In [6] it is shown that all locally finite solvable groups with the minimum condition on centralizers belong to the class $\mathfrak{U}$, and hence locally finite subgroups of solvable ω-stable groups are all in this class. Hence everything proved in [8] has consequences for solvable ω-stable groups, which we shall now explore.

In [8] it is shown that the groups in $\mathfrak{U}$ allow a satisfactory theory of sylow bases, basis normalizers, formations $\mathfrak{F}$, and $\mathfrak{F}$-projectors ($\mathfrak{F}$-covering groups) along the general lines of [7]. In particular, a special case of the formation theory of [8] runs as follows.

NOTATION. 1. Let $\mathfrak{X} \leqslant \mathfrak{U}$ be a class of solvable groups, closed under formation of quotients. The class $\mathfrak{X}$ is called a $\mathfrak{U}$-*formation* if any residually-$\mathfrak{X}$ group in $\mathfrak{U}$ is in $\mathfrak{X}$.

2. Let π be a set of primes. A $\mathfrak{U}$-*formation function* on π is a function $\mathfrak{f}$ which associates to each prime $p \in \pi$ a $\mathfrak{U}$-formation $\mathfrak{f}(p)$.

3. The *saturated formation* $\mathfrak{F}$ associated with a $\mathfrak{U}$-formation function $\mathfrak{f}$ with domain π is the class of π-groups G in $\mathfrak{U}$ which satisfy $G/O_{p'p}G$ is in $\mathfrak{f}(p)$ for all $p \in \pi$.

4. An $\mathfrak{F}$-*projector* in a group G is a subgroup $X \in \mathfrak{F}$ such that whenever $X \leqslant H \leqslant G$ and $K \lhd H$ with $H/K \in \mathfrak{F}$, we have $H = KX$.

The following is a special case of [8, Theorem 5.4].

FACT 5. *If* $\mathfrak{F}$ *is a saturated* $\mathfrak{U}$-*formation then every group in* $\mathfrak{U}$ *has a unique conjugacy class of* $\mathfrak{F}$-*projectors*.

To extend this to a solvable ω-stable group we may simply restrict to locally finite subgroups of G throughout. In other words, we use the formation theory available in the maximal locally finite subgroups of G; since these are all conjugate, there is a unique conjugacy class of $\mathfrak{F}$-projectors in G in this sense. However, as the referee remarked with regard to an earlier version of the present paper, in order to get a satisfactory theory one wants to know the following.

THEOREM 5. *Let G be a solvable ω-stable group. Let H_1 and H_2 be maximal locally finite subgroups of G, and suppose that K is an $\mathfrak{F}$-projector for H_1 which is contained in H_2. Then K is an $\mathfrak{F}$-projector for H_2.*

This will follow readily from the following facts found in [**8**] as Lemmas 5.2 and 5.3.

FACT 6. *Let G be a group, let $\mathfrak{F}$ be a class of groups closed under formation of quotients, and let X, N be subgroups of G with $N \lhd G$.*

1. *If X is an $\mathfrak{F}$-projector of G, then*
   (i) *for any group H between X and G, it follows that X is an $\mathfrak{F}$-projector of H;*
   (ii) *XN/N is an $\mathfrak{F}$-projector of G/N.*
2. *If XN/N is an $\mathfrak{F}$-projector of G/N and X is an $\mathfrak{F}$-projector of XN, then X is an $\mathfrak{F}$-projector of G.*

We also need the following, which is a standard consequence of the Hall Theorem, adapted to our context.

LEMMA 1. *Let G be ω-stable and solvable, $N \lhd G$, and let H be a maximal π-subgroup of G for some set π of primes. Then,*

1. *$H \cap N$ is a maximal π-subgroup of N, and all maximal π-subgroups of N are of this form;*
2. *if N satisfies one of the following conditions;*
   (i) *N is a π-group;*
   (ii) *N is a π-divisible $\pi^{\perp}$-group satisfying the descending chain condition on centralizers in G;*
   (iii) *N is definable;*

then HN/N is a maximal π-subgroup of G/N, and all maximal π-subgroups of G/N are of this form.

Proof. Part (1) is a formal consequence of the Hall Theorem for G, as is the latter half of Part (2) if the first half is granted.

The first half of Part (2) is contained in the proof of the Hall Theorem: (i) is clear, (ii) uses a version of Schur–Zassenhaus, and (iii) follows from (i) and (ii).

Notice that some restriction on the subgroup N is clearly required in (2) as the example $G = \mathbb{Q} \rtimes (\mathbb{Z}/2\mathbb{Z})$, $N = \mathbb{Z}$ shows.

Proof of Theorem 5. We have G, H_1, H_2, $\mathfrak{F}$, and K (an $\mathfrak{F}$-projector for H_1, contained in H_2). We show that K is an $\mathfrak{F}$-projector for H_2. We proceed by induction on the length c of a normal series for G with abelian quotients. The groups G_i occurring in this series may be taken to be definable (as noted for example in [**16**]). If the group G is abelian then $H_1 = H_2$ (the torsion subgroup) and there is nothing to prove.

We assume that $c > 1$ and that A is the first nontrivial group in the given series; so $A \lhd G$ is definable and abelian, and G/A has a normal series of length $c-1$ with abelian quotients. Let A_t be the torsion subgroup of A, which in general is not definable but is normal in G. For $i = 1$ or 2, we have $A_t \subseteq H_i$ by the maximality of H_i, and hence $H_i \cap A = A_t$, $H_i A/A \simeq H_i/A_t$.

By Lemma 1 H_iA/A is a maximal locally finite subgroup of G/A for $i = 1$ or 2. Under the isomorphism $H_iA/A \simeq H_i/A_t$, the quotient KA/A corresponds to KA_t/A_t and by Fact 6.1(ii) KA_t/A_t is an $\mathfrak{F}$-projector for H_1/A_t, or, equivalently, KA/A is an $\mathfrak{F}$-projector for H_1A/A. Since $KA/A \subseteq H_2A/A$, it follows by induction that KA/A is an $\mathfrak{F}$-projector for H_2A/A, or, equivalently, that KA_t/A_t is an $\mathfrak{F}$-projector for H_2/A_t. Therefore by Fact 6.2, in order to show that K is an $\mathfrak{F}$-projector for H_2 it suffices to show that K is an $\mathfrak{F}$-projector for KA_t. However KA_t is contained in H_1, so Fact 6.1(i) applies. This concludes the argument.

In the case in which $\mathfrak{F}$ is the class of locally nilpotent groups (in $\mathfrak{U}$) the $\mathfrak{F}$-projectors are called *Carter subgroups*. These are characterized in the finite case as the self-normalizing nilpotent subgroups, with analogous characterizations in more general contexts. We shall show that the version of this characterization given in [**8**] applies in our case. Before entering into this, it seems worth pointing out that even in the case of Carter subgroups some proof of Theorem 5 is needed; specifically, that it is possible for a Carter subgroup to belong to more than one maximal locally finite subgroup of a solvable ω-stable group G of finite Morley rank. On the other hand we shall show that if the group G is connected then this cannot happen.

EXAMPLE. Take the natural representation of the symmetric group Σ on three letters acting on a rational vector space V of dimension 3 and form the semidirect product $V \rtimes \Sigma$. The maximal locally finite subgroups are the conjugates of Σ. Take an involution $\tau \in \Sigma$ and a fixed vector v of τ. Then $\Sigma \cap \Sigma^v = \langle \tau \rangle$ and this is a Carter subgroup of Σ and Σ^v.

On the other hand, when G is a connected solvable group of finite Morley rank then G' is nilpotent by Fact 3, and in this case any $\mathfrak{F}$-projector for any saturated $\mathfrak{U}$-formation $\mathfrak{F}$ with $\pi(\mathfrak{F}) \supseteq \pi(G/G')$ belongs to a unique maximal locally finite subgroup. This holds rather more generally.

REMARK. Suppose that G is a solvable ω-stable group having a nilpotent normal subgroup N with G/N nilpotent, and let $\mathfrak{F}$ be a saturated $\mathfrak{U}$-formation such that $\pi(\mathfrak{F}) \supseteq \pi(G/N)$. If H is a maximal locally finite subgroup of G and K is an $\mathfrak{F}$-projector for H, then the only maximal locally finite subgroup of G containing K is H.

Proof. We may take N definable. Notice that any locally finite subgroup of G/N is in $\mathfrak{F}$ (the definitions force $\mathfrak{F}$ to contain every locally nilpotent $\pi(\mathfrak{F})$-group in $\mathfrak{U}$).

The group N has a torsion subgroup N_t which is contained in every maximal locally finite subgroup of G. Furthermore H/N_t belongs to $\mathfrak{F}$, so by definition $H = KN_t$. Then any maximal locally finite subgroup containing K contains $KN_t = H$.

Now we discuss the following characterization of the Carter subgroups.

THEOREM 6. *The Carter subgroups of a solvable ω-stable group are the locally finite locally nilpotent subgroups K such that K contains every element of finite order in its normalizer.*

Another way of putting this, of course, is that K is self-normalizing in every maximal locally finite subgroup of G containing K.

This follows rapidly from considerations in [**8**], where a general theorem of this type is proved. The setting there is as follows. Let $\mathfrak{X}$ be a subgroup-closed subclass of $\mathfrak{U}$ such that every locally nilpotent group N in $\mathfrak{U}$ satisfies the normalizer condition: the normalizer of any proper subgroup K of N is a proper extension of K. Then [**8**, Lemma 5.8], in part, gives us the following fact.

FACT 7. *If H is in $\mathfrak{X}$, then the Carter subgroups of H are the self-normalizing locally nilpotent subgroups of H.*

For our application we take $\mathfrak{X}$ to be the class of solvable locally finite groups with the minimal condition on centralizers. This satisfies our conditions, including the normalizer condition; by [**5**, Corollary 2.9] these groups are hypercentral, which implies that the normalizer condition holds [see, for example, **18**, p. 49]. So the Carter subgroups of a locally finite subgroup H of a solvable ω-stable group G are the self-normalizing locally nilpotent subgroups of H. Hence Theorem 6 follows from Theorem 5.

5. *More remarks on sylow theory*

In conclusion we discuss a couple of technical points connected with the developments in [**8**]. We reproduce an interesting remark of a referee on the analog of the theory of sylow bases as given in [**8**], answering a question raised in an earlier version of this paper. We also give an example of a useful principle which is valid in locally finite subgroups of ω-stable groups but which fails miserably in a more global form (lifted to the whole ω-stable group), and we 'rescue' the principle by an extra definability hypothesis.

A *sylow basis* is a family $\mathfrak{G} = (S_p)$ of sylow subgroups (one for each prime p) of the given group such that for any set π of primes, the group $\langle S_p : p \in \pi \rangle$ is a π-group. However it is proved in [**8**, Corollary 2.6 and Lemma 2.7(iv)] that within the class $\mathfrak{U}$ this is equivalent to the pair of conditions:

$$S_p S_q = S_q S_p \text{ for all } p, q, \tag{1}$$

$$\prod_{p \in \pi} S_p \text{ is a } \pi\text{-group for each set } \pi \text{ of primes.} \tag{2}$$

FACT [**8**, Theorem 2.10]. *Every $\mathfrak{U}$-group possesses a unique conjugacy class of sylow bases.*

This depends in part on the following result.

FACT [**8**, Corollary 2.6]. *Let $\mathfrak{G}$ be a complete set of sylow subgroups of the $\mathfrak{U}$-group G, one for each prime p. Then the following two conditions are equivalent:*

1. *$\mathfrak{G}$ is a sylow basis of G,*
2. *$S_p S_q = S_q S_p$ for all p, q.*

It is clear what the 'global' versions of these notions will be in a solvable ω-stable group: a sylow basis will be a sylow basis for some maximal locally finite subgroup, and the second condition remains as stated: $S_p S_q = S_q S_p$. The following was pointed out by a referee.

LEMMA 2. *Let $\mathfrak{S}$ be a complete set of sylow subgroups of the solvable ω-stable group G, one for each prime p. Then the following two conditions are equivalent:*

1. *$\mathfrak{S}$ is a sylow basis of (some maximal locally finite subgroup of) G,*
2. *$S_p S_q = S_q S_p$ for all p, q.*

Proof. That 1 implies Part 2 is proved in [**8**]. For the converse, we need to show, for every finite ordered set of primes π, that $S_\pi = \prod_{p\in\pi} S_p$ is a π-group. We proceed by induction on $|\pi|$, and on the length of a normal series

$$(1) \lhd G_1 \lhd \ldots \lhd G_n = G \qquad (*)$$

having quotients G_i/G_{i-1} which are either π-groups, or π-divisible $\pi^\perp$-groups with descending chain condition on centralizers in G/G_{i-1}. We write $\pi = \pi_1 \dot\cup \pi_2$ with π_1 a proper initial segment of π. Thus S_{π_i} is a maximal π_i-subgroup of G for $i = 1, 2$, by induction. It suffices to show that $S_{\pi_1} S_{\pi_2}$ is a π-subgroup of G.

According to Lemma 1 induction applies in G/G_1 to show that $S_{\pi_1} S_{\pi_2} G_1/G_1$ is a π-group. If G_1 is a π-group there is nothing to prove, so assume that G_1 is a π-divisible, $\pi^\perp$-group with the descending chain condition on centralizers of subgroups of G. If $S_{\pi_1} S_{\pi_2}$ is not a π-group, then $S_{\pi_1} S_{\pi_2} \cap G_1 \neq (1)$. Consider $H = S_{\pi_2}(S_{\pi_1} S_{\pi_2} \cap G_1)$. This is a $\pi_1^\perp$-group contained in $S_{\pi_1} S_{\pi_2}$. Then $H = (S_{\pi_1} \cap H) S_{\pi_2} = S_{\pi_2}$ and thus $H \cap G_1 = (1)$, a contradiction. Hence $S_{\pi_1} S_{\pi_2}$ is a π-group.

It follows that the groups S_p generate a (maximal) locally finite subgroup of G, and form a sylow basis for that subgroup.

Finally we give a simple example which shows that a line of argument on which much of [**8**] depends is not available to us in a global form. Consider the semidirect product $\mathbb{Q} \rtimes (\mathbb{Z}/2\mathbb{Z})$ with the generator of $\mathbb{Z}/2\mathbb{Z}$ acting by inversion. This is an ω-stable group, and the complements of $\mathbb{Q}$ are the sylow 2-subgroups. The subgroup $\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$ is not ω-stable, and indeed its sylow 2-subgroups are not all conjugate. In particular, the following principle, which is trivial when one operates in a subgroup-closed context, fails in the context of solvable ω-stable groups:

Any two maximal π-subgroups S, T are conjugate by an element of $\langle S, T\rangle$. $(**)$

The problem of course is caused by undefinable subgroups, but at the same time one cannot simply restrict attention to definable subgroups, as the maximal π-subgroups themselves are usually undefinable.

Since $(**)$ is of some technical importance, we shall formulate a version of it that does hold; the proof is contained in the proof of the Hall Theorem in §1.

LEMMA 3. *Let G be solvable and ω-stable, H a definable normal subgroup of G, and π a set of primes. If S_1, S_2 are maximal π-subgroups of G such that $HS_1 = HS_2$, then they are conjugate under the action of H.*

These issues are rather marginal. It would be more interesting to transfer parts of the theory given in [**11**, Chapter VI] to our setting.

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