On central extensions of algebraic groups

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In this paper the following theorem is proved regarding groups of finite Morley rank which are perfect central extensions of quasisimple algebraic groups.

Theorem 1 *Let G be a perfect group of finite Morley rank and let $C_\circ$ be a definable central subgroup of G such that $G/C_\circ$ is a universal linear algebraic group over an algebraically closed field; that is G is a perfect central extension of finite Morley rank of a universal linear algebraic group. Then $C_\circ = 1$.*

Contrary to an impression which exists in some circles, the center of the universal extension of a simple algebraic group, as an abstract group, is not finite in general. Thus the finite Morley rank assumption cannot be omitted.

Corollary 1 *Let G be a perfect group of finite Morley rank such that $G/Z(G)$ is a quasisimple algebraic group. Then G is an algebraic group. In particular, $Z(G)$ is finite* ([*3*] *Section 27.5*).

An understanding of central extensions of quasisimple linear algebraic groups which are groups of finite Morley rank is necessary for the classification of *tame* simple K^*-*groups* of finite Morley rank, which constitutes an approach to the Cherlin-Zil'ber conjecture. For this reason the theorem above and its corollary were proven in [5] (Theorems 4.1 and 4.2) under the assumption of *tameness*, which simplifies the argument considerably. The result of the present paper shows that this assumption can be dropped. The main line of argument is parallel to that in [5]; the absence of the tameness assumption will be countered by a model-theoretic result and results from K-theory. The model-theoretic result places limitations on definability in stable fields, and may possibly be relevant to eliminating certain other uses of tameness.

Our terminology and notation with regard to groups of finite Morley rank conforms to that in [2]; for algebraic groups the reader is referred to [3] or [9].

The first part of the paper is devoted to the model-theoretic definability result which will be needed to eliminate the use of tameness in the theory of central extensions. The argument we give is carried out in greater generality than is needed in the case of finite Morley rank, at the cost of a few extra preparatory definitions, which would be irrelevant in that case. This seems appropriate as the definability issues raised here may be of interest in connection with the general study of stable fields, a mysterious but intriguing subject.

If $\phi(\bar{x}, \bar{y})$ is a formula with two sets of free variables, by a *ϕ-formula* we will mean either a formula of the form $\phi(\bar{x}, \bar{b})$ with $\bar{b}$ taken from some ambient model, or the negation of such a formula. Note that this notion depends not only on the formula ϕ, but on the manner in which its free variables have been partitioned.

Definition 1 ([6], Definition 6.13, p. 72) *Let $\mathcal{L}$ be a first-order language and $\phi(\overline{x}, \overline{y})$ ($l(\overline{x}) = n$) be an $\mathcal{L}$-formula. The ordinal valued rank $R^{\phi}_{\aleph_\circ}$ of an n-formula $\psi(\overline{x})$ (possibly with parameters) is defined as follow:*

(i) *If $\psi(\overline{x})$ is consistent, then $R^{\phi}_{\aleph_\circ}(\psi) \geq 0$*

(ii) *If δ is a limit ordinal. then $R^{\phi}_{\aleph_\circ}(\psi) \geq \delta$ if $R^{\phi}_{\aleph_\circ}(\psi) \geq \alpha$ for all $\alpha < \delta$.*

(iii) *If $\alpha = \beta + 1$ then $R^{\phi}_{\aleph_\circ}(\psi) \geq \alpha$ if for each $i < \omega$, there exists Ψ_i, a finite collection of ϕ-formulas, such that*

- **(a)** *For $i < j < \omega$, Ψ_i and Ψ_j are contradictory (i.e. either there are $\phi(\overline{x}, \overline{b}) \in \Psi_i$ and $\neg\phi(\overline{x}, \overline{b}) \in \Psi_j$ or there are $\neg\phi(\overline{x}, \overline{b}) \in \Psi_i$ and $\phi(\overline{x}, \overline{b}) \in \Psi_j$),*
- **(b)** *For each $i < \omega$, $R^{\phi}_{\aleph_\circ}(\psi \wedge \bigwedge \Psi_i) \geq \beta$.*

(iv) *If $R^{\phi}_{\aleph_\circ}(\psi) \geq \alpha$ for all ordinals α, then we say $R^{\phi}_{\aleph_\circ}(\psi) = \infty$, and, if $R^{\phi}_{\aleph_\circ} \geq \alpha$ but $R^{\phi}_{\aleph_\circ} \not\geq \alpha + 1$ for an ordinal α, we say $R^{\phi}_{\aleph_\circ} = \alpha$.*

Fact 2 ([6], Proposition 6.21) *A first-order theory T is stable if and only if for all $\phi(\overline{x}, \overline{y})$, $R^{\phi}_{\aleph_\circ}(\overline{x} = \overline{x})$ is finite.*

Definition 3 *If p is an n-type then then $R^{\phi}_{\aleph_\circ}(p)$ is defined as $min\{R^{\phi}_{\aleph_\circ}(\psi(\overline{x})) : \psi(\overline{x}) \in p\}$.*

Note that by the definition of the $R^{\phi}_{\aleph_\circ}$ rank, in a stable theory, a formula $\psi(\overline{x})$ is contained only in finitely many ϕ-types (i.e. restrictions of types to the ϕ-formulas that they contain) of maximal rank, i.e. the *multiplicity is finite.*

Fact 4 ([6], Lemma 6.24, p. 74) *Let $\psi_1(\overline{x})$ and $\psi_2(\overline{x})$ be formulas with parameters. Let $\phi(\overline{x}, \overline{y})$ be an $\mathcal{L}$-formula. Then $R^{\phi}_{\aleph_\circ}(\psi_1 \vee \psi_2) \leq max(R^{\phi}_{\aleph_\circ}(\psi_1), R^{\phi}_{\aleph_\circ}(\psi_2))$.*

Definition 5 *Let T be a stable theory. Let $\psi_1(\overline{x})$ and $\psi_2(\overline{x})$ be formulas with parameters. Let $\phi(\overline{x}, \overline{y})$ be a $\mathcal{L}$-formula. We say $\psi_1 \sim \psi_2$ if $R^{\phi}_{\aleph_\circ}(\psi_1) = R^{\phi}_{\aleph_\circ}(\psi_2)$ and $R^{\phi}_{\aleph_\circ}(\psi_1 \triangle \psi_2) < R^{\phi}_{\aleph_\circ}(\psi_1)$, where $\triangle$ denotes the symmetric difference.*

The following lemma follows from Facts 2 and 4.

Lemma 6 *In a stable environment, the relation $\sim$ defines an equivalence relation.*

Fact 7 ([7], Lemme 1.1, p.20) *A stable monoid which is left and right-cancellable, or which is left-cancellable and has a right identity, is a group.*

Fact 8 ([4]) *An infinite ω-stable field F is algebraically closed. Furthermore, its additive and multiplicative groups, F_+ and F^*, are both connected groups – that is, they have no proper definable subgroups of finite index.*

It is useful to know that in the more specific context of ω-stable structures of finite Morley rank, we never encounter pairs of algebraically closed fields $K \le F$. This is because the structure $(F; K)$ (with the inclusion of K into F) has Morley rank ω; in any case it is quite easy to see that the rank is infinite. Cf. [7], Lemme 3.2.

To state our result on definability we need one further definition.

Definition 9 ([7], Chap. 5, §a.) *Let G be a stable group, and X a definable subset of G. Then X is generic in G if G can be written as the union of finitely many translates of X. (It is proved that it makes no difference whether one allows right translates, left translates, or both together.). A type p over G is said to be generic if it contains generic formulas only.*

Fact 10 ([7], Lemme 5.1) *Let G be a stable group. If A is a definable subset of G, then either A or $\neg A$ is generic.*

As we will be using the notion of generic set within a stable field F, and there are two group structures available (deletion of (0) being an inessential alteration), it simplifies matters further to know that the notion of genericity is not dependent on which of the group structures is considered. (This is not known to hold for arbitrary stable group structures, but is correct for the additive and multiplicative structures on stable fields, ultimately because of the linkage provided by the distributive law.) Cf. [7], Theorem 5.10.

A stable group G acts on its 1-types: if p is such a type realized by x in an elementary extension of G and $g \in G$, then gp is the type of gx over G. This action was investigated by Poizat in [7] where he proves the following fact:

Fact 11 ([7], Corollaire 5.6) *Every type can be translated to the neighborhood of a generic, i.e. if p is a 1-type and $\theta(x)$ is a generic formula, then there exists $g \in G$ such that gp satisfies θ.*

The following result on definability is of a type frequently occurring in stability theory. It is crucial in the proof of Theorem 1:

Theorem 2 *Let F be a stable field and $F_\circ$ an infinite subfield of F, not assumed definable. Suppose that A is a definable subset of F which contains $F_\circ$, and A is not a generic subset of F. Then F has an infinite proper definable subfield containing $F_\circ$.*

Proof. Let $\psi(x, \bar{y})$ be a formula such that some instance $\psi(x, \bar{c})$ defines A, and let $\phi(x; \bar{y}, z_1, z_2)$ be the formula $\psi(z_1 x + z_2, \bar{y})$. Let $\mathcal{A} = \{a + bA : a \in F, b \in F^*, F_\circ \subseteq a + bA\}$ and $\mathcal{A}^* = \{A_1 \cap \ldots \cap A_k : A_1, \ldots, A_k \in \mathcal{A}\}$. The sets in $\mathcal{A}$ are all definable by instances of ϕ.

Let X be a set in $\mathcal{A}^*$ of minimal rank and multiplicity. Let $R = \{r \in F : r + X \sim X\}$ and $S = \{r \in R : rR \subseteq R\}$. R acts on the finite set of ϕ-types of maximal rank which contain the partial ϕ-type corresponding to X. In particular, the pointwise stabilizer $R_\circ$ of this set is of finite index in R. The definability of types in stable theories implies that $R_\circ$ is a definable subgroup (see p. 162 of [7]). Therefore, so is R, and hence also S.

Let $a \in F_\circ \setminus \{0\}$. Since X is of the form $(a_1 + b_1 A) \cap \ldots \cap (a_k + b_k A)$, where the a_i are in F and the b_i are in F^*, and each $a_i + b_i A$ contains $F_\circ$, $a + X = (a + a_1 + b_1 A) \cap \ldots \cap (a + a_k + b_k A) \in \mathcal{A}^*$. Similarly, we have $aX \in \mathcal{A}^*$. Since X is a set in $\mathcal{A}^*$ of minimal rank and multiplicity, $(a+X) \cap X$ and $aX \cap X$, which are elements of $\mathcal{A}^*$ also, have the same rank and multiplicity as X. This implies $a + X \sim X$ and $aX \sim X$. The first of these equivalences shows that $F_\circ \subseteq R$. Using the second one we will show that $F_\circ \subseteq S$. Let $r \in R$. Then $ar + X \sim a(r + a^{-1}X) \sim a(r + X) \sim aX \sim X$. Thus $ar \in R$ and $a \in S$. Since 0 is clearly in S, we conclude $F_\circ \subseteq S$.

As R is an additive group, S is a subring of F. Thus S is a stable integral domain, and we conclude that it is a field using Fact 7. It remains to be seen that it is a proper subfield. As $S \leq R$, it suffices to show that $R \neq F$. Suppose towards a contradiction that $R = F$. Note that R stabilizes a ϕ-type containing X. Let p be a complete type containing this ϕ-type. As X is not generic, its complement is by Fact 10. By Fact 11, there exists $r \in R$ such that $r + p$ contains $\neg X$. But $r + p$ contains X as well, a contradiction. This shows that $R < F$ and finishes the proof. $\Box$

Corollary 2 *Let F be an ω-stable field of finite Morley rank and $F_\circ$ an infinite subfield of F. Suppose that A is a definable subset of F which contains $F_\circ$. Then $RM(A) = RM(F)$, where RM is the Morley rank.*

Proof. If $RM(A) < RM(F)$ then evidently A is not a generic subset of F, and thus F contains a proper infinite definable subfield K containing $F_\circ$. But as the rank of the pair $(F; K)$ is infinite, and K is definable, the rank of F is infinite, a contradiction. $\square$

The second ingredient in the proof of Theorem 1 is the theory of central extensions of linear algebraic groups as explained in [9], blended with a dose of model theory needed for definability arguments. This ingredient was already present in [5], but we will give an overview here as well. Our notation and terminology for linear algebraic groups follows [9] except where stated otherwise. It is worth emphasizing that the phrase "simple group" is used in the sequel for a group which is simple as an *abstract* group. For any group G and x, $y \in G$, (x, y) denotes $xyx^{-1}y^{-1}$.

For a field k and a root system Σ the following relations over a set of symbols $\{x_\alpha(t) : \alpha \in \Sigma, t \in k\}$ are considered:

(A) $x_\alpha(t)$ is additive.

(B) If α and β are roots and $\alpha+\beta \neq 0$, then $(x_\alpha(t), x_\beta(u)) = \prod x_{i\alpha+j\beta}(c_{ij}t^iu^j)$, where i and j are positive integers and the c_{ij} are integers depending on α, β, and the chosen ordering of the roots, but not on t or u.

(B′) $w_\alpha(t)x_\alpha(u)w_\alpha(-t) = x_{-\alpha}(-t^{-2}u)$ for $t \in k^*$, where

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$$

for $t \in k^*$.

(C) $h_\alpha(t)$ is multiplicative in t, where $h_\alpha(t) = w_\alpha(t)w_\alpha(-1)$ for $t \in k^*$.

Throughout the present paper, X_u will denote the groups presented by (A) and (B) if the rank of Σ is greater than 1 and by (A) and (B$'$) if the rank of Σ is equal to 1. If the relation (C) is added then the group presented is the universal Chevalley group ([9]). (The notation X_u is different from that used in [9]).

Fact 12 ([9], Lemma 39, p. 70) *Let α be a root and X_u be as above. In X_u, set $f(t,u) = h_\alpha(t)h_\alpha(u)h_\alpha(tu)^{-1}$. Then:*

(a) $f(t, u^2 v) = f(t, u^2) f(t, v)$.

(b) *If t, u generate a cyclic subgroup of k^* then $f(t,u) = f(u,t)$.*

(c) *If $f(t,u) = f(u,t)$, then $f(t,u^2) = 1$.*

(d) *If $t, u \neq 0$ and $t + u = 1$, then $f(t,u) = 1$.*

Fact 13 ([9], Theorem 9, p. 72) *Assume that Σ is indecomposable and that k is an algebraic extension of a finite field. Then the relations (A) and (B) (or (B$'$) if rank $\Sigma = 1$) suffice to define the corresponding universal Chevalley group, i.e. they imply the relations (C).*

Fact 14 ([9], Theorem 10, p. 78) *Let Σ be an indecomposable root system and k a field such that $|k| > 4$, and if rank $\Sigma = 1$, assume further that $|k| \neq 9$. If X is the corresponding universal Chevalley group (abstractly defined by the relations (A), (B), (B$'$), (C) above), if X_u is the group defined by the relations*

(A), (B), (B') ((B') is used only if rank $\Sigma = 1$), and if π is the natural homomorphism from X_u to X, then (π, X_u) is a universal covering extension of X.

Fact 15 ([9], Corollary 2, p. 82) *X_u is centrally closed. Each of its central extensions splits, i.e. its Schur multiplier is trivial. It yields the universal covering extension of all the Chevalley groups of the given type.*

Fact 16 ([9], Theorem 12 (Matsumoro, Moore)) *Assume that Σ is an indecomposable root system and k a field with $|k| > 4$. If X is the universal Chevalley group based on Σ and k, if X_u is the group defined by (A), (B), (B'), and if π is the natural map from X_u to X with $C = ker\pi$, the Schur multiplier of X, then C is isomorphic to the abstract group generated by the the symbols $\{t, u\}$ $(t, u \in k^*)$ subject to the relations:*

(a) $\{t, u\}\{tu, v\} = \{t, uv\}\{u, v\}$; $\{1, u\} = \{u, 1\} = 1$

(b) $\{t, u\}\{t, -u^{-1}\} = \{t, -1\}$

(c) $\{t, u\} = \{u^{-1}, t\}$

(d) $\{t, u\} = \{t, -tu\}$

(e) $\{t, u\} = \{t, (1 - t)u\}$

and in the case Σ is not of the type C_n $(n \geq 1)$ the additional relation

(ab′) $\{\ ,\ \}$ *is bimultiplicative .*

In this case relations (a)-(e) may be replaced by (ab') and

(c′) $\{\ ,\ \}$ *is skew.*

(d′) $\{t, -t\} = 1$.

(e′) $\{t, 1-t\} = 1$.

The isomorphism is given by $\phi : \{t, u\} \longmapsto h_\alpha(t)h_\alpha(u)h_\alpha(tu)^{-1}$*,* α *a fixed long root.*

We consider a perfect central extension G of finite Morley rank of a universal linear algebraic group X over an algebraically closed field K. Let (π, X_u) be the universal covering extension of X, $C = ker\pi$ and $C_\circ = ker\psi$ where ψ is the covering map from G onto X. By the universality of (π, X_u), there exists a map θ from X_u into G such that $\psi\theta = \pi$. Using the perfectness of G one can show that θ is surjective and $C = \theta^{-1}(C_\circ)$. As, by Fact 16, C is generated by $f(t, u)$ where f is as in Fact 12, it is important to prove the interpretability in G of $\theta \circ f$ in order to understand the structure of $C_\circ$. This was achieved in [5] by the following result:

Fact 17 ([5], Proposition 4.12) *Let G be a group of finite Morley rank. Assume that G is a perfect central extension of a universal linear algebraic group X, such that the kernel of the covering map from G onto X is a definable central subgroup of G. If X_u is the universal covering of X and $\theta : X_u \longrightarrow G$ is the unique induced map, and $f : K \times K \longrightarrow X_u$ is the function defined in Fact 12, then the function $\theta \circ f$ is interpretable in G.*

Before we start the proof of Theorem 1 we need one last ingredient. This comes from K-theory. The kernel of the covering map π is known in K-theory as $K_2(K)$ where K is the field over which the Chevalley group is defined. Fact 12 and Fact 16 describe how the group $K_2(K)$ is presented. The definition of K_2 can be generalized to rings and K_2 is actually a functor from rings to abelian groups. We will make use of some results about K_2 to show that in our case the algebraically closed field K contains an infinite subfield over which the generators f are trivial. This will be used to prove Theorem 1.

The characteristic of the field K plays an important role. If the characteristic of K is different from 0 then Fact 13 proves that over the algebraic closure of the prime field, the generators $f(t,u)$ are all equal to 1. This will provide the necessary infinite subfield. When the characteristic of K is 0, the following two results from K-theory imply that f is trivial on $\mathbb{Q} \times \mathbb{Q}$:

Fact 18 ([8], Theorem 4.4.9, p. 225) *$K_2(\mathbb{Q})$ is a direct limit of finite abelian groups.*

Fact 19 ([1]) *If F is an algebraically closed field then $K_2(F)$ is a divisible torsion-free group.*

Now we can prove Theorem 1.

Proof of Theorem 1. The arguments above show that in all characteristics K has an infinite subfield $K_\circ$ such that for t, $u \in K_\circ$, $f(t,u) = 1$. Let $t \in K^*$. We define $B_t = \{u \in K^* : f(t,u) = 1\}$. As K is an algebraically closed field,

B_t is a subgroup of K^* by Fact 12 (a). We will show that for any $t \in K^* \setminus \{1\}$, $B_t = K^*$. This will prove the theorem. First let $t \in K_\circ$. Since $B_t \geq K_\circ^*$, Corollary 2 implies that B_t is generic in K^*. But K^* is connected (Fact 8), therefore $B_t = K^*$. Now choose t to be any element of $K^* \setminus \{1\}$. For any $u \in K \setminus \{1\}$, $f(t,u) = f(u^{-1}, t)$ by Fact 16 (c). But if $u \in K_\circ^*$ then $f(u^{-1}, t) = 1$ by the first part of the argument. Hence, $B_t \geq K_\circ^*$ and we conclude again by Corollary 2 that $B_t = K^*$. $\square$

The derivation of Corollary 1 from Theorem 1 is as in [5].

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References

[1] H. Bass and J. Tate. The Milnor ring of a global field. In *Algebraic K-Theory II*, Lecture Notes in Mathematics, no. 342, pages 349–446. Springer Verlag, 1973.

[2] A. V. Borovik and A. Nesin. *Groups of Finite Morley Rank.* Oxford University Press, 1994.

[3] J. E. Humphreys. *Linear Algebraic Groups.* Springer Verlag, Berlin-New York, second edition, 1981.

[4] A. Macintyre. On ω_1-categorical theories of fields. *Fund. Math.*, 71:1–25, 1971.

[5] T. Altınel, A. Borovik, and G. Cherlin. Groups of mixed type. to be published in J. Algebra.

[6] A. Pillay. *An Introduction to Stability Theory.* Number 8 in Oxford Logic Guides. Oxford University Press, 1983.

[7] B. Poizat. *Groupes Stables.* Nur Al-Mantiq Wal-Ma'rifah, Villeurbanne, France, 1987.

[8] J. Rosenberg. *Algebraic K-Theory and its Application.* Number 147 in Graduate Texts in Mathematics. Springer Verlag, 1994.

[9] R. Steinberg. *Lectures on Chevalley Groups.* Yale University, 1967.