Groups of mixed type

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1 Introduction

The Cherlin-Zil'ber conjecture states that an infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field. The proofs of the following three conjectures would provide a proof of the Cherlin-Zil'ber conjecture:

Conjecture 1 *There exist no nonsolvable connected groups of finite Morley rank all of whose proper definable connected subgroups are nilpotent; such groups are called* bad groups.

Conjecture 2 *There exists no structure $\langle K, +, \cdot, A\rangle$ of finite Morley rank where K is an algebraically closed field and A is an infinite proper definable subgroup of the multiplicative group of K; such structures are called* bad fields.

Conjecture 3 *A simple group of finite Morley rank which does not have definable bad sections and in which no bad fields are interpretable is an algebraic group over an algebraically closed field.*

The first conjecture appears to be very difficult, and the state of the second conjecture is unclear, but in the last decade it has become clear that ideas from finite group theory are very helpful for the classification of the simple groups of finite Morley rank satisfying the properties stated in Conjecture 3, namely *tame groups*. Indeed, the first step towards the solution of Conjecture 3 is an idea from finite group theory. One analyzes a minimal counterexample, which in the context of groups of finite Morley rank means an example "of minimal rank". Infinite, proper, definable, simple sections of such a counterexample are then algebraic groups over algebraically closed fields. We make the following definitions for our convenience:

Definition 1.1 *A group of finite Morley rank whose infinite, definable, simple sections are algebraic groups over algebraically closed fields is called a K-*group*.*

Definition 1.2 *A group of finite Morley rank whose* proper*, definable subgroups are K-groups is called a K^*-group.*

Conjecture 3 would follow from the following conjecture:

Conjecture 3′ *An infinite, simple, tame K^*-group of finite Morley rank is an algebraic group over an algebraically closed field.*

Another reason why finite group theoretic ideas seem to be relevant in the context of tame groups of finite Morley rank is the impact of involutions on the structure of these groups. The following facts illustrates this principle:

Fact 1.3 ([7], [10], [9], [18]) *Bad groups do not have involutions.*

Fact 1.4 ([6]) *Let G be a connected, tame, K^*-group of finite Morley rank. Then any connected definable section of G which does not have involutions is nilpotent.*

In order to attack Conjecture 3′, one expects to follow the lead of finite group theory in analyzing the centralizers of involutions. As part of this analysis it is important to understand the structure of the Sylow 2-subgroups of a group of finite Morley rank. The main result in this direction is the following theorem of Borovik and Poizat (for the notation S° below cf. Definition 2.6):

Fact 1.5 ([8]) *If S is a Sylow 2-subgroup of a group of finite Morley rank then the following hold:*

i) *S is nilpotent by finite.*

ii) *$S^\circ = B * T$ is a central product of a definable, connected, nilpotent subgroup B of bounded exponent and a divisible, abelian 2-group T. Moreover, B and T are uniquely determined.*

It is helpful to compare the situation in Fact 1.5 to that of algebraic groups over algebraically closed fields. algebraic groups are defined over an algebraically closed base field K whose characteristic has a strong impact on the structure of the Sylow 2-subgroups. If $char(K) = 2$ then the Sylow 2-subgroups are nilpotent and of bounded exponent; if $char(K) \neq 2$ then the Sylow 2-subgroups are (divisible abelian)-by-finite. Consider for example $SL_2(K)$, where K is an algebraically closed field. If $char(K) = 2$ then any Sylow 2-subgroup of the group is isomorphic to the additive group K^+, e.g.:

$$\left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in K^+ \right\}.$$

On the other hand, if $char(K) \neq 2$ then the Sylow 2-subgroups are isomorphic to $K^* \rtimes \mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$ acting by inversion, e.g.:

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in K^*, \lambda^{2^n} = 1 \right\} \rtimes \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

Using the terminology in Fact 1.5, one can say that in algebraic group, either $B = 1$ or $T = 1$. In the context of groups of finite Morley rank, the general picture is somewhat different. For example, if K_1 is an algebraically closed field with $char(K_1) = 2$ and K_2 is an algebraically closed field with $char(K_2) \neq 2$ then $SL_2(K_1) \times SL_2(K_2)$ is a group of finite Morley rank (although not an algebraic group) each of whose Sylow 2-subgroups is isomorphic to the direct sum of a Sylow 2-subgroup of $SL_2(K_1)$ and a Sylow 2-subgroup of $SL_2(K_2)$. Therefore, in a group of finite Morley rank, the Sylow 2-subgroups are in general a mix of the possibilities which arise in algebraic groups. Accordingly, the following definitions are natural:

Definition 1.6 *A group of finite Morley rank is said to be of* even type *if its Sylow 2-subgroups are of bounded exponent.*

A group of finite Morley rank is said to be of odd type *if the connected component of any Sylow 2-subgroup is divisible abelian.*

A group of finite Morley rank is said to be of mixed type *if it is not of one of the above types.*

If the Cherlin-Zilber Conjecture is true, then there should be no simple group of finite Morley rank of mixed type:

Conjecture 3A *A simple group of finite Morley rank cannot be of mixed type.*

We prove the following special case, which is the version that will actually be needed in the context of Conjecture $3'$.

Theorem 1.7 *There exists no simple, tame, K^*-group of finite Morley rank of mixed type.*

Thus Conjecture $3'$ splits into two parts, which can be considered independently:

Conjecture 3′E *An infinite, simple, tame K^*-group of finite Morley rank of even type is an algebraic group over an algebraically closed field of even characteristic.*

Conjecture 3′O *An infinite, simple, tame K^*-group of finite Morley rank of odd type is an algebraic group over an algebraically closed field of characteristic different from 2.*

The proof of Theorem 1.7 is by contradiction. We analyze a simple, tame, K^*-group G of mixed type. There are two main steps in this analysis. We first show that G has a *weakly* *embedded subgroup*. The second main step of our analysis allows us to conclude that this weakly embedded subgroup is a *strongly embedded subgroup*. This yields a contradiction because groups of finite Morley rank with strongly embedded subgroups cannot be of mixed type. The notions of weakly and strongly embedded subgroups are defined in Sections 3 and 7 respectively.

Both the construction of a weakly embedded subgroup and the proof of the fact that this is a strongly embedded subgroup require a detailed analysis of some special proper, definable subgroups of G. These subgroups, which are necessarily K-groups, constitute the parts of G that are of "even type" or "odd type". While analyzing these subgroups we will encounter central extensions of quasi-simple algebraic groups which are perfect tame groups H of finite Morley rank. We prove that in such cases H is itself an algebraic group. It seems this result will be useful in the analysis of K-groups arising in other situations as well.

The organization of the paper is as follows: in the next section we review the background results which we need. In Section 3 we begin the discussion of groups of finite Morley rank with weakly embedded subgroups. Section 4 is devoted to central extensions. In Section 5 we analyze the "even" and "odd" parts of G and the interactions between these parts. In Section 6 we find a weakly embedded subgroup in G. The final section, Section 7, contains the proof of the result that the weakly embedded subgroup of Section 6 is strongly embedded. Section 7 also contains the results about strongly embedded subgroups which imply that this second main step of the proof yields a contradiction.

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2 Background

In this section we list various facts needed in the sequel. For any subset X of any group G, $I(X)$ will be used to denote the set of involutions in X. If $t \in G$ then $C_t = C_G(t)$.

Definition 2.1 *Let G be a group of finite Morley rank. A subgroup H of G is* definably characteristic *if it is invariant under definable group automorphisms.*

In the sequel, characteristic will mean definably characteristic.

Fact 2.2 ([6], Exercise 10 page 78) *Let G be a group of finite Morley rank. G° contains all connected definable subgroups of G.*

Proof. See the hints on page 78 of [6]. $\square$

Fact 2.3 ([26]) *Let G be a group of finite Morley rank. The subgroup generated by a set of definable connected subgroups of G is definable and connected and it is the setwise product of finitely many of them.*

Fact 2.4 ([26]) *Let $H \leq G$ be a definable connected subgroup. Let $X \subseteq G$ be any subset. Then the subgroup $[H, X]$ is definable and connected.*

Fact 2.5 ([26]) *Let G be a group of finite Morley rank. Then G^n and $G^{(n)}$ are definable. If G is connected, then G^n and $G^{(n)}$ are connected.*

Definition 2.6 *Let G be a group of finite Morley rank. If $X \subseteq G$, then the definable closure of X, denoted by $d(X)$ is the intersection of all the definable subgroups of G which contain X. Note that, as groups of finite Morley rank satisfy the descending chain condition on definable subgroups, this intersection is finite and therefore definable. If X is a subgroup of G then the connected component of X, denoted by X°, is defined to be $X \cap d(X)^\circ$.*

Fact 2.7 ([6], Lemma 5.34) *Let G be a group of finite Morley rank. Assume $X \leq G$. Then $C_G(X) = C_G(d(X))$.*

Proof. $d(X) \leq Z(C_G(X))$. $\square$

Fact 2.8 ([6], Lemma 5.36) *Let G be a group of finite Morley rank. If B is a definable normal subgroup of G and $B \subseteq X \subseteq G$, then $d(X/B) = d(X)/B$.*

Fact 2.9 ([6], Corollary 5.38) *Let G be a group of finite Morley rank and H be a subgroup of G. If H is solvable (resp. nilpotent) of class n, then $d(H)$ is solvable (resp. nilotent) of class exactly n.*

Fact 2.10 ([15]) *Let G be an abelian group of finite Morley rank. Then the following hold:*

i) *$G = D \oplus B$ where D is a divisible subgroup and B is a subgroup of bounded exponent.*

ii) *$D \cong \oplus_{p\ prime}(\oplus_{I_p}\mathbb{Z}_{p^\infty}) \oplus \oplus_I \mathbb{Q}$ where the index sets I_p are finite.*

iii) *$G = DC$ where D and C are definable characteristic subgroups of G, D is divisible, C has bounded exponent and their intersection is finite. The subgroup D is connected If G is connected, then C can be taken to be connected.*

Fact 2.11 ([22]) *Let G be a nilpotent group of finite Morley rank. Then $G = D * C$, where D and C are definable characteristic subgroups of G, D is divisible, and C is of bounded exponent.*

Fact 2.12 ([22]) *Let G be a divisible nilpotent group of finite Morley rank. Let T be the torsion part of G. Then T is central in G and $G = T \oplus N$ for some torsion-free divisible nilpotent subgroup N.*

Fact 2.13 ([6], Lemma 6.2) *If G is an infinite nilpotent group of finite Morley rank then $Z(G)$ is infinite.*

Fact 2.14 ([6], Lemma 6.3) *Let G be a nilpotent group of finite Morley rank . If $H < G$ is a definable subgroup of infinite index then $N_G(H)/H$ is infinite.*

Definition 2.15 *Let G be a group of finite Morley rank. Let $\sigma(G)$ be the subgroup generated by all the normal solvable subgroups of G, and $F(G)$ be the subgroup generated by all the normal nilpotent subgroups of G. $\sigma(G)$ is called the solvable radical of G. $F(G)$ is called the Fitting subgroup of G.*

Fact 2.16 ([1, 21]) *Let G be a group of finite Morley rank. Then $F(G)$ and $\sigma(G)$ are definable, and they are nilpotent and solvable respectively.*

Fact 2.17 ([27]) *Let $G = A \rtimes H$ be a group of finite Morley rank where A and H are infinite definable abelian subgroups and A is H-minimal. Assume $C_H(A) = 1$. Then the following hold:*

i) The subring $K = \mathbb{Z}[H]/ann_{\mathbb{Z}[H]}(A)$ of $End(A)$ is a definable algebraically closed field; in fact, there is an integer l such that every element of K can be represented as the endomorphism $\sum_{i=1}^{l} h_i$, where $(h_i \in H)$.

ii) $A \cong K^+$, H is isomorphic to a subgroup T of K^* and H acts on A by multiplication, in other words:

$$G = A \rtimes H \cong \left\{ \begin{pmatrix} t & a \\ 0 & 1 \end{pmatrix} : t \in T, a \in K \right\}.$$

iii) In particular, H acts freely on A, $K = T + \ldots + T$ (l times) and with the additive notation $A = \{\sum_{i=1}^{l} h_i a\ :\ h_i \in H\}$ for any $a \in A^*$.

Fact 2.18 ([27]) *Let G be a connected nonnilpotent solvable group of finite Morley rank. Then G interprets an algebraically closed field K. More precisely, a definable section of $F(G)$ is isomorphic to K^+ and a definable section of $G/F(G)$ is isomorphic to an infinite subgroup of K^*.*

Fact 2.19 ([20]) *Let G be a connected solvable group of finite Morley rank. Then $G/F(G)^\circ$ (hence, $G/F(G)$) is a divisible abelian group.*

Fact 2.20 ([12]) *Let G be a nilpotent group. For a given prime p, G has a unique Sylow p-subgroup. If all elements of G are of finite order then G is the direct sum of its Sylow p-subgroups.*

Fact 2.21 ([6], Exercises 11, 12 pages 13, 14) *If G is a nilpotent by finite p-group. Then the following hold:*

i) $Z(G) \neq 1$.

ii) *If H is a nontrivial normal subgroup of G then $Z(G) \cap H \neq 1$.*

iii) *For any $X < G$, $X < N_G(X)$; that is, G satisfies the normalizer condition.*

Proof. See [16]. □

Fact 2.22 ([4]) *Let G be group of finite Morley rank and H be a definable normal subgroup of G. If x is an element of G such that $\overline{x}$ is a p-element of $\overline{G} = G/H$ then the coset xH contains a p-element. In particular, if G is torsion-free, then G/H is torsion-free.*

Definition 2.23 *For any prime number p, a $p^{\perp}$-*element *g is an element whose order is either infinite or relatively prime to p. A p'-*element *is an element of finite order relatively prime to p.*

Fact 2.24 ([6], Exercise 11 page 72) *Let G be a $p^{\perp}$-group of finite Morley rank, where p is a prime number. Then G is p-divisible.*

Proof. See [16]. □

Fact 2.25 ([6], Exercise 12 page 72) *Let G a $p^{\perp}$-group of finite Morley rank. Then every element of G has a unique pth root.*

Proof. Solve the exercise. □

Fact 2.26 ([5]) *Let K and L be definable $p^{\perp}$-subgroups of a group G of finite Morley rank. Assume K normalizes L. Then KL is also a $p^{\perp}$-subgroup.*

Fact 2.27 ([8]) *Let G be a group of finite Morley rank. Let D be a divisible abelian subgroup of G. Then for every prime p, D has finitely many elements of order p.*

Fact 2.28 ([6], Exercise 1 page 97) *An infinite nilpotent p-group of finite Morley rank and of bounded exponent has infinitely many central elements of order p.*

Proof. See [16]. □

Definition 2.29 *A divisible abelian p-subgroup of a group of a finite Morley rank is called a* p-torus.

Fact 2.30 ([8]) *Let T be a p-torus in a group of finite Morley rank. Then $[N_G(T) : C_G(T)] < \infty$. Moreover, there exists $c \in \mathbb{N}$ such that $[N_G(T) : C_G(T)] \leq c$ for any torus in G.*

Fact 2.31 ([8]) *The Sylow 2-subgroups of a group of finite Morley rank are conjugate.*

Fact 2.32 ([6], Lemma 10.22) *Let S and T be as in fact 1.5 . If $X, Y \subseteq S^{\circ}$ and $X^g = Y$, where $g \in G$ then there exists $h \in N_G(T)$ such that $X^h = Y$ (that is, $N_G(T)$* controls fusion *in S°).*

Fact 2.33 ([17]) *Let G be ω-stable and solvable, $N \lhd G$, and let H be a Hall π-subgroup of G for some set π of primes. Then:*

i) *$H \cap N$ is a Hall π-subgroup of N, and all Hall π-subgroups of N are of this form.*

ii) *If N is definable then HN/N is a Hall π-subgroup of G/N, and all Hall π-subgroups of G/N are of this form.*

Fact 2.34 ([4]) *Let G be a connected solvable group of finite Morley rank. Then the Hall π-subgroups of G are connected.*

A special case of the following fact was proven in [16].

Fact 2.35 *Let Y be a connected solvable group of finite Morley rank and S be a Sylow 2-subgroup of Y. If B is the unique largest unipotent 2-subgroup of S as in Fact 1.5 then $B \leq F(Y)$. In particular, B is a characteristic subgroup of Y.*

Proof. By Fact 2.19, $Y/F(Y)$ is a divisible abelian group. Thus, by Fact 2.27, $Y/F(Y)$ has finitely many involutions. On the other hand a nontrivial unipotent 2-subgroup has infinitely many involutions by Fact 2.28. Therefore, $Y/F(Y)$ has no nontrivial unipotent 2-subgroups and this implies $B \leq F(Y)$. By Fact 2.20, B is contained in the unique Sylow 2-subgroup of $F(Y)$. Therefore, B is characteristic in Y. $\square$

Fact 2.36 ([6]) *Let Q and E be subgroups of a group of finite Morley rank such that Q is normal, $2^{\perp}$, connected, nilpotent, definable and E is a definable connected 2-group of bounded exponent. Then $[Q, E]$=1.*

Fact 2.37 ([19]) *Let α be a definable involutive automorphism of a group of finite Morley rank G. If α has finitely many fixed points then G has a definable normal subgroup of finite index which is abelian and inverted by α.*

Fact 2.38 ([19]) *Let α be a definable involutive automorphism of a group of finite Morley rank G. If α has no nontrivial fixed points then G is abelian and inverted by α.*

Fact 2.39 ([6], Theorem 8.4) *Let $\mathcal{G} = G \rtimes H$ be a group of finite Morley rank where G and H are definable, G is an infinite simple algebraic group over an algebraically closed field, and $C_H(G) = 1$. Then, viewing H as a subgroup of $Aut(G)$, we have $H \leq Inn(G)\Gamma$, where $Inn(G)$ is the group of inner automorphisms of G and Γ is the group of graph automorphisms of G.*

Fact 2.40 ([16]) *Let G be a connected nonsolvable K-group of finite Morley rank. Then $G/\sigma(G)$ is isomorphic to a direct sum of simple algebraic groups over algebraically closed fields.*

In Section 5, we will need the following facts about algebraic group s over algebraically closed fields. Apart from Fact 2.45, these facts are found in [25], [14] or [13], which are our main references for the theory of algebraic groups and related subjects.

Definition 2.41 ([14], Section 7.5) *Let G be an algebraic group over an algebraically closed field and M an arbitrary subset of G. $\mathcal{A}(M)$* (the group closure of M) *is the intersection of all closed subgroups of G containing M.*

Fact 2.42 ([14], Proposition 7.5) *Let H be an algebraic group over an algebraically closed field, I an index set, $f_i : X_i \longrightarrow G$ $(i \in I)$ a family of morphisms from irreducible varieties X_i, such that each $Y_i = f(X_i)$ contains the identity element of G. Set $M = \cup_{i \in I} Y_i$. Then:*

a) *$\mathcal{A}(M)$ is a connected subgroup of G.*

b) *For some finite sequence $a = (a(1), \cdots, a(n))$ in I,*

$$\mathcal{A}(M) = Y_{a(1)}^{e_1} \cdots Y_{a(n)}^{e_n} \ (e_i = \pm 1).$$

Fact 2.43 ([14], Theorem 17.6) *Let G be a connected solvable subgroup of $GL(V)$, $0 \neq V$ finite dimensional. Then G has a common eigenvector.*

Definition 2.44 *Let G be a group of finite Morley rank. $O(G)$ is the largest, definable, connected, normal $2^{\perp}$-subgroup of G.*

Fact 2.45 ([3]) *If H is a simple algebraic group over an algebraically closed field K of characteristic different from 2, and t is an involutive automorphism of H, then $F(C_t) = Z(C_t^\circ)$ is a finite extension of an algebraic torus over K and $E(C_t) = L_1 \cdots L_n$ is the central product of quasi-simple subgroups L_i which are algebraic groups over K. In particular, $O(C_t) = 1$ and $C_t^\circ = F(C_t)^\circ E(C_t)$.*

Fact 2.46 ([25], Corollary to Lemma 15) *Let α, β be roots with $\alpha + \beta \neq 0$. Let k be a field. Then*

$$(x_\alpha(t), x_\beta(t)) = \prod x_{i\alpha+j\beta}(c_{ij}t^i u^j)$$

where $(A, B) = ABA^{-1}B^{-1}$, where the product on the right is taken over all roots $i\alpha + j\beta$ ($i, j \in \mathbb{Z}$) arranged in some fixed order, and where the c_{ij} are integers depending on α, β and the chosen ordering, but not on t and u.

Fact 2.47 ([25], Example (a) on page 24) *If $\alpha + \beta$ is not a root, then the right side of the commutator formula in Fact 2.46 reduces to 1.*

Fact 2.48 ([13], Lemma 10.1) *Let Φ be a root system. If Δ is a base of Φ, then $(\alpha, \beta) \leq 0$ for $\alpha \neq \beta$ in Δ, and $\alpha - \beta$ is not a root.*

Fact 2.49 ([14], Theorem 27.3) *Let G be a reductive algebraic group, T a fixed maximal torus and $\Phi = \Phi(G, T)$. Let Δ be a base of Φ. Let Z_α denote $C_G(T_\alpha)$ where $T_\alpha = (ker\alpha)^\circ$ and $\alpha \in \Phi$. Then G is generated by the Z_α ($\alpha \in \Delta$), or equivalently by T along with all U_α ($\pm\alpha \in \Delta$).*

Fact 2.50 ([14], Corollary 32.3) *A semisimple algebraic group of* Lie rank 1 *is isomorphic to $SL_2(K)$ or $PGL_2(K)$.*

We conclude with some useful technical results on groups of finite Morley rank.

Proposition 2.51 *Let Q be a group of finite Morley rank and i an involution acting on Q. Suppose Q_1 is a 2-divisible, i-invariant, definable, normal subgroup of Q. If in Q_1 every element has a unique square root or if i inverts Q_1 then $C_{Q/Q_1}(i) = C_Q(i)Q_1/Q_1$.*

Proof. We have to show that for every $x \in Q$, $x^{-1}x^i \in Q_1$ implies that there exists $y \in Q_1$ such that $xy \in C_Q(i)$.

First, assume that i inverts Q_1. Let $x \in Q$ such that $x^{-1}x^i \in Q_1$. As Q_1 is 2-divisible, there is $y \in Q_1$ such that $x^{-1}x^i = y^2$. This implies that $xy = x^i y^{-1} = (xy)^i$. We are done.

Now assume that in Q_1 every element has a unique square root. Consider $Q_1^- = \{z \in Q_1 : z^i = z^{-1}\}$. Let $z \in Q_1^-$. As Q_1 is 2-divisible, z has a square root $z_1 \in Q_1$. $z^i = (z_1^2)^i = (z_1^i)^2$ and $z^{-1} = (z_1^2)^{-1} = (z_1^{-1})^2$. By the uniqueness of the square roots, $z_1^{-1} = (z_1)^i$. Hence, Q_1^- is a 2-divisible set. Let x be as above. Then $x^{-1}x^i \in Q_1^-$. Hence, $x^{-1}x^i = y^2$ for some $y \in Q_1^-$. Now, by the same argument as in the previous paragraph, we are done. $\square$

Proposition 2.52 *Let $G = d(S)Q$ be a group of finite Morley rank, where S is a Sylow 2-subgroup of G and Q is a normal, definable, solvable $2^{\perp}$-subgroup. Assume that Q has a definable, normal subgroup Q_1 which is normalized by $d(S)$ also. Assume also that $S \lhd d(S)$. Then $C_Q(d(S))Q_1/Q_1 = C_{Q/Q_1}(d(S))$.*

Proof. Let $L/Q_1 = C_{Q/Q_1}(d(S))$. It is sufficient to show that $L \leq C_Q(d(S))Q_1$. Let $x \in L$. The assumptions imply that $S^x \leq Q_1 d(S)$. By the conjugacy of Sylow 2-subgroups and the fact that $S \lhd d(S)$ we can find $y \in Q_1$ such that $S^x = S^y$. Thus $xy^{-1} \in N_Q(S) = C_Q(S) = C_Q(d(S))$ (Fact 2.7). This implies $L \leq C_Q(d(S))Q_1$. $\square$

Proposition 2.53 *Let $Q \rtimes V$ be a group of finite Morley rank, with Q a definable, connected, solvable $2^{\perp}$-group and V an elementary abelian 2-group of order at least 4 (i.e. the dimension of V as a vector space over $\mathbb{Z}_2$ is at least 2). Then $Q = \langle C_Q(V_\alpha) : V_\alpha \leq V$, and $dim(V_\alpha) = dim(V) - 1\rangle$.*

Proof. As Q is a $2^{\perp}$-group of finite Morley rank, every element has a unique square root by Fact 2.25. This allows us to use Proposition 2.52. As Q is solvable, by induction on the rank of Q, Proposition 2.52 and Fact 2.5, we may assume that Q is abelian.

We will prove the statement by induction on $n = dim(V)$. Assume first that $n = 2$. For $u \in V \setminus \{1\}$, the set $Q_u = \{[a, u] : a \in Q\}$ is the subgroup $[Q, u]$. As this subgroup is 2-divisible, for every $a \in Q$, there exists $b \in Q_u$ such that $[a, u] = b^2$. Equvialently $a^u b^{-1} = ab$. But b is inverted by u, thus $(ab)^u = ab$. Since $a = abb^{-1}$, we conclude that $Q = [Q, u]C_Q(u)$.

Note that any element of V normalizes $[Q, u]$. Thus, if v is another nontrivial element of V then an application of the argument in the above paragraph to $[Q, u]$ implies that $[Q, u] = [[Q, u], v]C_{[Q,u]}(v)$. The remaining nontrivial element V is nothing but uv. Since both u and v invert $[[Q, u], v]$, uv centralizes this last subgroup. Therefore, $Q = [[Q, u], v]C_{[Q,u]}(v)C_Q(u) \leq C_Q(uv)C_q(v)C_Q(u)$. This finishes the proof for $n = 2$.

Assume $n > 2$. $V = \langle v_1, v_2, v_3, \ldots, v_n\rangle$. We have $Q = [Q, v_1]C_Q(v_1)$. We may assume that $C_Q(v_1) \neq 1$. Note that the group $\langle v_2, \ldots, v_n\rangle$ normalizes $C_Q(v_1)$. By induction we conclude that $C_Q(v_1) = \langle C_Q(W) \cap C_Q(v_1) : W \leq \langle v_2, \ldots, v_n\rangle, dim(W) = n - 2\rangle$. Therefore, $C_Q(v_1) = \langle C_Q(W \oplus \langle v_1\rangle) \cap C_Q(v_1) : W \leq \langle v_2, \ldots, v_n\rangle, dim(W) = n - 2\rangle$.

We then analyze the action of $\langle v_1, \ldots, v_n\rangle$ on $[Q, v_1]$. $[Q, v_1] = [[Q, v_1], v_2]C_{[Q,v_1]}(v_2)$. As $\langle v_1, v_3, \ldots, v_n\rangle$ normalizes $C_{[Q,v_1]}(v_2)$, we obtain a generation statement for $C_{[Q,v_1]}(v_2)$ as in the last paragraph. Therefore we consider the action of $\langle v_1, \ldots, v_n\rangle$ on $[[Q, v_1], v_2]$.

Continuing in this manner we obtain

$$Q = [\cdots [Q, v_1] \cdots, v_n]C_{[\cdots[Q,v_1]\cdots,v_{n-1}]}(v_n) \ldots C_{[Q,v_1]}(v_2)C_Q(v_1)$$

Our arguments have shown that all the centralizers in the last statement are generated bu the centralizers of codimension 1 vector subspaces of V. As $[\cdots [Q, v_1] \cdots, v_n]$ is inverted by $v_1, \ldots, v_n$, it is centralized by $v_1v_2, v_1v_3, \ldots, v_1v_n$. This finishes the proof. $\square$

Corollary 2.54 *Let QV be a group of finite Morley rank, where Q is a definable, connected, solvable $2^{\perp}$-group and V is a definable subgroup whose Sylow 2-subgroups are 2-tori that contain $n \geq 2$ copies of $\mathbb{Z}_{2^\infty}$ (i.e. their Prufer rank is at least 2.) If T is a Sylow 2-subgroup of V then $Q = \langle C_Q(T_\alpha) : T_\alpha \leq T$, and the Prufer rank of $T_\alpha = n - 1\rangle$.*

Proof. It is sufficient to show that if $v \in I(T)$, then $C_Q(T_v) = C_Q(v)$, where T_v is the copy of $\mathbb{Z}_{2^\infty}$ in T which contains v. Let w be a squareroot v in T_v and $x \in C_Q(v)$. We have $w^2 = (w^2)^x = (w^x)^2$. Since T is abelian, $[w, x]^2 = 1$ and therefore $[w, x] = 1$ or $[w, x] = v$. If the first possibility holds then there is nothing to do, thus we assume that $[w, x] = v$. Then a small calculation shows that $w^{x^2} = w$. Therefore $x^2 \in C_Q(w)$. But Q is a $2^{\perp}$-group. Therefore by Fact 2.22 $x \in C_Q(w)$. We get $C_Q(w) = C_Q(v)$. Continuing this way we conclude that $C_Q(T_v) = C_Q(v)$. $\square$

3 Weakly Embedded Subgroups

Definition 3.1 *Let G be a group of finite Morley rank. A proper definable subgroup M of G is said to be* weakly embedded *if it satisfies the following conditions:*

i) *Any Sylow 2-subgroup of M is infinite.*

ii) *For any $g \in G \setminus M$, $M \cap M^g$ has finite Sylow 2-subgroups.*

We first prove a few basic properties of weakly embedded subgroups.

Proposition 3.2 *Let G be a group of finite Morley rank with a weakly embedded subgroup M. Then the following hold:*

i) *For any Sylow 2-subgroup S of M, $N_G(S) \leq M$.*

ii) *If S is a Sylow 2-subgroup of M then S is a Sylow 2-subgroup of G.*

Proof. i) This is immediate.

ii) Let S be a Sylow 2-subgroup of M and let suppose $T > S$ is a Sylow 2-subgroup of G. By Facts 1.5 i) and 2.21 iii), $N_T(S) > S$. But $N_T(S) \cap M = S$ as S is Sylow 2-subgroup of M. This contradicts part i). $\square$

Definition 3.3 *A definable, connected, 2-subgroup of bounded exponent of a group of finite Morley rank is called a* unipotent 2-subgroup.

Note that a unipotent 2-subgroup is nilpotent by Fact 1.5 i).

Now we give a characterization of weakly embedded subgroups.

Proposition 3.4 *Let G be a group of finite Morley rank. A proper definable subgroup M of G is a weakly embedded subgroup if and only if the following hold:*

i) *M has infinite Sylow 2-subgroups.*

ii) *For any unipotent 2-subgroup U and 2-torus T in M, $N_G(U) \leq M$, and $N_G(T) \leq M$.*

Proof.

It is clear that if M is a weakly embedded subgroup then it satisfies the above conditions. Therefore, we assume that M is a proper definable subgroup with the above properties and show that it is a weakly embedded subgroup. Let $g \in G$ and suppose S is an infinite Sylow 2-subgroup of $M \cap M^g$. We will show that $g \in M$.

Let S_1 be a Sylow 2-subgroup of M, such that $S \leq S_1$. By Fact 2.2 and Definition 2.6, $S^\circ \leq S_1^\circ$. By Fact 2.21 i), $Z(S_1^\circ) \neq 1$. $Z(S_1^\circ) \leq C_G(Z(S^\circ))$. By Fact 1.5, $S^\circ = B * T$ where B is a unipotent 2-subgroup and T is a 2-torus. If $T \neq 1$ then it is central in S° and therefore, $Z(S_1^\circ)$ centralizes a nontrivial 2-torus of M^g. On the other hand, if $T = 1$ then S° is a unipotent 2-subgroup and by Fact 2.13, $Z(S^\circ)^\circ$ is a nontrivial unipotent 2-subgroup. Therefore, if $T = 1$, then $Z(S_1^\circ)$ centralizes a nontrivial unipotent 2-subgroup of M^g. This implies that $Z(S_1^\circ) \leq M^g$.

By Fact 1.5, $S_1^\circ = B_1 * T_1$, where B_1 is a unipotent 2-subgroup and T_1 is 2-torus. If $T_1 \neq 1$ then being a subgroup of $Z(S_1^\circ)$, it forces $Z(S_1^\circ)$ to be infinite. If $T_1 = 1$ then S_1° is a unipotent 2-subgroup and has an infinite center by Fact 2.13. Therefore, $Z(S_1^\circ)$ contains a nontrivial characteristic subgroup which is either a unipotent 2-subgroup or a 2-torus. Hence, $S_1 \leq N_G(Z(S_1^\circ)) \leq M^g$. This means $S_1^g = S_1^x$, for some $x \in M^g$. Hence, $gx^{-1} \in N_G(S_1) \leq M^g$, and we get $g \in M^g$, forcing $g \in M$. $\square$

4 Central Extensions

It is possible to develop a theory of central extensions for tame groups. In this section, we show how to achieve this. Our proofs make use of the theory of central extensions of linear algebraic group s as explained in [25] and also of the "no bad fields" hypothesis.

We prove the following theorem.

Theorem 4.1 *Let G be a perfect tame group of finite Morley rank and let C be a definable central subgroup of G such that G/C is a universal linear algebraic group over an algebraically closed field; G is a central extension of finite Morley rank of a universal linear algebraic group. Then $C = 1$.*

In the proof of this theorem our notation and terminology will be the same as in [25] unless otherwise stated. We differ from [25], and indeed from the theory of linear algebraic groups in general, in the use of the word *simple*, which we apply only to groups which are simple as *abstract* groups.

Theorem 4.1 will be used in proving the following theorem:

Theorem 4.2 *Let G be a perfect tame group of finite Morley rank such that $G/Z(G)$ is a quasisimple algebraic group. Then G is an algebraic group. In particular, $Z(G)$ is finite ([14] Section 27.5).*

Before we start our argument, we would like to mention two results by Pillay and Sokolovic ([23]) which are related to Theorem 4.2 and Proposition 4.10. In the fact below *almost simple* means quasisimple.

Fact 4.3 ([23], Lemma 8) *Let G be an almost simple algebraic group over an algebraically closed field K. Then definable in the structure $(G, .)$ there are, a subset F of G, an algebraically closed field structure on F, and for some n a definable surjection from F^n onto G.*

Fact 4.4 ([23], Theorem 10) *Let M be an ω-stable structure. Let K be an algebraically closed field definable in M, and H an almost simple algebraic group over K. Let G be a definably almost simple group definable in M such that there is a definable (in M) homomorphism f from G onto H. The there is an algebraic group H_1 over K and a definable (in M) isomorphism of G with H_1.*

Throughout this section, for any group G and x, $y \in G$, (x, y) will denote $xyx^{-1}y^{-1}$ as in [25].

We start with an overview of some results from [25]. In [25], for a field k, and a root system Σ, the following relations over the set of symbols $\{x_\alpha(t) : \alpha \in \Sigma, t \in k\}$ are defined:

(A) $x_\alpha(t)$ is additive.

(B) If α and β are roots and $\alpha + \beta \neq 0$, then $(x_\alpha(t), x_\beta(t)) = \prod x_{i\alpha+j\beta}(c_{ij}t^i u^j)$, where i and j are positive integers and the c_{ij} are integers depending on α, β, and the chosen ordering of the roots, but not on t or u.

(B′) $w_\alpha(t)x_\alpha(u)w_\alpha(-t) = x_{-\alpha}(-t^{-2}u)$ for $t \in k^*$, where

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$$

for $t \in k^*$.

(C) $h_\alpha(t)$ is multiplicative in t, where $h_\alpha(t) = w_\alpha(t)w_\alpha(-1)$ for $t \in k^*$.

Let X_u denote the group presented by (A) and (B) if rank $\Sigma > 1$ and by (A) and (B′) if rank $\Sigma = 1$. If the the relation (C) is added, then we get the universal Chevalley group (see [25]; the notation X_u is different from the one used in [25].)

We quote the following lemmas and theorems:

Lemma 4.5 ([25], Lemma 39, p.70) *Let α be a root and X_u be as above. In X_u, set $f(t, u) = h_\alpha(t)h_\alpha(u)h_\alpha(tu)^{-1}$. Then:*
a) $f(t, u^2v) = f(t, u^2)f(t, v)$.
b) If t, u generate a cyclic subgroup of k^* then $f(t, u) = f(u, t)$.
c) If $f(t, u) = f(u, t)$, then $f(t, u^2) = 1$.
d) If $t, u \neq 0$ and $t + u = 1$, then $f(t, u) = 1$.

Theorem 4.6 ([25], Theorem 9, p.72) *Assume that Σ is indecomposable and that k is an algebraic extension of a finite field. Then the relations (A) and (B) (or (B′) if rank $\Sigma > 1$) suffice to define the corresponding universal Chevalley group, i.e. they imply the relations (C).*

The following theorems and corollaries about central extensions are proven in [25].

Theorem 4.7 ([25], Theorem 10, p. 78) *Let Σ be an indecomposable root system and k a field such that $|k| > 4$, and if rank $\Sigma = 1$, assume further that $|k| \neq 9$. If X is the corresponding universal Chevalley group (abstractly defined by the relations (A), (B), (B′), (C) above), if X_u is the group defined by the relations (A), (B), (B′) (we use (B′) only if rank $\Sigma = 1$), and if π is the natural homomorphism from X_u to X, then (π, X_u) is a universal covering extension of X.*

Corollary 4.8 ([25]) *X_u is centrally closed. Each of its central extensions splits, i.e. its Schur multiplier is trivial. It yields the universal covering extension of all the Chevalley groups of the given type.*

Theorem 4.9 ([25], Theorem 12 (Matsumoro, Moore)) *Assume that Σ is an indecomposable root system and k a field with $|k| > 4$. If X is the universal Chevalley group based on Σ and k, if X_u is the group defined by (A), (B), (B′), and if π is the natural map from X_u to X with $C = ker\pi$, the Schur multiplier of X, then C is isomorphic to the abstract group generated by the the symbols $\{t, u\}$ $(t, u \in k^*)$ subject to the relations:*

a) $\{t, u\}\{tu, v\} = \{t, uv\}\{u, v\}$; $\{1, u\} = \{u, 1\} = 1$

b) $\{t, u\}\{t, -u^{-1}\} = \{t, -1\}$

c) $\{t, u\} = \{u^{-1}, t\}$

d) $\{t, u\} = \{t, -tu\}$

e) $\{t, u\} = \{t, (1 - t)u\}$

and in the case Σ is not of the type C_n $(n \geq 1)$ the additional relation

ab′) $\{\ ,\ \}$ *is bimultiplicative .*

In this case relations a)-e) may be replaced by ab′) and

c′) $\{\ ,\ \}$ *is skew.*

d′) $\{t, -t\} = 1$.

The isomorphism is given by $\phi : \{t, u\} \longmapsto h_\alpha(t)h_\alpha(u)h_\alpha(tu)^{-1}$, α a fixed long root.

We have a perfect tame central extension G of finite Morley rank of a universal linear algebraic group X over an algebraically closed field K. Let X_u be the universal covering extension of X. Let $C_\circ = ker\psi$, where ψ is the covering map from G onto X. By the universality of (π, X_u), there exists a map θ from X_u into G such that $\psi\theta = \pi$. Then $\theta(X_u)C_\circ = G$. But $G = (G, G) = (\theta(X_u), \theta(X_u)) = \theta(X_u, X_u) = \theta(X_u)$. Hence, θ is surjective. Then $C = ker\pi = ker\psi\theta = \theta^{-1}ker\psi = \theta^{-1}(C_\circ)$.

If f is the function defined in Lemma 4.5, we will need to show that $\theta \circ f : K \times K \longrightarrow Z(G)$ is interpretable in G. The following fact is proved in [24]:

Fact 4.10 ([24], Corollaire 4.16) *In a simple algebraic group over an algebraically closed field, definability from the field and definability from the pure group coincide.*

We prove a generalization of this fact. The proof makes use of the following results from [24]:

Fact 4.11 ([24], Corollaire 4.2) *There are no constructible (in the sense of algebraic geometry) bad groups.*

Fact 4.12 ([24], Théorème 4.15) *An infinite field definable in a pure algebraically closed field F is definably (in F) isomorphic to F.*

Proposition 4.13 *In a quasisimple algebraic group over an algebraically closed field , definability from the field and definability from the pure group coincide.*

Proof. Let $X = X(F)$ be a quasisimple algebraic group over an algebraically closed field F. We will use $\overline{}$-notation to denote quotients by $Z(X)$. By Fact 4.11, no bad groups are definable in X. Therefore, we can interpret an algebraically closed field in $X/Z(X)$ using its Borel subgroups. Let us denote this field by K. As K is interpretable in F, these two fields are definably isomorphic in F by Fact 4.12. Let us denote this isomorphism by θ. Let $X(K)$ be a linear algebraic group over K isomorphic to $X(F)$ by an isomorphism ψ induced by θ. The isomorphisms $\psi_\alpha : X_\alpha(F) \longrightarrow X_\alpha(K)$ defined by $\psi_\alpha(x_\alpha(t)) = x_\alpha(\theta(t))$ are definable in X since they can be written as the composition of the isomorphism $X_\alpha(F) \longrightarrow \overline{X_\alpha(F)}$ induced by the canonical homomorphism $X \longrightarrow \overline{X}$ with the definable isomorphism $\overline{X_\alpha(F)} \longrightarrow X_\alpha(K)$ given by

$$\overline{x_\alpha(t)} \longmapsto t \longmapsto \theta(t) \longmapsto x_\alpha(\theta(t))$$

where the maps $\overline{x_\alpha(t)} \longmapsto t$ and $\theta(t) \longmapsto x_\alpha(\theta(t))$ are F-algebraic, hence definable in $\overline{X}$ by Fact 4.10.

The isomorphisms ψ_α are induced by the isomorphism $\psi : X(F) \longrightarrow X(K)$ and conversely ψ is definable from the ψ_α since every element of X is a product of a bounded number of elements from the root groups X_α. Therefore, ψ is definable.

Now, let A be a subset of $X(F)^n$ definable from F. $\psi(A)$ is definable in $X(K)^n$ from K and hence is definable in $\overline{X}$. But ψ and K are definable in X. Hence, $A = \psi^{-1}(\psi(A))$ is definable in X. □

Proposition 4.14 *Let G be a tame group of finite Morley rank. Assume that G is a perfect central extension of a universal linear algebraic group X, such that the kernel of the covering map from G onto X is a definable central subgroup of G. If X_u is the universal covering of X and $\theta : X_u \longrightarrow G$ is the unique induced map as in Theorem 4.7, and $f : K \times K \longrightarrow (X_u)$ is the function defined in Lemma 4.5, then the function $\theta \circ f$ is interpretable in G.*

Proof. In order to prove Theorem 4.7, Steinberg proves that the relations (A), (B) and (B′) can be lifted from a universal linear algebraic group X to any of its central extensions. To do so he starts with a central extension (ψ, G) of X and he constructs a map ϕ from the root subgroups of X into G. We will make use of this map in order to show that the function $\theta \circ f$ is interpretable in G.

The first step in the proof is to show that ϕ is interpretable in G. To do so, we need to look at the definition of ϕ. First, an element a of K^* is chosen so that $c = a^2 - 1 \neq 0$. In G/C, $(h_\alpha(a), x_\alpha(t)) = x_\alpha(ct)$ for all $\alpha \in \Sigma$, $t \in K$. Then $\phi(x_\alpha(t))$ is defined so that:
i) $\psi(\phi(x_\alpha(t))) = x_\alpha(t)$
ii) $(\phi(h_\alpha(a)), \phi(x_\alpha(t))) = \phi(x_\alpha(ct))$.

Steinberg observes that this determines ϕ as a map from the root group $X_\alpha = \langle x_\alpha(t) : t \in K\rangle$ into G. The x_α are definable from the field over which X is defined. Therefore, by Proposition 4.13, they are definable from the pure group G. On the other hand, the following formula defines ϕ:

$$\phi(x) = y$$
$$\text{if and only if}$$
$$\exists x_1, y_1(\psi(x_1) = y_1 \ \& \ (g_\circ, y_1) = y \ \& \ \exists t(x_1 = x_\alpha(t) \ \& \ x = x_\alpha(ct))),$$

where $g_\circ$ is the group element defined by $h_\alpha(a)$. As a result, we conclude that ϕ is an interpretable map from X_α into G. One can do the same thing for all roots and get a map ϕ which lifts (interpretably in G) the X_α from X to G.

Now we define the following functions from K into G:

$$\begin{array}{rcl}\overline{w_\alpha}(t) & = & \phi(x_\alpha(t))\phi(x_{-\alpha}(-t^{-1}))\phi(x_\alpha(t)) \\ \overline{h_\alpha}(t) & = & \overline{w_\alpha}(t)\overline{w_\alpha}(-1)\end{array}$$

As ϕ and the x_α are interpretable in G, so is $\overline{w_\alpha}$ and therefore, $\overline{h_\alpha}$. Hence, using Proposition 4.13, the following function also is interpretable in G:

$$\begin{array}{rccc}\overline{f} & : & K^* \times K^* & \longrightarrow & G \\ & & (t,u) & \longmapsto & \overline{h_\alpha}(t)\overline{h_\alpha}(u)\overline{h_\alpha}(tu)^{-1}\end{array} .$$

But $\overline{f} = \theta \circ f$ since i) and ii) hold in X_u for $x_\alpha(t)$ and are preserved by homomorphisms. This finishes the proof. $\square$

Now we can prove Theorem 4.1:

Proof of Theorem 4.1. Let $t \in K^* \setminus \{1\}$. We consider the set $B_t = \{u \in K^* : f(t,u) = 1\}$. As K is an algebraically closed field, by Lemma 4.5 a), f is multiplicative in the second component. Therefore, B_t is a subgroup containing t by Lemma 4.5 b) and c). If t is of infinite order, then B_t is infinite. Hence, using our assumption about the nonexistence of bad fields in the environment, we conclude that $B_t = K^*$. On the other hand, if t is of finite order, then fix $u \in K^*$ of infinite order. Then $f(u,t^{-1}) = 1$ by the first part of the argument. But $f(u,t^{-1}) = f(t,u) = 1$ by Theorem 4.9 c). Hence, again B_t contains elements of infinite order, and thus $B_t = K^*$. This finishes the argument. $\square$

Finally, we can prove Theorem 4.2.

Proof of Theorem 4.2. We start with a perfect tame group G of finite Morley rank such that $G/Z(G)$ is a quasisimple algebraic group. Let X be the universal group of the same type as $G/Z(G)$. Then we have the following diagram:

$$\begin{tikzcd} G \arrow[dr, "\pi_1"] & \\ & G/Z(G) \\ X \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

We form the pullback of this diagram:

$$\begin{tikzcd} & G \arrow[dr, "\pi_1"] & \\ Y \arrow[ur, "\theta_1"] \arrow[dr, "\theta_2"'] & & G/Z(G) \\ & X \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

Then $Y \cong \{(g,x) \in G \times X : \pi_1(g) = \pi_2(x)\}$. In this diagram, π_1 and $G/Z(G)$ are interpretable in G. On the other hand, as X is an algebraic group, it is interpretable in $G/Z(G)$ and hence in G. Moreover, the triple $(X, G/Z(G), \pi_2)$ is algebraic and hence interpretable in G, say as $(X^*, \overline{G}^*, \pi^*)$, where $\overline{G}^* \cong G/Z(G)$ definably; hence we may take $\overline{G}^* = G/Z(G)$ and $\pi^* : X^* \longrightarrow G/Z(G)$.

The pullback $\tilde{Y}$ of π_1 and π^* is interpretable in G. Hence it is of finite Morley rank. Since $\tilde{Y} \cong Y$, Y also has finite Morley rank. Moreover, Y is a definable central extension of X. Therefore, we can apply Theorem 4.1 to Y and X and conclude, using [25, (iii), p. 75] that $Y \cong X * A$, where A is abelian. Note that $\theta_1(Y') = (\theta_1(Y))' = G' = G$. But $Y' \cong X$ is an algebraic group. Therefore, G is a quotient of an algebraic group by a finite group. We conclude that G also is an algebraic group. $\square$

5 B-D

The proof of Theorem 1.7 is based on the analysis of the interaction between the unipotent 2-subgroups and the 2-tori of a counterexample to the theorem. In this section, we introduce concepts which will be used in this analysis and obtain some basic results.

If G is a group of finite Morley rank then $\mathcal{U}(G)$ will denote the set of its unipotent 2-subgroups and $\mathcal{T}(G)$ the set of its 2-tori.

Definition 5.1 *Let G be a group of finite Morley rank. Then $B(G) = \langle U : U \in \mathcal{U}(G)\rangle$ and $D(G) = \langle d(T) : T \in \mathcal{T}(G)\rangle$. A group G of finite Morley rank is said to be of* D-type *if $G = D(G)$ and of* B-type *if $G = B(G)$.*

Note that for any group of finite Morley rank G, $B(G)$ and $D(G)$ are definable and connected by Fact 2.3.

In the sequel, we will need information about $B(H)$ and $D(H)$, where H is a proper definable subgroup of a counterexample to Theorem 1.7. Such a subgroup is a K-group of finite Morley rank. Therefore, we prove some lemmas about K-groups. We start with some general statements:

Lemma 5.2 *Let X (Y) be a B-type (resp. D-type) group. Suppose X_1 (Y_1) is a normal definable subgroup of X (resp. Y). Assume that X_1 (Y_1) contains a maximal unipotent 2-subgroup (resp. a maximal 2-torus). Then $X = X_1$ ($Y = Y_1$).*

Proof. As X_1 is normal and definable in X, by Facts 1.5 ii) and 2.31, X_1 contains all the unipotent 2-subgroups of X. But X is a B-type group, hence $X = X_1$. The same argument works for a D-type group after one replaces unipotent 2-subgroups with the definable closures of 2-tori. □

Proposition 5.3 *Let R be a connected nonnilpotent group of finite Morley rank such that $F(R)^\circ$ is divisible. Then $[R, F(R)^\circ]$ is torsion-free.*

Proof. Let $F = F(R)^\circ$. By Fact 2.12, $F = R_1 * R_2$, where R_1 is torsion and R_2 is torsion-free. By Fact 2.12, R_1 is central in F. Therefore, $F' \leq R_2$ and, in particular is torsion-free.

We mimic the proof of Theorem 3 in [22]. Let $g \in R$. We define the following mapping:

$$\begin{array}{cccc} \gamma_g & : & F/d(R_1) & \longrightarrow & [R,F]/F' \\ & & ad(R_1) & \longmapsto & [g,a]F' \end{array}$$

We first check that γ_g is well-defined. Assume a, $b \in F$ such that $b^{-1}a \in d(R_1)$. R_1 and therefore $d(R_1)$ is abelian. By Fact 2.10, ii) R_1 is the direct sum p-tori for some prime numbers p. As R_1 is normal in R, by Fact 2.30, it is centralized by R. This forces $d(R_1)$ to be central in R also. Therefore, we have $1 = [g, b^{-1}a] = [g, a][g, b^{-1}]^a = [g, a]a^{-1}g^{-1}bgb^{-1}a$. This last expression is equal to $[g, a]a^{-1}b^{-1}ag^{-1}bg$ because $b^{-1}a \in d(R_1)$, $b^g \in F$ and $d(R_1)$ is central in F. As $[g, a]a^{-1}b^{-1}ag^{-1}bg = [g, a][a, b][b, g]$, $[g, b]$ and $[g, a]$ are in the same coset of F'. Therefore, γ_g is well-defined.

Next, we check that γ_g is group homomorphism. Let a, $b \in F$. Then $[g, ab] = [g, b][g, a]^b = [g, b][g, a][[g, a]^{-1}, b] \equiv [g, a][g, b]$ modulo F'.

As $F/d(R_1)$ is torsion-free, its image under γ_g is also torsion-free (Fact 2.22).

Now, for any $g_1, \cdots, g_n \in R$, let $\overline{g} = (g_1, \cdots, g_n)$ and define the following mapping:

$$\begin{array}{cccc} \gamma_{\overline{g}} & : & (F/d(R_1))^n & \longrightarrow & [R,F]/F' \\ & & (a_1d(R_1), \cdots, a_nd(R_1)) & \longmapsto & \prod_{k=1}^{n}[g_k, a_k]F' \end{array}$$

This mapping is a homomorphism because $[R, F]/F'$ is abelian. As $F/d(R_1)$ is torsion-free, so is the image of $\gamma_{\overline{g}}$ (Fact 2.22). This implies that $[R, F]/F'$ is torsion-free as every element of $[R, F]/F'$ is contained in the image of some $\gamma_{\overline{g}}$. But F' is a torsion-free definable subgroup. Therefore, we conclude that $[R, F]$ is also torsion-free. □

Lemma 5.4 *If R is a solvable D-type group of finite Morley rank then $R = d(T)F(R)$, where T is a maximal 2-torus of R.*

Proof. By Fact 2.19, $R/F(R)$ is a divisible abelian group.

$$R/F(R) = \langle d(T)F(R)/F(R) : T \in \mathcal{T}(R)\rangle.$$

Therefore, $R/F(R)$ also is a D-type group and since it is abelian, we have $R/F(R) = d(T_1/F(R))$, where $T_1/F(R)$ is a maximal 2-torus of $R/F(R)$. But by Fact 2.33 ii), $T_1/F(R) = TF(R)/F(R)$, where T is maximal 2-torus of R. Therefore, $R/F(R) = d(TF(R)/F(R)) = d(TF(R))/F(R) = d(T)F(R)/F(R)$. $\square$

Lemma 5.5 *Let Y be a D-type K-group. Then $\mathcal{U}(Y) = \emptyset$.*

Proof. We carry out a case-by-case analysis:

Y **is nilpotent:** In this case, $Y = d(T)$, where T is the maximal 2-torus of Y. By Fact 2.10, $d(T)$ is a divisible abelian group. Fact 2.27 implies Y has finitely many involutions. Therefore, using Fact 2.28, we conclude that $\mathcal{U}(Y) = \emptyset$.

Y **is solvable nonnilpotent:** We argue by induction on the rank of Y. Suppose Y is a counterexample of minimal rank. Let S be a Sylow 2-subgroup of Y. By Fact 2.34, S is connected. Hence, $S = B * T$ with $B \neq 1$ and $T \neq 1$. By Fact 2.35, $B \lhd Y$. Hence, B is a subgroup of every Sylow 2-subgroup of G and it is centralized by every 2-torus in G. Therefore, B is central in Y.

We claim that $F(Y)^\circ = B$. By Fact 2.11, $F(Y)^\circ = C * D$, where C is definable, connected and of bounded exponent, and D is definable and divisible. If $D \neq 1$, then $rk(Y/D) < rk(Y)$. This yields a contradiction because BD/D is nontrivial unipotent 2-subgroup of Y/D and Y/D is a D-type group. Thus, $F(Y)^\circ = C$.

Fact 2.20 implies $F(Y)^\circ$ is the direct sum of its Sylow p-subgroups, which are definable. If P is such a Sylow p-subgroup, where $p \neq 2$, then $P = 1$ because otherwise, P is infinite (it is connected by 2.34) and $rk(Y/P) < rk(Y)$, and arguing as above yields a contradiction. Therefore, $F(Y)^\circ = B$. In particular, $F(Y)^\circ$ is central in Y. This forces Y to be nilpotent, a contradiction.

Y **is nonsolvable:** We will use $\overline{}$ notation to denote quotients by $\sigma(Y)$. By Fact 2.40, $\overline{Y} = \overline{Y_1} \oplus \cdots \oplus \overline{Y_m}$, where the Y_i are simple algebraic groups over algebraically closed fields of odd characteristic.

Let S be a Sylow 2-subgroup of Y. Then $S^\circ = B * T$, where B is definable, connected, of bounded exponent and T is divisible abelian. Both B and T are nontrivial. We claim $\sigma(Y)^\circ = F(Y)^\circ = Z(Y)^\circ = B$. As $\overline{Y}$ is the direct sum of simple algebraic groups over algebraically closed fields, $B(\overline{Y}) = 1$. Therefore, $B \leq \sigma(Y)^\circ$. By Fact 2.35, B is characteristic in $\sigma(Y)$, and thus $B \lhd Y$. As B is the largest unipotent 2-subgroup of every Sylow 2-subgroup of Y, it is centralized by the 2-tori of Y. But Y is a D-type group, therefore B is central in Y. The same argument as in the case where Y is solvable implies that $F(Y)^\circ = B$. But by Fact 2.19, $\sigma(Y)^\circ/F(Y)^\circ$ is an abelian group and B is central in Y. This forces $\sigma(Y)^\circ$ to be nilpotent and in particular $\sigma(Y)^\circ = F(Y)^\circ$.

We have $B = Z(Y)^\circ$ and $Y/B = Y_1/B * \cdots * Y_m/B$, where the Y_i/B are finite central extensions of simple algebraic groups over algebraically closed fields of odd characteristic. We will do component analysis in the style of [21]. If $i \neq j$, then $[Y_i, Y_j] \leq B$. For every i, Y_i' is connected and $Y_i'B/B \lhd Y_i/B$. Therefore, we have $Y_i = Y_i'B$ and $Y = Y_1' \cdots Y_m'B$. The subgroup $Y_1' \cdots Y_m'$ contains a maximal 2-torus of Y. Therefore, by Lemma 5.2, $Y = Y' = Y_1' \cdots Y_m'$. By continuing to take commutators we may assume that $Y = Y_1 * \cdots * Y_m$, where the Y_i are perfect groups. We also have $Z(Y) = Z(Y_1) \cdots Z(Y_m)$. Therefore, each $Y_i/Z(Y_i)$ is a perfect group which is a central extension of a simple algebraic group by a finite center. By Theorem 4.2 each $Y_i/Z(Y_i)$ is a quasisimple algebraic group. By Theorem 4.2 , Y_i is an algebraic group. Therefore, $Z(Y_i)$ is finite. This last conclusion proves that Y cannot have unipotent 2-subgroups. $\square$

Lemma 5.6 *Let $G = K \rtimes H$ be a group of finite Morley rank, where H and K are connected definable subgroups of G, and K is a K-group of B-type. Assume that H centralizes all definable simple and B-type solvable sections of K normalized by H. Then H centralizes K. (A similar statement can be made for D-type groups.)*

Proof. We prove the statement for B-type groups. The same proof works for D-type groups "mutatis mutandis". By Fact 2.40, $K/\sigma(K)$ is isomorphic to the direct sum of simple algebraic groups over algebraically closed fields of characteristic 2. Since H is connected, H normalizes each of these simple algebraic groups. Thus by the hypotheses, H centralizes $K/\sigma(K)$. Therefore, H centralizes $U\sigma(K)/\sigma(K)$ where U is a unipotent 2-subgroup of K. This implies that H normalizes $U\sigma(K)$. $U\sigma(K)$ is a definable solvable subgroup of K. Therefore, H acts trivially on $B(U\sigma(X))$, and in particular, U. As K is B-type group, H centralizes K. $\square$

Lemma 5.7 *If B is a unipotent 2-subgroup acting on a D-type K-group Y, then the action is trivial.*

Proof. We carry out a case-by-case analysis:

Y **is nilpotent:** Y is nilpotent implies that $Y = d(T)$, where T is the 2-torus of Y. By Fact 2.30, B centralizes T. Therefore, it also centralizes $d(T)$.

Y **is solvable and nonnilpotent:** By Lemma 5.4, $Y = d(T)F(Y)^\circ$, where T is maximal 2-torus of Y. The group B acts on $F(Y)^\circ$. We claim that the group $BF(Y)^\circ$ is nilpotent. Suppose this is not the case. As $BF(Y)^\circ$ is solvable and connected, by Fact 2.18, an algebraically closed field K can be interpreted in $BF(Y)^\circ$, and a definable section of B is isomorphic to an infinite definable subgroup of K^*. But then by Fact 2.28, K^* has infinitely many involutions. This contradicts Fact 2.27. Therefore, $BF(Y)^\circ$ is nilpotent. Then by Fact 2.11, B centralizes $F(Y)^\circ$.

By replacing T by one of its conjugates in Y, we may assume that B and T are in the same Sylow 2-subgroup of BY. But this implies that B centralizes T and therefore $d(T)$. By the above paragraph, B centralizes $F(Y)^\circ$. Thus, we conclude that B centralizes $Y = d(T)F(Y)^\circ$.

Y **is nonsolvable:** This follows from Fact 2.39, the result for Y solvable and Lemma 5.6. $\square$

Corollary 5.8 *If H is a K-group then $B(H)$ and $D(H)$ commute.*

Lemma 5.9 *If X is a B-type K-group then $\mathcal{T}(X) = \emptyset$.*

Proof. We carry out a case-by-case analysis.

X **is solvable:** By Fact 2.35, X is a unipotent 2-group and we are done in this case.

X **is nonsolvable:** We assume that X is a counterexample of minimal rank to the statement. Therefore, $D(X) \neq 1$. $X/\sigma(X)$ is the direct sum of simple algebraic groups over algebraically closed fields of characteristic 2.

We claim that $\sigma(X)^\circ = F(X)^\circ = Z(X)^\circ$. By Lemma 5.7, $D(X)$ is central in X and $D(X) = d(T)$, where T is the maximal 2-torus of X. By Fact 2.11, $F(X)^\circ = C * D$, where C is definable, connected, of bounded exponent and D is definable and divisible. By induction on rank, we may assume $C = 1$. If $F(X)^\circ < \sigma(X)^\circ$ then by Fact 2.18, an algebraically closed field K is interpretable in $\sigma(X)^\circ$. Since $\sigma(X)^{\circ\prime} \subseteq F(X)^\circ$, K^+ is isomorphic to a definable section of $F(X)^\circ$, and a definable section of $\sigma(X)^\circ/F(X)^\circ$ is isomorphic to an infinite subgroup of K^*. The "no bad fields" assumption implies that this definable section is isomorphic to K^*.

The characteristic of K is 0 because $F(X)^\circ$ is divisible. Therefore, K^* contains a nontrivial 2-torus. Fact 2.33 implies that $\sigma(X)^\circ/F(X)^\circ$ contains a nontrivial 2-torus. This implies that $\sigma(X)^\circ$ contains a noncentral 2-torus, a contradiction. Therefore, $F(X)^\circ = \sigma(X)^\circ$.

Let U be a unipotent 2-subgroup. We claim that the group $F(X)^\circ U$ is nilpotent. If this is not the case then, as $F(X)^\circ U$ is connected and solvable, by Fact 2.18, an algebraically closed field K can be interpreted in $F(X)^\circ U$ and a definable section of U is isomorphic to an infinite subgroup of K^*. But then Fact 2.28 implies that K^* has infinitely many involutions although this is impossible by Fact 2.27. Therefore, $F(X)^\circ U$ is nilpotent and Fact 2.11 implies that $F(X)^\circ$ centralizes U. Since X is B-type group, we conclude that $F(X)^\circ$ is central in X; $\sigma(X)^\circ = F(X)^\circ = Z(X)^\circ$.

Now we can finish the argument as in the proof of Lemma 5.5. □

Lemma 5.10 *If X is a B-type K-group and C is a nilpotent, connected, $2'$-group of bounded exponent of finite Morley rank acting definably on X, then the action is trivial.*

Proof. We carry out a case-by case analysis:

X **is solvable:** By Facts 2.35 and 1.5 i), X is nilpotent. The rest of the argument is an application of Fact 2.18.

X **is nonsolvable:** This follows from Fact 2.39, the result for X solvable and Lemma 5.6. over algebraically closed fields □

Lemma 5.11 *If X is a B-type K-group and T is a torsion-free nilpotent group of finite Morley rank acting on X definably then this action is trivial.*

Proof. We carry out a case-by-case analysis:

X **is solvable:** By Fact 2.35, X is nilpotent. We claim that the group XT is nilpotent. If this is not the case then as XT is a connected solvable group of finite Morley rank, by Fact 2.18, an algebraically closed field K of characteristic 2 is interpretable in XT. The "no bad fields" hypothesis implies that a definable section of T is isomorphic K^*. But T is torsion-free, a contradiction. Therefore, XT is nilpotent. Therefore X is centralized by T.

X **is nonsolvable:** This follows from Fact 2.39, the result for X solvable and Lemma 5.6. □

Lemma 5.12 *Let T be a group such that $T \cong F^*$, where F is an algebraically closed field of characteristic 2. If T acts on a D-type K-group Y, then the action is trivial.*

Proof. The argument is again a case-by-case analysis:

Y **is nilpotent:** Y is the definable closure of its sole 2-torus.

Y **is solvable and nonnilpotent:** Suppose the action of T is nontrivial. By Fact 2.18, we can interpret an algebraically closed field of characteristic 2 in the group $H = YT$. This forces Y to have a definable section which is a unipotent 2-group. But then $B(Y) \neq 1$, a contradiction.

Y **is nonsolvable:** This follows from Fact 2.39, the result for Y solvable and Lemma 5.6. □

Lemma 5.13 *If an involution i acts on a D-type K-group Y, and $C_Y(i)$ does not contain nontrivial 2-tori, then Y is solvable. If, in addition, Y is nilpotent, then i inverts Y.*

Proof. We first prove the second statement. Therefore, we assume that Y is nilpotent. As $Y = D(Y)$, $Y = d(T)$, where T is the maximal 2-torus in Y. In particular, Y is abelian. By Fact 2.37, it is enough to show that under the assumptions of the statement of the fact, $C_Y(i)$ is finite.

We let $K = \{g^{-1}g^i : g \in Y\}$. Note that the elements of K are inverted by i. As Y is abelian, K is a definable subgroup and the following map is a homomorphism from Y onto K:

$$\begin{array}{rccc} \theta: & Y & \longrightarrow & K \\ & g & \longmapsto & g^{-1}g^i \end{array}$$

As the fibers of this map correspond to the cosets of $C_Y(i)$ in Y, we have $rk(Y) = rk(K) + rk(C_Y(i))$. The assumptions on Y imply that $rk(C_Y(i)) < rk(Y)$. Therefore, K is an infinite divisible subgroup of Y.

Let $g \in G$. Then $g^{-1}g^t = k^2$ for some $k \in K$. This implies $g^t k^{-1} = gk$. But $k^{-1} = k^t$. Therefore, $(gk)^t = gk$. As $g = gkk^{-1}$, we conclude that $Y = C_Y(i)K$.

As Y is connected, we have $Y = C_Y(i)^\circ K$. The assumption on $C_Y(i)$ implies that $C_Y(i)^\circ$ is $2^\perp$. But then K contains the Sylow 2-subgroup of Y. This forces $C_Y(i)^\circ = 1$.

Now we will show that Y is solvable. Assume towards a contradiction that Y is nonsolvable. We will first show that $\sigma(Y)^\circ = 1$. Suppose this is not the case. We first analyze $F(Y)^\circ$. If $F(Y)^\circ$ is $2^\perp$ then, by Proposition 2.51, $C_{Y/F(Y)^\circ}(i) = C_Y(i)F(Y)^\circ/F(Y)^\circ$. Therefore, $C_{Y/F(Y)^\circ}(i)$ does not contain nontrivial 2-tori. By induction on rank, $Y/F(Y)^\circ$ is solvable, forcing Y to be solvable, a contradiction. Therefore, $F(Y)^\circ$ has a nontrivial 2-torus. Let T_1 be the maximal 2-torus of $F(Y)^\circ$. T_1 is i-invariant. By the assumption on $C_Y(i)$, and using Fact 2.37, i inverts T_1. It follows that i inverts $d(T_1)$. By Proposition 2.51, $C_{Y/d(T_1)}(i) = C_Y(i)d(T_1)/d(T_1)$. By induction on rank, we conclude that $Y/d(T_1)$ is solvable, which forces Y to be solvable, a contradiction. As a result, we have $F(Y)^\circ = 1$, and therefore, $\sigma(Y)^\circ = 1$.

The conclusion of the above paragraph and Fact 2.40 imply that $Y = Y_1 \cdots Y_n$ where the Y_i are central extensions of simple algebraic groups over algebraically closed fields of odd characteristic. Moreover, these can be taken to be perfect subgroups. Then the results on central extensions imply that the Y_i are quasisimple algebraic groups over algebraically closed fields of odd characteristic.

Note that, since i is an involution, n in the above paragraph can be taken to be at most 2. Hence, we have $Y = Y_1 * Y_2$. Now, we will argue that $n = 1$. i is contained in a Sylow 2-subgroup S of $Y \rtimes \langle i \rangle$. $T = S^\circ$ is a 2-torus. The hypotheses on $C_Y(i)$ imply that i inverts T. But this means that $T \leq Y_1 \cap Y_1^i$, in particular, this intersection is infinite. This forces $Y_1 = Y_1^i = Y_2$. Hence, we have reduced our nonsolvable configuration to the case where Y is a quasisimple algebraic group over an algebraically closed field of odd characteristic. Now, the conclusion follows from Facts 2.38 and 2.45. □

Definition 5.14 *Let G be a group of finite Morley rank. Define a graph on $\mathcal{U}(G)$ as follows. The vertices of the graph are the elements of $\mathcal{U}(G)$. Two vertices U_1 and U_2 are connected by an edge if U_1 and U_2 normalize each other; this will be denoted by $U_1 \sim U_2$.*

We will investigate some basic properties of this graph. Note that two vertices U and V are in the same connected component of $\mathcal{U}(G)$ if and only if there exists a sequence $A_\circ, \ldots, A_n$ of unipotent 2-subgroups such that $U = A_\circ$, $V = A_n$ and for $0 \leq i \leq n-1$, A_i and A_{i+1} normalize each other.

Proposition 5.15 *Let G be a tame K^*-group, and $U_1, U_2 \in \mathcal{U}(G)$. If $U_1 \sim U_2$, then $D(C_G(U_1)) = D(C_G(U_2))$.*

Proof. As U_1 normalizes U_2, U_1 normalizes $D(C_G(U_2))$. Since, by Lemma 5.5, U_2 is not in $D(C_G(U_2))$, $D(C_G(U_2))$ is a K-group. Then by Lemma 5.7 U_1 centralizes $D(C_G(U_2))$. Hence, $D(C_G(U_2)) \leq C_G(U_1)$ and we get $D(C_G(U_2)) \leq D(C_G(U_1))$. By symmetry, we obtain equality. □

Corollary 5.16 *Let G be a tame K^*-group, and $U_1, U_2 \in \mathcal{U}(G)$. If U_1 and U_2 are in the same connected component of $\mathcal{U}(G)$, then $D(C_G(U_1)) = D(C_G(U_2))$.*

Corollary 5.17 *Let G be a tame K^*-group, and $U_1, U_2 \in \mathcal{U}(G)$. If $\mathcal{U}(G)$ is a connected graph then $D(C_G(U)) \lhd G$, for any unipotent 2-subgroup U of G.*

Proposition 5.18 *Let G be a group of finite Morley rank. Let $\mathcal{W}$ denote a connected component of $\mathcal{U}(G)$, and $M = N_G(\langle \mathcal{W} \rangle)$. Then $\mathcal{U}(M) = \mathcal{W}$.*

Proof. We need to show that $\mathcal{U}(M) \subseteq \mathcal{W}$. Let $U, V \in \mathcal{U}(M)$ such that $U \in \mathcal{W}$. We will show that $V \in \mathcal{W}$. First, we argue that U and V can be replaced by maximal unipotent 2-subgroups. $U \leq S_1$ and $V \leq S_2$, where S_1 and S_2 are two Sylow 2-subgroups of M. By Fact 1.5, $S_1^\circ = B_1 * T_1$ and $S_2^\circ = B_2 * T_2$, and $U \leq B_1$ and $V \leq B_2$. Note that $B_1 \sim Z(B_1)^\circ \sim U$

($Z(B_1)^\circ \neq 1$ by Facts 1.5 i), and 2.21 i)) and $B_2 \sim Z(B_2)^\circ \sim V$. Hence, we can replace U and V by B_1 and B_2 which are maximal unipotent 2-subgroups.

By the Sylow theorem, there exists $g \in M$ such that $B_1^g = B_2$. Hence, $B_2 \leq \langle \mathcal{W} \rangle$ and g can be taken to be in $\langle \mathcal{W} \rangle$. Proceeding inductively, it suffices to treat the case in which $g \in W \in \mathcal{W}$. In this case, $W^g = W$, so $\mathcal{W}^g = \mathcal{W}$ and $B_2 = B_1^g \in \mathcal{W}$.

□

Corollary 5.19 *Let G be a group of finite Morley rank. Let $\mathcal{W}$ be a connected component of $\mathcal{U}(G)$. $N_G(\langle \mathcal{W} \rangle) = Stab(\mathcal{W})$. In particular, $Stab(\mathcal{W})$ is a definable subgroup.*

In the sequel we will have certain groups which are isomorphic to either $PSL_2(K)$ or $SL_2(K)$ over an algebraically closed field K. We will denote this situation by $(P)SL_2(K)$.

Lemma 5.20 *If H is a nonsolvable, tame, connected K-group with a weakly embedded subgroup, then $H/O(H) \cong (P)SL_2(K)$, where K is an algebraically closed field.*

Proof. Let M be a weakly embedded subgroup of H. Note that $F(H)^\circ = O(H)$. We will show that either $MO(H)/O(H)$ is a weakly embedded subgroup of $\overline{H} = H/O(H)$ or $H = MO(H)$. We will use Proposition 3.4. We use $\overline{}$-notation to denote quotients by $O(H)$. Clearly, $\overline{M}$ has infinite Sylow 2-subgroups. Let U be a subgroup of H, which contains $O(H)$, such that $\overline{U}$ is a unipotent 2-subgroup of $\overline{M}$. Suppose $\overline{U}^{\overline{x}} = \overline{U}$ for some $\overline{x} \in \overline{H}$. This implies that $U^x = U$, and therefore $x \in N_H(B(U))$. Clearly, $U \leq MO(H)$ and we have $U = O(H)(M \cap U)$. As U is solvable and its Sylow 2-subgroups are of bounded exponent, by Fact 2.35, U has a unique Sylow 2-subgroup, which is a subgroup of $M \cap U$ by Fact 2.33. Hence, $B(U) \leq M$. But $x \in N_H(B(U))$ and M is a weakly embedded subgroup. This implies $x \in M$ forcing $\overline{x} \in \overline{M}$. Now let T be a subgroup of H such that $\overline{T}$ is a 2-torus of $\overline{M}$ and $\overline{x} \in \overline{H}$ such that $\overline{T}^{\overline{x}} = \overline{T}$. Hence, $T^x = T$, and therefore $x \in N_H(D(d(T)))$. If $MO(H) < H$ then $MO(H)$ is a weakly embedded subgroup of H. Therefore, replacing M by $MO(H)$, we may assume that M contains $O(H)$ and therefore T. But then $x \in N_H(D(d(T)))$ forces $x \in M$, and $\overline{x} \in \overline{M}$. If $\overline{M} < \overline{H}$ then $\overline{M}$ is a weakly embedded subgroup of $\overline{H}$, otherwise $H = MO(H)$.

We first analyze the possibility $\overline{M} < \overline{H}$. In this case, $\sigma(H)^\circ = O(H)$ and therefore, by Fact 2.40, $\overline{H} = H/O(H)$ is the central product of quasisimple groups of finite Morley rank. As $\overline{H}$ has a weakly embedded subgroup, we conclude that $\overline{H}$ is a quasisimple group. By the results on central extensions, $\overline{H}$ is a quasisimple algebraic group with a weakly embedded subgroup.

We claim that $\overline{H} \cong (P)SL_2(K)$, where K is an algebraically closed field. Let $X = \overline{H}$. We will show that X has Lie rank 1. Then the result will follow from Fact 2.50. Let M denote a weakly embedded subgroup of X. By Proposition 3.2, M has a Sylow 2-subgroup S which is also a Sylow 2-subgroup of X.

We first assume that $\mathrm{char}(K) = 2$. In this case S is a maximal unipotent subgroup of X. We may assume that $S = \langle X_\alpha : \alpha \in \Phi^+ \rangle$. Let $S^- = \langle X_\alpha : \alpha \in \Phi^- \rangle$. By Proposition 3.4, $N_X(U) \leq M$ for any unipotent 2-subgroup in M. In particular, $N_X(U_\alpha) \leq M$ for every $\alpha \in \Phi^+$, where U_α is the root subgroup corresponding to α (by Proposition 4.13 U_α is definable in X). Suppose X has Lie rank greater than 1. Then Φ^+ has a base with at least 2 distinct roots α and β. By Fact 2.48, $\alpha - \beta$ is not a root. Fact 2.46 implies that the root subgroups U_α and $U_{-\beta}$ commute. Therefore, $U_{-\beta}$ centralizes a unipotent 2-subgroup of M and $U_{-\beta} \leq M$. This forces $Z(S^-) \leq M$. But then, as $Z(S^-)$ is a nontrivial unipotent 2-subgroup, $S^- \leq M$. Hence, by Fact 2.49, $M = G$, a contradiction. Therefore, if $\mathrm{char}(K) = 2$ then the Lie rank of X is 1.

Now we assume $\mathrm{char}(K) \neq 2$. Then the Sylow 2-subgroups of X are contained in maximal tori. Let T_α and Z_α be as in Fact 2.49. Suppose the rank is greater than 1. Then $T_\alpha \neq 1$ for any $\alpha \in \Delta$. Moreover, $Z_\alpha \leq M$ by Proposition 3.4. This forces $M = X$, a contradiction. Therefore, in this case also the rank of X is 1.

Now we analyze the possibility $H = MO(H)$. Note that under this assumption $B(H) = 1$ because otherwise H has a nontrivial unipotent 2-subgroup U which normalizes $O(H)$. But then

Fact 2.36 implies that U centralizes $O(H)$ forcing $O(H) \leq M$ and thus $H = M$. This contradicts that M is a proper subgroup of H. Another implication of the assumption $H = MO(H)$ is that the Sylow 2-subgroups of H have Prufer rank 1. If the Prufer rank of the Sylow 2-subgroups of H is bigger than 1 then Corollary 2.54 implies that $O(H)$ is generated by the centralizers large 2-tori in M. This forces $O(H) \leq M$ and $H = M$, which contradicts that M is weakly embedded subgroup of H.

We claim that $\sigma(H)^\circ = O(H)$. We first show $F(H)^\circ = O(H)$. If $F(H)^\circ \neq O(H)$ then as H is tame $F(H)^\circ > O(H)$. As a result $F(H)^\circ$ has an infinite Sylow 2-subgroup (Fact 2.34), which is unique in $F(H)^\circ$ (Fact 2.20). Therefore it is normal in H. This contradicts that H has a weakly embedded subgroup. If $\sigma(H)^\circ > O(H)$, then $\sigma(H)^\circ/O(H)$ has a connected Sylow 2-subgroup (Fact 2.34). This nontrivial 2-torus is contained in the connected component of every Sylow 2-subgroup of $H/O(H)$. But the Prufer rank is 1. Therefore, the Sylow 2-subgroup of $\sigma(H)^\circ/O(H)$ is the connected component of every Sylow 2-subgroup of $H/O(H)$, a contradiction.

Fact 2.40, the fact that the Prufer rank is 1 and $\sigma(H)^\circ = O(H)$ imply that $H/O(H)$ is isomorphic to a quasisimple algebraic group. This group is over an algebraically closed field of characteristic different from 2 as $B(H) = 1$. We will show that this group is isomorphic to $(P)SL_2(K)$. By replacing H with $N_H(M \cap O(H))$ we may assume that $M \cap O(H) \lhd H$. We will use $\tilde{}$-notation to denote quotients by $M \cap O(H))$.

We consider the following action of $\tilde{H}/\widetilde{O(H)}$ on $Z(\widetilde{O(H)}) = Z_1/M \cap O(H))$. If $x \in Z_1$ and $h \in H$, then $\tilde{x}^{\tilde{h}}$ is defined to be $\widetilde{x^h}$. This action is well-defined because if $x_1, h_1 \in M \cap O(H)$ then $(xx_1)^{h_1h} = x^{h_1h}x_1^{h_1h} = x^h[x, h_1]^h x_1^{h_1h}$.

Let $\tilde{A}$ be a $\tilde{H}/\widetilde{O(H)}$-minimal subgroup of $Z(\widetilde{O(H)})$. We claim that the action of $\tilde{H}/\widetilde{O(H)}$ on $\tilde{A}$ is not trivial. Suppose towards a contradiction that the action is trivial. Then for any $x \in A$ and $h \in H$, $\widetilde{x^h} = \tilde{x}$. This implies $[x, h] \in M \cap O(H)$. If S is a Sylow 2-subgroup of M then $[S, x] \subseteq M \cap O(H)$. Hence, for $s \in S$, $s^x \in s(M \cap O(H))$ and therefore x normalizes $S(M \cap O(H))$ which is forced to be a subgroup of $M \cap M^x$. As M is a weakly embedded subgroup of H, $x \in M$, thus $xM \cap O(H)$. But then $\tilde{A} = 1$, a contradiction.

Let $X = \tilde{H}/\widetilde{O(H)}$. As X is a quasisimple algebraic group and $C_X(\tilde{A}) \lhd X$, we may assume that $C_X(\tilde{A}) = 1$. Let B be a Borel subgroup of X. B is definable and connected. Let $\widetilde{A_1}$ be a B-minimal subgroup of $\tilde{A}$. Then, by Fact 2.17, $\widetilde{A_1} \rtimes B/C_B(A_1) \cong L_+ \rtimes L_1$, where L is an algebraically closed field of characteristic different from 2 and $L_1 \leq L^*$. By the "no bad fields" hypothesis, $L_1 = L^*$. As L is interpretable in K where K is the base field of $\tilde{H}/\widetilde{O(H)}$, $L \cong K$ by Fact 4.12. Applying the same argument to $\tilde{A}/\tilde{A}_1$ we eventually get a finite-dimensional vector space structure on $\tilde{A}$ on which X acts.

We will show that the Lie rank of X is 1. B has a common eigenvector v by Fact 2.43. Bv is a 1-dimensional vector subspace of $\tilde{A}$. One can choose a basis for $\tilde{A}$, which contains v, in such a way that B has a representation with upper triangular matrices. If the Lie rank of X is bigger than 1 then Bv is fixed pointwise by at least one 1-dimensional torus. We may assume $Bv = \widetilde{A_1}$. This implies that there is an infinite 2-subgroup T such that $[T, A_1] \leq M \cap O(H)$. In particular, for any $x \in A_1$ and $t \in T$, $t^x \in t(M \cap O(H))$. Therefore, x normalizes $T(M \cap O(H))$ which forces $T(M \cap O(H))$ to be a subgroup of $M \cap M^x$ and $x \in M \cap O(H)$, a contradiction to $\tilde{A}_1 \neq 1$. Hence, the Lie rank of X is 1 and by Fact 2.50, $X \cong (P)SL_2(K)$ over an algebraically closed field of characteristic different from 2. This finishes the proof. □

Proposition 5.21 *If X is a B-type K-group and $\mathcal{U}(X)$ is not connected, then $X \cong PSL_2(K)$ where K is an algebraically closed field of characteristic* 2.

Proof. As $\mathcal{U}(X)$ is not connected, Fact 2.35 implies that X is not solvable. Similarly, $B(\sigma(X)) = 1$. Therefore, using Lemma 5.9, we conclude that $\sigma(X)^\circ$ is a $2^\perp$-subgroup. By Fact 2.18, the action of any unipotent 2-subgroup on $\sigma(X)^\circ$ is trivial. Therefore, $\sigma(X)^\circ = Z(X)^\circ$.

We claim that $\mathcal{U}(X/Z(X)^\circ)$ is not a connected graph. We will use $\overline{}$-notation to denote quotients by $Z(X)^\circ$. Let $\overline{U}, \overline{V} \in \mathcal{U}(\overline{X})$ be such that $\overline{U}$ and $\overline{V}$ normalize each other and

$Z(X)^\circ \leq U, V$. Let $\overline{x} \in \overline{U}$. Then $V^x = V$. V is a definable nilpotent subgroup of X. Therefore, by Fact 2.20, it has a unique Sylow 2-subgroup normalized by x. This argument applies to U also. Therefore, U and V have a single unipotent 2-subgroup each, and these subgroups normalize each other. This implies that $\mathcal{U}(\overline{X})$ is not connected.

The above paragraph forces $X/\sigma(X)$ to be a simple algebraic group over an algebraically closed field of characteristic 2. This implies $X = X'Z(X)^\circ$. But X is a B-type group, therefore, $X = X'$. Using Theorem 4.2, we conclude that X is a quasisimple algebraic group over an algebraically closed field of characteristic 2.

If we can prove that X has a weakly embedded subgroup then we are done by Lemma 5.20. We claim that the stabilizer of a connected component of $\mathcal{U}(X)$ is a weakly embedded subgroup. Let $\mathcal{W}$ be such a connected component and $M = \mathrm{Stab}(\mathcal{W})$. M is definable by Corollary 5.19. As $\mathcal{U}(X)$ is not connected, by Proposition 5.18, $M < X$. M has infinite Sylow 2-subgroups. Let $g \in X \setminus M$. Then $M \cap M^g$ does not contain a nontrivial unipotent 2-subgroup because otherwise $\mathcal{W} = \mathcal{W}^g$ by Proposition 5.18 and $g \in M$, a contradiction to the choice of g. Using Lemma 5.9, we conclude that $M \cap M^g$ has finite Sylow 2-subgroups. Therefore M is a weakly embedded subgroup. $\square$

Definition 5.22 *Let G be a group of finite Morley rank. Define a graph on $\mathcal{T}(G)$ as follows. The vertices of the graph are the elements of $\mathcal{T}(G)$. Two vertices T_1 and T_2 are connected by an edge if and only if they normalize (equivalently by Fact 2.30, centralize) each other; this will be denoted by $T_1 \approx T_2$.*

Note that two vertices T_1 and T_2 of $\mathcal{T}(G)$ are in the same connected component of $\mathcal{T}(G)$ if and only if there exists $R_\circ, \ldots, R_n \in \mathcal{T}(G)$ such that $T_1 = R_\circ$ and $T_2 = R_n$, and for $1 \leq i \leq n-1$, $[R_i, R_{i+1}] = 1$.

Proposition 5.23 *Let G be a tame, K^*-group of finite Morley rank. Let $T_1, T_2 \in \mathcal{T}(G)$ such that $T_1 \approx T_2$. Then $B(C_G(T_1)) = B(C_G(T_2))$.*

Proof. Let T_1 and T_2 be as in the statement of the proposition. Then T_1 normalizes $B(C_G(T_2))$. By Lemma 5.8, T_1 centralizes $B(C_G(T_2))$. This forces $B(C_G(T_2)) \leq B(C_G(T_1))$. We get equality by symmetry. $\square$

Corollary 5.24 *Let G be a tame, K^*-group of finite Morley rank. If T_1 and T_2 are in the same connected component of $\mathcal{T}(G)$, then $B(C_G(T_1)) = B(C_G(T_2))$.*

Corollary 5.25 *Let G be a tame, K^*-group of finite Morley rank. If $\mathcal{T}(G)$ is connected then $B(C_G(T)) \lhd G$ for every $T \in \mathcal{T}(G)$.*

The next lemma is an analog of Proposition 5.18.

Lemma 5.26 *Let G be a group of finite Morley rank. Let $\mathcal{W}$ denote a connected component of $\mathcal{T}(G)$ and $M = N_G(\langle \mathcal{W} \rangle)$. Then $\mathcal{T}(M) = \mathcal{W}$.*

Proof. Let T_1 and T_2 be two 2-tori in M such that $T_1 \in \mathcal{W}$. We will show that $T_2 \in \mathcal{W}$. We may assume that T_1 and T_2 are maximal since any 2-torus is in the same connected component of $\mathcal{T}(M)$ as a maximal 2-torus which contains it. By the Sylow theorem, there exists $g \in M$ such that $T_1^g = T_2$. Therefore, $T_2 \leq \langle \mathcal{W} \rangle$ and g can be taken to be in $\langle \mathcal{W} \rangle$. Proceeding inductively it suffices to treat the case in which $g \in T_3 \in \mathcal{W}$. In this case T_1, $T_3 \in \mathcal{W}$ and therefore their conjugates $T_2 = T_1^g$ and $T_3^g = T_3$ are in $\mathcal{W}$. $\square$

Corollary 5.27 *Let G be a group of finite Morley rank. Let $\mathcal{W}$ be a connected component of $\mathcal{T}(G)$. Then $N_G(\langle \mathcal{W} \rangle) = Stab(\mathcal{W})$. In particular, $Stab(\mathcal{W})$ is definable.*

Corollary 5.28 *If Y is a nonsolvable D-type K-group and $\mathcal{T}(Y)$ is not connected then $Y/O(Y) \cong SL_2(K)$ or $PSL_2(K)$, where K is an algebraically closed field of characteristic different from* 2.

Proof. Let $\mathcal{W}$ be a connected component of $\mathcal{T}(Y)$ and $M = N_G(\langle \mathcal{W} \rangle)$. By Corollary 5.27, $M = \mathrm{Stab}(\mathcal{W})$. We claim that M is a weakly embedded subgroup. M has infinite Sylow 2-subgroups. If $g \in X \setminus M$, then $M \cap M^g$ does not have infinite Sylow 2-subgroups because otherwise $\mathcal{W} = \mathcal{W}^g$ and $g \in M$. Now the result follows from Lemma 5.20. $\square$

Before we finish this section, we would like to mention an interesting property of B-type and D-type subgroups of a tame K^*-group of finite Morley rank. Let G be a group of finite Morley rank. Let $\mathcal{B}$ and $\mathcal{D}$ denote the posets of B-type and D-type subgroups of G respectively, ordered by inclusion. One can define the following mappings:

$$\begin{array}{rclcl} BC & : & \mathcal{D} & \longrightarrow & \mathcal{B} \\ & & Y & \longmapsto & B(C_G(Y)) \end{array} \qquad \begin{array}{rclcl} DC & : & \mathcal{B} & \longrightarrow & \mathcal{D} \\ & & X & \longmapsto & D(C_G(X)) \end{array}$$

These mappings define a *Galois connection* (see [2]) between $\mathcal{B}$ and $\mathcal{D}$ because they satisfy the following properties:

Proposition 5.29 **i)** *The mappings BC and DC are order-reversing.*

ii) *If $X \in \mathcal{B}$ then $X \leq BCDCX$ and if $Y \in \mathcal{D}$ then $Y \leq DCBCY$.*

The following properties of BC and DC (or any Galois connection) can be checked easily:

Proposition 5.30 *Let $X \in \mathcal{B}$ and $Y \in \mathcal{D}$. Then $DCX = DCBCDCX$ and $BCY = BCDCBCY$.*

Corollary 5.31 *$BCDC$ and $DCBC$ are* closure operations *on $\mathcal{B}$ and $\mathcal{D}$ respectively, i.e.*

i) *For any $X \in \mathcal{B}$ and $Y \in \mathcal{D}$, $X \leq BCDCX$ and $Y \leq DCBCY$.*

ii) *For any $X \in \mathcal{B}$ and $Y \in \mathcal{D}$, $BCDCX = BCDCBCDCX$ and $DCBCY = DCBCDCBCY$.*

iii) *Both $BCDC$ and $DCBC$ are order-preserving.*

The closed elements of $\mathcal{B}$ and $\mathcal{D}$ are of the form BCY and DCX respectively.

In the sequel we will use these properties several times and refer to them under the rubric of Galois connection.

6 Construction of a Weakly Embedded Subgroup

In this section, we assume that G is a simple, tame, K^*-group of mixed type. We will show that G has a weakly embedded subgroup. Let S be a Sylow 2-subgroup of G. Then $S^\circ = B * D$ where B is a maximal unipotent 2-subgroup and D is a 2-torus, with $B \neq 1$ and $D \neq 1$.

As we have assumed that G is simple, $\mathcal{U}(G)$ is not a connected graph by Corollary 5.17. Let $\mathcal{W}$ be the connected component of $\mathcal{U}(G)$ containing B and $M_1 = \mathrm{Stab}(\mathcal{W})$. By Corollary 5.19, M_1 is definable. Also note that $\mathcal{W} = \mathcal{U}(M_1)$ by Proposition 5.18.

Proposition 6.1 $N_G(B) \leq M_1$.

Proof. Let $g \in N_G(B)$. Then $B = B^g \in \mathcal{W} \cap \mathcal{W}^g$, so $\mathcal{W} = \mathcal{W}^g$ and $g \in M_1$. $\square$

Corollary 6.2 *$S \leq M_1$.*

Now we prove that M_1 is a reasonable candidate for a weakly embedded subgroup of G.

Proposition 6.3 *For $g \in G \setminus M_1$, $M_1 \cap M_1^g$ does not contain nontrivial unipotent 2-subgroups. In particular, for any $U \in \mathcal{U}(M_1)$, $N_G(U) \leq M_1$.*

Proof. Suppose U is a unipotent 2-subgroup of $M_1 \cap M_1^g$. Then $U \in \mathcal{W} \cap \mathcal{W}^g$ as $\mathcal{W} = \mathcal{U}(M_1)$ by Proposition 5.18. But then $\mathcal{W} = \mathcal{W}^g$, contradiction. □

Propositions 3.4 and 6.3 show that if the normalizer of every 2-torus of M_1 were in M_1 then M_1 would be a weakly embedded subgroup of G. Therefore, for the remainder of this section, we analyze the case in which this does not happen. We assume that M_1 has a 2-torus R such that $N_G(R) \not\leq M_1$. Note that by Fact 2.30, $B(N_G(R)) = B(C_G(R))$.

Proposition 6.4 *$\mathcal{U}(N_G(R))$ is not connected.*

Proof. Suppose $\mathcal{U}(N_G(R))$ is connected. Corollary 6.2 implies that $R \leq S_1 \leq M_1$, where S_1 is a Sylow 2-subgroup of G. Therefore, $\mathcal{U}(N_G(R)) \cap \mathcal{W} \neq \emptyset$. This forces $\mathcal{U}(N_G(R)) \subseteq \mathcal{W}$. Thus $\mathcal{W} \cap \mathcal{W}^g \neq \emptyset$ for every $g \in N_G(R)$. As a result $N_G(R) \leq M_1$, which contradicts our assumption. □

By Proposition 5.21, $B(C_G(R)) \cong PSL_2(K)$, where K is an algebraically closed field of characteristic 2.

R centralizes a unipotent 2-subgroup (say A) of M_1. This implies $R \leq D(C_G(A))$. But then, as $\mathcal{U}(M_1) = \mathcal{W}$, $R \leq D(C_G(A)) = D(C_G(B))$. This implies that $B \leq B(C_G(R))$. As B is a maximal unipotent 2-subgroup of G, we conclude that B is a Sylow 2-subgroup of $B(C_G(R))$. As $B(C_G(R)) \cong PSL_2(K)$, B is an elementary abelian 2-group and $N_{B(C_G(R))}(B) = BT$ where T is a maximal torus of $B(C_G(R))$. Let w be an involution in $B(C_G(R))$ which inverts T. T normalizes B and B^w, and therefore normalizes $C_G(B)$, $C_G(B^w)$, $D(C_G(B))$, and $D(C_G(B^w))$. Lemma 5.12 implies that T centralizes $D(C_G(B))$ and $D(C_G(B^w))$. Hence, $\langle D(C_G(B)), D(C_G(B^w))\rangle \leq D(C_G(T))$.

We let $L = D(C_G(T))$ and $Q = B(C_G(R))$. Then $B \leq Q$ and the discussion in the above paragraph can be summarized as follows:

$$D(C_G(Q)) \leq D(C_G(B)) \leq D(C_G(T)) = L.$$

Before we state the next lemma, we remark that the group $QD(C_G(Q)) = Q \times D(C_G(Q))$. Indeed, Q and $D(C_G(Q))$ centralize each other, and $Q \cap C_G(Q) = Z(Q) = 1$ as $Q = PSL_2(K)$ where K is an algebraically closed field of characteristic 2.

Lemma 6.5 *If $D(C_G(Q)) = D(C_G(B))$ then $N_G(Q \times D(C_G(Q))) = N_G(Q)$ is a weakly embedded subgroup.*

Proof. Let $X = N_G(Q \times D(C_G(Q)))$.

We claim that $B(Q \times D(C_G(Q))) = Q$. Let $U \in \mathcal{U}(Q \times D(C_G(Q)))$. Then as $D(C_G(Q)) \lhd Q \times D(C_G(Q))$, U acts on $D(C_G(Q))$. But Lemma 5.7 implies that this action is trivial. Hence, U centralizes $D(C_G(Q))$ and in particular, it centralizes R, forcing $U \leq B(C_G(R)) = Q$. This also implies that $X = N_G(Q)$.

On the other hand, Q acts on $D(Q \times D(C_G(Q)))$ trivially by Lemma 5.7. This forces $D(Q \times D(C_G(Q))) = D(C_G(Q))$.

Next, we claim that if $U \in \mathcal{U}(X)$ then $U \leq Q$. Let $U \in \mathcal{U}(X)$. Then U acts on $Q \times D(C_G(Q))$, and hence, on $D(Q \times D(C_G(Q))) = D(C_G(Q))$. But by Lemma 5.7, this action is trivial. In particular, U centralizes R forcing $U \leq Q$.

Similarly, as Q acts on $D(X)$ trivially by Lemma 5.7, if $T \in \mathcal{T}(X)$, then $T \leq D(C_G(Q))$.

Now, we can prove the lemma. We will make use of the characterization obtained in Proposition 3.4. Let $B_1 \in \mathcal{U}(X)$. It follows from the above discussion that $B_1 \in \mathcal{U}(Q)$. Let $g \in N_G(B_1)$.

By Fact 2.31, $B_1 \leq B^x$, where x can be taken to be in Q as $B(Q \times D(C_G(Q))) = Q$. As B^x and B_1 are in the same connected component of $\mathcal{U}(G)$, $D(C_G(B_1)) = D(C_G(B))^x = D(C_G(Q))^x = D(C_G(Q))$. Hence, g normalizes $D(C_G(Q))$. But $Q = B(C_G(D(C_G(Q))))$, so g normalizes Q.

Let $R_1 \in \mathcal{T}(X)$. By the above $R_1 \in \mathcal{T}(D(C_G(Q)))$. Let $g \in N_G(R_1)$. This implies $B(C_G(R_1))^g = B(C_G(R_1)$. $R_1 \leq R^x$, where x can be taken to be in $D(C_G(Q))$ as $D(Q \times D(C_G(Q))) = D(C_G(Q))$. As $R_1 \approx R^x$, $B(C_G(R_1)) = B(C_G(R))^x = Q^x = Q$. Hence, g normalizes Q and thus $g \in N_G(Q \times D(C_G(Q))$. This finishes the proof. □

Lemma 6.6 *Let B be a maximal unipotent 2-subgroup of G, such that $B < Q$. Suppose B_1 is another unipotent 2-subgroup of G such that $B \sim B_1$. Then $B_1 \leq B$.*

Proof. By Proposition 5.15, $D(C_G(B)) = D(C_G(B_1))$ and this implies that B_1 centralizes $D(C_G(B))$, in particular, B_1 centralizes $d(R)$. Therefore, $B_1 \leq Q$. But $Q \cong PSL_2(K)$. This forces $B_1 \leq B$. □

Corollary 6.7 *Let B be a maximal unipotent 2-subgroup of G, which is in Q. Suppose B_1 is another unipotent 2-subgroup of G such that B and B_1 are in the same connected component of $\mathcal{U}(G)$. Then $B_1 \leq B$.*

In the remainder of our argument we consider various possibilities for the structure of L and come back to Lemma 6.5 in all cases. After some preliminary analysis, the main argument begins with Proposition 6.22 below.

Proposition 6.8 *If $\mathcal{T}(L)$ is connected then G has a weakly embedded subgroup.*

Proof. Let R_1 be any 2-torus in L. Then, $Q = B(C_G(R)) = B(C_G(R_1))$ by Corollary 5.24 since $\mathcal{T}(L)$ is connected, In particular, R_1 centralizes Q. Therefore, $d(R_1)$ centralizes Q. But L is a D-type group. Hence, L centralizes Q. Therefore, $L \leq D(C_G(Q)) \leq D(C_G(B)) \leq L$, and we have equality. Now, by Lemma 6.5, $N_G(Q \times D(C_G(Q)))$ is a weakly embedded subgroup of G. □

Proposition 6.9 *If $L = D(C_G(Q))O(L)$ and $D(C_G(Q)) < D(C_G(B)) < L$, then $C_{Z(O(L))}(B) = 1$.*

Proof. Let $A_1 = C_{Z(O(L))}(B)$. Suppose $A_1 \neq 1$. Then $B(C_G(A_1)) < G$. The subgroup $D(C_G(B))$ centralizes B and normalizes $Z(O(L))$. Hence, $D(C_G(B))$ normalizes A_1. But $L = D(C_G(B))O(L)$. Hence, L normalizes A_1. Therefore, L normalizes $B(C_G(A_1))$. By Lemma 5.8, L centralizes $B(C_G(A_1))$. In particular, L centralizes B as $B \leq B(C_G(A_1))$. But this forces $L = D(C_G(B))$, which contradicts our assumptions. □

Lemma 6.10 *If $L = D(C_G(Q))O(L)$ and $D(C_G(Q)) < D(C_G(B)) < L$, then $B = B(C_G(D(C_G(B))))$.*

Proof. Clearly, $B(C_G(D(C_G(B)))) \leq B(C_G(D(C_G(Q))))$. By the Galois connection, $B(C_G(D(C_G(Q)))) = Q$. As $D(C_G(Q)) < D(C_G(B))$, we have the following strict inequality: $B(C_G(D(C_G(B)))) < B(C_G(D(C_G(Q))))$. This forces $B(C_G(D(C_G(B)))) = B$. □

Proposition 6.11 *If $L = D(C_G(Q))O(L)$ and $D(C_G(Q)) < D(C_G(B)) < L$, then $C_{O(L)}(B)^\circ$ is divisible.*

Proof. By Fact 2.11, $O(L) = E * D$, where E is of bounded exponent and D is divisible. We may assume that E is connected. Let $A_1 = C_E(B)^\circ$. Suppose $A_1 \neq 1$. Let $A_2 = N_E(A_1)^\circ$. $D(C_G(B))$ normalizes A_1. Therefore, $D(C_G(B))$ normalizes $B(C_G(A_1))$. By Lemma 5.8, this action is trivial. Hence, $B(C_G(A_1)) \leq B(C_G(D(C_G(B))))$. By Lemma 6.10,

$B(C_G(D(C_G(B)))) = B$. But $B \leq B(C_G(A_1))$. Therefore, we conclude $B = B(C_G(A_1)) = B(C_G(D(C_G(B))))$.

A consequence of the last paragraph is that A_2 normalizes B. $B \rtimes A_2$ is a nilpotent group, because otherwise, using Fact 2.18 we can interpret in $B \rtimes A_2$ an algebraically closed field K and K^+ is isomorphic to a definable section of B. Therefore K is of characteristic 2. But a definable section of A_2 is isomorphic to an infinite subgroup of K^* and the "no bad fields" hypothesis implies that this subgroup is K^* itself. But as A_2 is a nilpotent $2^\perp$-group of bounded exponent, the multiplicative subgroup of this algebraically closed field would have Sylow p-subgroups with infinitely many elements of order p (Fact 2.28). This is impossible by Fact 2.27. Hence, $B \rtimes A_2$ is nilpotent and by Fact 2.20, A_2 centralizes B. Therefore, $A_2 = A_1$. As E is nilpotent of finite Morley rank, we conclude that $E = A_2 = A_1$. This forces $Z(E) \leq C_{Z(O(L))}(B)$. But this last group is trivial by Proposition 6.9, a contradiction. As a result, $A_1 = 1$ and therefore $C_{O(L)}(B)^\circ$ is divisible. □

Corollary 6.12 *If $L = D(C_G(Q))O(L)$ and $D(C_G(Q)) < D(C_G(B)) < L$, then $(O(L) \cap D(C_G(B)))^\circ$ is divisible and torsion free.*

Proof. By Fact 2.11, $(O(L) \cap D(C_G(B)))^\circ = E * D$, where E is a definable, connected group of bounded exponent and D is a definable, divisible group. If $E \neq 1$ then $C_{O(L)}(B)^\circ$ has a nontrivial definable subgroup of bounded exponent. But this contradicts Proposition 6.11. Therefore, $(O(L) \cap D(C_G(B)))^\circ$ is divisible. The torsion part of $(O(L) \cap D(C_G(B)))^\circ$ is also contained in the torsion divisible part of $O(L)$. But, by Fact 2.12, this last subgroup of $O(L)$ is central in $O(L)$. Now, Proposition 6.9 forces the torsion part of $(O(L) \cap D(C_G(B)))^\circ$ to be trivial. □

Proposition 6.13 *If $L = D(C_G(Q))O(L)$ and $D(C_G(Q)) < D(C_G(B)) < L$, then $B = B(C_G(C_{O(L)}(B)))$.*

Proof. As $D(C_G(B))$ normalizes $C_{O(L)}(B)$, it normalizes $B(C_G(C_{O(L)}(B)))$. Thus, by Lemma 5.8, $D(C_G(B))$ centralizes $B(C_G(C_{O(L)}(B)))$. Therefore,

$$B(C_G(C_{O(L)}(B))) \leq B(C_G(D(C_G(B)))).$$

But by Lemma 6.10, this last subgroup is equal to B. □

Proposition 6.14 *If $L = D(C_G(Q))O(L)$, $D(C_G(Q)) < D(C_G(B)) < L$, and the group $N_{O(L)}(C_{O(L)}(B))^\circ$ is torsion-free, then $B \rtimes N_{O(L)}(C_{O(L)}(B))^\circ$ is nilpotent.*

Proof. Let $A = N_{O(L)}(C_{O(L)}(B))^\circ$. Clearly $B \rtimes A$ is a solvable group. Suppose it is not nilpotent. Then we can interpret in $B \rtimes A$ an algebraically closed field K using Fact 2.18. K^+ is isomorphic to a definable section of B. Therefore, K is of characteristic 2. As there are no bad fields in the environment, we conclude that a definable section of A is isomorphic to the multiplicative group of this field, in particular A is not torsion-free by Fact 2.22. This contradiction shows that $B \rtimes A$ is nilpotent. □

Lemma 6.15 *If L is solvable, $D(C_G(Q)) < D(C_G(B)) < L$ and $\mathcal{T}(L)$ is not connected, then $L = d(R)O(L)$, $D(C_G(B)) = d(R)(O(L) \cap D(C_G(B)))^\circ$ and $O(L) \neq 1$. L and $D(C_G(B))$ are not nilpotent.*

Proof. $\mathcal{T}(L)$ is not connected implies that L is not nilpotent. By Lemma 5.4, we conclude that $L = d(R)F(L)^\circ$. As $\mathcal{T}(L)$ is not connected, $F(L)^\circ = O(L)$. Hence, we have $L = d(R)O(L)$. This implies $D(C_G(B)) = d(R)(O(L) \cap D(C_G(B)))^\circ$. Also $(O(L) \cap D(C_G(B)))^\circ \neq 1$ as $d(R) \leq D(C_G(Q)) < D(C_G(B))$. We also claim that $D(C_G(B))$ is not nilpotent because otherwise $D(C_G(B)) = d(R)$ and $D(C_G(Q)) = D(C_G(B))$, a contradiction. □

Proposition 6.16 *If L is nonsolvable, $\mathcal{T}(L)$ is not connected and $D(C_G(Q)) < D(C_G(B)) < L$, then $L = D(C_G(B))O(L) = D(C_G(Q))O(L)$.*

Proof. By Corollary 5.28, $L/O(L) \cong (P)SL_2(K)$, where K is an algebraically closed field of characteristic different from 2. We will use $\overline{}$-notation to denote quotients by $O(L)$.

As $\overline{R} \cong R$, $\overline{R}$ is a 2-torus of $\overline{L}$. Moreover, as the 2-tori of $(P)SL_2(K)$, where K is an algebraically closed field of characteristic different from 2, are all 1-dimensional, $\overline{R}$ is a maximal 2-torus of $\overline{L}$. $\overline{R} \leq \overline{D(C_G(Q))}$.

We claim that $\overline{D(C_G(Q))}$ is a Zariski closed subgroup of $\overline{L}$. Let R_1 be a 2-torus of $D(C_G(Q))$. Clearly $d(\overline{R_1}) \leq \overline{d(R_1)}$. For the other inclusion we argue as follows. $d(\overline{R_1}) = \overline{d(R_1O(L))}$ by Fact 2.8. But $d(R_1O(L)) = d(R_1)O(L)$. Therefore, $d(\overline{R_1}) = \overline{d(R_1)}$. Therefore, $\overline{D(C_G(Q))} = \langle d(\overline{R_1}) : R_1$ is a 2-torus centralizing $Q\rangle$. Thus, by Fact 2.42, it is sufficient to show that $d(\overline{R_1})$, where R_1 is a 2-torus centralizing Q, is a closed subgroup of $\overline{L}$. The Zariski closure of $\overline{R_1}$ in $\overline{L}$ is isomorphic to K^* and thus the "no bad fields" hypothesis implies that $d(\overline{R_1})$ and the Zariski closure of $\overline{R_1}$ are the same group.

Therefore, $\overline{D(C_G(Q))}$ can be equal to three different closed subgroups of $\overline{L}$: $d(\overline{R})$, a Borel subgroup $\overline{U}d(\overline{R})$, or $\overline{L}$. We will show that the first two possibilities cannot happen.

i) $\overline{D(C_G(Q))} = \overline{U}d(\overline{R})$.

Let $\overline{v} \in I(\overline{L})$ such that $\overline{v}$ inverts $d(\overline{R})$. The element v acts on $RO(L)$. By Fact 2.31, there exists $g \in O(L)$ such that $R^v = R^g$ and $vg^{-1} \in N_L(R)$. This implies that vg^{-1} normalizes $B(C_G(R)) = Q$. Hence, vg^{-1} normalizes $D(C_G(Q))$, and so $\overline{v}$ normalizes $\overline{D(C_G(Q))}$. But $\overline{v} \notin \overline{U}d(\overline{R})$, which contradicts the fact that Borel subgroups are self-normalizing. Hence, $\overline{D(C_G(Q))} \neq \overline{U}d(\overline{R})$.

ii) $\overline{D(C_G(Q))} = d(\overline{R})$.

Let v be as in part **i)**. Then we also get $g \in O(L)$ such that vg^{-1} normalizes R, Q and $D(C_G(Q))$.

In $\overline{L}$, we can find $\overline{W} < \overline{R}\langle\overline{v}\rangle$, such that $\overline{W} \cong Q_8$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, depending on whether $\overline{L} \cong SL_2(K)$ or $PSL_2(K)$ respectively. We take an element $\overline{x}$ of order 3 from $N_{\overline{L}}(\overline{W})$.

We claim that $\overline{L} = \langle d(\overline{R}), d(\overline{R})^{\overline{x}}\rangle$.

To illustrate we give an example in the $SL_2(K)$ situation.

$$\overline{W} = \left\langle \overline{u} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \quad \overline{v} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad \overline{w} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right\rangle,$$

$$\overline{x} = \frac{1}{2}\begin{pmatrix} -1+i & -1-i \\ 1-i & -1-i \end{pmatrix}$$

where $i^2 = -1$. One can choose $\overline{x}$ so that

$$\overline{u}^{\overline{x}} = \overline{v},\ \overline{v}^{\overline{x}} = \overline{w},\ \overline{w}^{\overline{x}} = \overline{u}.$$

Let $L_1 = \langle d(\overline{R}), d(\overline{R})^{\overline{x}}\rangle$. We have $N_{\overline{L}}(d(\overline{R})) = d(\overline{R}) \rtimes \langle\overline{v}\rangle \leq L_1$. Therefore,$L_1$ cannot be a Borel subgroup of $\overline{L}$. As L_1 is connected, $L_1 = \overline{L}$.

In L, we may assume that $WO(L) = O(L) \rtimes W$, where $W \cong \overline{W}$. As $\overline{x}$ normalizes $\overline{W}$, x normalizes $O(L) \rtimes W$. By Fact 2.31, we can find $h \in O(L)$ such that $xh^{-1} \in N_L(W)$.

By taking conjugates, we may assume $W < R\langle v\rangle$. Then W normalizes R. This implies W normalizes Q. Therefore, using Fact 2.39, we may assume W is a group of inner automorphisms of Q. As W centralizes T, W can be seen as a subgroup of T (maximal tori are self-centralizing in reductive algebraic groups). But Q is an algebraic group over an algebraically closed field of characteristic 2, thus T is a $2^{\perp}$-group. Therefore, the action of W on Q is trivial. Hence, we get $Q \leq B(C_G(W))$.

We claim that $Q = B(C_G(W))$. Suppose $Q < B(C_G(W))$. Then there is a unipotent 2-subgroup B_1 in $B(C_G(W)) \setminus Q$. By Corollary 6.7, B_1 cannot be in the same connected component of $\mathcal{U}(G)$ as any unipotent 2-subgroup of Q. Therefore by Proposition 5.21,

$B(C_G(W)) \cong PSL_2(L)$, where L is an algebraically closed field of characteristic 2. But Q contains maximal unipotent 2-subgroups of G. In particular, these are maximal unipotent 2-subgroups of $B(C_G(W))$. As $B(C_G(W)) \cong PSL_2(L)$, where L is an algebraically closed field of characteristic 2, these subgroups correspond to root subgroups of $B(C_G(W))$. But then Q contains two opposite root subgroups of $B(C_G(W))$. This forces $Q = B(C_G(W))$, a contradiction.

Since xh^{-1} normalizes W and vg^{-1} normalizes R, $\langle d(R), xh^{-1}, vg^{-1}\rangle$ normalizes Q. Let $L_2 = \langle d(R), xh^{-1}, vg^{-1}\rangle$. Then $\overline{L_2} = \overline{L}$. As L_2 normalizes Q, L_2 normalizes $D(C_G(Q))$. Therefore, $\overline{L_2}$ normalizes $\overline{D(C_G(Q))}$. But $\overline{D(C_G(Q))} = d(\overline{R})$ and $\overline{L_2} = \overline{L} \cong (P)SL_2(K)$. This is a contradiction because $(P)SL_2(K)$ cannot have a large normal subgroup. □

Proposition 6.17 *If L is solvable, $\mathcal{T}(L)$ is not connected, and $D(C_G(Q)) < D(C_G(B)) < L$, then $d(R)$ has p-torsion for every prime p.*

Proof. By Lemma 6.15 and Corollary 6.12, $(O(L) \cap D(C_G(B)))^\circ$ is divisible and torsion-free. As $D(C_G(B))$ is nonnilpotent (Lemma 6.15) and solvable, by Fact 2.18 an algebraically closed field K is interpretable in it. K^+ is a definable section of $(O(L) \cap D(C_G(B)))^\circ$. Since $(O(L) \cap D(C_G(B)))^\circ$ is torsion-free, K is of characteristic 0. The "no bad fields" hypothesis implies that a definable section of $d(R)$ is isomorphic to K^* and, using Fact 2.22, we conclude that $d(R)$ has p-torsion for every prime p. □

Lemma 6.18 *Let G be a D-type tame group of finite Morley rank. Assume $G = G'C$, where C is a definable $2^\perp$-subgroup of G. Then G is perfect.*

Proof. G/G' is a $2^\perp$-group. Hence, all the tori are in G', forcing $G = G'$. □

Proposition 6.19 *Let G be a tame group of finite Morley rank, T a definable divisible abelian subgroup of G, and O a definable, connected, nilpotent, torsion-free subgroup of G. Assume T normalizes O. Assume also that T does not have p-torsion for some prime p. Then $H = TO$ is nilpotent.*

Proof. Suppose that H is not nilpotent. Then we can interpret in H an algebraically closed field K, using Fact 2.18. As no bad fields are interpretable in the environment, a definable section of T will be isomorphic to K^*. This forces $char(K) = p$, where p is a prime missed by T. K^+ is isomorphic to a definable section of O. Hence, it is torsion-free by Fact 2.22. This yields a contradiction. Therefore, H is nilpotent. □

Proposition 6.20 *If L is nonsolvable, $\mathcal{T}(L)$ is not connected and $D(C_G(Q)) < D(C_G(B)) < L$, then $L/O(L) \cong (P)SL_2(K)$ where K is an algebraically closed field of characteristic 0.*

Proof. We will use $\overline{}$-notation to denote quotients by $O(L)$. By Proposition 6.16, $\overline{D(C_G(B))} \cong \overline{L}$. By Corollary 5.28 we know that $L/O(L) \cong (P)SL_2(K)$ where the characteristic of K is different from 2. Suppose the characteristic of K is not 0. Then it is at least 3. $\overline{D(C_G(B))} = \langle d(S)O(D(C_G(B)))/O(D(C_G(B))) : S \text{ is a 2-torus of } D(C_G(B))\rangle$. Proposition 6.19 and Corollary 6.12 imply that $d(S)O(D(C_G(B)))$ is nilpotent. Thus $SO(D(C_G(B))) = S \oplus O(D(C_G(B)))$. As S is an arbitrary 2-torus of $D(C_G(B))$, and $D(C_G(B))$ is a D-type group, $O(D(C_G(B)))$ is a central subgroup of $D(C_G(B))$. As $\overline{D(C_G(B))} \cong (P)SL_2(K)$, we conclude that $O(D(C_G(B))) = Z(D(C_G(B)))^\circ$. Therefore,

$$D(C_G(B))/Z(D(C_G(B)))^\circ \cong (P)SL_2(K)$$

where K is an algebraically closed field of characteristic at least 3. By Lemma 6.18, $D(C_G(B))$ is a perfect group. Theorem 4.2 implies that $O(D(C_G(B))) = 1$. But this forces $D(C_G(B)) = D(C_G(Q))$, which contradicts our assumptions. This finishes the proof of the claim. □

Lemma 6.21 *Let $G = d(T)E$ be connected solvable group of finite Morley rank, where T is a p-torus and E is a nilpotent group of bounded exponent a power of p. Then G is nilpotent.*

Proof. Suppose G is nonnilpotent. Then by Fact 2.18, we can interpret an algebraically closed field K in G so that a definable section of E is isomorphic to K^+. This implies that the characteristic of K is p. On the other hand, a definable section of $d(T)$ is isomorphic to an infinite subgroup of K^*. As the characteristic of K is p, T centralizes E. Therefore, $d(T)$ centralizes E. This forces G to be nilpotent, a contradiction. □

Proposition 6.22 *If $\mathcal{T}(L)$ is not connected and $D(C_G(Q)) < D(C_G(B)) < L$, then $O(L)$ is divisible.*

Proof. We carry out a case-by-case analysis.

L **is solvable:** By Lemma 6.15, L is not nilpotent. By Theorem 2.11, $O(L) = E * D$, where E is of bounded exponent and D is divisible. $d(R)E$ is a connected solvable group. Suppose $E \neq 1$. $d(R)$ has p-torsion for every prime p by Proposition 6.17. Suppose p is an odd prime such that E has a nontrivial Sylow p-subgroup, say E_p. By Lemma 6.21, E_p centralizes the p-torus T_p of $d(R)$. As E_p centralizes T_p, it normalizes $B(C_G(T_p))$. By Lemma 5.10, E_p centralizes $B(C_G(T_p))$. In particular, E_p centralizes B. But $C_{Z(O(L))}(B) = 1$ by Proposition 6.9 and Lemma 6.15. This is a contradiction. Therefore $E = 1$.

L **is nonsolvable:** Let $D_\circ$ be the divisible part of $O(L)$. The quotient $(L/D_\circ)/O(L/D_\circ) = (L/D_\circ)/(O(L)/D_\circ) \cong L/O(L) \cong (P)SL_2(K)$, where K is an algebraically closed field of characteristic 0, by Proposition 6.20. $L/D_\circ$ is generated by $d(S)D_\circ/D_\circ$, where S ranges over all the 2-tori of L. Consider the action of $d(S)D_\circ/D_\circ$ on $O(L/D_\circ)$. The group $d(S)D_\circ/D_\circ O(L/D_\circ)$ is nilpotent, because otherwise, using Fact 2.18 and Proposition 6.20, we can interpret an algebraically closed field of characteristic 0 in it in such a way that a definable section of $O(L/D_\circ)$ is isomorphic to the additive group of this field. But $O(L/D_\circ)$ is of bounded exponent. Therefore, the action of $d(S)D_\circ/D_\circ$ on $O(L/D_\circ)$ is trivial. As S is an arbitrary 2-torus in L, we conclude that $O(L/D_\circ)$ is central in $L/D_\circ$. Arguments similar to the ones used in the proof of Proposition 6.20 show that $O(L/D_\circ) = 1$. This implies that $O(L) = D_\circ$. □

Proposition 6.23 *If $\mathcal{T}(L)$ is not connected and $D(C_G(Q)) < D(C_G(B)) < L$, then $O(L)$ is torsion-free.*

Proof. We carry out a case-by-case analysis.

L **is solvable:** By Lemma 6.15, L is not nilpotent. By Proposition 6.22, $O(L)$ is divisible. By Proposition 5.3, $[L, O(L)]$ is torsion-free. Clearly, $O(L)/[L, O(L)]$ is central in $L/[L, O(L)]$. Also $(L/[L, O(L)])/(O(L)/[L, O(L)]) \cong L/O(L)$ and this last group is abelian because $L = d(R)O(L)$ by Lemma 6.15. Therefore, $L/[L, O(L)]$ is a nilpotent group. Since L is a D-type group, so is $L/[L, O(L)]$ and $L/[L, O(L)] = d(R_1/[L, O(L)])$, where $R_1/[L, O(L)]$ is a maximal 2-torus of $L/[L, O(L)]$. As L is solvable, Fact 2.33 ii) implies that $L/[L, O(L)] = d(R)[L, O(L)]/[L, O(L)]$. Thus $L = d(R)[L, O(L)]$ and $O(L) = (O(L) \cap d(R))^\circ[L, O(L)]$. By Corollary 6.12, $(O(L) \cap d(R))^\circ$ is torsion-free. Thus Fact 2.26 implies that $O(L)$ is torsion-free.

L **is nonsolvable:** Proposition 6.22 and Fact 2.12 imply that $O(L) = A_1 \times A_2$, where A_1 and A_2 are the torsion and the torsion-free parts of $O(L)$ respectively. By Proposition 5.3, $[L, O(L)]$ is torsion free. We have by Proposition 6.20

$$(L/[L, O(L)])/(O(L)/[L, O(L)]) \cong L/O(L) \cong (P)SL_2(K),$$

where K is an algebraically closed field of characteristic 0. Clearly, $O(L)/[L, O(L)]$ is central in $L/[L, O(L)]$. We have a central extension of finite Morley rank of a simple algebraic group by a group of finite Morley rank. By Proposition 6.18, $L/[L, O(L)]$ is a perfect group. Using arguments similar to the ones in the proof of Proposition 6.20, we conclude that $O(L)/[L, O(L)] = 1$. This implies that $L/[L, O(L)] \cong (P)SL_2(K)$ But $L/[L, O(L)]$ has an isomorphic copy of A_1. Therefore, $A_1 = 1$ and $O(L)$ is torsion-free. □

Theorem 6.24 *A simple tame, K^*-group G of mixed type contains a weakly embedded subgroup.*

Proof. Let M_1 denote the stabilizer of a connected component of $\mathcal{U}(G)$. By Proposition 6.3, if $N_G(T) \leq M_1$ for any 2-torus of M_1, then M_1 is a weakly embedded subgroup. If this is not the case then there is a 2-torus R in M_1 such that $N_G(R) \not\leq M_1$. By Proposition 6.4, $\mathcal{U}(B(N_G(R)))$ is not connected. Fact 5.21 implies that $Q = B(C_G(R)) \cong PSL_2(K)$, where K is an algebraically closed field of characteristic 2. Then, as it was discussed after Proposition 6.4, $N_Q(B) = BT$, where T is a maximal 2-torus in Q and

$$D(C_G(Q)) \leq D(C_G(B)) \leq L = D(C_G(T)).$$

By Proposition 6.8, if $\mathcal{T}(L)$ is a connected graph then G has a weakly embedded subgroup. Therefore, we may assume that $\mathcal{T}(L)$ is not a connected graph.

If $D(C_G(Q)) = D(C_G(B))$ then by Lemma 6.5, G has a weakly embedded subgroup. Therefore, we may assume that $D(C_G(Q)) < D(C_G(B))$. If $D(C_G(B)) = L$ then $L = D(C_G(B^w))$ also where w is an involution in Q inverting T. This implies that L centralizes Q as $Q = \langle B, B^w \rangle$. Therefore, $L = D(C_G(Q))$. This contradicts our assumption that $D(C_G(Q)) < D(C_G(B))$. Hence, $D(C_G(Q)) < D(C_G(B)) < L$.

We will show that $D(C_G(Q)) < D(C_G(B)) < L$ cannot happen. Let $A = N_{O(L)}(C_{O(L)}(B))$. A is connected since by Proposition 6.23, $O(L)$ is torsion-free. We have $C_{O(L)}(B) < O(L)$ because otherwise $L = D(C_G(B))$, contrary to our assumptions. Hence, $[O(L) : C_{O(L)}(B)] = \infty$ by Lemma 6.23. As $O(L)$ is nilpotent, by Fact 2.14, $[N_{O(L)}(C_{O(L)}(B)) : C_{O(L)}(B)] = \infty$. By Proposition 6.23, A is torsion-free. This, together with Proposition 6.14, implies that $B \rtimes A = B \oplus A$. Hence, $[N_{O(L)}(C_{O(L)}(B)) : C_{O(L)}(B)] = 1$, a contradiction. □

7 From Weak to Strong Embedding

In this section we will finish the proof of Theorem 1.7. We will continue working in the hypothetical simple tame K^*-group G of mixed type. In the previous section we showed that such a group has a weakly embedded subgroup. The main result in this section will show that this weakly embedded subgroup is a *strongly embedded subgroup*. This will yield a contradiction for reasons we will mention shortly.

Definition 7.1 *A proper definable subgroup M of a group G of finite Morley rank is said to be a* strongly embedded subgroup *if it satisfies the following conditions:*

i) *M contains involutions.*

ii) *For every $g \in G \setminus M$, $M \cap M^g$ does not contain involutions.*

The following are two characterizations of strongly embedded subgroup s:

Fact 7.2 ([6], [11]) *Let G be a group of finite Morley rank with a proper definable subgroup M. Then the following are equivalent:*

i) *M is a strongly embedded subgroup.*

ii) *$I(M) \neq \emptyset$, $C_G(i) \leq M$ for any $i \in I(M)$, and $N_G(S) \leq M$ for any Sylow 2-subgroup S of M.*

iii) *$I(M) \neq \emptyset$, and $N_G(S) \leq M$ for any nontrivial 2-subgroup S of M.*

The following fact is crucial for our argument:

Fact 7.3 ([6], [11]) *Let G be a group of finite Morley rank with a strongly embedded subgroup M. Then $I(G)$ is a single conjugacy class in G, and $I(M)$ is a single conjugacy class in M.*

The following proposition shows that the proof of Theorem 1.7 will be over once we have shown that G has a strongly embedded subgroup.

Proposition 7.4 ([16]) *If a group G of finite Morley rank has a single conjugacy class of involutions then the connected component of any Sylow 2-subgroup is either of bounded exponent or divisible.*

Proof. Let S be Sylow 2-subgroup of G. $S^\circ = B * T$ of by Fact 1.5 ii). We will show that we cannot have $B \neq 1$ and $T \neq 1$. Suppose towards a contradiction that $B \neq 1$ and $T \neq 1$. By Fact 2.27, T contains finitely many involutions As $I(B)$ is infinite, we can find $u \in I(S^\circ \setminus T)$ and $v \in I(T)$. Since these are conjugate in G, by Fact 2.32, there exists $g \in N_G(T)$ such that $u = v^g$. But this implies that $u \in T$, contradicting the choice of u. □

Theorem 7.5 *Let G be a simple tame K^*-group of mixed type with a weakly embedded subgroup M. Then M is a strongly embedded subgroup.*

Proof. Proposition 3.4 and Fact 7.2 ii) imply that it is sufficient to prove that for any $t \in I(M)$, $C_G(t) \leq M$. Suppose towards a contradiction that there is an involution $t \in I(M)$ such that $C_G(t) \not\leq M$. Let $H = C_G(t)$.

1 *$H \cap M$ has an infinite Sylow 2-subgroup.*

Proof of 1. Let S be a Sylow 2-subgroup of M such that $t \in S$. $S^\circ = B * D$, where $B \neq 1$ and $D \neq 1$. The involution t acts on the definable subgroup B. As there are infinitely many involutions in B, t cannot fix finitely many elements of B (Fact 2.38). □

It follows from (1) that $M \cap H$ is a weakly embedded subgroup of H, and in particular contains a Sylow 2-subgroup of H.

Note that it follows from the proof of (1) that $B \cap H$ and thus H contains an infinite elementary abelian 2-group. In particular, $B(H) \neq 1$.

2 *H is not solvable.*

Proof of 2. Suppose H is solvable. As $B(H) \leq H^\circ$, we conclude using Fact 2.35 that $B(H)$ is a unipotent 2-subgroup normal in H. As $B(H) \lhd H$, $B(H)$ is contained in every Sylow 2-subgroup of H and hence, in $H \cap M$. As $H \cap M$ is a weakly embedded subgroup of H, $H \leq N_H(B(H)) \leq H \cap M$, a contradiction. □

3 *$H^\circ = L \times O(H)$, where $L \cong PSL_2(K)$ and the characteristic of K is 2. In particular, H does not contain 2-tori and $t \in H \setminus H^\circ$.*

Proof of 3. Lemma 5.20, (2) and the fact that H has nontrivial unipotent 2-subgroups imply that $H^\circ/O(H) \cong PSL_2(K)$, where K is an algebraically closed field of characteristic 2. Therefore, $H^\circ = B(H)O(H)$. $B(H) \cap M$ is a weakly embedded subgroup of $B(H)$. In particular, $\mathcal{U}(B(H))$ is not connected. Therefore, by Proposition 5.21, $B(H) \cong PSL_2(K)$, where K is an algebraically closed field of characteristic 2. This proves the claim. □

Let $Y = D(M)$.

4 *Y is solvable.*

Proof of 4. As $C_M(t) \leq H$, $C_M(t)$ does not contain 2-tori. Hence, $C_Y(t)$ does not contain 2-tori either. Therefore, by Lemma 5.13, we conclude that Y is solvable. □

5 *$M \cap L$ contains a Sylow 2-subgroup of L.*

Proof of 5. (1) and the fact that $H^\circ = L \times O(H)$, where $L \cong PSL_2(K)$ with $char(K) = 2$, imply that $L \cap M$ contains a Sylow 2-subgroup which is an infinite elementary abelian 2-group A. If $A \leq A_1$, where A_1 is a Sylow 2-subgroup of L, then A_1 centralizes A as A_1 is abelian. But A is a subgroup of M and M is a weakly embedded subgroup. Therefore, A_1 is a subgroup of M. But A is a Sylow 2-subgroup of $L \cap M$. Therefore, $A_1 = A$. □

We fix A, a Sylow 2-subgroup of L in $M \cap L$. $N_L(A) = A \rtimes T$, where T is a torus. Let $w \in N_L(T)$ be such that w inverts T. By Lemma 5.7, $D(C_G(A)) \geq Y$. Moreover, as T normalizes both A and A^w, we have $\langle D(C_G(A)), D(C_G(A^w))\rangle \leq D(C_G(T))$. We will use R to denote $D(C_G(T))$. We can summarize the above discussion as follows:

$$D(M) = Y = D(C_G(A)) \leq R = D(C_G(T))$$

6 *R is solvable.*

Proof of 6. As $T \leq L \leq C_G(t)$ and R is a characteristic subgroup of $C_G(T)$, $t \in N_G(R)$. Moreover, since $C_R(t) \leq C_G(t)$, $C_R(t)$ does not contain 2-tori. By Lemma 5.13, R is solvable. □

7 *$R = Y$.*

Proof of 7. By (6), R and Y are both solvable. By Lemma 5.4, R and Y can be factored as a product of their Fitting subgroups and the definable closure of one of their maximal 2-tori. As M is a weakly embedded subgroup, Proposition 3.2 ii) implies that Y contains a maximal 2-torus T_1 of G. Therefore, we have $Y = d(T_1)F(Y)^\circ$ and $R = d(T_1)F(R)^\circ$.

Suppose that $Y < R$. We will reach a contradiction. If R is nilpotent, then we immediately conclude that $R = Y$. Therefore, R is nonnilpotent.

Note that $R \not\leq M$ because otherwise $R = Y$. This forces $F(R)^\circ$ to be a $2^\perp$-group because otherwise, by Lemma 5.5, $F(R)^\circ$ would contain a nontrivial 2-torus T_c. As $T_c \lhd R$, T_c is centralized by R, and therefore by T_1. This forces $T_c \leq M$ and thus $R \leq M$, a contradiction to $R \not\leq M$. This implies $R = d(T_1)O(R)$ and $Y = d(T_1)(Y \cap O(R))$.

7.1 *If t inverts T_1 (equivalently $d(T_1)$), then $C_{O(R)}(t)$ is infinite and $C_{O(R)}(t)^\circ \leq M$.*

Proof of 7.1. The involution t centralizes T. Therefore, it acts on R. As $O(R)$ is characteristic in R, t acts on $O(R)$ also. If $C_{O(R)}(t)$ is finite then t inverts $O(R)$ by Fact 2.37. Then by Proposition 2.51, $C_{R/O(R)}(t) = C_R(t)O(R)/O(R)$. As t inverts $d(T_1)$, t inverts $R/O(R)$. Hence, $C_{R/O(R)}(t)$ is finite. But

$$C_R(t)O(R)/O(R) \cong C_R(t)/C_{O(R)}(t).$$

Hence, $C_R(t)/C_{O(R)}(t)$ is finite. As $C_{O(R)}(t)$ is finite, $C_R(t)$ is finite also, which forces R to be abelian. This contradicts our assumption that R is not nilpotent. Hence, $C_{O(R)}(t)$ is infinite.

The connected group $C_{O(R)}(t)^\circ$ acts on L and centralizes T. The action of $C_{O(R)}(t)^\circ$ on L is by inner automorphisms. Therefore, $C_{O(R)}(t)^\circ$ normalizes A. This forces $C_{O(R)}(t)^\circ \leq M$ as M is a weakly embedded subgroup. □

7.2 $C_{Z(O(R))}(A) = 1$.

Proof of 7.2. Let $C_1 = C_{Z(O(R))}(A)$. Suppose $C_1 \neq 1$. Then $B(C_G(C_1)) < G$. Note that $A \leq B(C_G(C_1))$. As $R = d(T_1)O(R)$ and $d(T_1) \leq C_G(A)$, R normalizes C_1. Hence, by Lemma 5.8, R centralizes $B(C_G(C_1))$, and in particular it centralizes A. But M is a weakly embedded subgroup, which forces $R \leq M$. This contradicts our assumption on R. □

7.3 *$O(R)$ is a divisible group.*

Proof of 7.3. By Fact 2.11, $O(R) = C * D$, where C is definable, connected and of bounded exponent and D is definable and divisible. Suppose $C \neq 1$. Then the action of $d(T_1)$ on C is nontrivial because, otherwise, $d(T_1)D$ is a definable subgroup of R which contains all the 2-tori of R. But R is a D-type group and thus by Lemma 5.2, $R = d(T_1)D$. Therefore, $O(R) = (d(T_1) \cap O(R))D$ which forces $C = 1$ (Facts 2.11, 2.27 and 2.28). This contradicts our assumption on C. Therefore, C has an infinite p-subgroup C_p on which $d(T_1)$ acts nontrivially.

We claim that $d(T_1)$ does not have p-torsion. Suppose T_p is the nontrivial p-torus of $d(T_1)$. By Lemma 6.21, C_p centralizes T_p. Thus C_p normalizes $B(C_G(T_p))$. This last subgroup contains A. By Lemma 5.10, C_p centralizes $B(C_G(T_p))$. Hence, C_p centralizes A. But $C_p \cap Z(O(R)) \neq 1$. This contradicts (7.2). Therefore $T_p = 1$.

As $d(T_1)$ has no p-torsion, an application of Fact 2.18 and the "no bad fields" assumption imply that $d(T_1)D$ is nilpotent. T_1 is central in $d(T_1)D$. This implies $d(T_1)D$ centralizes $d(T_1)$. Thus $d(T_1)C$ contains all the 2-tori of R. But R is a D-type group. Therefore, by Lemma 5.2 $R = d(T_1)C$. In particular, this implies that $D < d(T_1)$.

We claim that $(C \cap M)^\circ \neq 1$. Suppose to the contrary that $(C \cap M)^\circ = 1$. Then $R \cap M = d(T_1)(C \cap M)$. The subgroup $C \cap M$ is normal in $R \cap M$ because $d(T_1)$ normalizes both C and M. Therefore, the connected group $d(T_1)$ centralizes $M \cap C$. We already know that $Y = d(T_1)(Y \cap O(R)) = d(T_1)(Y \cap CD)$. $D \leq d(T_1) \leq Y$ implies that $Y = d(T_1)D(Y \cap C) = d(T_1)(Y \cap C)$. $Y \cap C \leq M \cap C$ which we have assumed is finite. But $Y = D(M)$ is connected. Therefore, $Y = d(T_1)$. This implies that t acts on $d(T_1)$. As $H = C_G(t)$ does not contain nontrivial 2-tori, Lemma 5.13 implies that t inverts $d(T_1)$. By (7.1), $C_{O(R)}(t)$ is infinite and $C_{O(R)}(t)^\circ \leq M$. As $O(R) = CD$ and $C \cap M$ is finite, any element of $C_{O(R)}(t)^\circ$ can be written as dc_i where $d \in D$ and $c_i \in C \cap M = \{c_1, \cdots, c_m\}$. There exists $c \in C \cap M$ such that $\{d_j c : c \in \mathbb{N}\}$ is an infinite set of distinct elements of $C_{O(R)}(t)^\circ$. $(d_1 c)(d_j c)^{-1} = d_1 d_j^{-1} \in C_{O(R)}(t)^\circ \cap D$ for all $j \in \mathbb{N}$. This implies that $C_{O(R)}(t)^\circ \cap D \neq 1$, a contradiction. Therefore, $(C \cap M)^\circ \neq 1$.

The subgroup $(C \cap M)^\circ$ centralizes A by Lemma 5.10. The subgroup $N_C(C_C(A))$ normalizes $B(C_G(C_C(A)))$. On the other hand, $Y \leq D(N_G(C_C(A)))$ because Y centralizes $B(M)$ and Y normalizes C. This implies $B(C_G(C_C(A)))$ centralizes Y and therefore, $B(C_G(C_C(A))) \leq M$. But then $N_C(C_C(A)) \leq C \cap M$. This forces $[N_C(C_C(A)) : C_C(A)] < \infty$, so by Fact 2.14, $C_C(A) = C$, $C < M$ and thus $R < M$, a contradiction. This final contradiction shows that $O(R)$ is divisible. $\square$

7.4 $C_{O(R)}(A) = 1$.

Proof of 7.4. Suppose $C_{O(R)}(A) \neq 1$. Then $A \leq B(C_G(C_{O(R)}(A))) < G$. By Corollary 5.8, $d(T_1)$ centralizes $B(M)$ and in particular A. As $d(T_1)$ normalizes $O(R)$, we conclude that $d(T_1)$ normalizes $C_{O(R)}(A)$. Therefore, $d(T_1)$ normalizes $B(C_G(C_{O(R)}(A))$. By Corollary 5.8, $d(T_1)$ centralizes $B(C_G(C_{O(R)}(A))$. As M is weakly embedded, $B(C_G(C_{O(R)}(A)) < M$. Therefore, $N_{O(R)}(C_{O(R)}(A))$, which also normalizes $B(C_G(C_{O(R)}(A))$, is a subgroup of M.

An immediate consequence of the above paragraph is that $Z(O(R)) < M$. This also implies that $O(R)$ is not abelian. Therefore, $(O(R)' \cap Z(O(R)))^\circ$ is an infinite, definable, connected subgroup of $M \cap O(R)$. By (7.3) and Fact 2.10, $(O(R)' \cap Z(O(R)))^\circ$ is divisible abelian. Moreover, $(O(R)' \cap Z(O(R)))^\circ$ is torsion-free, because $[R, O(R)]$ is torsion-free by Proposition 5.3. By Lemma 5.11, $(O(R)' \cap Z(O(R)))^\circ$ centralizes $B(M)$ and in particular A. This contradicts (7.2). Therefore, $C_{O(R)}(A) = 1$. $\square$

(7.4) implies that $O(R) \cap Y = 1$ because $O(R) \cap Y \leq C_{O(R)}(A)$. This implies that $Y = d(T_1)$. Therefore, t acts on $d(T_1)$. As $H = C_G(t)$ does not contain nontrivial 2-tori, Lemma 5.13 implies that t inverts $d(T_1)$. By (7.1), $C_{O(R)}(t)$ is infinite and $C_{O(R)}(t)^\circ \leq M$.

Being a subgroup of $H = C_G(t)$, $C_{O(R)}(t)^\circ$ normalizes L. As $C_{O(R)}(t)^\circ \leq M$ also, it normalizes $M \cap L$. $M \cap L$ has a unique Sylow 2-subgroup, namely A. Therefore, we conclude that $C_{O(R)}(t)^\circ$ normalizes A.

Another consequence of (7.4) is that the action of $C_{O(R)}(t)^\circ$ on A is faithful. The facts that the action of T on $A \setminus \{1\}$ is transitive and that T normalizes $C_O(R)(t)$ imply that the action of $C_{O(R)}(t)^\circ$ on A is fixed-point-free. Therefore, we have $A \rtimes C_{O(R)}(t)^\circ$, a centerless, hence nonnilpotent solvable group. By Fact 2.18, we can interpret in this group an algebraically closed field K of characteristic 2. A definable section of $C_{O(R)}(t)^\circ$ is isomorphic to K^* ("no bad fields"). This implies that $C_{O(R)}(t)^\circ$ has some torsion in it (Fact 2.22). We will show that this is not possible.

Now, we consider the group $[R, O(R)]$. It is nontrivial as R is assumed to be nonnilpotent. By Proposition 5.3, $[R, O(R)]$ is torsion-free. Clearly, $O(R)/[R, O(R)]$ is central in $R/[R, O(R)]$.

$$(R/[R, O(R)])/(O(R)/[R, O(R)]) \cong R/O(R) \cong d(T_1).$$

This forces $R/[R, O(R)]$ to be a nilpotent group. But this is also a D-type group, so $R/[R, O(R)]$ is the definable closure of a 2-torus. Hence, using the solvability of R and Fact 2.33 ii), we conclude $R/[R, O(R)] = d(T_1)[R, O(R)]/[R, O(R)]$. This implies $R = d(T_1)[R, O(R)]$. As $O(R) \cap Y = 1$, $d(T_1) \leq Y$ and $[R, O(R)] \leq O(R)$, $O(R) = [R, O(R)]$. Therefore, $O(R)$ is torsion-free, which contradicts the above conclusion on $C_{O(R)}(t)^\circ$. This last contradiction finishes the proof of (7). □

(7) implies that $D(C_G(A)) = Y = R$. Therefore, $D(C_G(A^w)) = R$. Hence, w normalizes Y forcing $w \in M$. This means $L = \langle A, w \rangle \leq M$. On the other hand, $O(H) \leq M$ because $O(H)$ centralizes L and in particular A, which is a unipotent 2-subgroup of M. Therefore, we conclude that $H^\circ \leq M$.

H° contains the connected components of all the Sylow 2-subgroups of H. Therefore, by the Frattini argument $H = N_H(A)H^\circ$, where A is a Sylow 2-subgroup of H°. As M is a weakly embedded subgroup, we conclude that $H \leq M$. This contradiction finishes the proof of the theorem. □

Now Theorem 1.7 follows from Proposition 7.4 and Theorem 7.5.

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