GRAPHS OMITTING A FINITE SET OF CYCLES REVISED, 2022

GREGORY CHERLIN AND NIANDONG SHI

Abstract. For $\mathcal{C}$ a finite set of cycles, we show that if there is a universal $\mathcal{C}$-free graph then $\mathcal{C}$ consists precisely of all the odd cycles of order less than some specified bound. The converse was proved by Komjáth, Mekler, and Pach (*Israel J. Math* **64** (1988), 158–168).

1. Introduction

R. Rado observed that among the countable graphs there is a universal one [9]. That is, there is a countable graph such that every countable graph is isomorphic to one of its induced subgraphs. Many similar problems have been considered in the graph theoretic literature. Most results of this type concern classes $\mathcal{R}$ of countable graphs which omit a class $\mathcal{C}$ of forbidden subgraphs. We may call a such a class $\mathcal{C}$ a *Rado constraint* if the class of countable $\mathcal{C}$-free graphs has a universal member.

The investigation in this paper is motivated by the following: Komjáth, Mekler, and Pach proved that for every odd positive integer $k \geq 3$, there is a universal countable graph for the class of graphs omitting odd cycles of lengths at most k. We will show that the only Rado constraints consisting of finitely many cycles are of this form. It is known however that there is also a universal $\mathcal{C}$-free graph if $\mathcal{C}$ is the family of cycles of length greater than some fixed lower bound, so the question of the characterization of Rado constraints consisting of infinitely many cycles remains open.

In particular the only Rado constraint consisting of a single cycle is $\{C_3\}$. Henson observed the existence of such a univeral graph (more generally, any clique gives a Rado constraint) and Pach showed that C_4 is not a Rado constraint [5]. The negative result was extended to C_n for $n \geq 4$ in [I]. The method of that paper will be applied once more here.

First author supported in part by NSF Grant DMS 9208302.
Second author supported in part by ESU Grant FDR 9405.

The result accordingly is as follows.

Theorem 1. *Let $\mathcal{C}$ be a finite set of cycles. If there is a countable universal graph for the class of graphs omitting $\mathcal{C}$, then $\mathcal{C} = \{C_3, C_5, \dots, C_{2k+1}\}$ for some k .*

It was also shown previously that the particular constraint sets

$$\{C_3, \dots, C_n\}$$

for fixed $n > 3$ are not Rado constraints. Goldstern and Kojman dealt with the case of n even [4], while Komjáth dealt with n odd (presonal communication).

Theorem 1 may be more conveniently phrased as follows.

Theorem 2. *Let $\mathcal{C}$ be a finite set of cycles, i.e., $\mathcal{C} = \{C_n \mid n \in I\}$, where $I \subseteq \mathbb{N}\backslash\{1, 2\}\}$ is finite.*

If $m \in I$ but $rn - 2 \notin I$ for some $m \geq 4$, then there is no countable universal $\mathcal{C}$-free graph.

More broadly, we ask whether there is an effective criterion for a finite set of forbidden finite subgraphs be a Rado constraint.[1] A necessary condition for this is *joint embedding*: any two $\mathcal{C}$-free graphs must embed in a third. This will hold whenever the constraints in $\mathcal{C}$ are connected, via disjoint unions, and in some other cases. A more natural formulation allowing for the failure of joint embedding is to ask for finite complexity in the sense of [**?**], which is equivalent to the universality problem when the class involved has joint embedding. In this direction, our paper [2] used a model theoretic argument to prove the finite complexity of the class of graphs omitting a specified finite family of disjoint unions of complete graphs, answering a question of [**?**].

The term "effective criterion" may mean many things. It is an open question whether the problem is decidable at all; on the other hand it is possible that there is a completely explicit classification of the Rado constraints (which tend to be exceptional, as we see here).

The point of the present paper is that for classes of graphs determined by specifying finitely many forbidden cycles, the only positive results of this kind are the ones covered by a theorem of [6].

There is another result of this kind which calls for a similar systematic analysis. In [6] it is shown that classes of graphs determined by specifying a forbidden (finite) path size have universal elements.[2]

[1]Forbidding infinite graphs is quite interesting but tends to lead toward set theoretic considerations.

[2]In fact, any finite class of finite connected forbidden graphs which contains some path will also be a Rado constraint.

The natural generalization of this would be to consider forbidden finite trees, or more generally finite sets of finite trees. Very little is known about this case, though one suspects that there are relatively few cases, apart from the case of forbidden paths.[3]

As far as the present work is concerned, the main point is that the method of [I] can be applied to close the gap between [I] and [6]. The issues that arise in adapting the method used for one cycle to a set of finitely many cycles are as follows: first, the formulation of Theorem 2 allows us to concentrate primarily on one cycle size; second, the simple construction used in Lemma 3.5, which varies slightly from the one given in [I], avoids any interference from the other cycle sizes. The rest of the argument requires some additional attention to the other cycles, but it turns out that the main lemma of [**?**] applies to this more general situation without modification.

2. Preliminaries

A (simple, loopless) graph $G = (V, E)$ is a structure consisting of a set V of vertices and an irreflexive and symmetric binary relation E on V. All graphs in this paper will be assumed to be countable without further mention.

An *embedding* from a graph $G_1 = (V_1, E_l)$ into a graph $G_2 = (V_2, E_2)$ is an isomorphism f with a subgraph: that is, $f : V_I \hookrightarrow V_2$ and for all $x, y \in V_l$, if $E_1(x, y)$ holds then $E_2(f(x), f(y))$ also holds.. If the converse also applies, so that f is an isomorphism with an induced subgraph, we say that f is a *strong embedding.*

Definition 2.1. For graphs G, H, we say that G *weakly omits* H if H cannot be strongly embedded into G . We say that G *strongly omits* H if H cannot be embedded into G (as a subgraph). We also say in this case that the graph G is H-free (we do not say *strongly* in this case, because we do not use any similar terminology in the weak case).

For a finite set $\mathcal{C}$ of graphs, we say that G weakly or strongly omits $\mathcal{C}$ if G weakly or strongly omits all graphs in $\mathcal{C}$. In the strong case, we may also say that G is $\mathcal{C}$-free.

We come now to universality.

---

[3]This problem was eventually settled in the case of one tree in Universal graphs with a forbidden subtree, G. Cherlin and S. Shelah, *J. Comb. Theory,* Series B, **97** (2007), 293-333. There are in fact few Rado trees other than paths, and the main work comes in proving the negative results.

Definition 2.2. We say that a graph G is weakly or strongly universal for a class $\mathcal{R}$ of graphs if each graph in the class $\mathcal{R}$ weakIy or strongly embeds into G.

Both the strong and the weak notions of universality are of interest, and are distinct. For the class of graphs, a complete graph is weakly universal but the existence of a strongly universal graph requires more attention. Whenever possible we prefer to prove results in whichever form is strongest: that is, for existence we prefer to find strongly universal graphs, and for non-existence we prefer to rule out weakly universal graphs, which may take some additional attention.

In the case at hand, the positie results have already been proved in the context of strong universality, so our objective now will be to prove Theorem 2 in its sharp form: namely, under the stated conditions, there is no weakly universal $\mathcal{C}$-free graph.

The standard approach to such problems is to construct an uncountable family of pairwise *incompatible* $\mathcal{C}$-free graphs. The appropriate notion of incompatibility is that no two of the graphs embed weakly into a larger $\mathcal{C}$-free graph. One can vary the construction a little, taking e.g. graphs with a distinguished basepoint rather than graphs, which leads to the same conclusion. We will give the concluding argument in detail just below.

3. The proof

We fix the notation associated with Theorem 2. Let $\mathcal{C}$ be a finite set of cycles and $C_m \in \mathcal{C}$, $C_{m-2} \notin \mathcal{C}$ for some $m \geq 4$.

Definition 3.1. If G is a $\mathcal{C}$-free graph and $S \subseteq X \subseteq V(G)$, we will call the pair (S, X) $\mathcal{C}$*-rigid* relative to G if the following holds.

> *For any two embeddings $f, g : G \hookrightarrow H$ of G into a $\mathcal{C}$-free graph H such that f and g agree on S, f and g must agree on X.*

The purely formal part of the argument is the following.

Lemma 3.2. *Suppose that G is a $\mathcal{C}$-free graph containing a $\mathcal{C}$-rigid pair (S, X). Suppose that there is an uncountable family of graphs X_i, with the following properties.*

1. *The induced graph on X is a subgraph of X_i.*
2. *$G \cup X_i$ is $\mathcal{C}$-free.*
3. *There is no $\mathcal{C}$-free graph containing $X_i \cup X_j$ as a subgraph, for distinct i, j. (Note here that X is a subgraph of both.)*

Then there is no weakly universal $\mathcal{C}$-free graph.

Proof. If G is weakly universal—and countable, as we suppose throughout—then we have embeddings of all G_i into G, and as S is finite there must be two indices i, j for which the embedding of S is the same. By the $\mathcal{C}$-rigidity the embeddings agree on X and so G contains a copy of $X_i \cup X_j$. But by hypothesis this cannot occur. □

The following lemma contains the key step in our argument.

Lemma 3.3. *There exists an infinite countable $\mathcal{C}$-free graph G with bounded vertex degrees and a $\mathcal{C}$-rigid pair (S, X) of vertices in G with S finite and X infinite.*

We will first show how to derive Theorem 2 from this lemma, and then give the proof of the lemma.

Proof of Theorem 2. Suppose C_n is the cycle in $\mathcal{C}$ of maximum length. Let G be as in Lemma 3.3.

Since the vertex degrees of G are bounded, we can choose an infinite sequence of vertices

$$x_0, x_l, \dots \in X$$

such that the distance in G between any two distinct vertices in this sequence is at least n.

For every string $\epsilon = (\epsilon_0, \epsilon_1, \dots)$ of zeros and ones, extend G to a graph $G(\epsilon)$ as follows: join the pair of vertices x_{2i} and x_{2i+1} by an edge if $\epsilon_i = 0$, and by a path of length $n-1$ if $\epsilon_i = 1$. In the latter case the path will contain $n-2$ new vertices of degree 2.

It will be convenieent to write $G(\epsilon) = G \cup X(\epsilon)$ where $X(\epsilon)$ is the extension made of X. To conclude, we will apply Lemma 3.2 to the family of all such graphs $G(\epsilon)$

Claim 1. The graphs $G(\epsilon)$ are $\mathcal{C}$-free.

As the vertices x_{2i}, x_{2i+1} are at distance at least n in G, and the same applies to two such pairs of vertices, there is no new cycle of length at most n in the extension of G by the new edges joining some of these pairs. Since the added paths have length $n-1$ and they are adjoined to pairs of vertices that are not edges of $G(\epsilon)$, thre is no such new cycle in $G(\epsilon)$. Thus $G(\epsilon)$ remains $\mathcal{C}$-free.

Claim 2. There is no $\mathcal{C}$-free graph containing $X(\epsilon) \cup X(\epsilon')$ for distinct ϵ, ϵ'.

We consider an index i for which $\epsilon_i \neq \epsilon'_i$. Then x_{2i} and x_{2i+1} are joined by an edge and by a path of length $n-1$ in the union of the two graphs, and hence by a cycle of length n.

Thus Lemma 3.2 applies. □

Thus it suffices to prove Lemma 3.3.

We introduce an auxilary graph T_m.

Notation 3.4. Let T_m be a graph with three distinguished vertices a, b, c, with c adjacent to a and to b, and with c also joined to a and b by paths P_a, P_b of length $m-3$, disjoint apart from the specified intersections. Thus T_m is the union of two cycles of length $m-2$, with intersection c, and with a, b adjacent to c on the corresponding cycle. However, for $m = 4$, this reduces to a path acb.

Lemma 3.5. *(a, b, c) is a $\mathcal{C}$-rigid pair in T_m.*

Proof. Suppose that H is a graph containing images of T_m under two maps agreeing on a, b, but not on c. That is, H contains vertices a, b, c', c" with c', c'' adjaent to a, b, distinct, and having the paths of length $m-3$ connecting c' or c'' to each of a and b. We will find an m-cycle and conclude that H is not $\mathcal{C}$-free.

Though the argument is general, as the case $m = 4$ is somewhat degenerate we may notice in this case that (a, c', b, c) is already a 4-cycle, and set this aside.

For the general argument we consider the paths P_a, P_b of length $m-3$ linking c' to a and b which are inherited from T_m, and which meet in c'. Now c'' cannot lie on both paths, so we may suppose it does not lie on P_a. Now P_a together with the path $ac''bc'$ gives a cycle of length m, as required. □

To prove Lemma 3.3 we will glue together several copies of T_m in the manner described in [I]. At each stage of the construction we take two vertices which are sufficiently far in the path metric, and treat them as a, b in a new copy of T_m, thereby adding another vertex to our rigid set. We need to show that this construction can be iterated infinitely often without running out of suitable pairs a, b.

Again, let n be the maximum length of a cycle in $\mathcal{C}$. We let $N = (14" - 1)/13$ and quote the following from [I].

Fact 3.6. *If G is a graph with all vertex degrees at most 12, and X is a set of $4N+1$ vertices in G, then there is a cycle H of length $4N+1$ on the set X such that $E(G) \cap E(H) = \emptyset$, and in $G \cup H$ there is no cycle C_k with $k \leq n$ which contains an edge of H.*

For our present purposes, the numbers N, 12 and (below) 6 could by more appropriately replaced by $(10^n - 1)/9$, 8, and 4, but we prefer to quote the fact as previously given and apply it in that form.

The proof of Fact 3.6 is a slight variation of a well known argument of G. Dirac relating to Hamiltonian cycles.

Proof of Lemma 3.3. We construct a sequence of $\mathcal{C}$-free graphs $(G_s \mid s \in \omega)$ together with a distinguished set X_s of $4N+1$ vertices in each G_s with the following properties.

(1) G_s is an induced subgraph of G_t for $s \leq t$.
(2) The vertex degrees in G_s are all at most 12.
(3) The vertex degrees in G_s for vertices in X_s are all at most 6.
(4) The pair (X_{s-l}, X_s) is $\mathcal{C}$-rigid in G_s for $s \geq 1$.
(5) There is a cycle H_s on the set of vertices X_s , such that $E(G) \cap E(H) = \emptyset$, and $G_s \cup H_s$ contains no cycles of length $k \leq n$ which contain an edge of H.

The construction is inductive. Let G_0 be a graph with $4N+1$ vertices and no edge, and $X_0 = V(G_0)$. If G_s, X_s, and H_s satisfy conditions (1)-(5), enumerate the vertices of H_s as $(x_i^s \mid i \in \mathbb{Z}/(4N+1)\mathbb{Z})$ in cyclic order, and for each $i \in \mathbb{Z}/(4N+l)\mathbb{Z}$, adjoin a copy $T_m^{i,s}$ of T_m to G_s, identifying the vertices a, b with x_i^s, x_{i+1}^s respectively. Let c_i^s be the vertex corresponding to c in $T_m^{i,s}$. Let G_{s+1} be the resulting graph

$$G_s \cup \bigcup T_m^{i,s} \mid i \in \mathbb{Z}/(4N+1)\mathbb{Z},$$

with no further identifications. Let $X_{s+1} = \{c_i^s \mid i \in \mathbb{Z}/(4N+l)\mathbb{Z}\}$

P.roperties (1)-(4) of G_{s+1} and X_{s+1} are clear, and Fact 3.6 allows us to choose a cycle H_{s+1} satisfying condition (5). We must check that G_{s+1} is $\mathcal{C}$-free.

Suppose that C is a cycle of length t in G_{s+1}, where $t \in I$. So $t \leq n$. As G_s is $\mathcal{C}$-free, C does not lie wholly in G_s. Thus if we replace all the edges of C which occur in some $T_m^{i,s}$ by the corresponding edges in the cycle H_s, the result is a contraction C' of C lying in $G_s \cup H$, and containing an edge of H_s. Note that C' is a cycle with length not larger than t , so not larger than n (or perhaps a doubled edge). But this contradicts the induction hypothesis.

Thus the induction proceeds.

Now to conclude let $G = \bigcup_{s\in\omega} G_s$, $S = X_0$, $X = \bigcup_{s\in\omega} X_s$. Then G is $\mathcal{C}$-free and as (X_s, X_{s+l}) is $\mathcal{C}$-rigid in G for all s, it follows that (Xo, X_s) is $\mathcal{C}$-rigid in G_s and thus (S, X) is $\mathcal{C}$-rigid in G_s, as desired. $\square$

References

[I] G. Cherlin and P. KomjBth, There is no universal countable pentagon-free graph, *J. Graph Theory* **18** (1994), 337–341.

[2] G. Cherlin and N. Shi, Graphs omitting sums of complete graphs, *J. Graph Theory* **24** (1997), 237–247.

[3] G. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* (3) **2** (1952), 69–81.

[4] M. Goldstern and M. Kojman, Universal arrow-free graphs. *Acta Mathematica Hungarica* **73** (1996), no. 4, 319–326. MR1428038.

[5] A. Hajnal and J. Pach, Monochromatic paths in infinite graphs, in **Finite and Infinite Sets,** Coll. Math. SOC. J. Bolyai, Eger, Hungary, (1981), 359–369.

[6] P. Komjhth, A. Mekler, and J. Pach, Some universal graphs, *Israel J. Math.* **64** (1988), 158–168.

[7] 7] P. Komjith and J. Pach, Universal elements and the complexity of certain classes of infinite graphs, *Discrete Math.* **95** (1991), 255-270.

[8] A. Lachlan and R. Woodrow, Countable ultrahomogeneous undirected graphs, *Trans. American Mathematical Soc.* **262** (1980), 51–94.

[9] R. Rado, Universal graphs and universal functions, *Acta Arithmetica* **9** (1964), 331–340.

Department of Mathematics, Rutgers University; East Stroudsburg University

Email address: `nshi@esu.edu`