AN INTEGER LATTICE ARISING IN THE MODEL THEORY OF WREATH PRODUCTS

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Introduction

While attempting to find methods of some generality for computing a model-theoretic invariant of finite structures (the *arity*, or *relational complexity*, as defined in §1, we found it useful to compute a number of examples by making use of a related integer lattice L_r depending on a single parameter r. These computations have led to a plausible formula for this invariant which is at least a correct lower bound, and is exact in the few cases we were able to check directly. For more details, see §3.

In order to make explicit machine computations in the lattice L_r we were led to prove a few results of a general character about it. In particular we determine its rank and we give an explicit basis which has some properties which are computationally convenient.

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Our computations involve certain sublattices of L_4 for which it would also be interesting to know the ranks, and more interesting to know the discriminants. We will give our results on the lattice L_r here and indicate their relevance to machine computations bearing on the original model theoretic problem. The work that led to these computations, and was in turn influenced by them, is reported in detail in [CM].

Our model theoretic problem may be recast in the following purely combinatorial form. Fix an integer r and a set X with r elements. For $i \in X$ we say that two relations on the set X *agree modulo* i if their restrictions to $X - \{i\}$, and similarly that two multi-sets $\{R_1, \dots, R_d\}$ and $\{R'_1, \dots, R'_d\}$ agree modulo i if the restrictions to $X \setminus \{i\}$ agree as multi-sets (i.e., up to some re-ordering).

The combinatorial problem is then to determine under which conditions we can find two distinct multi-sets of equivalence relations on the set X which agree modulo each element of X. More precisely, the question is the following.

Problem 1. *Given parameters r and n with $r \geq n \geq 2$, find the least value d for which there are two disjoint multi-sets each consisting of d equivalence relations on a fixed set X of r elements, satisfying the following conditions.*

(1) Each of the equivalence relations occurring has at most n classes.
(2) At least one of the equivalence occurring (in at least one of the two multi-sets) has exactly n classes.
(3) The two multi-sets agree modulo each element of X.

Notation 0.1. We denote by $\delta(r, n)$ the least d satisfying the conditions of Problem 1.

Example 1. We have $\delta(3, 3) = 3$. This is illustrated (as an upper bound) by the following two multi-sets of equivalence relations on the set $\{1, 2, 3\}$, where the relations are represented by their classes.

E_1	E_2	E_3	E'_1	E'_2	E'_3
$\{1\}\{2\}\{3\}$	$\{1\}\{2\}\{3\}$	$\{1,2,3\}$	$\{1\}\{2,3\}$	$\{1,3\}\{2\}$	$\{1,2\}\{3\}$

We will describe the original model theoretic problem in §1 which leads to this combinatorial problem, and is almost equivalent to it. Our focus here will be on the determination of the function δ rather than the model theoretic information to be extracted from it. Readers more interested in combinatorics than in model theory will find that the results and conjectures of the remaining sections can be read independently of §1.

One may wonder whether $\delta(n, r)$ is in fact defined. In fact for $n = 2$ and for r odd, no suitable value of d exists, and we set $\delta(n, r) = \infty$ in that case. One can easily show that $\delta(r+2, n+2) \leq 2\delta(r, n)$, and there is also some evidence that this inequality may be an equation. If this is correct then the values of δ are determined by $\delta(r, 3)$ and either $\delta(r, 2)$ (for r even) or $delta(r, 4)$ (for r odd). It can easily be shown that

$$\delta(r, 2) = 2^{r-2}$$

for r even, but for $n = 3$ or $n = 4$ the precise values are conjectural (see §3 for a precise conjecture).[1]

In this context machine computation of explicit values can be useful. We will see that the computation of the function δ amounts to the determination of a vector of minimal length in an appropriate integer lattice. With r fixed, consider the space V_r (over some field of characteristic 0) having as basis the set of all equivalence relations on a fixed set X with r elements. Associate to a pair of multi-sets of equivalence relations the difference of their characteristic vectors in V_r. If the two multi-sets are disjoint then they can be recovered from the associated vector. In the case of interest here, where the two multi-sets have equal cardinality d, that cardinality is half the ℓ^1 norm of the associated vector.

The vectors of interest are the integer vectors that represent a pair of multi-sets whose restrictions modulo each element of X coincide up to order. This condition is expressed by a set of homogeneous linear equations. Let L_r be the lattice of all integral solutions to these equations. Let $L_{r,n}$ be the sublattice of vectors with all entries corresponding to an equivalence relation with more than n classes vanishing. Then $\delta(n, r)$ is half the ℓ^1-norm of the shortest vector in $L_{r,n}$ and not in $L_{r,n-1}$.

Our approach to explicit computation has been to generate a convenient basis for L_r and to hunt for suitable short vectors in $L_{r,n}$ out to a specified radius d, making all computations relative to the basis for L_r. This is a computation which might not appear to be feasible over a significant range of the parameters. In fact we have been limited to $r \leq 7$.

The key parameter here is the rank of the lattice L_r. The radius d and (to some slight extent) the ambient dimension $v_r = \dim(V_r)$ (the Bell numbers, which grow rapidly) also come into consideration. We give the first few values of these parameters.

r:	1	2	3	4	5	6	7	8	9	10
v_r:	1	2	5	15	52	203	877	4140	21147	115475
ℓ_r:	0	1	1	4	11	41	162	715	3425	17722

[1] The value of δ was determined subsequently by Daniel Saracino in a series of three papers, and turned out to be quite complicated.

In §2 we discuss the lattice L_r and give a particular basis for it. In particular we work out the rank of the lattice and show that the basis found is spares and the vectors have coordinates of small size. These properties facilitate computations of the sort we have discussed.

It would be interesting to have similar information for the sublattices $L_{r,n}$, and to know their discriminants, but we do not even know the discriminant of L_r.

We thank Thomas Müller for directing our attention to [Be].

1. The arity of wreath products

1.1. Types and arity. We work with finite structures, for which model theory and permutation group theory provide closely related points of view. Any finite structure gives rise to an associated permutation group, namely the automorphism group with its action on the structure, and the relations invariant under that group are those definable in the original structure. So we will give our main definition in permutation group theoretic terms.

We consider a permutation group G acting (faithfully) on a set X. G acts on each cartesian power X^s coordinatewise. The orbits under this action are called *s-types.* Two elements of the same orbit will be said to be *conjugate* (or G-conjugate).

In particular when G is the full symmetric group then the s-types correspond to equivalence relations on the coordinates $1, \ldots, s$: to specify such an orbit is to specify which entries should be equal.

The key definition, which corresponds model theoretically to quantifier elimination, is the following.

Definition 1.1. Given a permutation group (G, X) and $r \leq s$, we say that "r-types determine s-types" if the following criterion for conjugacy of s-tuples is valid.

> Two s-tuples $\mathbf{a}$, $\mathbf{b}$ are conjugate if their respective restrictions to an arbitrary r-set of coordinates are conjugate.

There are other ways in which r-types might determine s-types; this is the strongest such notion, and corresponds to quantifier-free definability of relations in s variables from relations in r variables.

If (G, X) is a permutation group with $|X| = n$ then it is easy to see that N-types determine s-types for all $s \geq n$. We define the *arity* of G as the least r such that r-types determine s-types for $s \geq r$; or, equivalently, so that r-types determine n-types.[2] We may also define

[2] In more modern usage this tends to be called the *relational complexity.*

the arity as the maximal r for which $(r-1)$-types do not determine r-types. This is the way we tend to view it in practice.

Some examples are in order. The arity of the natural action of $\mathrm{Sym}(n)$ for $n > 1$ is 2 (which in model theoretic terms emphasizes the importance of the equality relation). For the natural action of $\mathrm{Alt}(n)$ it is $n-1$. Group theoretically these are quite similar actions but model theoretically they lie at the two possible extremes.[3] The arity of the regular action of any non-trivial group on itself is 2.

A more interesting family of examples is given by the following.

Fact 1.2 ([CM]). *Let $G = \mathrm{Sym}(n)$, take $k \geq 1$ with $2k \leq n$. Let $\begin{bmatrix} n \\ k \end{bmatrix}$ be the family of all k-subsets of $\{1, \ldots, n\}$.*

(1) The arity of the natural action of G on $\begin{bmatrix} n \\ k \end{bmatrix}$ is
$$2 + \lfloor \log_2 k \rfloor.$$
(2) The arity of the natural action of G on $\mathcal{P}(\{1, \ldots, n\})$ is
$$1 + \lfloor \log_2 |X| \rfloor.$$

As a concrete illustration take $n = 5$ and $k = 2$. Then $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ may be thought of as the *Petersen graph*. That is, if we connect two k-sets by an edge when they are disjoint, we get a graph whose automorphism group is $\mathrm{Sym}(5)$ with the natural action on pairs. The arity of the action is not 2, but 3. That is, there are two 3-types which are not determined by their constituent 3-types. If we identify the pairs in $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ with edges in the complete graph on 5 points, then these triples consist of edges which meet pairwise in one point, but form either a triangle or a star inside the complete graph on 5 vertices.

1.2. **Wreath products and powers.** If (G, X) and (H, Y) are two permutation groups then there is a natural action of the wreath product $G \wr H$ on X^Y. Here $G \wr H$ is the subdirect product $G^Y \rtimes H$ with H permuting the factors of G^Y, and the subgroup G^Y acts coordinatewise on X^Y while H again permutes the factors. Determining the arities of such wreath product (power) actions in terms of the constituents G and

[3]In fact the bound $r \leq n$ mentioned above becomes $r \leq n-1$, bearing in mind that a permutation whose action is known on $n-1$ points is fully determined).

H seems very challenging, even when H is a symmetric group $\mathrm{Sym}(d)$ acting naturally on d elements: $G \wr \mathrm{Sym}(d)$ acting on X^d, a power in the most straightforward sense. We generally denote this action simply by X^d when the context allows it.

We have in particular the case in which G is also a symmetric group acting naturally. We may then write n^d for the action of $\mathrm{Sym}(n) \wr \mathrm{Sym}(d)$ on $\{1. \dots, n\}^d$. This is the case to be considered here, in fact.

Examples of the form n^2 were already considered in connection with the study of homogeneous finite graphs [Gar]. These are grid-like graphs where the edge relation is as follows: *the pairs $a, b \in n^2$ have exactly one coordinate in common.* However the rows and columns of these grids are themselves complete graphs. They are also called the *line graphs* of the complete bipartite graphs $K_{n,n}$. For $n \leq 3$ the graphs have arity 2, that is the orbits on s-tuples are given by the isomorphism types of the induced graphs. For $n \geq 4$ the arity is 4 and the critical 4-types consist of pairs of disjoint edges which either parallel or orthogonal in the sense of the grid. We have found a qualitatively similar behavior for n^d: with d fixed, there is a "generic" value for large d and a gradual increase to it depending on the value of n, with exact values not yet determined.

For general wreath product actions on powers, we have an upper bound in terms of the arity of $G, X)$ and the arity of the action of H on $\mathcal{P}(Y)$ induced by its action on Y. This of course also raises the question of the determination of the latter. It may be computed explicitly in some non-trivial cases. We have the following bounds in general [CM].

Fact 1.3. *Let (G, X) and (H, Y) be permutation groups. Let r_G and r_H^* denote the arities of G on X and of H in the induced action on $\mathcal{P}(Y)$. Let $\rho(G, H)$ denote the arity of $G \wr H$ acting on X^Y. Then*

$$\max(r_G, r_H^*) \leq \rho(G, H) \leq r_G \cdot r_H^*.$$

Fact 1.4. *Let $G = \mathrm{Sym}(n)$ acting naturally on $\begin{bmatrix} n \\ k \end{bmatrix}$ and let (H, Y) be any permutation group. Let $r_{G,k}$ be the arity of G in the action on k-sets, r_H^* the arity of H acting on $\mathcal{P}(Y)$, and $\rho(G, k, H)$ the arity of $G \wr H$ acting on $\begin{bmatrix} n \\ k \end{bmatrix}^Y$.*

Suppose that

$$n \geq 2k \cdot r_H^*.$$

Then

$$\rho(G, k, Y) = r_{G,k} \cdot r^*_H.$$

Combining Facts 1.2 and 1.4 gives a more explicit result when H the full symmetric group on Y, which for $k = 1$ takes on the following form.

Corollary 1.5. *For $n \geq 2 \cdot \lfloor 1 + \log_2 d \rfloor$ the arity of n^d is*

$$2\lfloor 1 + \log_2 d \rfloor.$$

The behavior for n below the given bound is more erratic. For $n = 2$ we have an explicit formula for the arity of 2^d [CM]:

$$2\lfloor 1 + \log_4 d \rfloor,$$

or in other words, roughly half the value for large n. At this point one would like to see some detailed data.

In the first table on the next page we give some conjectured values for the arity of n^d in all cases for which $d \leq 36$ and the value of n is below the generic range. These numbers give valid lower bounds. None of the predicted values with $r \geq 8$ are supported computationally as exact values. The values with $r \leq 7$ are supported to varying degrees. In particular all of those in the range $d \leq 7$ were checked. As we have indicated, the first and last (zigzag) rows of the table, corresponding to $n = 2$ and to $n = 2\lfloor 1 + \log_2 d \rfloor$, are known to be exact in general. The actual conjectures behind the table are given afterward, in §3.

We present the data in what is ultimately a more useful way in a second table that gives d as a function of n and r, namely, the smallest value of d which, for a given value of n, suffice to raise the arity of n^d to exactly r (when it exists). This leads to more transparent conjectures and to a useful way of analyzing the original problem. We will make this more formal in the next subsection.

1.3. The function $\delta(r, (G, X))$.

Definition 1.6. Given a permutation group (G, X), let $\delta(r, (G, X))$ denote the least d for which $r - 1$-types do not determine r-types in $(G, X)^d$, if there is one, or ∞ otherwise.

We can formulate our results and conjectures in a straightforward manner in terms of this function, and this gives a determination of the arity of each $(G, X)^d$, at least implicitly. That is, the arity of $(G, X)^d$ is the least r satisfying

$$\delta(s, (G, X)) > d \text{ for all } s > r.$$

n \ d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
2	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
3		2	4	4	4	4	4	4	5	5	5	5	5	5	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
4		4	4	4	4	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	8	8	8	8	8
5				4	4	6	6	6	6	6	6	6	6	6	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	8	8	8
6				6	6	6	6	6	6	6	6	6	6	7	7	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7								6	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	9
8								8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	9	9	9	9	10	10	10	10	10
9																8	8	8	8	8	8	8	8	10	10	10	10	10	10	10	10	10	10	10	10	10
10																10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
11																																10	10	10	10	10
12																																12	12	12	12	12

n \ r	2	3	4	5	6	7	8	9	10	11	12
2	1		4		16	...					
3	2		3	9	16	...					
4			2	7	8	32	...				
5			4		6	18	32	...			
6					6	14	16	...			
7					8		12	36	...		
8							8	28	32	...	
9							16		24	...	
10									16	...	
11									32	...	
12											32

One might reasonably anticipate that in the case $G = \mathrm{Sym}(n)$ acting naturally, this would turn out to be the function $\delta(r,n)$ we discussed in the introduction, but this is not quite right. In fact, we have

$$\delta(r, \mathrm{Sym}(n)) = \inf(r \mid \delta(r', n') > d \text{ for } r' > r \text{ and } n' \leq n).$$

for the natural action of $\mathrm{Sym}(n)$, because our definition of $\delta(r,n)$ is more precise in its measurement of the effect of the parameters, notably the parameter n.

A more practical version of the formula just given restricts r' to values satisfying

$$r < r' \leq 2\lfloor 1 + \log_2 d \rfloor.$$

We give an example to clarify these considerations. The arity of 3^{16} should be 6, as in 2^{16}. In particular 5-types do not determine 6-types in 3^{16}. But if we want an example that shows this, and that actually makes use of the fact that we have 3 distinct points in the base, we have to go to 3^{25} rather than 3^{16}. This point does not affect the computation of the arity but does affect our understanding of how the parameters n, r, d all interact in this context.

After these preliminaries we can connect the model theoretic and combinatorial invariants directly. We focus on the following condition.

(∗) $r - 1$-types in $(G, X)^d$ do not determine r-types.

View the elements of $(G, X)^d$ lying in a given r-tuple as the rows of a table of shape $r \times d$ with entries in X. To conjugate one such table to another in $G \wr \mathrm{Sym}(d)$ we must permute the columns by $\mathrm{Sym}(d)$ so that corresponding columns have the same r-type in (G, X). So we may associate to one such table the multi-set of r-types represented by its columns, and two such tables represent the same r-type if these multi-sets coincide.

When we restrict to $(r-1)$-tuples we omit the row corresponding to one index from $\{1, \ldots, r\}$. Thus to show that $(r-1)$-types do not determine r-types in $(G, X)^d$ we require two distinct multi-sets of size d consisting of r-types from (G, X), which coincide when restricted to $(r-1)$-types by in each of the r possible ways.

This becomes considerably more concrete when (G, X) is itself the natural action of $\mathrm{Sym}(n)$. Then the r-types are given by the equivalence relation on $\{1, \ldots, r\}$ specifying when two entries are the same, and so we must have distinct multi-sets of equivalence relations on the set $\{1, \ldots, r\}$ which coincide when any single element is deleted. For this to be realized in n^d, the equivalence relations in question must have at most n classes. We come back to the original definition of $\delta(r, n)$, apart from the additional restriction that at least one of the equivalence relations actually has n classes, so that the same example does not arise in connection with $(n')^d$ for some $n' < n$. As we have mentioned, this additional constraint is not essential in theory, but is useful in practice.

To sum up then, consideration of the arity of wreath product actions on powers leads to the consideration of a general function $\delta(r,(G,X))$ and a slightly more precise variant $\delta(r,n)$ in the particular case of wreath products of natural actions of symmetric groups. This approach seems well suited to the study of arity in $(G,X)^d$ for general (G,X), where the full symmetric group $\mathrm{Sym}(d)$ acts on the factors.

In principle one can also compute in an associated integral lattice, but the rank tends to increase very rapidly. From this point on we restrict our attention to the powers n^d.

2. The lattice L_r

We have explained our motivation for studying the function $f(r,n)$. Now we introduce the associated integer lattice L_r and its sublattices $L_{r,n}$, which are used in explicit computations, and we give the results described in the introduction.

2.1. **The space V_r and the lattice L_r.**

We may take the rational field as base field and let V_r be the vector space with basis vectors all equivalence relations E on the set $\{1,\ldots,r\}$. Let $V_{r,n}$ be the subspace generated by the equivalence relations with at most n classes. The dimension of V_r is called the *Bell number* B_r.[4] Bell gave the following estimate for these numbers.

$$a_r^{r-a_r}e^{a_r-1}(\log a_r)^{-1/2}$$

where a_r is very roughly r, but is defined by the condition

$$a_r \ln a_r = r - 1/2.$$

Given two multi-sets of equivalence relations of the same size, $\mathcal{E} = \{E_1,\ldots,E_d\}$ and $\mathcal{E}' = \{E'_1,\ldots,E'_d\}$, we encode each by the vector of its multiplicities and let $v(\mathcal{E},\mathcal{E}')$ be the difference of these two vectors. We wish to impose the condition of equality of restrictions for $1 \le i \le r$:

(†) The restrictions of $\mathcal{E}$ and $\mathcal{E}'$ to $\{1,\ldots,\hat{\imath},\ldots,r\}$ coincide.

More explicitly, for each index i and each equivalence relation E on $\{1,\ldots,\hat{\imath},\ldots,r\}$, the condition is that the restrictions of $\mathcal{E}$ and $\mathcal{E}'$ to

[4] We learn that this was a number of particular interest in classical Japanese mathematics—and art—due to its association with The Book of Genji and a game (genjiko) involving discriminating among five possibly distinct fragrances of incense. Stylized representations of equivalence relations on 5 elements are a recurrent motif.

that set contain E with the same multiplicity. This is a condition on the coordinates of $v(\mathcal{E}, \mathcal{E}')$ which restrict on that set to E, and is a homogeneous linear equation.

Let L_r be the integer lattice defined by those homogeneous linear equations, and let $L_{r,n}$ be $L_r \cap V_{r,n}$.

This may also be expressed as follows. For $1 \le i \le r$ let $\alpha_i : V_{r-1} \to V_r$ be the linear map taking the basis element E to the sum of its extensions of $\{1, \ldots, r\}$, where E is viewed as a relation on $\{1, \ldots, \hat{\imath}, \ldots, r\}$. Then we are taking the intersection of the subspaces $(\alpha_i[V_{r-1}])^{\perp}$. and L_r consists of the integer points in $(\sum_i \operatorname{im} \alpha_i)^{\perp}$.

2.2. **The rank of L_r.** If $X \subseteq \{1, \ldots, \hat{\imath}, \ldots, r\}$, we define a map $\alpha_X : V_{r-|X|} \to V_r$ as we defined α_i (in particular, $\alpha_{\{i\}} = \alpha_i$). It may also be defined as the composition of the corresponding maps α_i (modulo the natural identifications of vector spaces), taken in any order. The composition with the restriction map back to $V_{r-|X|}$ is diagonal and invertible so the map is invertible. Accordingly the dimension of the image of α_X is $B_{r-|X|}$ and this space is contained in the intersection of the images of the maps α_i.

In fact we claim that

$$\operatorname{im} \alpha_X = \bigcap_{i \in X} \operatorname{im} \alpha_i.$$

Let $v \in \operatorname{im} \alpha_i$ for all $i \in X$ and let E, E' be two equivalence relations which agree on $\{1, \ldots, r\} \backslash X$. We claim $v \in \operatorname{im} \alpha_X$. It suffices to check the equality of coordinates

$$v_E = v_{E'}$$

This reduces to the case in which E' can be obtained from E by moving one index out of one E-class and into another (possibly a new one). As $v \in \operatorname{im} \alpha_i$ the claim holds in this case.

Thus we have

$$\dim \bigcap_{i \in X} \operatorname{im} \alpha_i = B_{r-|X|}$$

and thus by a version of inclusion/exclusion we find

$$\dim(\sum_i \operatorname{im} \alpha_i) = \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} B_{r-i}$$

The dimension of L_r is then

$$\dim(\sum_i \operatorname{im} \alpha_i)^{\perp} = \sum_{i=0}^{r} (-1)^{i} \binom{r}{i} B_{r-i}$$

We remark that by inclusion/exclusion the number $\sum_{i=0}^{r}(-1)^i\binom{r}{i}B_{r-i}$ also counts the number B'_r of equivalence relations with no singleton classes (associate to any set X the set of equivalence relations having each element of X as a singleton class). So the rank of L_r may also be written as B'_r. We remark that this is also the number of equivalence relations on $r-1$ points which do have singleton classes: for such a relation, we can group its singletons together and adjoin r to get a relation on r points with no singletons, and this is reversible.

For computational purposes we define a particular ordering on the equivalence relations on the set $\{1,\ldots,r\}$. Individual subsets are compared by first comparing cardinality, and then taking the lexicographic order for subsets of equal size. Equivalence relations are then viewed as ordered partitions, and compared lexicographically. In particular, relations having some singleton class appear before those which do not.

Now define a $0,1$-matrix with rows indexed by equivalence relations on $\{1,\ldots,r\}$ having at least one singleton class, and with columns indexed by all equivalence relations on that set (taken in order). The entry is 1 if the relations agree modulo the first singleton in the relation indexing the row, and is 0 otherwise. The matrix is upper triangular with 1s on the main diagonal. We illustrate for $r=3$.

	[1][2][3]	[1][23]	[2][13]	[3][12]	[123]
[1][2][3]	1	0	1	1	0
[1][23]	0	1	0	0	1
[2][13]	0	0	1	0	1
[3][12]	0	0	0	1	1

We claim that the rows form a basis for $\sum_i \operatorname{im}\alpha_i$. They are linearly independent and we have the correct number, so it suffices to check that each row lies in the image of some α_i.

In the row corresponding to the relation E let $\{i\}$ be the first singleton occurring, and let E_0 be the restriction of E to $\{1,\ldots,\hat{\imath},\ldots,r\}$. Them the row is $\alpha_i(E_0)$. Thus our claim holds.

To get a basis for L_r we therefore solve the corresponding linear system by row reduction to the form $[I|A]$ and take as our basis the columns of $\begin{bmatrix}-A\\ I\end{bmatrix}$ with I the identity matrix of appropriate size.

For $r=4$ this yields the following basis, writing rows rather than columns for the sake of the display.

	1 2 3 4	1 2 34	1 3 24	1 4 23	2 3 14	2 4 13	3 4 12	1 234	2 134	3 124	4 123	12 34	13 24	14 23	1234
[12][34]	1	-1	0	0	0	0	0	0	-1	0	0	1	0	0	0
[13][24]	1	0	-1	0	0	0	-1	0	0	0	0	0	1	0	0
[14][23]	1	0	0	-1	0	-1	0	0	0	0	0	0	0	1	0
[1234]	-3	1	1	1	-1	1	1	-1	1	-1	-1	0	0	0	1

Given the form of the initial incidence matrix it is easy to see what these entries represent in general. Thinking of this in its original form, as a set of columns v_E indexed by equivalence relations which do not have singleton classes, with entries indexed by arbitrary equivalence relations E' on the same set, the coordinate of v_E at position E' will be nonzero if and only if E can be converted into E' by removing some elements from one or more classes and making them singletons. In that case we must count the number of ways this transformation can be made if it is broken down into a series of steps in which each class is transformed in turn by removing a decreasing sequence of elements from that class, with the minimal element the last one removed (when only two remain). Since E' is specified, and we know which elements must become singletons, there is usually only one way to do this; but if the class C has k elements and it is entirely broken down into singletons in E', then there are $k-1$ ways to do this. This leads to the following formula for the entry (when non-zero).

$$\prod_C n_C(E, E')$$

where C varies over classes of E and $n_C(E, E')$ depends on the size of C and the number k of singletons of E' lying in C, as follows.

$$n_C(E, E') = \begin{cases} (-1)^k & \text{if } k < |C| \\ (-1)^k(k-1) & \text{if } k = |C| \end{cases}$$

Here the decreasing order of removal of elements corresponds to the usual order of row reduction.

In particular the coordinates of these vectors are not too large as a function of r, of the order of magnitude not more than $e^{r/e}$. In particular, for $r \leq 8$ the coordinates have absolute value at most 9. Furthermore, the number of non-zero coordinates in a basis vector indexed by an equivalence relation with c classes, of sizes $n_1, \ldots, n_c$, is

$$\prod_i (2^{n_i} - n_i),$$

at most $2^r - r$ in any case, and usually much less.

For example, in L_7 the ambient space has dimension 877, the lattice has rank 162, and the basis vectors have at most 121 nonzero coordinates, and in fact only one has more than 60. This comparative sparsity allows us to sort the basis so that vectors are determined by short initial segments. As a result one can search for short vectors in the lattice in a direct fashion, terminating unprofitable branches rapidly. And the sparsity has other advantages in terms of efficient storage and processing of large bases.

These computations serve their intended purposes, as we describe in the next section, and suggest profitable lines to pursue more conceptually. The lattices which arise seem to have their own combinatorial interest as an expression of relations among equivalence relations satisfying various constraints.

3. Conjectures on δ

The computations described in §2 gave the values of $\delta(r, n)$ for $n \leq\leq 6$ as well as some upper bounds for $n \leq r = 7$ which we think are probably the correct values (the largest being $\delta(7, 3) \geq 45$). More useful than the numerical values are the examples encoded by the corresponding vectors, namely specific pairs of multi-sets of equivalence relations satisfying our conditions, which suggest general constructions and upper bounds for δ which are valid generally. These correspond to lower bounds on the arity of the sort discussed in §1.

There is a general relation

$$\delta(r + 2, n + 2) \leq 2 \cdot \delta(r, n)$$

which may be seen as follows.

if $d = \delta(r, n)$ and $\mathcal{E}$, $\mathcal{E}'$ are distinct multi-sets of d equivalence relations on r elements with the largest number of equivalence classes occurring equal to n, and with the usual condition that the restrictions modulo any of the r elements agree, then we add two elements a, b and form new multi-sets $\hat{\mathcal{E}}$ and $\hat{\mathcal{E}}'$ on the $r+2$ elements $\{1, \ldots, r\} \cup \{a, b\}$ by taking $\hat{\mathcal{E}}$ to consist of the relations of $\mathcal{E}$ extended by a and b as singleton classes, and the relations of $\mathcal{E}'$ extended by the single class $\{a, b\}$, and taking $\hat{\mathcal{E}}'$ similarly, with the roles of $\mathcal{E}$ and $\mathcal{E}'$ reversed. We then have multi-sets of size $2d$ which give the stated bound on $\delta(r + 2, n + 2)$.

We conjecture that the inequality is an equation in general. Small sporadic counterexamples would not be surprising but they have not appeared as yet in the computations of small values. This conjecture then leads to the following.

Conjecture 1.

(1) $\delta(r, n) = 2^{(n-2)/3}\delta(r - n + 2, 2)$ *for* $r \geq n \geq 2$ *both even.*
(2) $\delta(r, n) = 2^{(n-3)/2}\delta(r - n + 3, 3)$ *for* $r \geq n \geq 3$ *and* n *odd.*
(3) $delta(r, n) = 2^{(n-4)/2}\delta(r - n + 4, 4)$ *for* $r \geq n \geq 4$ *and* n *even (which is relevant if* r *is odd).*

We also have the following, which is easily checked.

Fact 3.1 ([CM])**.** $\delta(r, 2) = 2^{r-2}$ *for* r *even.*

So the conjecture becomes

$$\delta(r, n) = 2^{(r-n/2)-1}$$

for r, n both even. In the other cases the formulas will be messier and it is better simply to write out the conjectures for the minimal cases required.[5]

Conjecture 2.

(1) $\delta(r, 3) =$

$$\begin{cases} 3 \cdot (2^{r-3} - 1) & \textit{for } r \geq 5 \textit{ odd.} \\ 2^{r-1} - 2^{r/2} + 1 & \textit{for } r \equiv 2 \pmod 4. \\ (1/2)(6^{r/2-1}) & \textit{for } r \equiv 0 \pmod 4. \end{cases}$$

(2) $\delta(r, 4) =$

$$\begin{cases} 12(2^{r-5} - 1) & \textit{for } r \equiv 3 \pmod 4,\ r \geq 7. \\ 2^{r-2} + 2^{r-3} - 2^{(r-3)/2} & \textit{for } r \equiv 1 \pmod 4. \end{cases}$$

One also requires the value $\delta(3, 3)$, which appears to be an isolated case.

The five equations conjectured represent valid upper bounds for the values of δ. This is proved by constructing the corresponding examples.

Example 2. Suppose r is divisible by 4. Divide r elements into $r/2$ pairs. Consider all equivalence relations with at most 3 classes such that no class contains any of the distinguished pairs. Fixing one pair, that determines two of the classes. Each of the remaining $r/2 - 1$ pairs

[5] These conjectures were substantially modified already in [CMS].

may then be assigned to one of these classes, or a third, in 6 ways. So there are $6^{r/2-1}$ such equivalence relations.

One then checks that these may be divided into two disjoint families whose restrictions on removal of any vertex coincide. One way to get at this is to start with the relation that has only two classes and then consider the effect of successively displacing elements from one class to another (which may be the third). Details will be given in [CM].

The net result of this construction is that

$$\delta(r,3) \le \frac{1}{2}6^{r/2-1}$$

Qualitatively, our conjectures would indicate that the arity of n^d should be approximately $\log_2 d) + n/2 + 1$ over the critical range $n \le 2\lfloor 1 + \log_2 d \rfloor$, or somewhat more precisely $2\lfloor \log_4 d + n/4 + 1/2 \rfloor$, with the value exact when d is a power of 2.

4. Postscript

This version of the paper incorporates a few scattered remarks about later developments but does not address the various kinds of progress made since its original publication. For that, one should look for articles on the topic of relational complexity in the group theoretic, combinatorial, or model theoretic literature. There is a great deal that is still not known in the context of natural actions of the symmetric group, notably the determination of the relational complexity of powers of the action on k-sets for $k > 1$, or the action on partitions of fixed shape, which, though puzzling questions, are not beyond all conjecture.[Bro]

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