Linear Algebra Theory: Explanations

Facts

Refer to the review sheet for a list of 20 important theoretical facts. Here, we explain items 1, 2, 5, 8, 14, 17, and 19.

1. `The rank of a matrix is equal to the rank of the transpose.`

This is a deep fact.

We know that the rank of a matrix A gives the dimension of both its row space and its column space! (This takes a lot of theory.)

Applying this to A^T, and noticing that the row and column spaces of A^T are the column and row spaces of A, we get the result.

2. `The rank of a matrix gives the dimension of both its row space and its column space.`

There are two different claims here.

(i) Row space: this is relatively easy. Gaussian elimination does not change the row space of a matrix. Hence we may suppose the matrix is in row echelon form, in which case the rank (number of pivots) is the same as the number of nonzero rows, and since these are linearly independent they form a basis for the row space.

Column space: much trickier. Gaussian elimination *does* change the column space, but it does not change the null space, and the null space gives the linear relations among the columns. Now the pivot columns in the *row echelon form* of the matrix form a basis for *its* column space, so the pivot columns in the *original* matrix form a basis for the *original* column space.

A further comment: we use the column space algorithm to test linear independence and to find relationships among linearly independent vectors. Simply make the given fectors the columns of a matrix, and find the pivot columns. Any columns which are not pivot columns can be expressed as linear combinations of the pivot columns.

5. `For any subspace` V `of` $\mathbb{R}^n$ `we have` $\dim(V) + \dim(V^\perp) = n$.

For this you need to think about the matrix A whose columns consist of a basis for V. If V is d-dimensional then this will be a matrix of shape $n \times d$ and rank d, and the transpose A^T will have shape $d \times n$ and rank d. Then $V^\perp$ is the null space of A^T, and its dimension is the nullity of A^T, which will be $n - d$.

So now our claim is $n = d + (n - d)$, which is correct.

8. `A diagonalizable matrix has all of its eigenvalues real.`

A diagonalizable matrix is similar to a diagonal matrix, and therefore has the same eigenvalues as the diagonal matrix. But the eigenvalues of the diagonal matrix are its diagonal entries, which are real numbers.

14. `The set of eigenvectors for a square matrix` A `corresponding to a particular eigenvalue, with the zero vector` *included*`, forms a subspace.`

Short version:

the eigenspace (with the zero vector included) is the null space of $A - \lambda I$, and the null space is a standard example of a subspace.

Detailed version:

If $A\mathbf{u} = \lambda\mathbf{u}$ and $A\mathbf{v} = \lambda\mathbf{v}$ then

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \lambda\mathbf{u} + \lambda\mathbf{v} = \lambda(\mathbf{u} + \mathbf{v})$$

and thus the eigenspace is closed under addition.

Similarly if $c \in \mathbb{R}$ then $A(c\mathbf{u}) = cA\mathbf{u} = c\lambda\mathbf{u} = \lambda(c\mathbf{u})$, and the eigenspace is closed under scalar multiplication.

17. `A set of homogeneous linear equations involving more variables than equations must have a nonzero solution.`

Elementary version:

The coefficient matrix is $m \times n$ with $m < n$, so there are at most m basic variables and at least $n - m$ free variables. So there are in fact some free variables, and by setting the free variables to nonzero values and solving for the basic variables, we can find nonzero solutions.

Advanced version:

If we have m equations in n unknowns, then the homogeneous system describes the orthogonal complement in $\mathbb{R}^n$ of a space V spanned by m vectors. So the dimension of V is at most m, and the dimension of $V^\perp$ is at least $n - m$, which is greater than 0. So $V^\perp$ contains nonzero vectors, since its dimension is positive.

19. `Similar matrices have the same characteristic polynomial, the same eigenvalues, and the same algebraic multiplicities.`

First choose notation: the similar matrices are called A and B, and they are related by $B = P^{-1}AP$ with P invertible.

Now compute the characteristic polynomials:

$$\det(B-\lambda I) = \det(P^{-1}AP-\lambda I) = \det(P^{-1}(A-\lambda I)P) = \det(P)^{-1}\det(A-\lambda I)\det(P) = \det(A-\lambda I)$$

So A and B have the same characteristic polynomial.

If they have the same characteristic polynomial, then they have the same eigenvalues and algebraic multiplicities, since these just depend on the factorization of that polynomial.

Food for thought

`Prove` $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$

You have to make the calculation. Letting A and B be $[a_{ij}]$ and $[b_{kl}]$, one can show that the trace of the product AB is $\sum_{i,j} a_{ij}b_{ji}$. Then the trace of the product BA is given by the corresponding formula $\sum_{j,i} b_{ji}a_{ij}$ and these are clearly the same. (If this is unreadable, just work through the 2×2 case for yourself.)

`What are the eigenvalues of a projection matrix, and what are their algebraic and geometric multiplicities? Is every projection matrix diagonalizeable?`

If we project orthogonally onto V, then the vectors in V are eigenvectors with eigenvalue 1, and the vectors in $V^\perp$ are eigenvectors with eigenvalue 0. So the geometric multiplicities are the dimensions of V and $V^\perp$, the matrix is diagonalizable, and consequently the algebraic multiplicities are the same. (And the characteristic polynomial must be $(1-\lambda)^d(-\lambda)^{d'}$ where $d = \dim V$ and $d' = \dim V^\perp$.)