Linear Algebra: Review for first Midterm (March 3, 2003)

Emphasis on the first midterm is on algorithms. This continues through the course, but we will test a bit more theory on the second midterm and final. However, we need to take stock of the theory at this point as well, so this review sheet contains a summary of that, too.

Topics and sample problems

1. Linear Systems

Coefficient matrix, augmented matrix, consistency, general solution in vector form; solution via matrix inversion or LU-decomposition

p. 80, Problem 28: Consistency, general solution
p. 173, Problem 24: Using the inverse to solve a system
p. 148, Problems 3,7: LU-decomposition, used to solve a system

2. Vectors

Span, linear dependence

p. 80, Problems 19,21,34: Span
p. 77, Problem 25: Span
p. 80, Problem 45: Linear dependence

3. Matrix multiplication

Matrix-vector product, matrix product, block form, inverse, transpose

p. 173, Problem 16: Block form
p. 173, Problem 19: inverse
p. 80, Problem 18: Matrix-vector product

Let $A = \begin{bmatrix} 0.5 & 0.3 & 0 \\ 0.4 & 0.5 & 0.6 \\ 0.1 & 0.2 & 0.4 \end{bmatrix}$. Calculate $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A^5$

Let $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Calculate D^5 and D^{-1}.

4. Elementary matrices

LU-decomposition

p. 148, Problems 3,5: LU-decomposition

Write the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ as a product of elementary matrices. (Method: reduce to row echelon form—which turns out to be the identity matrix—keeping track of the elementary matrices used at each step. Then A^{-1} is the product of these matrices, and A is the product of their inverses, in the reverse order.)

5. Applications

Rotation matrices, flow problems, population shifts

p. 80, Problem 16: Rotation
If A is a rotation matrix, calculate $A^T A$.
p. 109, Problem 5: Leslie matrix

Theory

I. Homogeneous Equations

A homogeneous equation with more variables than equations has a nonzero solution.

II. Theorem 1.9 page 75: span and linear independence

If $k+1$ vectors are contained in the span of k vectors, then they are linearly dependent – as for example, $n+1$ vectors in $\mathbb{R}^n$.

III. Gaussian elimination and matrix multiplication

Gaussian elimination *is* matrix multiplication (on the left) in two senses:

(a) It is the same thing as a sequence of multiplications by elementary matrices;

(b) It is the same thing as multiplication by one invertible matrix.

IV. Linear correspondence property (page 120)

The columns of a matrix, and the columns of any other matrix obtained by Gaussian elimination, have the same linear dependencies.

V. Uniqueness of reduced row echelon form

Any matrix has a unique *reduced* row echelon form

VI. Consequences of Gaussian Elimination

(1) Any matrix has an LU-decomposition

(2) Any invertible matrix is a product of elementary matrices.

Explanations – Summarizing the book and class discussion

I. Homogeneous equations

If there are more variables than equations, then there are *free* variables, and so one can make a nonzero solution by taking a free variable to be nonzero.

II. Span and linear independence

Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are in the span of $\mathbf{u}_1, \ldots, \mathbf{u}_m$ with $n > m$. Let A be the matrix whose columns are the $\mathbf{u}_i$ and B the matrix whose columns are the $\mathbf{v}_i$. Then the relation is $B = AX$ with X some $m \times n$ matrix. By part I we can solve $X\mathbf{c} = \bar{0}$ with $\mathbf{c}$ nonzero. Then $B\mathbf{c} = AX\mathbf{c} = \bar{0}$ so $c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \bar{0}$.

III. Gaussian elimination is matrix multiplication

One must realize first that an elementary row operation is given by multiplication by one elementary matrix. It then follows that repeated elementary row operations are given by products of elementary matrices. But elementary matrices are invertible (since the operations are reversible) and hence a product of elementary matrices is also invertible.

IV. Linear Correspondence Property

Applying elementary row operations to a matrix does not affect the linear relations among its columns.

Express this in terms of matrix multiplication:

(a) If B comes from A by elementary row operations, we have $B = XA$ with X invertible.

(b) Linear relations among the columns of A are given by equations of the form: $A\mathbf{c} = \bar{0}$. Multiplying on the left by X, this is the same as $B\mathbf{c} = \bar{0}$.

V. Uniqueness of reduced row echelon form

This depends on the Linear Correspondence Property. The point is: if you know all the linear relations among the columns of a matrix in reduced row echelon form, then you know the matrix.

VI. LU-decomposition and decomposition into elementary matrices

LU-decomposition: Applying the first half of Gaussian elimination to a matrix A reduces it to upper triangular form: $LA = U$. This can be rewritten as $A = L^{-1}U$ and L^{-1} is the product of the corresponding elementary matrices, inverted, in reverse order.

Matrix inverse: reduction of $[AI]$ to $[IA^{-1}]$ involves multiplication by a series of elementary matrices; so A^{-1} is the product of those elementary matrices and A is the product of their inverses (in the opposite order).