Notes on Uniform Integrability for Math 501

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1 Introduction

There is an important sense in which integrable functions are "almost bounded", "almost supported on sets of finite measure" and "cannot concentrate mass on too small a set". The first theorem in this section makes this precise.

1.1 THEOREM (Concentration Properties of Integrable Functions)**.** *Let f be an integrable function on $(X, \mathcal{M}, \mu)$. Then, for all $\epsilon > 0$:*

(1) There is a $\lambda < \infty$ so that

$$\int_{\{x\ :\ |f(x)|>\lambda\}} |f(x)| \mathrm{d}\mu \leq \epsilon \ . \tag{1.1}$$

(2) There is a set $A \in \mathcal{M}$ with $\mu(A) < \infty$ so that

$$\int_{A^c} |f(x)| \mathrm{d}\mu \leq \epsilon \ . \tag{1.2}$$

(3) There is a $\delta > 0$ so that for all $E \in \mathcal{M}$, whenever $\mu(E) < \delta$

$$\int_{E} |f(x)| \mathrm{d}\mu \leq \epsilon \ . \tag{1.3}$$

Proof. For $n \in \mathbb{N}$, define $f_n = f 1_{\{x\ :\ |f(x)| \geq n\}}$. Then $|f_n| \leq |f|$ for all n, and $|f_n| \to 0$ a.e., Therefore, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_X |f_n| \mathrm{d}\mu = 0 \ .$$

Thus, for all sufficiently large n, (1.1) is true.

Next, for $n \in \mathbb{N}$, define let $B_n = \{x\ :\ |f(x)| \geq 1/n\}$. Since $|f(x)| \geq \dfrac{1}{n} 1_{B_n}(x)$,

$$\int_X |f| \mathrm{d}\mu \geq \int_{B_n} |f| \mathrm{d}\mu \geq \frac{1}{n} \int_X 1_{B_N} \mathrm{d}\mu = \frac{\mu(B_N)}{n} \ .$$

Thus, for all n $\mu(B_n)$ is finite:

$$\mu(B_n) \leq n \int_X |f| \mathrm{d}\mu < \infty \ .$$

Next, $1_{B_n^c}|f| \le |f|$ and $1_{B_n^c}|f| \le 1/n$ for all $n \in \mathbb{N}$. The latter inequality shows that $1_{B_n^c}|f| \to 0$ a.e., and then the former allows us to apply the Dominated Convergence Theorem to show that

$$\lim_{n\to\infty} \int_{B_n^c} |f| \mathrm{d}\mu = 0 \ .$$

This shows that with B_n in place of A, (1.2) is true for all sufficiently large n, and taking any such n we have proved (1.2).

Finally, since we have proved (1.1), we know there is an $n \in \mathbb{N}$ so that

$$\int_{\{x\ :\ |f(x)|>n\}} |f(x)| \mathrm{d}\mu \le \frac{\epsilon}{2} \ . \tag{1.4}$$

Then, for any $E \in \mathcal{M}$,

$$\begin{aligned} \int_E |f| \mathrm{d}\mu &= \int_{E\cap\{x\ :\ |f(x)|>n\}} |f| \mathrm{d}\mu + \int_{E\cap\{x\ :\ |f(x)|\le n\}} |f| \mathrm{d}\mu \\ &\le \int_{\{x\ :\ |f(x)|>n\}} |f| \mathrm{d}\mu + \int_E n \mathrm{d}\mu \\ &\le \frac{\epsilon}{2} + n\mu(E) \ . \end{aligned}$$

Thus, provided n is chosen so that (1.4) is satisfied, and then we set $\delta = \epsilon/(2n)$, (1.3) is satisfied. □

We now turn to the followiing question: Consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ of integrable functions such that $f_n \to f$, either almost everyhwhere or in measure. What else is required to ensure that $f_n \to f$ in L^1?

According to Vitali's Theorem that we state and prove below, the answer is that the properties listed in Theorem 1.1 must hold *uniformly* for all the functions in the sequence. First we make the relevant definition.

1.2 DEFINITION (Uniform Integrability)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space, and $\mathcal{F}$ a set of measurable functions on X. Then $\mathcal{F}$ is *uniformly integrable* in case

(1) There is a $C < \infty$ such that for all $f \in \mathcal{F}$,

$$\int_X |f| \mathrm{d}\mu \le C \ . \tag{1.5}$$

(2) For all $\epsilon > 0$, there is a set $A_\epsilon \in \mathcal{M}$ so that for all $f \in \mathcal{F}$,

$$\int_{A_\epsilon^c} |f| \mathrm{d}\mu \le \epsilon \ . \tag{1.6}$$

(3) For all $\epsilon > 0$, there is a $\delta_\epsilon > 0$ so that when $E \in \mathcal{M}$ and $\mu(E) \le \delta_\epsilon$, then for all $f \in \mathcal{F}$,

$$\int_E |f| \mathrm{d}\mu \le \epsilon \ . \tag{1.7}$$

1.3 EXAMPLE. *Let g be a non-negative integrable function, and let $\mathcal{F}$ be the set of measurable functions satisfying*

$$|f| \le g \ .$$

Then by Theorem 1.1, $\mathcal{F}$ is uniformly integrable.

Indeed, given $\epsilon > 0$ let A_ϵ and δ_ϵ be such that $\mu(A_\epsilon) < \infty$, $\int_{A_\epsilon^c} |g| \mathrm{d}\mu < \epsilon$, and $\mu(E) < \delta_\epsilon \Rightarrow \int_E |g| \mathrm{d}\mu < \epsilon$. Since $|f| \le |g|$, the same A_ϵ and δ_ϵ work for each f in $\mathcal{F}$, and of course $\int |f| \mathrm{d}\mu \le \int g \mathrm{d}\mu =: C$ for all $f \in \mathcal{F}$.

1.4 THEOREM (Vitali's Theorem)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\mathcal{F}$ be a uniformly integrable set of functions on $(X, \mathcal{M}, \mu)$. Let $\{f_n\}$ be a sequence of functions in $\mathcal{F}$ and suppose that $\lim_{n\to\infty} f_n = f$ either almost everywhere or in measure. Then*

$$\lim_{n\to\infty} \int_X |f_n - f| \mathrm{d}\mu = 0 \ . \tag{1.8}$$

Conversely, Suppose that $\{f_n\}$ is any sequence of integrable functions and that (1.8) holds. Then the set $\mathcal{F}$ consisting of the functions f_n in the sequence, together with the limit f, is uniformly integrable

Proof. Fix $\epsilon > 0$, and let C, A_ϵ and δ_ϵ be such that (1.5), (1.6) and (1.7) hold for all g in $\mathcal{F}$, and each f_n in our sequence. By Fatou's Lemma, or its analog for convergence in measure,

$$\int_X |f| \mathrm{d}\mu \le \liminf_{n\to\infty} \int_X |f_n| \mathrm{d}\mu \le C \ ,$$

$$\int_{A_\epsilon^c} |f| \mathrm{d}\mu \le \liminf_{n\to\infty} \int_{A_\epsilon^c} |f_n| \mathrm{d}\mu < \epsilon$$

and

$$\int_E |f| \mathrm{d}\mu \le \liminf_{n\to\infty} \int_E |f_n| \mathrm{d}\mu$$

so that $\mu(E) \le \delta_\epsilon \Rightarrow \int_E |f| \mathrm{d}\mu \le \epsilon$.

Now use (1.6) in the definition of uniform integrability to reduce the proof to that of the special case in which $\mu(X) < \infty$:

$$\begin{aligned} \int_X |f_n - f| \mathrm{d}\mu &= \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu + \int_{A_\epsilon^c} |f_n - f| \mathrm{d}\mu \\ &= \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu + \int_{A_\epsilon^c} (|f_n| + |f|) \mathrm{d}\mu\epsilon \\ &\le \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu + 2\epsilon \ . \end{aligned} \tag{1.9}$$

It therefore suffices to show that

$$\lim_{n\to\infty} \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu = 0 \ . \tag{1.10}$$

Suppose first that $f_n \to f$ a.e. Since $\int_X |f| \mathrm{d}\mu \le C$, f is finite almost everywhere. By Egoroff's Theorem, there is a subset $E \subset A_\epsilon$ with $\mu(E) \le \delta_\epsilon$, and such that $f_n \to f$ uniformly on $A_\epsilon \backslash E$.

Thus,

$$
\begin{aligned}
\int_{A_\epsilon} |f_n - f| \mathrm{d}\mu &\leq \int_E |f_n - f| \mathrm{d}\mu + \int_{A_\epsilon \backslash E} |f_n - f| \mathrm{d}\mu \\
&\leq \int_E (|f_n| + |f|) \mathrm{d}\mu + \mu(A_\epsilon) \sup\{ \ |f_n(x) - f(x)| \ : \ x \in A_\epsilon \backslash E \ \} \\
&\leq 2\epsilon + \mu(A_\epsilon) \sup\{ \ |f_n(x) - f(x)| \ : \ x \in A_\epsilon \backslash E \ \}
\end{aligned}
\tag{1.11}
$$

Since $\epsilon > 0$ is arbitrary and $\lim_{n\to\infty} \sup\{ \ |f_n(x) - f(x)| \ : \ x \in A_\epsilon \backslash E \ \} = 0$, we have proved (1.10) in this case.

Next, suppose that $f_n \to f$ in measure. Fix $\eta > 0$, and define $E_\eta(n) = \{ \ x \ : \ |f_n(x) - f(x)| > \eta \ \}$. Since $f_n \to f$ in measure, $\lim_{n\to\infty} \mu(E_\eta(n)) = 0$. Now observe that for all n such that $\lim_{n\to\infty} \mu(E_\eta(n)) \leq \delta_\epsilon$,

$$
\begin{aligned}
\int_{A_\epsilon} |f_n - f| \mathrm{d}\mu &\leq \int_{E_\eta(n)} |f_n - f| \mathrm{d}\mu + \int_{A_\epsilon \backslash E_\eta(n)} |f_n - f| \mathrm{d}\mu \\
&\leq \int_{E_\eta(n)} (|f_n| + |f|) \mathrm{d}\mu + \mu(A_\epsilon)\eta \\
&\leq 2\epsilon + \mu(A_\epsilon)\eta
\end{aligned}
\tag{1.12}
$$

Since $\epsilon, \eta > 0$ are arbitrary, this proves (1.10) in this case as well.

Now we prove the converse part of the thoerem. For any set B,

$$
\int_B |f_n| \mathrm{d}\mu \leq \int_B |f| \mathrm{d}\mu + \int_B |f_n - f| \mathrm{d}\mu \leq \int_B |f| \mathrm{d}\mu + \int_X |f_n - f| \mathrm{d}\mu \ .
$$

For any fixed $\epsilon > 0$, choose N_ϵ so that

$$
n > N_\epsilon \Rightarrow \int_X |f_n - f| \mathrm{d}\mu < \epsilon/2 \ .
$$

We then have that for all $n > N_\epsilon$,

$$
\int_B |f_n| \mathrm{d}\mu \leq \int_B |f| \mathrm{d}\mu + \epsilon/2 \ .
$$

Since $\{f\}$ itself is uniformly integrable, there is a number $\tilde{\delta}_\epsilon > 0$ so that

$$
\mu(B) \leq \tilde{\delta}_\epsilon \Rightarrow \int_B |f| \mathrm{d}\mu \leq \epsilon/2 \ .
$$

Hence, for all $n > N_\epsilon$,

$$
\mu(B) \leq \tilde{\delta}_\epsilon \Rightarrow \int_B |f_n| \mathrm{d}\mu \leq \epsilon \ .
$$

Finally, using the fact that for each $n \leq N_\epsilon$, $\{f_n\}$ is uniformly integrable, there is a $\delta_\epsilon^{(n)} > 0$ so that

$$
\mu(B) \leq \delta_\epsilon^{(n)} \Rightarrow \int_B |f_n| \mathrm{d}\mu \leq \epsilon \ .
$$

Define

$$\delta_\epsilon = \min\{\delta_\epsilon^{(1)}, \delta_\epsilon^{(2)}, \ldots, \delta_\epsilon^{(N_\epsilon)}, \tilde{\delta}_\epsilon\} \ .$$

Since the minimum of a *finite* set of strictly positive numbers is strictly positive, we have that $\delta_\epsilon > 0$ Also,

$$\mu(B) \le \delta_\epsilon \Rightarrow \int_B |f_n| \mathrm{d}\mu \le \epsilon$$

for all n and for f as well. Thus, condition (1.6) is satisfied. The other two condtions are easily proved in the same way. □

Vitali's Theorem implies a generalized for of the Dominated Convergence Thoerem:

1.5 THEOREM (A Generalized Dominated Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of measurable functions on $(X, \mathcal{M}, \mu)$, and let $\{g_n\}$ be a sequence of integrable functions on $(X, \mathcal{M}, \mu)$ such that for sime $g \in L^1(X, \mathcal{M}, \mu)$, $\lim_{n\to\infty} \|g_n - g\|_1 = 0$.*

Suppose that

$$|f_n| \le |g_n|$$

a.e. for all n, and that for some f, $f_n \to f$ either a.e. or in measure. Then

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)| \mathrm{d}\mu = 0 \ .$$

Proof. By the converse to Vitali's Theroem, the sets $\{g_n\}_{n\in\mathbb{N}}$ is uniformly integrable. But then since $|f_n| \le |g_n|$ for all n, $\{f_n\}_{n\in\mathbb{N}}$ is also uniformly integrable with the same C, A_ϵ and δ_ϵ as $\{g_n\}_{n\in\mathbb{N}}$. Then the first psrt of Vitali's Theorem yields $f_n \to f$ in L^1. □

1.6 Remark. The special case in which $g_n = g$ for all n gives us the Dominated Converge Theorem since

$$\left| \int_X f_n \mathrm{d}\mu - \int_X f \mathrm{d}\mu \right| \le \int_X |f_n(x) - f(x)| \mathrm{d}\mu \ .$$

Not all applications of Vitali's Theorem involve a dominating function. A situtation that frequently arrises in applications is that one has a sequence of functions $\{f_n\}$ for which one has an *a priori* bound on

$$\int_X \phi(|f_n|) \mathrm{d}\mu$$

for some function ϕ that grows faster than linearly at infinity; for example $\phi(t) = t \log_+(t)$ or $\phi(t) = t^2$.

1.7 THEOREM (Integral Limits on Concentration)**.** *Let ϕ be a monotone increasing function on $[0, \infty)$ with values in $[0, \infty)$ such that*

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \infty \ .$$

Then for any measure space $(X, \mathcal{M}, \mu)$ and any $C > 0$, let $\mathcal{F}_C$ be the set of functions satisfying

$$\int_X \phi(|f|) \mathrm{d}\mu \le C \ .$$

Then

$$\lim_{\delta \to 0} \left(\sup \left\{ \int_E |f| \mathrm{d}\mu \;\middle|\; \mu(E) \le \delta \ , \ f \in \mathcal{F}_C \ \right\} \right) = 0 \ . \tag{1.13}$$

In particular, if $\mu(X) < \infty$, $\mathcal{F}_C$ is uniformly integrable.

Proof. Let E be any measurable set, and f any memeber of $\mathcal{F}_C$. Then for any $a > 0$, let

$$B_a = \{ x \mid |f(x)| > a \ \} \ ,$$

and let a_0 be such that $\phi(a)$ is strictly positive for $a > a_0$. Then since ϕ is monotone increasing, for all $a > a_0$,

$$\begin{aligned}
\int_E |f| \mathrm{d}\mu &= \int_{E \cap B_a} |f| \mathrm{d}\mu + \int_{E \cap B_a^c} |f| \mathrm{d}\mu \\
&\le \int_{E \cap B_a} |f| \frac{\phi(|f|)}{\phi(a)} \mathrm{d}\mu + \int_{E \cap B_a^c} a \mathrm{d}\mu \\
&\le \int_X |f| \frac{\phi(|f|)}{\phi(a)} \mathrm{d}\mu + \int_E a \mathrm{d}\mu \\
&\le \frac{C}{\phi(a)} + a\mu(E) \ .
\end{aligned}$$

Now given $\epsilon > 0$, choose a so that $C/\phi(a) < \epsilon/2$, and then choose $\delta_\epsilon = \epsilon/(2a)$. It then follows that

$$\mu(E) < \delta_\epsilon \Rightarrow \int_E |f| \mathrm{d}\mu < \epsilon$$

and f was an arbirary member of $\mathcal{F}_C$. Since $\epsilon > 0$ was arbitrary, this proves (1.13), which is another way of stating condition *(3)* in the definition of uniform integrability.

Now suppose $\mu(X) < \infty$. Let a_1 be such that $\phi(t) \ge t$ for all $t \ge a_1$. Then for $f \in \mathcal{F}_C$,

$$\begin{aligned}
\int_X |f| \mathrm{d}X &\le \int_{\{|f| \le a_1\}} |f| \mathrm{d}\mu + \int_{\{|f| \ge a_1\}} |f| \mathrm{d}\mu \\
&\le \int_{\{|f| \le a_1\}} a_1 \mathrm{d}\mu + \int_{\{|f| \ge a_1\}} \phi(|f|) \mathrm{d}\mu \\
&\le a_1 \mu(X) + C,
\end{aligned}$$

so that *(1)* is satisfied. Finally, for *(2)*, we can simply take $A_\epsilon = X$; the second requirement in the definition of uniform integrability is vacuous in case $\mu(X) < \infty$. □