Solutions to the Topology Exercsises

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1 Solutions to the Exercises

1. Let (X, d) be a separable metric space. Let Y be any subset of X, and define d_Y to be the restriction of d to $Y \times Y$. Show that (Y, d_Y) is separable.

SOLUTION Let $\{x_n\}_{n\in\mathbb{N}}$ be a dense sequence in X. Pick any $y_0 \in Y$. For each $k \in \mathbb{N}$ construct a sequence $\{y_n^{(k)}\}_{n\in\mathbb{N}}$ as follows: If there exists some $y \in Y$ such that $d(x_n, y) < (2k)^{-1}$, choose $y_n^{(k)} = y$ for some such y. Otherwise, choose $y_n^{(k)} = y_0$.

Now, given any $y \in Y$, and any k, there is some n so that $d(y, x_n) < (2k)^{-1}$. But then $d(y_n^{(k)}, x_n) < (2k)^{-1}$. By the triangle inequality, and the definition of d_Y,

$$d_Y(y_n^{(k)}, y) = d(y_n^{(k)}, y) \le d(y_n^{(k)}, x_n) + d(x_n, y) < \frac{1}{k} \ .$$

Thus we have constructed a sequence in Y that passes within a distance $1/k$ of every $y \in Y$.

Now the set of all pairs $(k, n) \in \mathbb{N}\times\mathbb{N}$ is countable. Choose some ordering, and using it arrange all of the terms of the sequences $\{x_n\}_{n\in\mathbb{N}}$, $k \in \mathbb{N}$ into a single sequence. This sequence is clearly dense in Y.

2. Suppose that (X, d) is a complete metric space with a finite diameter; i.e., there exists $D < \infty$ such that $d(x, y) \le D$ for all $x, y \in X$. Is it true that every continuous real valued function on X is bounded? Prove this assertion or give a counterexample.

SOLUTION This assertion is false. If X is compact, then every continuous function on X is bounded. So to find a counterexample, we must find a metric space (X, d) of finite diameter that is compete, but not compact. We have seen that the unit ball in ℓ_2 is such a space, and that the unit vectors $e^{(n)}$ in it satisfy

$$d_{\ell_2}(e^{(m)}, e^{(n)}) = \sqrt{2}$$

for all $m \ne n$ – and hence the sequence $\{e^{(n)}\}_{n\in\mathbb{N}}$ has no convergent subsequence.

Now define functions f_n, $n \in \mathbb{N}$, on ℓ_2 (and hence the unit ball in ℓ_2) by

$$f_n(x) = \begin{cases} n(1 - 4d_{\ell_2}(x, e^{(n)})) & d_{\ell_2}(x, e^{(n)})) \le 1/4 \\ 0 & d_{\ell_2}(x, e^{(n)})) > 1/4 \end{cases}$$

Note that each of the functions f_n is continuous, being constructed out of the metric using operations that preserve continuity. Also, since $f_n(x) = 0$ unless $d_{\ell_2}(x, e^{(n)}) < 1/4$, $f_n(x)$ can be non-zero for at most one $n \in \mathbb{N}$, and moreover, if $f_n(x) \neq 0$, then for all y with $d_{\ell_2}(y, x) < 1/4$, $f_m(y) = 0$ for all $m \neq n$. This is true since by the triangle inequality $d_{\ell_2}(y, e^{(n)}) < 1/2$, and so, again by the triangle inequality,

$$d_{\ell_2}(y, e^{(m)}) \geq d_{\ell_2}(e^{(n)}, e^{(m)}) - d_{\ell_2}(e^{(n)}, y) \geq \sqrt{2} - \frac{1}{2} > \frac{1}{4} \ .$$

Thus, the function $g(x)$ given by

$$g(x) = \sum_{n=1}^{\infty} f_n(x)$$

is well-defined and continuous since for each x, there is neighborhood U of x such that for all $y \in U$ (including s), $f_n(y) \neq 0$ for at most one $n \in N$. But clearly $g(e^{(n)}) = n$, so g is not bounded.

3. Let (X, d) be a compact metric space.

(a) Show that if $f : X \to X$ is continuous but not onto, there is some $x_0 \in X$ and some $r > 0$ so that $d(f(x), x_0) \geq r$ for all $x \in X$.

(b) Let f be an isometry from X into itself; i.e., a function with the property that

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$. Show that f is necessarily one to one and onto, and hence invertible.

SOLUTION (a) Suppose f is not onto. then for some x_0, $f(x) \neq x_0$ for any x. Define $\varphi(x) = d(x, x_0)$. This is continuous and X is compact, so there exists $y \in X$ such that $\varphi(x) \geq \varphi(y)$ for all $x \in X$. Let $r = \varphi(y)$. Since $f(y) \neq x_0$, $r = d(f(y), x_0) > 0$. But then by the definition of φ, $d(f(x), x_0) \geq r$ for all $x \in X$.

(b) Continuing with the assumption, define a sequence $\{x_n\}_{n \in N}$ by $x_1 = f(x_0)$ and $x_{n+1} = f(x_n)$ for all $n \geq 1$. This sequence has a convergent subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$, and like all convergent subsequences, this sequence is Cauchy, so that or all $\ell > k$ with k sufficiently large,

$$d(x_{n_k}, x_{n_\ell}) < \frac{r}{2} \ .$$

However, by the isometry property,

$$d(x_{n_k}, x_{n_\ell}) = d(x_0, x_{n_\ell - n_k}) \geq r \ .$$

This contradiction shows that f must be onto.

4. Let (X, d) be a compact metric space, and let $f : X \to \mathbb{C}$ be continuous. Show that for all $\epsilon > 0$, there exists $L < \infty$ so that

$$|f(x) - f(y)| \leq L d(x, y) + \epsilon$$

for all $x, y \in X$.

SOLUTION Let $C \subset X \times X$ be given by

$$C = \{(x, y) \ : \ |f(x) - f(y)| \geq \epsilon\} \ .$$

Since the function $(x,y) \mapsto |f(x)-f(y)|$ is continuous on $X \times X$, the set C is closed. Since $X \times X$ is compact, C is compact.

The function

$$g(x,y) = \frac{|f(x)-f(y)|}{d(x,y)}$$

is continuous on C. (The denominator is continuous and bounded from below on C by a positive number since if (x_1,y_1) minimizes $d(x,y)$ on C, $|f(x_1)-f(y_1)| \geq \epsilon$, hence $x_1 \neq y_1$ and hence $d(x_1,y_1) \neq 0$.)

Define

$$L = \max_{(x,y)\in C} \frac{|f(x)-f(y)|}{d(x,y)} \ ,$$

which is well defined since continuous functions have maxima on compact sets. Then

$$(x,y) \in C \Rightarrow |f(x)-f(y)| \leq Ld(x,y) \qquad \text{and} \qquad (x,y) \in C^c \Rightarrow |f(x)-f(y)| \leq \epsilon \ .$$

Thus for all (x,y),

$$|f(x)-f(y)| \leq Ld(x,y) + \epsilon \ .$$

5. (a) Let (X,d) be a complete metric space in which bounded sets are totally bounded. Let $A \subset X$ be closed and $B \subset X$ be compact. Show that there exist $x_1 \in A$ and $x_2 \in B$ such that

$$d(x_1,x_2) \leq d(x,y) \qquad \text{for all } x \in A \ , y \in B \ .$$

(b) Show by example that this is false if we weaken the assumption to only suppose that B is closed.

SOLUTION (a) This is easy if both A and B are compact since then $d(x,y)$ is continuous on $A \times B$, which is compact, and hence there exist $(x_1,x_2) \in A \times B$ so that $d(x_1,x_2) \leq d(x,y)$ for all $(x,y) \in A \times B$.

To reduce to the compact case, pick any $y_0 \in B$, and define

$$R = \max\{d(x,y) \ : x \in A \ \}$$

which exists and is finite since the distance to y is continuous, and A is compact. Define C by

$$C = \{z \in B \ : d(z,y) \leq 2R\} \ .$$

Note that C is closed and bounded, and hence is compact by our hypotheses on (X,d). If $z \in B \cap C^c$, and $x \in A$, then by the triangle inequality, $d(z,y) \leq d(z.x) + d(x,y)$, so that

$$d(z,x) \geq d(y,z) - d(x,z) \geq 2R - R = R \ .$$

Since $d(y,x) \leq R$, it follows that

$$\inf_{(x,w)\in A\times B}\{d(x,w)\} \geq \inf_{(x,w)\in A\times C}\{d(x,w)\} \ .$$

But since C is compact, were are now reduced to the compact case.

(b) Take A to be the real axis in $\mathbb{R}^2$. Take B to be the graph of $y = e^x$ in $\mathbb{R}^2$. Both are closed and have empty intersection, but $(x,0)\ inA$, $(x,e^x) \in$ and $|(x,0)-(x,e^x)| = e^x \to 0$ as $x \to -\infty$. Finally, $\mathbb{R}^2$ is complete, and bounded subsets in it are totally bounded.

6. Define ℓ_1 to be the set of complex valued sequences $\{x_j\}_{j\in\mathbb{N}}$ such that $\sum_{j=1}^{\infty}|x_j| < \infty$. Define a function on d_{ℓ_1} on ℓ_2 by

$$d_{\ell_1}(\{x_j\}\ ,\ \{y_j\}) = \sum_{j=1}^{\infty}|x_j - y_j|\ .$$

(a) Show that (ℓ_1, d_{ℓ_1}) is a metric space.

(b) Show that the metric space (ℓ_1, d_{ℓ_1}) is complete.

SOLUTION (a) It is clear that for all $x, y \in \ell_1$, $d_{\ell_1}(x,y) = d_{\ell_1}(y,x)$ and that $d_{\ell_1}(x,y) = 0$ if and only if $x = y$. Finally, let x, y and z be in ℓ_1. Then for all n,

$$|x_n - z_n| \leq |x_n - y_n|||y_n - z_n|$$

by the triangle inequality in the complex plane. Summing, we obtain the triangle inequality for d_{ℓ_1}. Thus, d_{ℓ_1} is a metric.

(b) Let $\{x^{(n)}\}_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ_1. Choose an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that

$$m > n_k \Rightarrow d_{\ell_1}(x^{(n_k)}, x^{(m)}) < 2^{-k}\ .$$

Then

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}|x_j^{n_{k+1})} - x_j^{(n_k)}|\right) = \sum_{k=1}^{\infty} d_{\ell_1}(x^{n_{k+1})}, x^{n_k)}) \leq \sum_{k=1}^{\infty} 2^{-k} = 1\ .$$

Therefore, since sums of countably many positive terms can be summed in any order,

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|x_j^{n_{k+1})} - x_j^{(n_k)}|\right) \leq 1\ .$$

Define $z_j := \sum_{k=1}^{\infty}|x_j^{(n_{k+1})} - x_j^{(n_k)}|$. It follows that $\sum_{k=1}^{\infty}|x_j^{n_{k+1})} - x_j^{(n_k)}| = \sum_{j=1}^{\infty} z_j \leq 1$. In particular, the sequence z whose jth entry is z_j, belongs to ℓ_1.

By the telescoping sum formula, $x_j^{(n_m)} = x_j^{(n_1)} + \sum_{k=1}^{m}(x_j^{(n_{k+1})} - x_j^{(n_k)})$ and then since since absolutely convergence series converge,

$$x_j := \lim_{m\to\infty} x_j^{(n_m)} = \lim_{m\to\infty}\left[x_j^{(n_1)} + \sum_{k=1}^{m}(x_j^{(n_{k+1})} - x_j^{(n_k)})\right]$$

exists, and $|x_j| \leq |x_j^{(n_1)}| + z_j$. Since $x^{(n_1)} \in \ell_1$ and $z \in \ell_1$, $x \in \ell_1$.

Next, for each j,

$$|x_j^{(n_k)} = x_j| \leq \sum_{\ell=k}^{\infty}|x_j^{(n_{\ell+1})} - x_j^{(n_\ell)}|$$

and then summing on j, we obtain $d_{\ell_1}(x^{(n_k)}, x) \leq \sum_{j=k}^{\infty} z_j$, and hence

$$\lim_{k\to\infty} d_{\ell_1}(x^{(n_k)}, x) = 0\ .$$

Thus, the subsequence $\{x^{(n_k)}\}_{k\in\mathbb{N}}$ converges in the ℓ_1 metric to x, and then since the original sequence is Cauchy, it too must converge to the same limit.

7. Let (ℓ_2, d_{ℓ_2}) Show that a bounded subset X of ℓ_2 is totally bounded if and only if for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{k>N_\epsilon} |x_j|^2 < \epsilon^2$$

for all $\{x_j\} \in X$.

SOLUTION Let X be a bounded subset of ℓ_2, and let $\epsilon > 0$. Suppose that for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{k>N_\epsilon} |x_j|^2 < \frac{\epsilon^2}{16}$$

for all $\{x_j\} \in X$. Define the map Φ from X ro $\mathbb{C}^N$ by

$$\Phi(x) = (x_1, \ldots, x_N)$$

The image $\Phi(X)$ is bounded in $\mathbb{C}^N$,and bounded sets in $\mathbb{C}^N$ are totally bounded. Hence there exists a finite cover $\{U_1, \ldots, U_M\}$ of $\Phi(X)$ by open sets of diameter no more than $\epsilon/2$.

If $x, y \in X$,

$$\|x-y\|^2 - \|\Phi(x) - \Phi(y)\|^2_{\mathbb{C}^N} = \sum_{j>N} |x_j - y_j|^2 \le 2\sum_{j>N} |x_j|^2 + 2\sum_{j>N} |y_j|^2 \le \frac{\epsilon^2}{4} \ .$$

Hence if $x, y \in X \cap \Phi^{-1}(U_j)$ for any j,

$$\|x - y\| \le \frac{\epsilon}{\sqrt{2}} \ .$$

Therefore,

$$\{X \cap \Phi^{-1}(U_1), \ldots, X \cap \Phi^{-1}(U_M)\}$$

is a finite cover of X by sets of diameter $\epsilon/\sqrt{2}$. Now define

$$V_j = \bigcup_{x\in\Phi^{-1}(U_j)} B_{\epsilon/10}(x) \ .$$

Then V_j is open and have diameter no more than $\epsilon/\sqrt{2} + \epsilon/5 < \epsilon$, and $\{V_1, \ldots, V_M\}$ covers X. Since $\epsilon > 0$ is arbitrary, this proves that X is totally bounded.

Conversely, suppose that X is totally bounded. Pick $\epsilon > 0$. Cover X by a finite set $\{V_1, \ldots, V_M\}$ of open sets of diameter no more than $\epsilon/4$. For each j, pick $x^{(j)}\ inV_j \cap X$.

Pick N_j such that

$$\sum_{k>N_j} |x^{(j)}_k|^2 < \frac{\epsilon^2}{4} \ .$$

Define $N = \max\{N_1, \ldots, N_M\}$.

For any $x \in X$, $x \in V_j$ for some j.

$$\begin{aligned} \sum_{k>N} |x_k|^2 &\le 2\sum_{k>N} |x^{(j)}_k - x_k|^2 + 2\sum_{k>N} |x^{(j)}_k|^2 \\ &\le \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2 \ . \end{aligned}$$

8. Let (ℓ_1, d_{ℓ_1}) be defined as in Exercise 6. Show that $X \subset \ell_1$ is totally bounded if and only if for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{k > N_\epsilon} |x_j| < \epsilon$$

for all $\{x_j\} \in X$. Then show that $B_1(\{0\})$, the ball of radius 1 about the zero sequence, is not totally bounded, and hence that the closed ball of radius 1 about the zero sequence is not compact.

SOLUTION Let X be a bounded subset of ℓ_1, and let $\epsilon > 0$. Suppose that for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{k > N_\epsilon} |x_j| < \frac{\epsilon}{4}$$

for all $\{x_j\} \in X$. Define the map Φ from X ro $\mathbb{C}^N$ by

$$\Phi(x) = (x_1, \dots, x_N)$$

The image $\Phi(X)$ is bounded in $\mathbb{C}^N$,and bounded sets in $\mathbb{C}^N$ are totally bounded. Hence there exists a finite cover $\{U_1, \dots, U_M\}$ of $\Phi(X)$ by open sets of diameter no more than $\epsilon/(4\sqrt{N})$.

If $x, y \in X \cap \Phi^{-1}(U_j)$ for some j,

$$\begin{aligned} d_{\ell_1}(x,y) &= \sum_{j=1}^{N} |\Phi(x)_j - \Phi(y)_j| + \sum_{j>N} |x_j - y_j \\ &\leq \sqrt{N}\|\Phi(x) - \Phi(y)\|_{\mathbb{C}^N} + \sum_{j>N} |x_j| + \sum_{j>N} |y_j| \\ &\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{3}{4}\epsilon \ . \end{aligned}$$

$$\{X \cap \Phi^{-1}(U_1), \dots, X \cap \Phi^{-1}(U_M)\}$$

is a finite cover of X by sets of diameter $\epsilon/\sqrt{2}$. Now define

$$V_j = \bigcup_{x \in \Phi^{-1}(U_j)} B_{\epsilon/10}(x) \ .$$

Then V_j is open and have diameter no more than $\epsilon/\sqrt{2} + \epsilon/5 < \epsilon$, and $\{V_1, \dots, V_M\}$ covers X. Since $\epsilon > 0$ is arbitrary, this proves that X is totally bounded.

Conversely, suppose that X is totally bounded. Pick $\epsilon > 0$. Cover X by a finite set $\{V_1, \dots, V_M\}$ of open sets of diameter no more than $\epsilon/2$. For each j, pick $x^{(j)} \ inV_j \cap X$.

Pick N_j such that

$$\sum_{k > N_j} |x_k^{(j)}|^2 < \frac{\epsilon}{2} \ .$$

Define $N = \max\{N_1, \dots, N_M\}$.

For any $x \in X$, $x \in V_j$ for some j, and then

$$\begin{aligned} \sum_{k>N} |x_k| &\leq \sum_{k>N} |x_k^{(j)} - x_k|^2 + \sum_{k>N} |x_k^{(j)}|^2 \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ . \end{aligned}$$

For the second part, note that the closed unit ball B contains the sequences $e^{(n)}$ with

$$e_j^{(n)} = \begin{cases} 1 & j = n \\ 0 & j \neq n \ . \end{cases}$$

Since for any N, when $n > N$,

$$\sum_{j>N} |e_j^{(n)}| = 1$$

B is not totally bounded, and hence not compact.

9. Let $X = [0,1]$. Each $x \in X$ has a binary expansion

$$x = \sum_{n=1}^{\infty} b_n(x) 2^{-n}$$

with each $b_n(x) \in \{0,1\}$. We stipulate that if x is a dyadic rational, only finitely many of the $b_n(x)$ are non-zero, and under this condition, the $b_n(x)$ are uniquely determined, so that $b_x : X \to \{0,1\} \subset X$ is a well-defined function for each n.

(a) Show that no subsequence of $\{b_n\}_{n\in\mathbb{N}}$ converges pointwise.

(b) Equip X^X with its product topology and note that each b_n is a function from X to X, and hence is an element of X^X. Show that no subsequence of $\{b_n\}_{n\in\mathbb{N}}$ converges in the product topology, and thus that the analog of Tychonov's Theorem for sequential compactness is false.

SOLUTION (a) Suppose $\{b_{n_k}\}_{k\in\mathbb{N}}$ is a pointwise convergent subseqeunce. Define x so that the nth bit of x is zero if $n \neq n_k$ for some even k, and is 1 otherwise. Then $b_{n_k}(x) = 0$ for infinitely many k and $b_{n_k}(x) = 1$ for infinitely many k. Hence $\lim_{k\to\infty} b_{n_k}(x)$ does not exist.

(b) By its construction, if $\{b_{n_k}\}_{n\in\mathbb{N}}$ converges in the product topology, then for each $x \in X$, $\{b_{n_k}(x)\}_{n\in\mathbb{N}}$ converges in X. But by part **(a)**, this is impossible.

10. Let (X,d) be a compact metric space. Then $X^{\mathbb{N}}$ consists of all sequences $\{x_k\}_{k\in\mathbb{N}}$ in X. Define a function d on $X^{\mathbb{N}} \times X^{\mathbb{N}}$ by

$$d(\{x_k\}_{k\in\mathbb{N}}, \{y_k\}_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty} 2^{-k} d(x_k, y_k) \ .$$

(a) Show that d is a metric on $X^{\mathbb{N}} \times X^{\mathbb{N}}$.

(b) Show that the metric topology in $X^{\mathbb{N}}$ induced by d is at least as strong as the product topology.

(c) Show that with the metric topology induced by d, $X^{\mathbb{N}}$ is sequentially compact.

(d) Show directly, without invoking Tychonov's Theorem that $X^{\mathbb{N}}$ compact in the product topology.

SOLUTION (a) It is clear that

$$d(\{x_k\}_{k\in\mathbb{N}}, \{y_k\}_{k\in\mathbb{N}}) = d(\{y_k\}_{k\in\mathbb{N}}, \{k_k\}_{k\in\mathbb{N}})$$

and that $d(\{x_k\}_{k\in\mathbb{N}}, \{y_k\}_{k\in\mathbb{N}}) = 0$ if and only if $d(x_k, y_k) = 0$ for each k,which is the ca if and only if $x_k = y_k$ for al k. i.e;, $\{x_k\}_{k\in\mathbb{N}} = \{y_k\}_{k\in\mathbb{N}}$. As for the triangle inequality, for any $\{x_k\}_{k\in\mathbb{N}}$, $\{y_k\}_{k\in\mathbb{N}}$ and $\{z_k\}_{k\in\mathbb{N}}$, for each k,

$$d(x_k, z_k) \leq d(x_k, y_k) + d(y_k, z_k) \ ,$$

and now summing we obtain the triangle inequality for d. Thus, d is a metric.

(b) We must show that every set that is open in the product topology is open in the metric topology. We know that every open set in the product topology is a union of sets of the form

$$U = \cap_{j=1}^{N}\{\{x_k\}_{k\in\mathbb{N}} \ : x_{k_j} \in B_{r_j}(y_j)\}$$

for some $N \in \mathbb{N}$, some $k_1 < k_2 < \cdots < k_N$, and some $y_1, \ldots y_N \in X$ and some $r_1, \ldots, r_N > 0$.

Therefore, it suffices to show that for each $j \in \mathbb{N}$, $y \in X$ and $r > 0$,

$$V := \{\{x_k\}_{k\in\mathbb{N}} \ : x_j \in B_r(y)\}$$

id open in the metric topology. Fix any $\{x_k\}_{k\in\mathbb{N}} \in V$. For all $\{z_k\}_{k\in\mathbb{N}} \in X^{\mathbb{N}}$, if

$$d(\{x_k\}_{k\in\mathbb{N}}, \{z_k\}_{k\in\mathbb{N}}) < s$$

then certainly

$$d(x_j, z_j) \leq 2^j s \ .$$

So as long as $s < 2^{-j}[r - d(x_j, y_j)]$, $\{z_k\}_{k\in\mathbb{N}}) \in V$. Thus, V contains the open ball of radius $2^{-j}[r - d(x_j, y_j)]$ about $\{x_k\}_{k\in\mathbb{N}}) \in V$, and this shows that V is open in the metric topology.

(c) This follows easily by a "Cantor diagonal" argument.

(d) By **(c)**, $X^{\mathbb{N}}$ is sequentially compact in the metric topology, and hence it is compact in the metric topology. But then it is compact in any weaker topology. By **(a)**, it is compact in the product topology.

11. Let (X, d_X) and (Y, d_Y) be two compact metric spaces. Let $\mathcal{C}(X \times Y, \mathbb{R})$ be the set of all valued functions on $X \times Y$ continuous real that are continuous with respect to the product topology. Let $\mathcal{A}$ be the set of functions f on $X \times Y$ of the form

$$f(x, y) = \sum_{j=1}^{n} g_j(x)h_j(y)$$

for some $n \in \mathbb{N}$ and some $\{g_1, \ldots, g_n\} \subset \mathcal{C}(X, \mathbb{R})$ and some $\{h_1, \ldots, h_n\} \subset \mathcal{C}(Y, \mathbb{R})$. Show that $\mathcal{A}$ is dense in $\mathcal{C}(X \times Y, \mathbb{R})$ in the uniform topology. (Note: The usual notation for $\mathcal{A}$ is $\mathcal{C}(X, \mathbb{R}) \otimes \mathcal{C}(Y, \mathbb{R})$, and it is called the tensor product of $\mathcal{C}(X, \mathbb{R})$ and $\mathcal{C}(Y, \mathbb{R})$.)

SOLUTION It is easy to check that $\mathcal{A}$ is an algebra that separates points and included the constant functions. The result then follows immediately from the Stone-Wierstrass theorem.

12. A topological space is *locally compact* in case every point has a neighborhood whose closure is compact. Let (X, d_X) and (Y, d_Y) be locally compact metric spaces, and suppose that $f : X \to Y$ is continuous and bijective. Show that f^{-1} is continuous if and only if $f^{-1}(K)$ is compact for all compact $K \subset Y$.

SOLUTION Let $g : Y \to X$ denote f^{-1}. Suppose that g is continuoust. Since the image of a compact set under a continuous function is always compact, $g(K)$ is compact in X for all compact $K \subset Y$. For this part, local compactness plays no role.

To get an idea of how to approach the converse, let us first consider a special case: Suppose that $X$and Y are *compact* metric spaces. Then the closure of *every* neighborhood is compact, so this is certinaly a special case that we have to be able to treat.

To do so, fix any open set V in X. We must show that $f(V)$ is open in Y. let $C = V^c$, this is a closed, and hence compact subset of X. since the image of a compact set under a continuous function is compact, $f(C)$ is compact, and hence closed. But $f(V) = [f(C)]^c$, which shows $f(V)$ to be open.

To do the general case, we will have to use the local compactness to restrict to the conisderation of open sets V whose closure is compact, and then to "cut down" both X and Y to open sets containing V and $f(V)$ respectively, whose closeures are compact. In the proof bellow, these "cut down" replacements for X and Y are the sets $g(W)$ and W.

Now we turn to the general case. Suppose that $g(K)$ is compact in X for all compact $K \subset Y$. Let U be any open set in X. We must show that $f(U)$ is open in Y. Since X is locally compact, each $x \in U$ is contained in an open subset V_x of U such that the closure of V_x, $\overline{V_x}$, is compact. Since $f(U) = \cup_{x \in U} f(V_x)$, it suffices to show that whenever $V \subset X$ is open and $\overline{V}$ is compact, then $f(V)$ is open. Consider such an open set V.

Since f is continuous, $f(\overline{V})$ is compact, and so $f(\overline{V})$ may be covered by finitely many open sets whose closures are compact. Let W be the union of these finitely many open sets. Then $\overline{W}$ is compact, and

$$f(V) \subset f(\overline{V}) \subset W \subset \overline{W} \ .$$

Moreover, $g(\overline{W})$ is the union of finitely many images of compact sets, so $g(\overline{W})$ is compact, and

$$V \subset g(W) \subset g(\overline{W}) \ .$$

Let $C = g(\overline{W}) \cap V^c$, which is a closed, and being contained in a compact set, it is compact. Since f is continuous, $f(C)$ is compact. But $f(C) = \overline{W} \cap f(V^c)$, and hence , using $f(V) \subset W$,

$$f(V) = \overline{W} \cap [f(C)]^c = W \cap [f(C)]^c \ ,$$

and this displays $f(V)$ as the intersection of two open sets.

13. Let $(X, \mathcal{O})$ be a compact topological space. Let A and B be non-empty closed and disjoint subsets of X. Suppose that for every $b \in B$, there exist a continuous function $f_b : X \to [0, 1]$ such that $f_b(b) = 1$ and $f_b(a) = 0$ for all $a \in A$. Show that there exist open sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

SOLUTION For each $g\ inB$, let f_B be a function as given. Define $V_x = f_b^{-1}((2/3\infty))$, and note that V_b is open and contains b. Hence $B \subset \cup_{b \in B} V_b$, and since B is compact, there exist a finite subset $\{b_1, \ldots, b_N\}$ such that

$$B \subset \cup_{j=1}^N V_{b_j} \ .$$

Now define

$$U_j = f_{b_j}^{-1}((-\infty, 1/3) \ .$$

Then for each j, U_j is open, $A \subset U_j$ and $U_{b_j} \cap U_j = \emptyset$. Now define $A = \cap_{j=1}^N U_j$ and $V = \cup_{j=1}^N U_{b_j}$. Then U and V are open, $A \subset U$, $B \subset V$, and $V \cap U = \emptyset$.

14. Let $(X, \mathcal{O})$ be a compact topological space, and let $\mathcal{F}$ be a set of functions real valued on X that is equicontinuous and uniformly bounded. Define

$$g(x) = \sup_{f \in \mathcal{F}} f(x) \ .$$

Is $g(x)$ necessarily continuous? Prove that your answer is correct.

SOLUTION The Arzela-Ascoli Theorem says that $\mathcal{F}$ is compact in the uniform metric. Fix $n \in \mathbb{N}$. There exists a finite set $\{f_1, \ldots, f_M\} \subset \mathcal{F}$ such that

$$\mathcal{F} \subset \bigcup_{j=1}^{M} B_{1/n}(f_j) \ .$$

Then define

$$g_n(x) = \max\{\ f_1(x), \ldots, f_M(x)\} \ .$$

Then g_n is continuous (since we are taking the maximum over a finite set), and clearly for all x,

$$g_n(x) \leq g(x) \leq g_n(x) + 1/n \ .$$

It follows that the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges to g uniformly, and thus g is continuous.

15. Let (X, d_X) and (Y, d_Y) be metric spaces with Y complete. For $L \in (0, \infty)$, a function $f : X \to Y$ is L-Lipschitz in case $d_Y(f(x), f(y)) \leq L d_X(x, y)$ for all $x, y \in X$.

Let S be a dense subset of X. Let $g : S \to Y$ satisfy

$$d_Y(g(x), g(y)) \leq L d_X(x, y) \quad \text{for all} \quad x, y \in S \ .$$

Show that there exists a unique L-Lipschitz function $f : X \to Y$ such that the restriction of f to S is g.

SOLUTION Let $x \in X$, Suppose there is such an extension of g. Let $\{s_n\}$ be any sequence in S that converges to x. Then

$$L d_X(x, s_n) = d_Y(f(x), f(s_n)) = d_Y(f(x), g(s_n)) \ .$$

Since the left side goes to 0 as $n \to \infty$, $\lim_{n\to\infty} d_Y(f(x), g(s_n))$, so if there is any such extension, it must be given by

$$f(x) = \lim_{n\to\infty} g(s_n) \ .$$

We now show that this formula does define such an extension. The first issue is to show that f does not depend on the choice of the approximation sequence.

Let $\{s_n\}$ and $\{\tilde{s}_n\}$ be any two sequence in S that converges to x. Then both are Cauchy in X. Then

$$d_Y(g(s_n), g(s_m)) \leq L d_X(s_n, s_m) \ ,$$

so that $\{g(s_n)\}_{n\in\mathbb{N}}$ is Cauchy in Y, and the same reasoning applies to $\{g(tildes_n)\}_{n\in\mathbb{N}}$. Since Y is complete (or taking the completion of Y to do the extension), $\{g(s_n)\}_{n\in\mathbb{N}}$ converges to an element $y\ inY$. Then since $d_Y(\tilde{s}_n, s_n) \leq L d_X(\tilde{s}_n, s_n)$, $\{\tilde{g}(s_n)\}$ converges to the same limit y, no matter what the approximating sequence is. Thus we may define $f(x) = y$.

Now consider any $x_1, x_2 \in X$. and take $\{s_n\}$ and $\{t_n\}$ converging to x_1 and x_2 respectively. Then

$$d_Y(f(x_1), f(x_2)) = \lim_{n\to\infty} d_Y(f(s_n), f(t_n)) \leq L \limsup_{n\to\infty} d_X(s_n, t_n) = L d_X(x_1, x_2) \ .$$

Thus, the extended function is L-Lipschitz.

16. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous real valued functions on $[0,1]$ that are continuously differentiable on $(0,1)$. Suppose that $f_n(0) = 0$ for all n and that there is a continuous function $g : [0,1] \to [0,\infty)$ such that $|f_n'(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and all $x \in (0,1)$. Show that there exists a uniformly convergent subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$.

SOLUTION Note that for all n, and all $x < y \in [0,1]$,

$$|f(y) - f(x)| = \left|\int_x^y f'(t)\mathrm{d}t\right| = \int_x^y g(t)\mathrm{d}t \ .$$

Since g is continuous, for some L, $g(t) \leq L$ for all $t \in [0,1]$. Then we have that

$$|f_n(y) - f_n(x)| \leq L|y - x| \ ,$$

for all n.

Moreover, since $f_n(0) = 0$, $|f_n(x)| \leq L$ for all x and n. Hence the functions $\{f_n\}_{n\in\mathbb{N}}$ are uniformly bounded (and hence pointwise bounded) and equicontinuous, and the result now follows from the Arzela-Ascoli Theorem.