Solutions forTest One, Math 501 Fall 2014

October 24, 2014

1. Let (X, d) be a metric space. Let $\{U_k\}_{k\in\mathbb{N}}$ be a sequence of open dense subsets of X. For each $n \in \mathbb{N}$ define $V_n = \cap_{k=1}^n U_k$. Show that each V_n is open and dense.

SOLUTION Let U and V be two open dense subsets of X. We claim that $U \cap V$ is open and dense. It is open since the finite intersection of open sets is open.

To see that is is dense, we must show that for all $x \in X$ and all $\epsilon > 0$,

$$B_\epsilon(x) \cap U \cap V \neq \emptyset \ . \tag{0.1}$$

Pick x and $\epsilon > 0$. Since U is dense, we have $B_\epsilon(x) \cap U \neq \emptyset$, and $B_\epsilon(x) \cap U$ is open, being the intersection of two open sets. Hence for any $y \in B_\epsilon(x) \cap U$ there is an $r > 0$ so that $B_r(y) \subset B_\epsilon(x) \cap U$, and then since V is dense, there exists $z \in B_r(y) \cap V$. But since $B_r(y) \subset B_\epsilon(x) \cap U$, $z \in B_\epsilon(x) \cap U \cap V$. This proves (0.1).

From here, the rest is a simple induction: Assuming, for $n \geq 2$ that V_{n-1} is open and dense, we apply the above with V_{n-1} in place of V and U_n in place of U to conclude that V_n is open and dense.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $\{f_n\}$ be a sequence of real valued measurable (with respect to $(\mathbb{R}, \mathcal{B}_\mathbb{R})$) functions on X. Suppose that there is a set $E \in \mathcal{M}$ with $\mu(E) = 0$, and for all $x \in E^c$,

$$\sup_{n\in\mathbb{N}} |f_n(x)| < \infty \ .$$

Show that for every $\epsilon > 0$, there is a constant $C_\epsilon < \infty$ such that

$$\mu\left(\{x \ : \ \sup_{n\in\mathbb{N}} |f_n(x)| > C_\epsilon \ \}\right) < \epsilon \ .$$

Is this still true without the assumption that $\mu(X) < \infty$? Either prove the extension or provide a counterexample.

SOLUTION This is a "continuity from above" problem. Define

$$E_n = \{x \ : \ \sup_{n\in\mathbb{N}} |f_n(x)|\} \ .$$

Since $\sup_{n\in\mathbb{N}} |f_n$ is a measurable function, E_n is a measurable set. Note that $E_{n+1} \subset E_n$ for all n, and since $\sup_{n\in\mathbb{N}} |f_n(x)| < \infty$, $\cap_n E_n = \emptyset$. Since $\mu(E_1) \leq \mu(X) < \infty$, continuity from above gives

$$\lim_{n\to\infty} \mu(E_n) = 0 \ .$$

Now given any $\epsilon > 0$, let n_ϵ be such that $\mu(E_{n_\epsilon}) < \epsilon$. Then we have the desired inequality with $C_\epsilon = n_\epsilon$.

The hypothesis that $\mu(X) = \infty$ is essential. Let $X = \mathbb{N}$, $\mathcal{M} = 2^{\mathbb{N}}$ and let μ be counting measure. Take $f_n(j) = j$ for all n and j. Then $\sum_n f_n(j) = j < \infty$ for all j. But for all $C > 0$, the measure of the set on which $\sup_n f_n$ exceeds C is infinite.

3. Let $\{f_n\}$ be a sequence in $L^1(X, \mathcal{M}, \mu)$ such that $f_n \to f$ pointwise, and such that for some $g \in L^1(X, \mathcal{M}, \mu)$, $|f_n| \leq g$ for all n. Then for every $\epsilon > 0$, there is a set $E_\epsilon \in \mathcal{M}$ such that $\mu(E_\epsilon) \leq \epsilon$ and such that $f_n \to f$ uniformly on the complement of E_ϵ.

SOLUTION The statement to be proved is very much like that of Egoroff's Theorem. We must use the domination by g to eliminate the requirement that $\mu(X) < \infty$ from Egoroff's Theorem.

Proceeding as in the proof of Egoroff's Theorem, from $m, k \in \mathbb{N}$, define

$$E_{m.k} = \cup_{n \geq m} \{x \ : \ \mathcal{F}_n(x) - f(x)| > 1/k \ \} \ .$$

The for all k and n, $E_{n+1,k} \subset E_{n,k}$ and $\cap_{n \geq 0} E_{n,k} = \emptyset$. We need to show that

$$\mu(E_{1,k}) < \infty$$

for all $k \in \mathbb{N}$ and then we can apply continuity from above.

To do this note that $|f| = \lim_{n\to\infty} |f_n| \leq g$, and then for all n,

$$|f - f_n| \leq 2g \ .$$

Therefore $2kg \geq 1_{E_{1,k}}$ and hence

$$\mu(E_{1,k}) = \int_X 1_{E_{1,k}} \mathrm{d}\mu \leq 2k \int_X g \mathrm{d}\mu < \infty \ .$$

With this done, applying continuity from above, $\lim_{n\to\infty} \mu(E_{n,k}) = 0$. We now proceed as with Egoroff's Theorem: Pick $\epsilon > 0$, and choose n_k so that $\mu(E_{n,k}) < 2^{-k}\epsilon$. Let $E = \cup_{k=1}^\infty E_{n_k.k}$. Then $\mu(E) < \epsilon$, and for all k, $E^c \subset E^c_{n_k.k}$ so that for all $x \in E^c$,

$$n > n_k \quad \Rightarrow \quad |f_n(x) - f(x)| < 1/k \ ,$$

which means that f_n converges to f uniformly on E^c.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of non–negative measurable functions. Suppose that $f_n(x) \geq f_{n+1}(x)$ for all n and x, and let f be the pointwise limit of $\{f_n\}$.

Does it follow that $\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu$? If so, prove it, and otherwise give a counterexample.

SOLUTION This is false. Let $X = \mathbb{N}$, $\mathcal{M} = 2^{\mathbb{N}}$ and let μ be counting measure. Let f_n be defined by $f_n(x) = 1/n$ for all x, and let f be defined by $f(x) = 0$ for all x. Then $\lim_{n\to\infty} f_n(x) = f(x)$ for all x, but for all n, $\int_X f_n \mathrm{d}\mu = \infty$ and, $\int_X f \mathrm{d}\mu = 0$.

If we were given the additional information that $\int_X f_1 \mathrm{d}\mu < \infty$, then the conclusion would follow, either from the Lebesgue Dominated Convergence Theorem with f_1 as the dominating function, or the Lebesgue Monotone Convergence Theorem applied to the sequence $\{g_n\}$ where $g_n := f_1 - f_n$.