Notes on σ-Algebras, Measures, and Measurable Functions

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December 9, 2014

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1 Introduction

Let f be a continuous real valued functions on the interval $[a,b]$. Since f is necessarily bounded, let us suppose that the range of f is contained in $[c,d]$. Riemann's approach to defining the integral is based on partitioning the domain $[a,b]$ into small pieces, and Lebesgue's approach is based on partitioning the range $[c,d]$ into small pieces.

It was not obvious, even to Lebesgue, just how advantageous the latter approach could be, until the theory was fully developed. Lebesgue's original motivation was pedagogical, and was based on his observation that the way French shopkeepers tallied up the day's sales was to arrange the coins in the till into piles according to their denomination, then to count the coins in each pile, multiply by the value of the denomination, and then to sum the results for each denomination.

In more mathematical terms, suppose we have a finite sequence $\{a_1, \ldots, a_N\}$ where each a_k takes its value in the discrete set $\{v_1, \ldots, v_m\}$. For $j = 1, \ldots, M$, define

$$I_j = \{ \ k \ : \ a_k = v_j \}$$

and define $m(I_j)$ to be the cardinality of this set. Then

$$\sum_{k=1}^{N} a_k = \sum_{j=1}^{M} v_j m(I_j) \ . \tag{1.1}$$

Directly adding up all of the terms on the left, one is computing the sum *a la Riemann*. Using the formula on the right, one is computing the sum *a la Lebesgue*. This is all very simple in a discrete setting, but it becomes more interesting in the context of integrals of continuous functions. What is more important is that the Lebesgue approach opens the way to a theory of integration for a broad class of integrands (measurable functions in place of continuous functions) in which simple theorems give conditions under which limits may be "taken under the integral sign".

1.1 The Lebesgue integral for continuous functions

We begin by explaining how to apply (1.1) to the integration of *continuos* functions. Let f be a continuous real valued function on $[a, b]$. Let us assume that its range lies in the interval $[c, d]$.

Let $\{y_1, \ldots, y_n\}$ be any finite sequence of numbers with

$$c < y_1 < \cdots < y_n < d \ .$$

Define $V_j = f^{-1}(y_j, \infty)$, so that each V_j is open and $V_j \subset V_{j-1}$ for $j = 1, \ldots, n-1$. Define $V_0 = [a, b]$, $V_{n+1} = \emptyset$, $y_0 = c$ and $y_{n+1} = d$. Recall that every open set in $[a, b]$ is a countable disjoint union of open intervals. For each open set $W \subset [a, b]$, define $m(W)$ to be the sum of the lengths of these disjoint intervals. We may think of $m(W)$ as representing the *total length* of W.

For $n = 0, \ldots, n$, define $E_n = V_n \backslash V_{n+1}$. Then the sets $\{E_1, \ldots E_n\}$ *partition* $[a, b]$. That is, they are mutually disjoint, and their union is $[a, b]$.

By construction, for $x \in E_j$, $x \in E_j = V_j \backslash V_{j+1}$, so $y_j \leq f(x) < y_{j+1}$. That is, for all $n = 0, \ldots, n$,

$$y_n 1_{E_n} \leq f 1_{E_n} \leq y_{n+1} 1_{E_n} \ .$$

Define the function φ by

$$\varphi(x) = \sum_{j=0}^{n} y_j 1_{E_j}(x) \ .$$

By what we have said above,

$$\varphi(x) \leq f(x) \leq \varphi(x) + \max\{y_{j+1} - y_j \ : \ j = 0, \ldots, n \ \} \ .$$

We are now in a position to adapt (1.1) to a continuous setting. The function φ has only finitely many values, $\{y_0, \dots, y_n\}$. The set on which φ takes on the value y_j is $E_j = V_j \backslash V_{j+1}$. Since $V_{j+1} \subset V_j$, it is natural to define the total length of E_j, $m(E_j)$, by

$$m(E_j) = m(V_j) - m(V_{j+1}) \ .$$

Then we define the integral of φ over $[a, b]$, $\int_{[a,b]} \varphi \mathrm{d}m$ to be

$$\int_{[a,b]} \varphi \mathrm{d}m = \sum_{j=1}^{M} y_j m(E_j) \tag{1.2}$$

where $m(E_j)$ is the total length of E_j, as defined above.

At this point we elaborate the notation for φ to record the finite sequence $\{y_1, \dots, y_n\}$ that generated it, and denote it by $\varphi_{\{y_1,\dots,y_n\}}$.

We may then define

$$\int_{[a,b]} f(x)\mathrm{d}m = \sup_{n\in\mathbb{N}, c<y-1<\cdots<y_n<d} \left\{ \int_{[a,b]} \varphi_{\{y_1,\dots,y_n\}} \mathrm{d}m \right\} \ .$$

This construction will give the correct value for $\int_{[a,b]} f(x)\mathrm{d}m$, but it is not *exactly* how the Lebesgue integral is defined. The definition will be much easier to work with – and will apply to a much broader class of integrands f – because before making it, we first extend the total length function $m(E)$ to a wider class of sets than simply differences of nested open sets. Once this is done, we can make sense of (1.2) in a very broad setting. As long as $m(E)$ is only defined when E is a difference of nested open sets, we will not able to integrate much more than continuous functions, which is what we already know how to do, using the Riemann integral.

So it is natural to ask whether we can extend the total length function to *all* subsets of $[a, b]$ in any reasonable answer. The answer is "no".

1.2 The Existence of Non-measurable Sets

Let $X = [0, 1)$ and equip it with the abelian group structure

$$x + y \bmod 1 = \begin{cases} x + y & x + y < 1 \\ x + y - 1 & x + y \geq 1 \end{cases} \ .$$

Note that $x + (1 - x) \bmod 1 = 0$, so $1 - x$ is the additive inverse of x for this group structure. We write

$$x - y \bmod 1 = x + (1 - y) \bmod 1 \ .$$

If $A \subset X$ and $x \in X$, we define

$$A + x = \{x + y \bmod 1 \ : y \ \in A\} \ .$$

It is clear that if A is a finite disjoint union of intervals, then so is $A + x$, and the total length of the intervals is the same for A and $A + x$. That is, the total length function $m(A)$, defined on sets A that are finite unions of disjoint intervals, is invariant under the operation $A \mapsto A + x$, which amounts to "translation by x with periodic boundary conditions".

1.1 THEOREM. *There does not exist a* $[0,1]$ *valued extension of the total length function* $m(A)$ *from the set of finite disjoint unions of intervals in* X *to the set of all subsets of* X *such that*
(1) $m(A+x) = m(A)$ *for all* $A \subset X$ *and all* $x \in X$*.*
(2) If $\{A_n\}_{n\in\mathbb{N}}$ *is any sequence of mutually disjoint sets in* X*,*

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)\ .$$

Proof. Define an equivalence relation $\sim$ on X by $x \sim y$ in case

$$x - y \bmod 1 \in \mathbb{Q}\ .$$

It is easily checked that this is an equivalence relation. Now define A to be a set containing exactly one representative of each equivalence class.

We now claim that for each $r \in (0,1)\cap\mathbb{Q}$, $(A+r)\cap A = \emptyset$. To see this, suppose $x \in (A+r)\cap A$. Then x and $x - r$ mod 1 both belong to A. But

$$x - (x-r) \bmod 1 = r\ ,$$

so that $x \sim x - r$ mod 1. By construction, A contains exactly one element in each equivalence class, and hence this is impossible.

It follows that for $r \neq q$, $r < s$ in $(0,1)\cap\mathbb{Q}$,

$$(A+r)\cap(A+s) = (A \cap A + (s-r)) + r = \emptyset\ .$$

We now claim that

$$\bigcup_{r\in[0,1)\cap\mathbb{Q}} A + r = [0,1)\ .$$

To see this, fix $y \in [0,1)$. Let $z \in A$ be the element of A that is equivalent to y. By definition,

$$y - z \bmod 1 = r \in [0,1)\cap Q\ .$$

But then $y \in A + r$.

Now suppose that there does exist a function m defined on the set of all subsets of $[0,1)$ that satisfies *(1)* and *(2)* and such that $m(A)$ is the total length of A when A is a finite disjoint union of intervals. Then

$$1 = m([0,1)) = m\left(\bigcup_{r\in[0,1)\cap\mathbb{Q}} (A+r)\right) = \sum_{r\in[0,1)\cap\mathbb{Q}} m(A+r) = \sum_{r\in[0,1)\cap\mathbb{Q}} m(A)\ .$$

This is impossible: If $m(A) = 0$, the sum on the right is zero. But if $m(A) \neq 0$, the sum on the right is infinite. □

We are left with two choices: We could give up either properties *(1)* or *(2)*, and try to forge ahead defining $m(a)$ for all subsets A of $[0,1)$, or we can retain properties *(1)* and *(2)*, and try to define the extension of the total length function on a smaller domain that the set of all subsets of $[0,1)$.

The second choice is the one that leads to a useful theory. We will develop this theory at first in an abstract setting in which there is no group structure, and so that translation invariance is not meaningful, but we shall require that whenever $\{A_m\}_{n\in\mathbb{N}}$ is any *disjoint* sequence of of sets, each with a well-defined "mass" $m(A_n)$, then $\bigcup_{n=!}^\infty A_n$ has a well defined "mass" $m(\bigcup_{n=!}^\infty A_n))$ and

$$m\left(\bigcup_{n=!}^\infty A_n\right) = \sum_{n=1}^\infty m(A_n) \ . \tag{1.3}$$

That is: *The "mass of whole is the sum of the mass of the parts" whenever the mass of the the parts are defined, and the whole is divided into countably many parts.* The restriction to countable unions is essential since $[0,1) = \bigcup_{x\in[0,1)}[x,x]$, and of course the length of $[x,x]$ is zero.

Because we shall require (1.3) to be true whenever $\{A_m\}_{n\in\mathbb{N}}$ is any *disjoint* sequence of of sets, each with $m(A_n)$ well defined, it is natural to require that the domain of our function m that measures the "mass" or "volume" of sets should be closed under countable disjoint unions.

Next, it is natural to require that the whole set X is in the domain of m. Then, if A is in the domain of m, it is natural to require that

$$m(X) = m(A) + m(A^c)$$

which would define $m(A^c)$ if both $m(A)$ and $m(X)$ are finite. Thus we shall require that the domain of our size function m is closed under taking complements. In the next section we study sets of subset of X that are closed under countable disjoint unions and complements. These are the sets of subsets of X on which we shall construct measures of the "mass" or "volume" of sets.

2 Algebras, σ-algebras, and measures

2.1 algebras of sets

2.1 DEFINITION (Algebra of sets)**.** *A set $\mathcal{A}$ of subsets of X is an* algebra *in case it is closed under differences and finite unions. That is, $\mathcal{A}$ is an algebra in case whenever A and B belong to $\mathcal{A}$, so do $A\backslash B$ and $A\bigcup B$.*

Note that if $\mathcal{A}$ is an algebra, and $A, B \in \mathcal{A}$, then

$$A \cap B = B\backslash(B\backslash A) \ ,$$

so algebras are always closed under finite intersections.

The symmetric difference of sets A and B, denoted $A\Delta B$, is defined by

$$A\Delta B = A\bigcup B\backslash(A\cap B) = (A\backslash B)\bigcup(B\backslash A) \ .$$

It follows that algebras are always closed under symmetric differences.

Note the identities

$$A\backslash B = (A\Delta B)\cap A \qquad \text{and} \qquad A\bigcup B = (A\Delta B)\Delta(A\cap B) \ .$$

Thus whenever a set of subsets of X is closed under symmetric differences and finite intersections, it is an algebra.

Let $\mathcal{A}$ be an algebra, and let $A, B \in \mathcal{A}$. Then

$$1_{A\cap B} = 1_A 1_B \qquad \text{and} \qquad 1_{A\Delta B} = 1_A + 1_B \text{ mod } 2\ .$$

Thus, a set $\mathcal{A}$ of subsets of X is an algebra if and only if the indicator functions of sets in $\mathcal{A}$ form an algebra over the field $\mathbb{Z}_2$ in the usual algebraic sense when this set is equipped with the usual multiplication of functions and addition of functions mod 2. This algebra has a multiplicative identity if and only if $X \in \mathcal{A}$.

Note that if $\mathcal{A}$ contains X, then $\mathcal{A}$ is closed under complements since $A^c = X\backslash A$. This is sometimes required in the definition of an algebra, but it will be convenient to distinguish between algebras containing X and those that do not. By what we have just explained, amounts to distinguishing between algebras of sets containing a multiplicative identity, and algebras of sets that do not.

We now introduce a special class of algebras, the class of σ-algebra. As we shall see, this is the "right" class of sets on which to attempt to define measures of the "mass" or "volume" of sets.

2.2 σ-algebras of sets

2.2 DEFINITION (σ-algebra of sets)**.** *A set $\mathcal{M}$ of subsets of X is a* σ-algebra *in case it is an algebra containing X that is closed under countable unions. A* measurable spaces *is a pair $(X, \mathcal{M})$ where $\mathcal{M}$ is a σ-algebra of subsets of X.*

Note that if $\mathcal{M}$ is a σ-algebra, then it is closed under complements and countable unions. Conversely, if $\mathcal{M}$ is closed under countable unions and complements, then it is a σ-algebra since if $A, B \in \mathcal{M}$, $A\backslash B = A \cap B^c$, so it is an algebra containing X that is closed under countable unions. Thus, $\mathcal{M}$ is a σ-algebra if and only if it is closed under complements and countable unions. This is often taken as the definition of a σ-algebra.

For examples of σ-algebras, let X be any set and let $\mathcal{M} = \{\emptyset, X\}$. This is clearly a σ-algebra, but is a trivial sort of example. Another trivial example is given by $\mathcal{M} = 2^X$, the set of all subsets of X. The σ-algebras that we shall be interested in generally lie between these two extremes. The σ-algebras that we are interested in are those that are the domains of interesting *measures*, as defined in the following subsection.

2.3 Measures

2.3 DEFINITION (Measure)**.** *Let $(X, \mathcal{M})$ be a measurable space. A* measure *μ on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that*

(1) $\mu(\emptyset) = 0$.

(2) If $\{E_n\}_{n\in\mathbb{N}}$ is an sequence of mutually disjoint sets belonging to $\mathcal{M}$,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)\ . \tag{2.1}$$

The property (2) is referred to as countable additivity*. A* measure space *is a triple $(X, \mathcal{M}, \mu)$ where $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure defined on $\mathcal{M}$.*

Foe example, fix $x_0 \in X$. Let $\mathcal{M}$ be any σ-algebra of subsets in X and for $E \in \mathcal{M}$, define $\mu(E) = 1$ in case $x_0 \in E$ and $\mu(E) = 0$ in case $x_0 \notin E$. It is east to see that this is a measure on $\mathcal{M}$, and though it is actually a useful example, it is very simple. Here is another very simple example:

2.1 EXAMPLE (Counting measure)**.** *Let X $\mathbb{N}$, and $\mathcal{M} = 2^{\mathbb{N}}$. For $E \subset \mathbb{N}$, define $\mu(E)$ to be the cardinality of E if E i a finite set, and define it to be ∞ otherwise. It is easy to check that this is a measure. Later we shall construct more interesting examples, but this simple example suffices to show that at least the definition is not empty.*

The definitions of σ-algebras and measures go hand-in-hand. Note for example that since σ-algebras are closed under complementation, $\mu(E^c)$ is defined whenever $\mu(E)$ is defined; i.e., whenever $E \in \mathcal{M}$. Next, countable additivity implies additivity: Simply take all but finitely many of the E_n in *(2)* to be the empty set, and then use *(1)* to eliminate these terms from the sum on the right. Hence, whenever $\mu(E)$ is defined, so is $\mu(E^c)$, and we have

$$\mu(X) = \mu(E) + \mu(E^c) \ ,$$

where, if $\mu(X) = \infty$, then at least one of $\mu(E)$ and $\mu(E^c)$ must be infinite also. However, if $\mu(X) < \infty$, then we have $\mu(E^c) = \mu(X) - \mu(E)$, as we would expect of a decent measure of the "mass" of subsets of X.

The following theorem gives several important properties of measures that follow directly from the definition.

2.1 THEOREM (Properties of measures)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then*

(1.) *(Monotonicity) If $E, F \in \mathcal{M}$, $E \subset F$, then $\mu(E) \le \mu(F)$.*

(2.) *(Subadditivity) $\{E_n\}_{n\in\mathbb{N}}$ is any sequence of sets in $\mathcal{M}$,*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} \mu(E_n) \ .$$

(3.) *(Continuity from below) If $\{E_n\}_{n\in\mathbb{N}}$ is any sequence of sets in $\mathcal{M}$ with $E_n \subset E_{n+1}$ for all n, then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) \ . \tag{2.2}$$

(4.) *(Continuity from above) If $\mu(X) < \infty$, and $\{F_n\}_{n\in\mathbb{N}}$ is any sequence of sets in $\mathcal{M}$ with $F_{n+1} \subset F_n$ for all n, then*

$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n) \ . \tag{2.3}$$

Proof. For *(1.)*, note that if $E \subset F$, $E, F \in \mathcal{M}$, then $F \backslash E \in \mathcal{M}$ and $F = E \bigcup (F \backslash E)$ is a disjoint union of sets in $\mathcal{M}$. Then since countable additivity implies additivity,

$$\mu(F) = \mu(E) + \mu(F \backslash E) \ge \mu(E) \ .$$

For *(2.)*, define $F_1 = E_1$ and recursively define $F_n = E_n \backslash (E_1 \bigcup \cdots \bigcup E_{n-1})$. Then $F_n \in \mathcal{M}$ for all n, and $\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty F_n$, and the latter union is disjoint by construction. Hence, by countable additivity and monotonicity,

$$\mu\left(\bigcup_{n=1}^\infty E_n\right) = \sum_{n=1}^\infty \mu(F_n) \leq \sum_{n=1}^\infty \mu(E_n) \ .$$

For *(3.)*, define $F_1 = E_1$ and for $n > 1$, $F_n = E_n \backslash E_{n-1}$. Then each $F_n \in \mathcal{M}$, and $\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty F_n$, and this last union is disjoint by construction. By countable additivity,

$$\mu\left(\bigcap_{n=1}^\infty E_n\right) = \mu\left(\bigcap_{n=1}^\infty F_n\right) = \sum_{n=1}^\infty \mu(F_n) = \lim_{n\to\infty} \sum_{m=1}^n \mu(F_m) = \lim_{n\to\infty} \mu(E_n) \ ,$$

where in the last line we have used the fact that $E_n = \bigcup_{m=1}^m (F_m)$ and that μ is additive.

For *(4.)*, define $E_n = F_n^c$. Then $\{E_n\}_{n\in\mathbb{N}}$ is an increasing nested sequence of measurable sets, and then by continuity from below, and additivity,

$$\mu\left(\bigcup_{n=1}^\infty E_n\right) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} [\mu(X) - \mu(F_n)] = \mu(X) - \lim_{n\to\infty} \mu(F_n) \ .$$

Since $\mu(X) < \infty$,

$$\lim_{n\to\infty} \mu(F_n) - \mu(X) - \mu\left(\bigcup_{n=1}^\infty F_n^c\right) = \mu(X) - \mu\left(\left(\bigcap_{n=1}^\infty F_n\right)^c\right) = \mu\left(\bigcap_{n=1}^\infty F_n\right) \ .$$

□

2.2 EXAMPLE (Necessity of finite measure for continuity from above)**.** *Let* $X = \mathbb{N}$*,* $\mathcal{M} = 2^{\mathbb{N}}$*, and let* μ *be counting measure. Let* $F_n = \{k \in \mathbb{N} \ : \ k \geq n \ \}$*. Then* $\mu(F_n) = \infty$ *for all* n*, but* $\cap_{n=1}^\infty F_n = \emptyset$*. Hence it is not the case that* $\mu(cap_{n=1}^\infty F_n) = \lim_{n=1}^\infty \mu(F_n)$*.*

Though the two concepts, that of measure, and that of σ-algebra, are conjugal twins, we will spend the next several sections studying σ-algebras alone, before returning to measures defined upon them. It will be quite some time before we actually construct any really interesting measures, such as Lebesgue measure. Before we do that, we shall thoroughly develop the *abstract theory* of measure and integration, showing that whenever one has constructed a measure space $(X, \mathcal{M}, \mu)$, there is a very effective theory of integration on it with a well developed set of theorems for taking limits under integral signs. This will show that constructing measures that satisfy the definitions given here is a worthwhile enterprise, and only then shall we undertake the construction of Lebesgue measure.

3 Generated σ-algebras

Let I be an arbitrary set. For each $\alpha \in I$, let $\mathcal{M}_\alpha$ be a σ-algebra. Let

$$\mathcal{M} = \bigcap_{\alpha\in I} \mathcal{M}_\alpha \ .$$

Then if $A \in \mathcal{M}$, $A \in \mathcal{M}_\alpha$ for all α. Since $\mathcal{M}_\alpha$ is closed under complements, $A^c \in \mathcal{M}_\alpha$ for all α, and hence $A^c \in \mathcal{M}$. Thus, $\mathcal{M}$ is closed under complements. The same sort of reasoning shows that $\mathcal{M}$ is closed under countable unions, and then by the remarks made above, $\mathcal{M}$ is a σ-algebra.

This has an important consequence: Let $\mathcal{E}$ be any set of subsets of X. Then the power set 2^X is a σ-algebra containing $\mathcal{E}$, and hence the set of σ-algebras containing $\mathcal{E}$ is not empty. The intersection over the set of all σ-algebras containing $\mathcal{E}$ is therefore a σ-algebra containing $\mathcal{E}$ that is contained in every other σ-algebra containing $\mathcal{E}$: In this sense it is the *smallest* σ-algebra containing $\mathcal{E}$.

3.1 DEFINITION (Generated σ-algebras)**.** *for any set $\mathcal{E}$ of subsets of X, $\sigma(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$.*

3.1 Borel σ-algebras

3.2 DEFINITION (Borel σ-algebra)**.** *Let $(X, \mathcal{O})$ be a topological space. The Borel σ-algebra $\mathcal{B}_X$ is σ-algebra containing*
mathcalO. That is, $\mathcal{B}_X = \sigma(\mathcal{O})$.

The same σ-algebra can be generated by different sets $\mathcal{E}$ of generators. It is often useful to identify "small" sets of generators, and the following theorem is useful in this regard.

3.1 THEOREM. *Let $\mathcal{E} \subset \mathcal{F} \subset 2^X$. If $\mathcal{F} \subset \sigma(E)$, then $\sigma(\mathcal{E}) = \sigma(\mathcal{F})$.*

Proof. It is evident that $\sigma(\mathcal{E}) \subset \sigma(\mathcal{F})$. On the other hand, $\sigma(\mathcal{E})$ is a σ-algebra containing $\mathcal{F}$. then by definition, $\sigma(\mathcal{F}) \subset \sigma(\mathcal{E})$. □

3.1 EXAMPLE. *Let $\mathcal{E}$ be the set of all subsets of R of the form (a, ∞), $a \in \mathbb{R}$. Then $\sigma(\mathcal{E})$ contains all sets of the form $(a, b]$, $a < b$, since*

$$(a, b] = (a, \infty) \cap (b, \infty)^c \ .$$

Next, $(a, b) = \bigcup_{n=1}^\infty (a, b - 1/n)$. But then every open set in $\mathbb{R}$ is the countable union of disjoint open intervals,and hence $\sigma(\mathcal{E})$ contains all open sets. By Theorem 3.1, $\sigma(\mathcal{E}) = \mathcal{B}_\mathbb{R}$.

4 Measurable functions

4.1 DEFINITION (Measurable function)**.** *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. Then a function $f : X \to Y$ is* measurable *in case*

$$f^{-1}(F) \in M \quad \text{for all} \quad F \in \mathcal{N} \ .$$

If we wish to emphasize the specific σ-algebras, we shall say that f is $\mathcal{M}, \mathcal{N}$ measurable.

4.1 THEOREM. *The composition of measurable functions is measurable. That is, let $(X, \mathcal{M})$, $(Y, \mathcal{N})$ and $(Z, \mathcal{P})$ be measurable spaces let $f : X \to Y$ and $g : YtoZ$ be measurable. Then $g \circ f : X \to Z$ is measurable.*

Proof. Let $H \in \mathcal{P}$. Since g is measurable, $g^{-1}(H) \in \mathcal{N}$. Then since f is measurable, $f^{-1}(g^{-1}(H)) \in \mathcal{M}$. This shows that $(g \circ f)^{-1}(H) \in \mathcal{M}$ for all H in $\mathcal{P}$ so that $g \circ f$ is measurable. □

There is a close parallel between the notion of measurability and continuity. In fact, as we show below, when X and Y are topological spaces, and $\mathcal{M}$ and $\mathcal{N}$ are their respective Borel σ-algebras

4.2 THEOREM. *Let X and Y be sets, and $f : X \to Y$ any function. Let $\mathcal{M}$ be any σ-algebra of sets in X.*
(1) Define $\mathcal{N}$ by

$$\mathcal{N} = \{F \subset Y \ : \ f^{-1}(F) \in \mathcal{M}\ \} \ .$$

Then $\mathcal{N}$ is a σ-algebra of sets in Y.
(2) Let $\mathcal{E} \subset 2^Y$. Then $f : X \to Y$ is $\mathcal{M}, \sigma(\mathcal{F})$ measurable if and only if $f^{-1}(F) \in \mathcal{M}$ for all $F \in \mathcal{F}$.

Proof. If $F \in \mathbb{N}$, $f^{-1}(F^c) = (f^{-1}(F))^c \in \mathcal{M}$, and so $\mathbb{N}$ is closed under complements. Likewise, for any sequence $\{F_n\}_{n\in\mathbb{N}}$ in $\mathcal{N}$,

$$f^{-1}\left(\bigcup_{n=1}^{\infty} F_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(F_n) \in \mathcal{M} \ .$$

Therefore, $\mathcal{N}$ is also closed under countable unions. This proves that $\mathcal{N}$ is a σ-algebra.

For the second part, the condition is obviously necessary. To see that it is sufficient, use the first part to see that $\mathcal{N}$ is a σ-algebra that contains $\mathcal{F}$. Hence $\sigma(\mathcal{F}) \subset \mathcal{N}$, and hence $f^{-1}(F) \in \mathcal{M}$ for all $F \in \sigma(\mathcal{F})$, which shows that f is $\mathcal{M}, \sigma(\mathcal{F})$ measurable. □

4.1 Borel measurability of continuous functions

4.3 COROLLARY. *Let $(X, \mathcal{O})$ and $(Y, \mathcal{U})$ be topological spaces, and let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be their respective Borel σ-algebras. If $f : X \to Y$ is continuous, it is $\mathcal{B}_X, \mathcal{B}_Y$ measurable*

Proof. Since $\mathcal{B}_Y = \sigma(\mathcal{U})$, and since when f is continuous, for all $U \in \mathcal{U}$,

$$f^{-1}(U) \in \mathcal{O} \subset \mathcal{B}_X \ .$$

The assertion now follows from part *(2)* of the previous theorem. □

The corollary shows that as far as Borel measurability is concerned, the class of measurable functions is at least as large as the class of continuous functions. In this sense, the notion of a measurable function constitutes a generalization of the notion of a continuous function.

5 Measurable functions with values in $\mathbb{R}^n$

Let $x = (x_1, \ldots, x_n)$ denote a point in $\mathbb{R}^n$. Let $\mathcal{E}$ denote the set of all subsets of $\mathbb{R}^n$ of the form

$$\{x \ : \ a < x_j < b\ \}$$

for some $a < b$ in $\mathbb{Q}$ and some $j \in \{!, \ldots, n\}$. The σ-algebra $\sigma(\mathcal{E})$ thus contains all sets of the form

$$\{x \ : \ a_j < x_j < b_j, \ j = 1, \ldots, n\ \}$$

where $a_j < b_j$ and $a_j, b_j \in \mathbb{Q}$ for $j = 1, \ldots, n$. Every open set in $\mathbb{R}^n$ is a countable union of such sets, and hence $\mathcal{B}_{\mathbb{R}^n} \subset \sigma(E)$. however, since each set in $\mathcal{E}$ is open, $\sigma(\mathcal{E}) \subset \mathcal{B}_{\mathbb{R}^n}$. Thus,

$$\mathcal{B}_{\mathbb{R}^n} = \sigma(E) \ .$$

5.1 THEOREM. *Let $(X, \mathcal{M})$ be a measurable space. Then a function $f : X \to \mathbb{R}^n$ where*

$$f(x) = (f_1(s), \dots, f_n(x))$$

is $\mathcal{M}, \mathcal{B}_{R^n}$ measurable if and only if each f_j is $\mathcal{M}, \mathcal{B}_{\mathbb{R}}$ measurable.

Proof. Suppose that f is $\mathcal{M}, \mathcal{B}_{R^n}$ measurable. Let $p_j : \mathbb{R}^n \to \mathbb{R}$ be given by

$$p_j((y_1, \dots, y_n)) = y_j \ .$$

Then p_j is continuous, and hence p_j is $\mathcal{B}_{\mathbb{R}^n}, \mathcal{B}_{\mathbb{R}}$ measurable. Now note that $f_j = p_j \circ f$, and then the measurability of f_j is a consequence of Theorem 4.1.

Now suppose that each f_j is measurable. Then for all $a < b$ and $j = 1, \dots, n$,

$$f_j^{-1}((a, b)) = f^{(-1)}(\{x \ : \ a < x_j < b\}) \ .$$

Since $f_j^{-1}((a, b)) \in \mathcal{M}$ by the measurability of f_j, $f^{(-1)}(\{x \ : \ a < x_j < b\}) \in \mathcal{M}$. But by the remarks made at the beginning of this section, the set $\mathcal{E}$ of all sets $\{x \ : \ a < x_j < b\}$, $a, b \in \mathbb{Q}$ and $j = 1, \dots, n$, generates $\mathcal{B}_{\mathbb{R}^n}$. Now the measurability of f follows from Theorem 4.2 □

The usual identification of $\mathbb{R}^2$ and $\mathbb{C}^2$ identifies not only the sets but the topologies, so $\mathcal{B}_{\mathbb{C}} = \mathcal{B}_{\mathbb{R}^2}$. It follows directly that for any measure space $(X, \mathcal{M})$ a function $f : X \to \mathbb{C}$ is $\mathcal{M}, \mathcal{B}_{\mathbb{C}}$ measurable if and only if the real and imaginary parts of f are $\mathcal{M}, \mathcal{B}_{\mathbb{R}}$ measurable.

5.2 THEOREM. *Let $(X, \mathcal{M})$ be an arbitrary measure space, and let $f, g : X \to \mathbb{R}$ be $\mathcal{M}, \mathcal{B}_{\mathbb{R}}$ measurable. Then both fg and $f + g$ are $\mathcal{M}, \mathcal{B}_{\mathbb{R}}$.*

Proof. Consider the function $h : X \to \mathbb{R}^2$ defined by $h(x) = (f(x), g(x))$. Then by Theorem 5.1, h is $\mathcal{M}, \mathcal{B}_{\mathbb{R}^2}$ measurable. But $\Phi : (s, t) \mapsto st$ and $\Psi : (s, t) \mapsto s + t$ are continuos from $\mathbb{R}^2$ to $\mathbb{R}$, and hence are $\mathcal{B}_{\mathbb{R}^2}, \mathcal{B}_{\mathbb{R}}$ measurable. Theorem 4.1 now implies that $fg = \Phi \circ h$ and $f + g = \Psi \circ h$ are $\mathcal{M}, \mathcal{B}_{\mathbb{R}}$ measurable.

□

5.1 Measurability of pointwise limits

5.3 THEOREM. *Let $(X, \mathcal{M})$ be a measurable space. Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of real valued functions on X that are each $\mathcal{M}, \mathcal{B}_{\mathbb{R}}$ measurable.*

(1) The functions

$$f(x) = \inf_{n \in \mathbb{N}} g_n(x)$$

and

$$h(x) = \sup_{n \in \mathbb{N}} g_n(x)$$

are both $M, \mathcal{B}_{\mathbb{R}}$ measurable.

(2) The functions

$$\limsup_{n \to \infty} g_n \qquad \text{and} \qquad \liminf_{n \to \infty} g_n$$

are both measurable.

(3) The set A on which $\lim_{n\to\infty} g_n(x)$ exists is $\mathcal{M},\mathcal{B}_{\mathbb{R}}$ measurable, and

$$g(x) := \lim_{n\to\infty} 1_A(x) g_n(x)$$

is $\mathcal{M},\mathcal{B}_{\mathbb{R}}$ measurable. In particular, if $\lim_{n\to infty} g_n(x)$ exists for all x, then $g(x) := \lim_{n\to infty} g_n(x)$ is $\mathcal{M},\mathcal{B}_{\mathbb{R}}$ measurable.

Proof. For any $a \in \mathbb{R}$, $f(x) > a$ if and only if $g_n(x) > a$ for all n, and hence

$$f^{-1}((a,\infty)) = \bigcap_{n=1}^{\infty} g^{-1}((a,\infty)) \in \mathcal{M}$$

Since the σ-algebra generated by the sets of the form (a,∞) is $\mathcal{B}_{\mathbb{R}}$, it follows from Theorem 4.2 that $f^{-1}(B) \in \mathcal{M}$ for all $B \in \mathcal{B}_{\mathbb{R}}$, and hence f is $M,\mathcal{B}_{\mathbb{R}}$ measurable. The proof for h is entirely analogous.

For *(2)*, note that

$$\limsup_{n\to\infty} g_n = \inf_{m>0}\left(\sup_{n>m} g_n(x)\right) \quad \text{and} \quad \liminf_{n\to\infty} g_n = \sup_{m>0}\left(\inf_{n>m} g_n(x)\right) ,$$

so that *(2)* is an immediate consequence of *(1)*.

For *(3)*, Define $\varphi(x) = \limsup_{n\to\infty} g_n(x) - \liminf_{n\to\infty} g_n(x)$ wich is a measurable function by *(2)* and Theorem 5.2. Then since $A = \varphi^{-1}(\{0\})$, $A \in \mathcal{M}$. Finally, on A,

$$\lim_{n\to\infty} g_n(x) = \limsup_{n\to\infty} g_n(x)$$

which is measurable, so

$$\lim_{n\to\infty} [1_A(x) g_n(x)] = 1_A(x)[\limsup_{n\to\infty} g_n(x)]$$

is a product of measurable functions and hence is measurable by Theorem 5.2 once more.

□

The final part of the theorem says that for an arbitrary measure space $(X,\mathcal{M})$, the class of $\mathcal{M},\mathcal{B}_{\mathbb{R}}$ measurable real valued functions on X is closed under pointwise limits. This will play a crucial role in proving limit theorems for integrals in what follows. Of course it is not true in general that if $(X,\mathcal{O})$ is a topological space, the set of real valued continuous functions on X is closed under pointwise limits.

5.2 Egoroff's Theorem

The following theorem shows an amazing property of pointwise limits of measurable real-valued functions: On set of finite measure, pointwise convergence is "almost" uniform convergence:

5.4 THEOREM (Egoroff)**.** *Ket $(X,\mathcal{M},\mu)$ be a measure space with $\mu(X) < \infty$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued measurable functions on X, and suppose that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. Then for all $\epsilon > 0$, there exists $E \in \mathbb{N}$ such that $\mu(E) < \epsilon$, and such that $f_n \to f$ uniformly on $X\backslash E$.*

Proof. For $m, k \in \mathbb{N}$, define $E_{m,k}$ by

$$E_{m,k} = \bigcup_{n=m}^{\infty} \{ \ x \ : \ |f_n(x) - f(x)| > 1/k \ \}|.$$

Note that for each k, $E_{m+1,k} \subset E_{m,k}$, and $\cap_{n=1}^{\infty} E_{m,k} = \emptyset$. Since $\mu(X) < \infty$, continuity from above implies

$$\lim_{n\to\infty} \mu(E_{m,k}) = 0 \ .$$

Fix $\epsilon > 0$, and choose n_k so that $\mu(E_{n_k}) < \epsilon 2^{-k}$. Let $E = \bigcup_{k=1}^{\infty} E_{n_k,k}$, so that $\mu(E) < \epsilon$. Then, if $x \in E^c$, $x \in E^c_{n_k,k}$ for all k, which means that $|f_n(x) - f(x)| < 1/k$ for all $n \geq n_k$, and hence the convergence is uniform on the complement of E. □

6 σ-algebras generated by algebras

The Borel σ-algebra of $\mathbb{R}^n$ is, by definition, generated by the set $\mathcal{O}$ of open sets in $\mathbb{R}^n$. But it is also generated by an algebra $\mathcal{A}$, namely the *algebra of half-open rectangles.*

We describe this algebra first in the simplest case, $n = 1$. Let $\mathcal{E}$ consist of all sets of the form $(a, b]$ with $a < b$ in $\mathbb{R}$, all sets of the form (a, ∞) with a in $\mathbb{R}$, and finally $\mathbb{R}$ itself and the empty set. These sets in $\mathcal{E}$ are the *half-open intervals*. The algebra $\mathcal{A}$ of half open intervals consists, by definition of all finite disjoint unions of sets in $\mathcal{E}$.

Let us check that $\mathcal{A}$ is indeed an algebra. For this purpose the following lemma is useful:

6.1 LEMMA. *Let X be any set, and let $\mathcal{F}$ be a set of subsets of X such that for all E and F in $\mathcal{F}$, $E\backslash F$ is a finite disjoint union of elements of $\mathcal{F}$. Then the set $\mathcal{A}$ of all finite disjoint union of elements of $\mathcal{F}$ is an algebra.*

Proof. We first claim that if $E \in \mathcal{A}$ and $F \in \mathcal{F}$, then $A\backslash F \in \mathcal{A}$. To see this, note that every $E \in \mathcal{A}$ has the form $E = \bigcup_{j=1}^{n} E_j$ where $E_1, \dots, E_m$ belong to $\mathcal{F}$ and are mutually disjoint and $m \in \mathbb{N}$. Then for any $F \in \mathcal{F}$,

$$E\backslash F = \bigcup_{j=1}^{n} E_j\backslash F \ . \tag{6.1}$$

Each $E_j\backslash F$ is a finite disjoint union of sets in $\mathcal{F}$. Since for $i \neq j$ E_i and E_j are disjoint, no subset of $E_i\backslash F$ intersects and subset of $E_j\backslash F$. Thus, $E\backslash F$ is a finite disjoint union of sets in $\mathcal{F}$, and hence belongs to $\mathcal{A}$.

Next we claim that $\mathcal{A}$ is closed under differences. Let $E, F \in \mathcal{A}$, and write $F = \bigcup_{j=1}^{n} F_j$ where $n \in \mathbb{N}$ and $F_1, \dots, F_n$ are disjoint elements of $\mathcal{F}$. Define $E_1 = E\backslash F_1$ and then for $j = 2, \dots, n$, recursively define

$$E_j = E_{j-1}\backslash F_j \ .$$

Then

$$E_n = E\backslash F \ .$$

By the first part, $E_1 \in \mathcal{A}$, and then by a simple induction, $E_j \in \mathcal{A}$ for each j.Thus $E\backslash F \in \mathcal{A}$, so $\mathcal{A}$ is closed under differences.

Finally, for $E, F \in \mathcal{A}$,

$$E \bigcup F = E \bigcup (F \backslash E) \ .$$

Since the union on the right is disjoint, and since the disjoint union of elements of $\mathcal{A}$ clearly belongs to $\mathcal{A}$, $E \bigcup F \in \mathcal{A}$. Now by a simple induction, A is closed under finite unions. This concludes the proof that $\mathcal{A}$ is an algebra. □

We turn to a fundamntal example:

6.1 DEFINITION (Rational half open rectangles in $\mathbb{R}^n$)**.** *A rational half open reactange in $\mathbb{R}^n$ is a set R of the form*

$$R = \{ \ (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ a_j < x_j \leq b_j \quad \text{for all } j = 1, \ldots, n \ \}$$

where for each $j = 1, dots, n$, a_j and b_j are rational numbers such that $-\infty \leq a_j \leq b_j \leq \infty$. Let $\mathcal{R}_n$ denote the set of all rational half open reactanges in $\mathbb{R}^n$. Note that $\emptyset$ and $\mathbb{R}^n$ belong to $\mathcal{R}_n$.

6.2 LEMMA (Half open rectangle algebra in $\mathbb{R}^n$)**.** *Let $E, F \in \mathcal{R}_n$. Then $E \backslash F$ is a finite disjoint union of sets in $\mathcal{R}_n$.*

Proof. Consider the case $n = 1$. Let $(a, b]$ and $(c, d]$ be bounded half open intervals. Then $(a, b] \backslash (c, d]$ is one of the following:

$$\emptyset \ , \quad (a, b], \quad (d, b], \quad (a, c] \bigcup (d, b], \quad \text{or} \quad (a, c] \ .$$

Likewise, $(a, \infty) \backslash (c, d]$ is one of the following:

$$(a, \infty), \quad (d, \infty), \quad \text{or} \quad (a, c] \bigcup (d, \infty).$$

$(a, b] \backslash (c, \infty)$ is one of the following:

$$\emptyset \ , \quad (a, c], \quad \text{or} \quad (a, b] \ .$$

Finally,$(a, \infty) \backslash (c, \infty)$ is either $\emptyset$ or $(a, c]$, and the cases if differences involving $\mathbb{R}$ or $\emptyset$ are trivial.

In the same way, one treats the cases $n > 1$; the only difference is notational. This is left as an exercise. □

Lemma 6.2 together with Lemma 6.1 tell us the that set of all finite disjoint unions of sets in $\mathcal{R}_n$ is an algebra. Since the rational numbers are countable, and since countable unions of countable sets are countable, it is clear that this algebra consist of countably many sets. This algebra plays a fundamental role in the theory of measure and integration on $\mathbb{R}^n$. We give it a name and establish a standard notation for it:

6.2 DEFINITION (The rational half open rectangle algebra in $\mathbb{R}^n$)**.** *The rational half open rectangle algebra in $\mathbb{R}^n$ is the algebra $\mathcal{A}_n$ consisting of all finite disjoint unions of sets in $\mathcal{R}_n$.*

6.3 THEOREM (Open sets and $\mathcal{A}_n$)**.** *Every open set in $\mathbb{R}^n$ is a countable disjoint union of sets in $\mathcal{R}_n$. In particular,*

$$\mathcal{B}_{R^n} = \sigma(A_n) \ .$$

Proof. Let U be open in $\mathbb{R}^n$. For every $r > 0$, and all $x \in \mathbb{R}^n$, $B_r(x)$ contains a rectangle R in $\mathcal{R}_n$ such that $x \in R$. For each $x \in U$, pick an $R_x \in \mathcal{R}_n$ such that $x \in R_x \subset U$. Let

$$\mathcal{J} = \{ \ R \in \mathcal{R}_n \ : \ R = R_x \quad \text{for some} \quad x \in U \ \} \ .$$

Note that $\mathcal{J}$ is countable since $\mathcal{R}_n$ is countable. Arrange the sets in $\mathcal{J}$ into a sequence $\{C_m\}_{m\in\mathbb{N}}$. Evidently, $U = \bigcup_{m\in\mathbb{N}} C_m$.

Next, define $B_1 = C_1$, and for $m \geq 1$ define

$$B_{m+1} = C_{m+1} \backslash \left(\bigcup_{j=1}^{m} C_j \right) \ .$$

Note that for each m, $\bigcup_{j=1}^m B_j = \bigcup_{j=1}^m C_j$, and hence $B_{m+1} = C_{m+1} \backslash \left(\bigcup_{j=1}^m B_j \right)$, which shows that $B_\ell \cap B_m = \emptyset$ for $\ell \neq m$. Since $C_j \in \mathcal{A}_n$ for all j, and since the algebra is closed under the operations used to form B_m, $B_m \in \mathcal{A}_n$ for all m, and hence is a finite dijoint union of sets in $\mathcal{R}_n$. Since a countable union of finite unions is countable, we have that U is the disjoint union of countably many rectangles. in $\mathcal{R}_n$.

For the final part, observe the each set in $\mathcal{R}_n$ is a Borel set, and hence $\sigma(\mathcal{A}_n) \subset \mathcal{B}_{\mathbb{R}^n}$. However, what we have just proved shows that every open set U in $\mathbb{R}^n$ belongs to $\sigma(\mathcal{A}_n)$. Hence $\mathcal{B}_{\mathbb{R}^n} \subset \sigma(\mathcal{A})$. □

6.1 Monotone classes

The main result of this section is a theorem that characterizes σ algebra generated by algebras including the whole set X as *monotone classes*:

6.3 DEFINITION (Monotone class)**.** *Let X be any set. A set $\mathcal{S}$ of subsets of X is a monotone class in case:*

(1) Whenever $\{A_n\}_{n\in\mathbb{N}}$ is an increasing sequence of sets in $\mathcal{S}$; i.e., $A_n \subset A_{n+1}$ for all n, then

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{S} \ .$$

(2) Whenever $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of sets in $\mathcal{S}$; i.e., $A_{n+1} \subset A_n$ for all n, then

$$\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{S} \ .$$

Just as with σ-algebra, if I is an arbitrary index set and for each $\alpha \in I$, $\mathcal{S}_\alpha$ is a monotone class, then $\cap_{\alpha\in I}\mathcal{S}_\alpha$ is a monotone class, and since 2^X is a monotone class, whenever $\mathcal{E}$ is any subset of 2^X, the intersection of all monotone classes containing $\mathcal{E}$ is a monotone class containing $\mathcal{E}$. The is *the smallest monotone class containing $\mathcal{E}$.*

6.4 LEMMA. *if a set $\mathcal{S} \subset 2^X$ is both an algebra and a monotone class, $\mathcal{S}$ is a σ-algebra.*

Proof. Suppose that $\mathcal{S} \subset 2^X$ is both an algebra and a monotone class. We must show that $\mathcal{S}$ is closed under arbitrary countable unions.

Let $\{A_n\}_{n\in N}$ be an arbitrary sequence of sets, and define $B_n = \bigcup_{j=1}^{n} A_n$ Then $B_n \subset B_{n+1}$ for all n, and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \ .$$

That is, every countable union of sets $\{A_n\}_{n\in N}$ can be written as the union of an increasing nested sequence of sets $\{B_n\}_{n\in N}$, and since $\mathcal{S}$ an algebra, each $B_n \in SS$ since B_n is a finite union of sets in $\mathcal{S}$, which is closed under finite unions. □

Clearly every σ-algebra is a monotone class since σ-algebras are closed under arbitrary countable unions and intersection, whether or not the sets are nested. However, monotone classes need not be closed under complements for example, and so the notion of a monotone class is much less restrictive than that of a σ-algebra: There are many monotone classes that are not σ-algebras. In fact, while an infinite σ-algebra must always have cardinality at least as large as the continuum, there are countably infinite monotone classes. Thus, there are subsets $\mathcal{E}$ of 2^X for which the smallest monotone class containing $\mathcal{E}$ is much smaller than $\sigma(E)$. However, this does not occur when $\mathcal{E}$ is an algebra continuing X.

6.5 THEOREM (Monotone Class Theorem)**.** *Let X be any set, and let $\mathcal{A}$ be any algebra of subsets of A such that $X \in \mathcal{A}$. Then the smallest monotone class containing $\mathcal{A}$ is also the smallest σ-algebra containing $\mathcal{A}$.*

Proof. Let $\mathcal{S}$ denote the smallest monotone class containing $\mathcal{A}$. By Lemma 6.4 it suffices to show that $\mathcal{S}$ is an algebra. Since $X \in \mathcal{A} \subset \S$, this amounts to showing that $\mathcal{S}$ is closed under complements and finite unions.

Define $\mathcal{C}$ by

$$\mathcal{C} = \{\ A \in \mathcal{S}\ :\ A^c \in \mathcal{S}\}\ . \tag{6.2}$$

Since $\mathcal{A}$ is closed under complementations and $\mathcal{A} \subset \mathcal{S}$, $A \subset \mathcal{C}$. We shall show that $\mathcal{C}$ is a monotone class. Since $\mathcal{S}$ is the smallest monotone class containing $\mathcal{A}$, this will show that $\mathcal{S} \subset \mathcal{C}$, and since by definition $\mathcal{C} \subset \mathcal{S}$, we shall have that $\mathcal{C} = \mathcal{S}$, which is precisely the statement that $\mathcal{S}$ is closed under complementation.

To see that $\mathcal{C}$ is a monotone class, let $\{A_n\}_{n\in N}$ be an arbitrary sequence of sets in $\mathcal{C}$. Notice that

$$\{A_n\}_{n\in N} \quad \text{is an increasing nested sequence} \quad \iff \{A_n^c\}_{n\in N} \quad \text{is a decreasing nested sequence}$$

Thus, if $\{A_n\}_{n\in N}$ is an increasing nested sequence in $\mathcal{C} \subset \mathcal{S}$,

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c \in \mathcal{S}$$

since $\{A_n^c\}_{n\in N}$ is a decreasing nested sequence in $\mathcal{C} \subset \mathcal{S}$, and $\mathcal{S}$ is a monotone class. Thus $\left(\bigcup_{n=1}^{\infty} A_n\right)^c \in \mathcal{C}$. An analogous argument shows that whenever $\{A_n\}_{n\in N}$ is a decreasing nested sequence in $\mathcal{C}$, then $\left(\bigcap_{n=1}^{\infty} A_n\right)^c \in \mathcal{C}$, and hence $\mathcal{C}$ is a monotone class.

It remains to show that for all $A, B \in \mathcal{S}$, $A \bigcup B \in \mathcal{S}$. This is done in two steps. First, fix $A \in \mathcal{S}$. Define $\mathcal{C}_A$ by

$$\mathcal{C}_A = \{ \ B \in \mathcal{S} \ : \ A \bigcup B \in \mathcal{S} \} \ . \tag{6.3}$$

Since $A \in \mathcal{A}$, which is an algebra contained in $\mathcal{S}$, $\mathcal{A} \subset \mathcal{C}_A$. As above, if we show that $\mathcal{C}_A$ is a monotone class, it will follow that $\mathcal{C}_A = \mathcal{S}$. To do this, suppose that $\{B_n\}_{n \in N}$ is an increasing nested sequence in $\mathcal{C}_A \subset \mathcal{S}$. Then $\{A \bigcup B_n\}_{n \in N}$ is an increasing nested sequence, and by the definition of $\mathcal{C}_A$, each $A \bigcup B_n \in \mathcal{S}$ for all n. Then

$$A \bigcup \left(\bigcup_{n=1}^{\infty} B_n \right) = \bigcup_{n=1}^{\infty} A \bigcup B_n \in \mathcal{S}$$

since $\mathcal{S}$ is a monotone class. Thus, $\mathcal{C}_A$ is closed under countable increasing unions. An exactly analogous argument shows that $\mathcal{C}_A$ is closed under countable decreasing intersections, and hence $\mathcal{C}_A$ is a monotone class. As noted above, this means that $\mathcal{C}_A = \mathcal{S}$, and hence $A \bigcup B \in \mathcal{S}$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{S}$. Switching the roles of A and B, this shows that $A \bigcup B \in \mathcal{S}$ for arbitrary $A \in \mathcal{S}$ and $B \in \mathcal{A}$.

Now fix $A \in \mathcal{S}$ and define $\mathcal{C}_A$ as before, but this time with $A \in \mathcal{S}$. By what we have just proved, it is still the case $\mathcal{A} \subset \mathcal{C}_A$. The argument made just above then shows that $\mathcal{C}_A$ is a monotone class, and hence that $\mathcal{C}_A = \mathcal{S}$. This then shows that $A \bigcup B \in \mathcal{S}$ for all $A, B \in \mathcal{S}$, and so $\mathcal{S}$ is closed under finite unions. □

6.2 Approximation of measurable sets

6.6 THEOREM. *Let $(X, \mathcal{M}, \mu)$ be a measure space, where $\mathcal{M} = \sigma(\mathcal{A})$ and $\mathcal{A}$ is an algebra of sets in X. Then for all $\epsilon > 0$ and all $E \in \mathcal{M}$ there is an $A \in \mathcal{A}$ such that $\mu(A \Delta E) < \epsilon$.*

Proof. Let $\mathcal{S}$ be the set of sets in $\mathcal{M}$ that have such an approximation. That is, $E \in \mathcal{S}$ if and only if for all $\epsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mu(A \Delta E) < \epsilon$. Clearly $A \in \mathcal{S}$ since we can approximate by A itself: $\mu(A \Delta A) = 0$. If $\mathcal{S}$ is a monotone class, it follows from the Monotone Class Theorem that SS contains a σ-algebra containing $\mathcal{A}$, and hence contains all of $\mathcal{M}$. Thus, it suffices to show that $\mathcal{S}$ is a monotone class,

Let $\{E_n\}_{n \in \mathbb{N}}$ be an increasing nested sequence in $\mathcal{S}$. Let $E = \bigcup_{n=1}^{\infty} E_n$. fix $\epsilon > 0$. We must find $A \in \mathcal{A}$ such that $\mu(A \Delta E) < \epsilon$. By continuity from below, and the fact that $\mu(E) < \infty$, there exists n such that $\mu(E_n) > \mu(E) - \epsilon/2$. Since $E_n \in \mathcal{S}$, there exists $A \in \mathcal{A}$ so that $\mu(E_n \Delta A) < \epsilon/2$ But then since

$$E \Delta A \subset (E \Delta E_n) \bigcup (E_n \Delta A) \ ,$$

$\mu(E \Delta A) < \epsilon$. Thus, $E \in \mathcal{S}$, and $\mathcal{S}$ is closed under unions of increasing nested sequences sets.

The exact same argument using continuity from above in place of continuity from below (and thus using the fact that $\mu(X) < \infty$ in one more way) shows that $\mathcal{S}$ is closed under intersections of decreasing nested sequences of sets. Thus, $\mathcal{S}$ is a monotone class. □

This theorem tells us something very useful: If we start from an algebra of sets $\mathcal{A}$, and then extend this to the σ-algebra $\mathcal{M} = \sigma(\mathcal{A})$, we may not have a very concrete description of what is in – or not in – $\mathcal{M}$. But if we equip $\mathcal{M}$ with any finite measure, then, up to a set of arbitrarily small measure, and set $E \in \mathcal{M}$ is "well approximated" by sets in $\mathcal{A}$ in the sense that we can find an $A \in \mathcal{A}$ that makes $\mu(E\Delta A)$ arbitrarily small. Thus, the extension from $\mathcal{A}$ to $\mathcal{M} = \sigma(\mathcal{A})$ does not "drag in" strange sets that have nothing much to do with sets in $\mathcal{A}$.

7 Exercises

1. Let (X, d) be a separable metric space. Let $f : X \times \mathbb{R} \to \mathbb{R}$ be a function such that for each $t \in \mathbb{R}$, $x \mapsto f(x,t)$ is continuous on X and such that for each $x \in X$, $\mapsto f(x,t)$ is measurable. Define

$$g(x) = \int \{f(x,t) \ \ t \in \mathbb{R}\} \ ,$$

so that g is a functions from X to $[-\infty, \infty)$. Show that g is a Borel measurable function.