Notes on the Construction of Measures for Math 501

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Contents

1 Measures, premeasures and outer measures

Up until this point in the course, we have not proved that any really interesting countably additive measures exist. We have asserted the existence of Lebesgue measure, but we have not actually constructed it. In this section of the notes, we finally turn to the problem of constructing a large class of countably additive measures including Lebesgue measure.

The starting point will be a *premeasure* on some algebra $\mathcal{A}$ of subsets of a set X.

Definition (premeasure) Let $\mathcal{A}$ be an algebra of subsets of X. A *premeasure* on $\mathcal{A}$ is a function $m : \mathcal{A} \to [0, \infty]$ such that:

(1) $m(\emptyset) = 0$

(2) If $A_1, \ldots, A_n$ is any finite collection of disjoint sets in $\mathcal{A}$,

$$m\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} m(A_j) \ .$$

For example, if $X = \mathbb{R}$, we have seen that the set of all finite disjoint unions of half open intervals, i.e., subsets of $\mathbb{R}$ of the form $\{x \in \mathbb{R} \ : \ a < x \leq b\}$ for some $a < b \in [-\infty, \infty]$, is an algebra. Every $A \in \mathcal{A}$ can be written as a disjoint union of half open intervals in infinitely many ways, but there is exactly one representation of the following form:

$$A = \bigcup_{j=1}^{n} (a_, b_j] \qquad \text{and} \qquad -\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \leq \infty \ . \tag{1.1}$$

We then define the *Lebesgue premeasure* m_L *on the half open interval algebra* $\mathcal{A}$ by

$$m_L(A) = \sum_{j=1}^{\infty} (b_j - a_j) \ .$$

It is easy to see that this is a premeasue; the proof is left to the reader.

Our goal is to extend a premeasure m on $\mathcal{A}$ to a countably additive measure μ on $\sigma(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$. Since every open set in $\mathbb{R}$ is a countable union of half open intervals, and thus a countable union of elements of $\mathcal{A}$, it follows that $\mathcal{B}_{\mathbb{R}} \subset \sigma(\mathcal{A})$. Also each half open interval is a Borel set in $\mathbb{R}$. It follows that $\sigma(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$. Therefore, once we succeed (as we shall) in our goal for this example, we shall have constructed a countably additive extension of the "sum of lengths of intervals" premeasure to the Borel σ-algebra of $\mathbb{R}$.

The construction of the countably additive measure μ_L on $\sigma(\mathcal{A})$ that extends m_L on $\mathcal{A}$ relies on a preliminary construction of an *outer measure*, which is a set function defined on the class of *all* subsets of X.

1.1 DEFINITION (outer measure)**.** An *outer measure* μ^* on a set X is a set function $\mu^* : 2^X \to [0, \infty]$ such that

(1) $\mu^*(\emptyset) = 0$

(2) (Monotonicity) If $B_1 \subset B_2$, then $\mu^*(B_1) \leq \mu^*(B_2)$.

(3) (Subadditivity) If $\{B_j\}$ is any sequence of subsets of X, then

$$\mu^*\left(\bigcup_{j=1}^{\infty} B_j\right) \leq \sum_{j=1}^{\infty} \mu^*(B_j) \ .$$

1.2 Remark. Consider a sequence of sets $\{B_j\}$ such that $B_j = \emptyset$ for all $j \geq n$, Then by *(1)* and *(3)*, we have that

$$\mu^*\left(\bigcup_{j=1}^{n} B_j\right) \leq \sum_{j=1}^{n} \mu^*(B_j) \ .$$

Thus, outer measures are *finitely sub-additive.*

The main theorem on outer measures is Caratheodory's Theorem:

1.3 THEOREM (Caratheodory's Theorem)**.** *Let* μ^* *be an outer measure on* X*. Let* $\mathcal{M}$ *be the class of subsets* X *defined as follows:*

$$\mathcal{M} = \{\ A \subset X \ :\ \text{for all } E \subset X,\ \mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E)\ \} \ . \tag{1.2}$$

Then $\mathcal{M}$ *is a* σ*-algebra, and the restriction of* μ^* *to* $\mathcal{M}$ *is a countably additive measure on* $\mathcal{M}$*.*

1.4 Remark. Since μ^* is an outer measure, and hence subadditive, for all $E, A \subset X$,

$$\mu^*(E) \le \mu^*(A \cap E) + \mu^*(A^c \cap E) \ .$$

Therefore, to show that $A \in \mathcal{M}$, it suffices to show that

$$\mu^*(E) \ge \mu^*(A \cap E) + \mu^*(A^c \cap E) \quad \text{for all } E \subset X \ . \tag{1.3}$$

Proof. We carry out the proof in several steps.
Step 1.: We show that $\mathcal{M}$ is an algebra. Every σ-algebra is an algebra, so we must show that $\mathcal{M}$ is an algebra. It is obviously closed under taking of complements, since the definition (1.2) is symmetric in A and A^c. Thus, to show $\mathcal{M}$ is an algebra, it suffices to show that $\mathcal{M}$ is closed under finite unions. Then by induction, it suffices to show that if $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$.

Let $A, B \in \mathcal{M}$, and let $E \subset X$. Then since $A \in \mathcal{M}$,

$$\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E) \ .$$

Let $F := A \cap E$. Since $B \in \mathcal{M}$, $\mu^*(F) = \mu^*(B \cap F) + \mu^*(B^c \cap F)$. Likewise, let $G := A^c \cap E$. Then since $B \in \mathcal{M}$, $\mu^*(G) = \mu^*(B \cap G) + \mu^*(B^c \cap G)$.

Combining these three identities we have

$$\begin{aligned} \mu^*(E) &= \mu^*(A^c \cap B^c \cap E) + \mu^*(A \cap B^c \cap E) + \mu^*(A^c \cap B \cap E) + \mu^*(A \cap B \cap E) \\ &= \mu^*((A \cup B)^c \cap E) + [\mu^*(A \cap B^c \cap E) + \mu^*(A^c \cap B \cap E) + \mu^*(A \cap B \cap E)] \ . \end{aligned} \tag{1.4}$$

Now,

$$(A \cup B) \cap E = (A \cap B^c \cap E) \cup (A^c \cap B \cap E) \cup (A \cap B \cap E) \ ,$$

and thus, by the subadditivity property of outer measures

$$\mu^*((A \cup B) \cap E) \le \mu^*(A \cap B^c \cap E) + \mu^*(A^c \cap B \cap E) + \mu^*(A \cap B \cap E) \ .$$

Therefore, (1.4) reduces to

$$\mu^*(E) \ge \mu^*((A \cup B) \cap E) + \mu^*((A \cup B)^c \cap E) \ . \tag{1.5}$$

By Remark 1.4, $A \cup B \in \mathcal{M}$. Thus, $\mathcal{M}$ is an algebra.

Step 2: We show that μ^* is finitely additive on $\mathcal{M}$, and somewhat more than that. Let $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$. Then since $A \in \mathcal{M}$, and using $E \cap (A \cup B)$ in place of E in (1.2), we have that

$$\begin{aligned} \mu^*(E \cap (A \cup B)) &= \mu^*((E \cap (A \cup B)) \cap A)) + \mu^*((E \cap (A \cup B)) \cap A^c)) \\ &= \mu^*(E \cap A) + \mu^*(E \cap B) \ . \end{aligned}$$

That is for all $A \in \mathcal{M}$, and all sets E, B with $B \cap A = \emptyset$,

$$\mu^*(E \cap (A \cup B)) = \mu^*(E \cap A) + \mu^*(E \cap B) \ . \tag{1.6}$$

In particular, taking $E = X$, we see that μ^* is finitely additive on $\mathcal{M}$.

Step 3: We show that $\mathcal{M}$ is a σ-algebra. Since $\mathcal{M}$ is an algebra, it suffices to show that whenever $\{A_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $A := \cup_{j=1}^{\infty} A_j \in \mathcal{M}$.

Let $E \subset X$ and let $\{A_j\}$ be a sequence of disjoint sets in $\mathcal{M}$, and let $A := \cup_{j=1}^{\infty} A_j \in \mathcal{M}$. Define

$$B_n := \bigcup_{j=1}^{n} A_j \ .$$

Then since $\mathcal{M}$ is an algebra, $B_n \in \mathcal{M}$, and so for all $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \ .$$

Since $B_n^c \subset A^c$, the monotonicity of outer measure implies that $\mu^*(E \cap B_n^c) \leq \mu^*(E \cap A^c)$. Thus,

$$\mu^*(E) \geq \mu^*(E \cap B_n) + \mu^*(E \cap A^c) \ . \tag{1.7}$$

Next, by a simple induction using (1.6), $\mu^*(E \cap B_n) = \sum_{j=1}^{n} \mu^*(E \cap A_j)$. Using this in (1.7), we have

$$\mu^*(E) \geq \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap A^c) \ .$$

Taking n to infinity, and using the subadditivity of μ^* in the last step, we have

$$\mu^*(E) \geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(E \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \geq \mu^*(E) \ . \tag{1.8}$$

This shows that $A \in \mathcal{M}$, and thus $\mathcal{M}$ is a σ-algebra.

Step 4: Keeping the notation of the previous step, note that all of the inequalities in (1.8) must be equalities for all $E \subset X$. Choosing $E = A$, we obtain

$$\mu(A) = \sum_{j=1}^{\infty} \mu^*(A_j) \ ,$$

which shows that μ^* is countably additive on A. □

1.5 DEFINITION (Caratheodory σ-algebra)**.** Let μ^* be an outer measure on X. Then the σ-algebra $\mathcal{M}$ specified by (1.2) is called the *Caratheodory σ-algebra determined by the outer measure μ^**.

The next theorem explains how to construct an outer measure out of a premeasure.

1.6 THEOREM (Outer measures from premeasures)**.** *Let $\mathcal{A}$ be an algebra of subsets of X, with $X \in \mathcal{A}$. Let m be a premeasure on $\mathcal{A}$. Then for every set $B \subset X$, define $\mu^*(B)$ by*

$$\mu^*(B) = \inf \left\{ \sum_{j=1}^{\infty} m(A_j) \ : \ E \subset \bigcup_{j=1}^{\infty} A_j \quad \text{and} \quad A_j \in \mathcal{A} \text{ for all } j \right\} \ .$$

Then μ^* is an outer measure on X. Moreover, if $\mathcal{M}$ is the σ-algebra of measurable sets determined by μ^* as in Caratheodory's Theorem, then $\mathcal{A} \subset \mathcal{M}$, and for all $A \in \mathcal{A}$,

$$\mu^*(A) \leq m(A) \ . \tag{1.9}$$

Proof. Taking $A_j = \emptyset$ for each j, we get a countable cover of $\emptyset$ for which $\sum_{j=1}^{\infty} m(A_j) = \sum_{j=1}^{\infty} m(\emptyset) = 0$. Thus, $\mu^*(\emptyset) = 0$. This shows that property *(1)* in Definition 1.1 is verified.

Second, if $B_1 \subset B_2$, any countable cover of B_2 is a countable cover of B_1. Hence condition *(2)* of the definition is verified.

Third, to show that condition *(3)* is satisfied, let $\{B_j\}$ be any sequence of sets on X. We must show that

$$\mu^*\left(\bigcup_{j=1}^{\infty} B_j\right) \le \sum_{j=1}^{\infty} \mu^*(B_j) \ . \tag{1.10}$$

To do this, fix $\epsilon > 0$. For each j, find a countable cover of B_j by a sequence of elements of $\mathcal{A}$, $\{A_{j,k}\}_{k\in\mathbb{N}}$, such that

$$\mu^*(B_j) + \frac{\epsilon}{2^j} \ge \sum_{k=1}^{\infty} m(A_{j,k}) \ .$$

But clearly

$$\bigcup_{j=1}^{\infty} B_j \subset \bigcup_{j,k=1}^{\infty} A_{j,k}$$

and

$$\sum_{j,k=1}^{\infty} m(A_{j,k}) \le \sum_{j=1}^{\infty} \left(\mu^*(B_j) + \frac{\epsilon}{2^j}\right) = \sum_{j=1}^{\infty} \mu^*(B_j) + \epsilon \ .$$

Since $\epsilon > 0$ is arbitrary, (1.10) is proved. Thus, μ^* is an outer measure.

To prove (1.9), consider the obvious countable cover of $A \in \mathcal{M}$ in which take $A_1 = A$ and for $j \ge 2$, take $A_j = \emptyset$. For this cover, it is clear that $\sum_{j=1}^{\infty} m(A_j) = m(A)$. The infimum cannot be larger than this.

It remains to show that $\mathcal{A} \subset \mathcal{M}$. Let $A \in \mathcal{A}$. We must show that for each $E \subset X$,

$$\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E) \ . \tag{1.11}$$

Fix $\epsilon > 0$. Let $\{A_j\}$ be a sequence of sets in $\mathcal{A}$ such that $E \subset \cup_{j=1}^{\infty} A_j$, and such that

$$\mu^*(E) + \epsilon \ge \sum_{j=1}^{\infty} m(A_j) \ .$$

Since m is additive on $\mathcal{A}$, $m(A_j) = m(A_j \cap A) + m(A_j \cap A^c)$, and hence

$$\begin{aligned} \mu^*(E) + \epsilon &\ge \sum_{j=1}^{\infty} m(A_j) \\ &= \sum_{j=1}^{\infty} m(A_j \cap A) + \sum_{j=1}^{\infty} m(A_j \cap A^c) \\ &\ge \sum_{j=1}^{\infty} \mu^*(A_j \cap A) + \sum_{j=1}^{\infty} \mu^*(A_j \cap A^c) \\ &\ge \mu^*(E \cap A) + \mu^*(E \cap A^c) \end{aligned}$$

where we have used (1.9) for the first inequality, and subadditivity of μ^* for the second. Since $\epsilon > 0$ is arbitrary, this proves that $\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Then by Remark 1.4, we have proved (1.11). □

There is one more point to be dealt with: At the present level of generality, we only have the inequality (1.9) relating m and μ^*. We would like to have μ^* be an extension of m; i.e.,

$$\mu^*(A) = m(A)$$

for all $A \in \mathcal{A}$. But as it stands, it might be that the following happens: For each $A \in A$, and each $\epsilon > 0$, one might find a countable cover of A by a sequence $\{A_j\}$ in $\mathcal{A}$ with $\sum_{j=1}^{\infty} m(A_j) < \epsilon$. If this were to happen, then we would have $\mu^*(A) = 0$ for all $A \in \mathcal{A}$. This in turn would imply that $\mu^*(E) = 0$ for all $E \subset X$, and then $\mathcal{M}$ would be 2^X, and all of our constructions would be trivial. This can happen.

1.7 EXAMPLE. *Let $\mathcal{A}$ be the algebra of half open intervals on $\mathbb{R}$. Let F be the function on $\mathbb{R}$ defined by*

$$F(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0\ . \end{cases} \ .$$

Define $m((a,b]) = F(b) - F(a)$, and extend m to $\mathcal{M}$ by making it finitely additive. It is easy to see that this procedure gives a premeasure on $\mathcal{A}$ for any *monotone non-decreasing function F. However, with the choice we have made, the outer measure associated to this premeasure is identically zero.*

To see this, note that we can cover $(-1,1]$ with half open intervals as follows:

$$(-1,1] = (-1,0] \bigcup \left(\bigcup_{j=2}^{\infty} \left(\frac{1}{j}, \frac{1}{j-1} \right] \right) \ .$$

Then $m((-1,0]) = F(0) - F(-1) = 0 - 0 = 0$, and for each $j \ge 2$, $m(1/j, 1/(j-1)] = F(1/j) - F(1/(j-1)) = 1 - 1 = 0$. Thus $\mu^*(-1,1]) = 0$. The same argument applies to show that $\mu^*((a,b]) = 0$ for any half open interval containing 0, and evidently $\mu^*((a,b]) = 0$ for any half open interval not containing 0. This implies that $\mu^*(A) = 0$ for all $A \in \mathcal{A}$.

To avoid the sort of disaster that arose in the example just above, we need to work with well behaved premeasures.

1.8 DEFINITION (continuity at the empty set)**.** Let $\mathcal{A}$ be an algebra of subsets of X, and let m be a premeasure on $\mathcal{A}$. Then m is *continuous at the empty set* if and only if whenever $\{A_j\}$ is a decreasing sequence of sets in $\mathcal{A}$ such that $m(A_1) < \infty$, then

$$\bigcap_{j=1}^{\infty} A_j = \emptyset \quad \Rightarrow \quad \lim_{j\to\infty} m(A_j) = 0\ .$$

1.9 DEFINITION (finite, σ-finite and semifinite)**.** Let $\mathcal{A}$ be an algebra of subsets of X, and let m be a premeasure on $\mathcal{A}$. Then m is *finite* if and only if $m(X) < \infty$. It is σ-finite if there is a sequence of sets $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $m(A_n) < \infty$ for all n and $\cup_{n=1}^{\infty} A_n = X$. It is *semifinite* if and only if whenever $A \in \mathcal{A}$ has $m(A) = \infty$, for each $r > 0$, there exists $B \in \mathcal{A}$ such that $B \subset A$ and $r < m(B) < \infty$.

Since for *any* countably additive finite measure μ defined on a σ-algebra including $\mathcal{A}$,

$$\bigcap_{j=1}^{\infty} A_j = \emptyset \quad \Rightarrow \quad \lim_{j\to\infty} \mu(A_j) = 0\ ,$$

our premeasure m *must* be continuous at the empty set if we are to have $m(A) = \mu^*(A)$ for all $A \in \mathcal{A}$. The following theorem says that this necessary condition, together with semifiniteness, is also sufficient.

1.10 THEOREM (Extensions of premeasures)**.** *Let $\mathcal{A}$ be an algebra of subsets of X. Let m be a semifinite premeasure on $\mathcal{A}$ that is continuous at the empty set. Let μ^* be the outer measure on X determined by m. Then for all $A \in \mathcal{A}$,*

$$\mu^*(A) = m(A) \ ,$$

so that μ^* extends m

Proof. We need only show that

$$\mu^*(A) \geq m(A) \qquad \text{for all } A \in \mathcal{A} \quad \text{such that} \quad m(A) < \infty \ . \tag{1.12}$$

Then by the semifiniteness property, whenever $m(A) = \infty$, for all $r > 0$, there is a $B \subset A$, $B \in \mathcal{A}$ with $r < m(B) < \infty$. By (1.12), $\mu^*(B) \geq m(B) > r$ and by monotonicity of μ^*, $\mu^*(A) > \mu^*(B) > r$. Since $r > 0$ is arbitrary, $\mu^*(A) = \infty$.

To show (1.12), pick any $\epsilon > 0$, and pick a sequence $\{A_j\}_{j\in\mathbb{N}}$ in A with $A \subset \cup_{j=1}^{\infty} A_j$ and

$$\mu^*(A) + \epsilon > \sum_{j=1}^{\infty} m(A_j) \ .$$

Define sequences of sets $\{B_j\}_{j\in\mathbb{N}}$ and $\{C^j\}_{j\in\mathbb{N}}$ in $\mathcal{A}$ as follows: For $j = 1$ define $B_1 = A \cap A_1$. For $j \geq 2$, define

$$B_j = (A \cap A_j)\backslash(B_1 \cup \cdots \cup B_{j-1}) \ .$$

Then $\{B_j\}_{j\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{A}$, and $\cup_{j=1}^{\infty} B_j = A$. Define C_j by

$$C_j = A \cap (B_1 \cup \cdots \cup B_j)^c \ .$$

Then for each j, $C_j \in \mathcal{A}$, and $\displaystyle\bigcap_{j=1}^{\infty} C_j = \emptyset$. Moreover, for each j,

$$A = B_1 \cup B_2 \cup \cdots B_j \cup C_j \ ,$$

and the right hand side is a disjoint union of sets in $\mathcal{A}$. Therefore, since m is additive on $\mathcal{A}$.

$$m(A) = \sum_{j=1}^{n} m(B_j) + m(C_j) \ .$$

Then, since m is continuous at the empty set, $\lim_{j\to\infty} m(C_j) = 0$, and so

$$m(A) = \sum_{j=1}^{\infty} m(B_j) \leq \sum_{j=1}^{\infty} m(A_j) < \mu^*(A) + \epsilon \ .$$

Since $\epsilon > 0$ is arbitrary, we have the desired result. □

Our next theorem states that when a premeasure m is σ-finite and continuous at the empty set, the extension to a measure on $\sigma(\mathcal{A})$ that we have just constructed is the *only* such extension.

1.11 THEOREM. *Let m be a σ-finite premeasure on an algebra $\mathcal{A}$. Then if μ and ν are two measures such that*

$$\mu(A) = \nu(A) \tag{1.13}$$

for all $A \in \mathcal{A}$, then (1.13) is valid for all $A \in \sigma(\mathcal{A})$. In particular, if m is continuous at the empty set, the countably additive extension of m to $\sigma(\mathcal{A})$ is the unique extension of m to $\sigma(\mathcal{A})$.

Proof. Let $B \in \mathcal{A}$ have finite measure. By continuity from below as the σ-finiteness of m, and hence any extension of m to $\sigma(\mathcal{A})$, it suffices to show that whenever $B \in \mathcal{A}$ has $m(B) < \infty$,

$$\mu(B \cap E) = \nu(B \cap E) \ . \tag{1.14}$$

for all $E \in \sigma(\mathcal{A})$. Let $\mathcal{S}$ be the sets of all sets $E \in \mathcal{M}$ such that (1.14) is true. Then $\mathcal{S}$ is a monotone class by continuity from below, and continuity from above together with $\mu(B) = \nu(B) < \infty$. Since by definition $\mathcal{A} \subset \mathcal{S}$, $\mathcal{S} = \sigma(\mathcal{A})$. □

2 Complete measure spaces and completion

Let m be a premeasure on an algebra $\mathcal{A}$ of subsets of X that is semifinite and continuous at the empty set. The countably additive measure μ extending m that we obtain from Caratheodory's Theorem is defined on a σ-algebra $\mathcal{M}$, the Caratheodory σ-algebra of μ^*, that contains $\sigma(\mathcal{A})$. As we shall see, it may be strictly larger. The measure space that we obtain form Caratheodory's construction has the property of being *complete*: it includes all subsets of sets of measure zero:

2.1 DEFINITION (Complete measure space)**.** A measure space $(X, \mathcal{M}, \mu)$ is *complete* in case whenever $E \subset F$, and $F \in \mathcal{M}$ with $\mu(F) = 0$, then $E \in \mathcal{M}$.

2.2 THEOREM (Completeness Theorem)**.** *Let μ^* be an outer measure on a set X. Let $\mathcal{M}$ be the corresponding Caratheodory σ algebra. Let $\mathcal{N}$ denote the set of all null sets in X. That is*

$$\mathcal{N} = \{ \ B \in X \ : \ \mu^*(B) = 0 \ \}$$

Then any subset of a null set is a null set, and

$$\mathcal{N} \subset \mathcal{M} \ .$$

Proof. Suppose that $\mu^*(B) = 0$. Then for any $E \subset X$,

$$\mu^*(E \cap B) + \mu^*(E \cap B^c) \leq \mu^*(B) + \mu^*(E) = \mu^*(E) \ .$$

This follows from property *(2)* in Definition 1.1 since $E \cap B \subset B$ and $E \cap B^c \subset E$. Thus, by Remark 1.4, $B \in \mathcal{M}$. □

The next thoerem says that any measure space may be *completed* in a simple and cannonical way:

2.3 THEOREM. *Let $(X,\mathcal{M},\mu)$ be a measure space. Let $\mathcal{N}$ denote the set of all subsets of sets of μ measure zero. That is, $E\in\mathcal{N}$ if and only if there is an $f\in\mathcal{M}$ with $E\subset F$ and $\mu(F)=0$. Define $\overline{\mathcal{M}}=\{G\cup E\ :\ G\in\mathcal{M}\ ,E\in\mathcal{N}\}$. Then*

(1) $\overline{\mathcal{M}}$ is a σ-algebra.

(2) If $G_1\cup E_1=G_2\cup E_2$ with $G_1,G_2\in\mathcal{M}$, and $E_1,E_2\in\mathcal{N}$, then $\mu(G_1)=\mu(G_2)$, and hence we may define a function $\overline{\mu}$ on $\overline{\mathcal{M}}$ by

$$\overline{\mu}(G\cup E)=\mu(G)\quad\text{for all}\quad G\in\mathcal{M}\ ,E\in\mathcal{N}\ .$$

This function $\overline{\mu}$ is a coiuntably additive measure on $\overline{\mathcal{M}}$.

(3) $\overline{\mu}$ is the unique extension of μ from $\mathcal{M}$ to $\overline{\mathcal{M}}$.

Proof. The proof of *(1)* is left as an exercise. Under the hypothesis of *(2)*, i $G_1\backslash G_2\subset E_2$, and hence $G_1\backslash G_2\in\mathcal{M}\cap\mathcal{N}$; i.e., $\mu(G_1\backslash G_2)=0$. Hence $\mu(G_1)\le\mu(G_2)$. By the symmetry, $\mu(G_2)\le\mu(G_1)$. Hence $\mu(G_1)=\mu(G_2)$. Therefore, $\overline{\mu}$ is well defined. Obviously $\overline{\mu}(\emptyset)=0$, and if $\{G_n\cup E_n\}_{n\in\mathbb{N}}$ is a sequence of mutually disjoint sets in $\overline{\mathcal{M}}$, with each $G_n\in\mathcal{M}$ and each $E_n\ in\mathcal{N}$, then

$$\bigcup_{n=1}^\infty G_n\cup E_n=\left(\bigcup_{n=1}^\infty G_n\right)\cup\left(\bigcup_{n=1}^\infty E_n\right)\ .$$

Since $\cup_{n=1}^\infty E_n$ is contained in a countable union of sets of μ-measure zero, $\cup_{n=1}^\infty E_n\in\mathcal{N}$. Thus,

$$\overline{\mu}\left(\bigcup_{n=1}^\infty G_n\cup E_n\right)=\mu\left(\bigcup_{n=1}^\infty G_n\right)=\sum_{n=1}^\infty\mu(G_n)=\sum_{n=1}^\infty\overline{\mu}(G_n\cup E_n)\ .$$

This proves the countable additivity of $\overline{\mu}$. The proof of *(3)* is also left as an exercise. □

2.4 DEFINITION (Completion of a measure space)**.** Let $(X,\mathcal{M},\mu)$ be an measure space. Let $(X,\overline{\mathcal{M}},\overline{\mu})$ be the measure space defined in Theorem 2.3 starting from $(X,\mathcal{M},\mu)$. Then $(X,\overline{\mathcal{M}},\overline{\mu})$ is the *completion of* $(X,\mathcal{M},\mu)$.

Now, let $\mathcal{A}$ be an algebra, and let m be a semifinite premeasure on $\mathcal{A}$ that is continuous at the empty set. Let m^* be the corresponding outer measure given by

$$m^*(E)=\inf\left\{\sum_{n=1}^\infty m(A_n)\ :\ \{A_n\}_{n\in\mathbb{N}}\in\mathcal{A}\quad\text{and}\quad E\subset\bigcup_{n=1}^\infty A_n\ \right\}\ , \tag{2.1}$$

Let μ be the restriction of m^* to $\sigma(\mathcal{A})$, and let ν be the resstriction of m^* to the Caratheodory σ-algebra of m^*, which we shall denote by $\mathcal{M}$. Then $(X,\sigma(\mathcal{A}),\mu)$ and $(X,\mathcal{M},\nu)$ are two measure spaces such that $\sigma(\mathcal{A})\subset\mathcal{M}$ and $\nu|_{\sigma(\mathcal{A})}=\mu$. The measure space $(X,\mathcal{M},\nu)$ is complete by Theorem 2.2. What is the relation between $(X,\mathcal{M},\nu)$ and the completion of $(X,\sigma(\mathcal{A}),\mu)$? At least when m is σ-finite, they are the same, as we shall now show. To do this, we introduce another outer measure which, in this context, turns out to coincide wtih m^*.

Let $(X,\mathcal{M},\mu)$ be any semifinite measure space, not necessarily complere. Since $\mathcal{M}$ is a σ-algebra, it is well-qualified as an algebra. Likewise, since μ is a countably additive measure on $\mathcal{M}$, it is certainly a premeasure on $\mathcal{M}$. Let μ^* be the associated outer measure:

$$\mu^*(E)=\inf\left\{\sum_{n=1}^\infty\mu(A_n)\ :\{A_n\}_{n\in\mathbb{N}}\subset\mathcal{M}\quad\text{and}\quad E\subset\bigcup_{n=1}^\infty A_n\ \right\}\ . \tag{2.2}$$

Define $\mathcal{M}_\mu$ to be the Caratheodory σ-algebra of μ^*. Then by Theorem 1.6, $\mathcal{M} \subset \mathcal{M}_\mu$. We refer to $\mathcal{M}_\mu$ as *the σ-algebra of μ-measurable sets.* Since continuity at the empty set is a direct consequnce of continuity from above, and hence the countable additivity of μ, μ is both semifinite and continuous at the empty set. Thus, by Theorem 1.10, $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{M}$. Define $\overline{\mu}$ to be the restriction of μ^* to $\mathcal{M}_\mu$. Then by Theorem 2.2, $\overline{\mu}$ is a complete measure that extends μ; i.e., $\overline{\mu}(A) = \mu(A)$ for all $A \in \mathcal{M}$.

There is an alternate formula for the outer measure μ^*:

2.5 LEMMA. *Let $(X, \mathcal{M}, \mu)$ be any semifinite measure space. Let μ^* be he outer measure on X defeined by (2.2). Then*

$$\mu^*(E) = \inf\{\mu(A) \ : \ E \subset A \quad \text{and} \quad A \in \mathcal{M}\ \} \ . \tag{2.3}$$

This outer measure has the property that $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{M}$, and if ν^* is any other outermeasure such that $\nu^*(A) = \mu(A)$ for all $A \in \mathcal{M}$, then $\mu^* \geq \nu^*$.

Proof. Fix any $E \subset X$ and $\epsilon > 0$. By definition, there exists a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ such that $\mu^*(E) + \epsilon \geq \displaystyle\sum_{n=1}^{\infty} \mu(A_n)$. Then, following a by now standard construction, we know that there exists a seuqence $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ of *disjoint* subsets such that for each n, $\cup_{j=1}^n B_j = \cup_{j=1}^n A_j$. Define $A = \cup_{n=1}^\infty A_n$., which is the same as $\cup_{n=1}^\infty B_n$. Then

$$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

and of course $E \subset \displaystyle\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} A_j = A \in \mathcal{M}$. Since $\epsilon > 0$ is arbitrary, this shows that

$$\mu^*(E) \geq \inf\{\mu(A) \ : \ E \subset A \quad \text{and} \quad A \in \mathcal{M}\ \} \ .$$

Then for any $A \in \mathcal{M}$ with $E \subset A$, consideration of the trivial covering of E by $\{A_n\}_{n\in\mathbb{N}}$ where $A_1 = A$ and $A_j = \emptyset$ for $j \geq 2$ shows the oppositie inequality, and this proves (2.3).

Next, we have already seen that μ^* extends μ. Let ν^* be any outer measure that extends μ. Let $E \subset X$, and $\epsilon > 0$. Then, by definition, there is a set $A \in \mathcal{M}$, $E \subset A$ such that $\mu^*(E) + \epsilon \geq \mu(A) = \nu^*(A)$. Then by monotonicity of ν^*, we have $\mu^*(E) + \epsilon \geq \nu^*(E)$. Since $\epsilon > 0$ is arbitraray, this proves $\mu^*(E) \geq \nu^*(E)$. □

2.6 LEMMA. *Let $\mathcal{A}$ be any algebra of subsets of A that includes X. Let m be a σ-finite premeasure on $\mathcal{A}$ that is continuous at the empty set. Let m^* be the outermeasure defined by (2.1). Let μ be the restriction of m^* to $\sigma(A)$. Let μ^* be the outer measure defined by in terms of μ by (2.2) with $\mathcal{M} = \sigma(\mathcal{A})$. Then $m^* = \mu^*$.*

Proof. Since μ^* extends μ, $\mu^*(B) = \mu(B)$ for all $B \in \sigma(\mathcal{A})$. Since μ is, by defnition, the restriction of m^* to $\sigma(A)$, we also have that $m^*(B) = \mu(B)$ for all $B \in \sigma(\mathcal{A})$. Hence $\mu^*(B) = m^*(B)$ for all $B \in \sigma(\mathcal{A})$.

It now follows from the final part of Lemma 2.5 that $m^* \leq \mu^*$. However, since in the definition of m^* we only consider coverings of a set E by countale unions of sets in $\mathcal{A}$, while in the definition of μ^*, we consider the wider class of covering of E by countale unions of sets in $\sigma(\mathcal{A})$, is is clear that $\mu^* \leq m^*$. Hence $\mu^* = m^*$. □

2.7 THEOREM (Completion of a σ-finite measure). *$(X, \mathcal{M}, \mu)$ be any σ-finite measure space. Let μ^* be the outer measure given by (2.2), let $\mathcal{M}_\mu$ be its Caratheodory σ-algebra. Then Then for all $F \in \mathcal{M}_\mu$, there exist sets E and $G \in \mathcal{M}$ such that*

$$E \subset F \subset G \qquad \text{and} \qquad \mu(G \backslash F) = 0 \ .$$

In particular, every E in $\mathcal{M}_\mu$ is the disjoint union of a set F in $\mathcal{M}$ and a set that is contained in set of μ-measure zero.

Proof. We claim that for all $E \in \mathcal{M}_\mu$, there exists $A \in \mathcal{M}$ such that $E \subset A$ and $\overline{\mu}(E) = \mu(A)$. If $\overline{\mu}(E) = \infty$, then we may take $A = X$. Suppose that $\overline{\mu}(E) < \infty$. Let μ^* be the outer measure on X defeined by (**??**). We have seen that μ^* is also given by (2.3). Then for each $n \in \mathbb{N}$, there exists $A_n \in \mathcal{M}$ with $E \subset A_n$ and $\overline{\mu}(E) + 1/n \geq \mu(A_n)$. Let $A = \cap_{n=1}^\infty A_n$. Define $B_n = \cap_{j=1}^n A_j$. Then $\{B_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of sets in $\mathcal{M}$ with $E \subset B_n$ for all n, $\cap_{n=1}^\infty B_n = A$, and $\overline{\mu}(E) + 1/n \geq \mu(B_n)$ for all n. By continuity from above, $\mu(A) = \lim_{n\to\infty} \mu(B_n)$. Hence $\overline{\mu}(E) \geq \mu(A)$. Since $E \subset A$, and $\mu(A) = \overline{\mu}(A)$, the opposite inequality is trivially true. Hence $\overline{\mu}(E) = \mu(A)$

Now suppose that $\mu(X) < \infty$. For any $E \in \mathcal{M}_\mu$, let $G \in M$ be such that $E \subset G$ and $\overline{\mu}(E) = \mu(G)$. Let $B \in M$ be such that $E^c \subset B$ and $\overline{\mu}(E^c) = \mu(B)$. Then $B^c \subset E$, and $B^c \in \mathcal{M}$. Since $\mu(X) < \infty$,

$$\mu(B^c) = \mu(X) - \mu(B) = \overline{\mu}(X) - \overline{\mu}(E^c) = \overline{\mu}(E) \ .$$

Then defining $F = B^c$, we have $F \subset E \subset G$ with $\mu(F) = \mu(G) = \overline{\mu}(E)$, which shows the existence of the equired sets in case $\mu(X) < \infty$.

In case μ is σ-finite. let $X - \cup_{n=1}^\infty A_n$ express X as the disjoint union of countably many sets in $\mathcal{M}$, each with finite measure. Applying the above to $E \cap A_n$, for each n, we find sets $F_n, G_n\ in\mathcal{M}$ and contianed in A_n such that $F_n \subset E \cap A_n \subset G_n$ and $\mu(G_n \backslash F_n) = 0$. Let $F = \cup_{n=1}^\infty F_n$ and let $G = \cup_{n=1}^\infty G_n$. Then

$$G \backslash F = \bigcup_{n=1}^\infty (G_n \backslash F_n)$$

expresses $G \backslash F$ as a countable union of sets of measure zero, and hence $\mu(G \backslash F) = 0$. □

2.8 THEOREM. *Let $\mathcal{A}$ be any algebra of subsets of A that includes X. Let m be a σ-finite premeasure on $\mathcal{A}$ that is continuous at the empty set. Let m^* be the outer measure defined by (2.1). Let μ be the restriction of m^* to $\sigma(\mathcal{A})$. Let ν be the restriction of m^* to ithe Caratheodory σ-algebra $\mathcal{M}$ of m^*. Then $(X, \mathcal{M}, \nu)$ is the completion of $(X, \sigma(\mathcal{A}), \mu)$. In particular, every set in $\mathcal{M}$ is the disjoint union of a set in $\sigma(A)$ and a subset of a set of μ-measure zero.*

Proof. By Lemma2.8, $m^* = \mu^*$, so $\mathcal{M}$ is the Caratheodory σ algebra of μ^*. It then follows from Theorem 2.7 that every set $F \in \mathcal{M}$ is the union (disjoint, even) of a set $E \in \sigma(\mathcal{A})$ and a set H that is contained in a set of μ-measure zero. Since μ and ν agree on $\sigma(\mathcal{A})$, any set of μ-measure zero is also a set of ν-measure zero, and, since $(X, \mathcal{M}, \nu)$ is complete by Theorem 2.2, $H\ in\mathcal{M}$ and $\nu(H) = 0$. Hence

$$\nu(F) = \nu(E \cup H) = \nu(E) + \nu(H) = \nu(E) = \mu(E) \ .$$

This shows that $\nu = \overline{\mu}$. □

3 Lebesgue-Stieltjes measures on $\mathbb{R}$

We are now ready to construct Lebesgue measure on $\mathbb{R}$. At the same time we shall construct a more general class of measures on $\mathbb{R}$. Our starting point is a premeasure on a. a subalgebra of the half open interval algebra. Recall that The *dyadic rational numbers* in $(0,1]$ are the numbers of the form $j2^{-n}$, $n \geq 0$, $j \in \mathbb{Z}$.

In the rest of this section, let $\mathcal{A}$ be the *dyadic half-open algebra*, which is the algebra consisting of all finite disjoint unions of intervals of the form $(a,b]$ where $a \leq b$ are dyadic rational numbers. It is easy to verify that this is an algebra. The length of such an interval is, of course, $b-a$. Moreover, since the dyadic rational numbers are dense in $\mathbb{R}$, it is easy to show that every open set in $\mathbb{R}$ is a countable union of dyadic half open intervals and of course each such interval is a Borel set. Thus, $\sigma(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$, the Borel σ-algebra of $\mathbb{R}$.

We begin with some definitions and terminology that will be used only in this section. If $(a,b]$ is any dyadic rational interval with finite endpoints, we may write

$$(a,b] = \left(\frac{j}{2^n}, \frac{k}{2^n}\right] \ ,$$

with the same power n in the denominator, and there is a unique least power of $n \geq 0$ for which we can do this. We call this integer n, which can be negative, the *rank* of the interval. For half open dyadic intervals of the form $(-\infty, a]$ or $(a, \infty]$ with a a dyadic rational. Let n be the least integer so that $a = k2^{-n}$ for some $k \in \mathbb{Z}$, we define the rank of such an interval to be n.

An *elementary dyadic rational interval* is one of the form

$$\left(\frac{i}{2^n}, \frac{i+1}{2^n}\right]$$

for some $i \in \mathbb{Z}$, and some non-negative integer n. Since either i or $i+1$ is odd, the rank of such an interval is n.

Next, if $A \in \mathcal{A}$, then A has a unique decomposition

$$A = \cup_{j=1}^{n}(a_, b_j] \qquad \text{where} \qquad -\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m \leq \infty \ . \tag{3.1}$$

we define the rank of A to be the maximum of the ranks of the intervals $(a_j, b_j]$. Now observe that every finite dyadic half open interval of rank n_0 can be written as a disjoint union of elementary dyadic half open intervals of rank n for any $n \geq n_0$:

$$\left(\frac{k}{2^{n_0}}, \frac{\ell}{2^{n_0}}\right] = \bigcup_{j=1}^{(\ell-k)2^{n-n_0}} \left(\frac{k2^{n-n_0}+j-1}{2^n}, \frac{k2^{n-n_0}+j}{2^n}\right] \ , \tag{3.2}$$

and making the obvious change in the indexing, this applies also to infinite dyadic half open interval of rank n_0. Then, if $A \in \mathcal{A}$ has rank n_0, and is given by (3.1) where each $(a_j, b_j]$ is a dyadic half open interval of rank n_0 with finite endpoints,

$$(a_j, b_j] = \bigcup \{ \text{ elementary dyadic half open intervals } I \text{ of rank } n \ : \ I \cap (a_j, b_j] \neq \emptyset \ \} \ , \tag{3.3}$$

since this follows from (3.2) applied to $(a_j, b_j]$.

3.1 LEMMA (Inclusion–exclusion property)**.** *Let A be any set in the dyadic half open interval algebra. Then there is a finite value of n_0 so that if I is any elementary dyadic interval of rank $n > n_0$, then either*

$$I \subset A \qquad \text{or else} \quad I \cap A = \emptyset \ .$$

Proof. Let n_0 be the rank of A. By the remarks made above, for each $n \geq n_0$, we can write A as a disjoint union of the elementary dyadic rationals of rank n, and to each of these (3.3) applies so that

$$A = \bigcup \{ \text{ elementary dyadic half open intervals } I \text{ of rank } n \ : \ I \cap A \neq \emptyset \ \} \quad .$$

□

With these preliminaries concerning our algebra dealt with, we define the premeasure we shall use:

Definition (Stieltjes premeasures on $\mathcal{A}$) Let F be an monotone non-decreasing function from $\mathbb{R}$ to $\mathbb{R}$ that is right continuous; i.e., for all $x \in \mathbb{R}$

$$F(x) = \lim_{h \downarrow 0} F(x+h) \ . \tag{3.4}$$

Let $F(-\infty) = \lim_{x \to -\infty} F(x)$ and $F(\infty) = \lim_{x \to \infty} F(x)$.

For any set A in $\mathcal{A}$ given by (3.1), define

$$m_F(A) = \sum_{j=1}^{m} [F(b_j) - F(a_j)] \ . \tag{3.5}$$

It is readily verified, using only the monotonicity of F, that m_F is a premeasure on $\mathcal{A}$. The set of premeasures obtained in this way is the set of *Stieltjes premeasures.*

In the special case in which $F(x) = x$for all x, m_F is the *Lebesgue premeasure*, and we simply write $m(A)$ in place of $m_F(A)$ for this choice of F.

All Stieltjes premeasure m_F are semifinite. Indeed, if $A \in \mathcal{A}$ and $m(A) = \infty$, then A contains a half open interval of the form $(a, \infty]$ or $(-\infty, b]$. If

$$\infty = m_F(((a, \infty]) = F(\infty) - F(a) \ ,$$

choose x so that $F(x) > F(x) + r$, $r > 0$ which is possible since $\lim_{x \to \infty} F(x) = \infty$. Then $(a, x] \subset (a, \infty]$ and $m_F((a, x]) > r$. The case of $(-\infty, b]$ is handled in the same manner.

It requires more effort to show that m is continuous at the empty set. That is the key to proving our next theorem.

3.2 THEOREM (Continuity at the empty set of Stieltjes premeasures)**.** *Let F be a monotone non-decreasing and right continuous function from $\mathbb{R}$ to $\mathbb{R}$. Then Stieltjes premeasure m_F on the dyadic interval algebra $\mathcal{A}$ in $\mathbb{R}$ is continuous at the empty set as well as semifinite, and thus extends to a countably additive measure μ that is defined on a $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A})$.*

Proof. Suppose that $\{A_j\}$ is a decreasing sequence of sets in $\mathcal{A}$; i.e., $A_{j+1} \subset A_j$ for all j, and suppose that

$$\lim_{j\to\infty} m_F(A_j) = c_0 \neq 0 \ , \tag{3.6}$$

and also $m(A_1) < \infty$. Then we must show that

$$\bigcap_{j=1}^{\infty} A_j \neq \emptyset \ .$$

If F is such that $-\lim_{x\to\infty} F(x) = \lim_{x\to\infty} F(x) = \infty$, then since $m_F(A_1) < \infty$ A_1 contains no infinite intervals, and hence is bounded. Thus, there is an $M \in \mathbb{N}$ so that

$$A_j \subset A_1 \subset (-2^M, 2^M] \tag{3.7}$$

for all j.

If $F(\infty) < \infty$, for all $\epsilon > 0$, there is a dyadic rational y so that $F(y) > F(\infty) - \epsilon$. Then for any $A \in \mathcal{A}$,

$$m_F(Acap(y,\infty]) = m_F((y,\infty]) - m_F(A^c cap(y,\infty]) < \epsilon \ .$$

Thus, picking $\epsilon < c_0/2$ we have $\cap_k(A_k \cap (y,\infty]) = \emptyset$ but $m(A_k \cap (y,\infty]) > c_0/2 > 0$ Thus replacing each A_k by $A_k \cap (y,\infty])$ we may assume without loss of generality that $A_1 \subset (-\infty, y]$ for some dyadic rational y.

Likewise, if $F(-\infty) > -\infty$, for all $\epsilon > 0$, there is a dyadic rational x such that $m_F((-\infty, x]) < \epsilon$, and arguing as above, we may assume without loss of generality that $A_1 \subset (x, y]$ for some dyadic rationals x and y. Pick N so that $|x|, |y| \leq 2^N$, and then (3.7) is true. Thus, we may assume that (3.7) is true for some $N \in \mathbb{N}$.

Let I_0 denote the interval $(-2^M, 2^M]$. We are going to produce, by a successive bisection procedure a nested sequence of half open dyadic intervals

$$I_0 \supset I_1 \supset I_2 \supset I_3 \ldots$$

in which I_{J+1} is obtained by bisecting I_j and keeping one of the two halves. That is, if $I_j = (a, b]$, we write

$$I_j = (a, (a+b)/2] \cup ((a+b)/2, b] \ ,$$

and choose (using a rule to be specified) I_{j+1} to be either the left or right interval. Since we halve the length each time, and since the length of I_0 is 2^{M+1}, I_{M+1} is an interval of unit length, and is of the form $(k, k+1]$ for some integer k. Thus it is an elementary dyadic interval of rank 0.

A bit of reflection shows that if one bisects an elementary dyadic rational interval in this way, each of the pieces is again an elementary dyadic rational interval whose rank is increased by one. Indeed,

$$\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right] = \left(\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right] \cup \left(\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right] \ .$$

Therefore, whatever rule we use to pick the left or right halves at each stage, I_{M+1+n} will be an elementary dyadic interval of order n.

It follows from our inclusion-exclusion lemma that for each fixed k, for all n large enough, either $I_n \subset A_k$, or $I_n \cap A_k = \emptyset$. *Thus, if we can show that for all n, $I_n \cap A_k \neq \emptyset$, we will have the much stronger statement that $I_n \subset A_k$ for all n sufficiently large.*

Let us suppose that we can do this, and moreover, can arrange that the interval we keep from each bisection is the one on the right infinitely many times. Let a_n and b_n be such that $I_n = (a_n, b_n]$. Then for all sufficiently large n (depending on k), $(a_n, b_n] \subset A_k$. Since we pick the right half infinitely often, we can find an n so that

$$(a_{n+1}, b_{n+1}] = ((a_n + b_b)/2, b + n] \subset (a_n, b_n] \subset A_k \ .$$

But then the *closed* interval $\overline{I_{n+1}}$ is contained in I_n, which is contained in A_k:

$$\overline{I_{n+1}} = [(a_n + b_b)/2, b + n] \subset (a_n, b_n] \subset A_k \ .$$

Since we produce the intervals by bisection, for all $k > n + 1$,

$$I_k \subset I_{n+1} \subset \overline{I_{n+1}} \subset A_k \ .$$

Note that $\{b_k\}_{k \in \mathbb{N}}$ is a decreasing sequence, and we have just shown that for all sufficiently large k, $b_k \in \overline{I_{n+1}}$, a *closed* subset of A_k. Therefore, $b := \lim_{k\to\infty} b_k$ exists and belongs to A_k. Since k is arbitrary, $b \in \cap_k = 1^\infty A_k$, and the intersection is not empty.

Thus, we will have accomplished our goal if we can devise a rule for carrying out the bisection so that:

(1) The nested sequence $\{I_j\}$ that we produce has the property that for all j and k, $I_j \cap A_k \neq \emptyset$.

(2) In producing our nested sequence $\{I_j\}$, we choose the right half interval infinitely often.

We now present the rule that accomplishes this.

Write I_0 as the union of its left and right halves

$$I_0 = J_0^{\text{left}} \cup J_0^{\text{right}} \ ,$$

bisecting as above. Since $A_k \subset I_0$ for all k, we have from (3.6) that

$$\lim_{k\to\infty} m_F(A_k \cap J_0^{\text{left}}) + \lim_{k\to\infty} m_F(A_k \cap J_0^{\text{right}}) = \lim_{k\to\infty} m_F(A_k) = c_0 > 0 \ .$$

If $\lim_{k\to\infty} m_F(A_k \cap J_0^{\text{right}}) > 0$, define this number to be c_1, and chose $I_1 = J_0^{\text{right}}$. Otherwise, $\lim_{k\to\infty} m_F(A_k \cap J_0^{\text{left}}) = c_0$, and we define $c_1 = c_0$, and $I_1 = J_0^{\text{left}}$. Either way, we have

$$\lim_{k\to\infty} m_F(A_k \cap I_1) = c_1 > 0 \ .$$

We proceed inductively as follows: At the nth stage, with I_n and $c_n > 0$ defined, we write $I_n = J_n^{\text{left}} \cup J_n^{\text{right}}$, and note that

$$\lim_{k\to\infty} m_F(A_k \cap J_n^{\text{left}}) + \lim_{k\to\infty} m_F(A_k \cap J_n^{\text{right}}) = \lim_{k\to\infty} m_F(A_k \cap I_n) = c_n > 0 \ .$$

If $\lim_{k\to\infty} m_F(A_k \cap J_n^{\text{right}}) > 0$, define this number to be c_n, and chose $I_{n+1} = J_n^{\text{right}}$. Otherwise, $\lim_{k\to\infty} m_F(A_k \cap J_n^{\text{left}}) = c_n$, and we define $c_{n+1} = c_n$, and $I_{n+1} = J_n^{\text{left}}$.

In this way, we produce a nested sequences of intervals $\{I_n\}$ such that $m_F(A_k \cap I_n) \geq c_n > 0$ for each k, and since $m_F(\emptyset) = 0$, this means that $A_k \cap I_n \neq \emptyset$ for any k and n. Thus *(1)* is accomplished.

Finally we show that the rule leads us to choose the right interval infinitely often. If this were not the case, then for some n_0, we would always choose the left interval in each later step. We would then have

$$\lim_{k\to\infty} m_F(A_k \cap I_n) = c_{n_0} > 0 \tag{3.8}$$

for all $n > n_0$. But since we always choose the left interval for $n > n_0$, for all such n, $I_n = (a, b_n]$ for some number a and some decreasing sequence $\{b_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} b_n = a$. Therefore, for all sufficiently large n,

$$m_F(A_k \cap I_n) \le m_F(I_n) = F(b_n) - F(a)\ ,$$

Since F is right continuous, the right hand side decreases to zero, (3.8) cannot hold for all $n \ge n_0$. Therefore *(2)* is verified, and the proof is complete. □

4 Locally finite Borel measures on $\mathbb{R}$

Let $(X, \mathcal{O})$ be a Hausdorff topological space. Then every compact set $K \subset X$ is closed, and hence is a Borel subset of X.

4.1 DEFINITION (Locally finite, inner and outer regular)**.** Let $(X, \mathcal{O})$ be a Hausdorff topological space, and let $\mathcal{B}$ be the corresponding σ-algebra. Let $\mathcal{M}$ be any σ-algebra containing $\mathcal{B}$, and Let μ be a measure on $\mathcal{M}$. Then:

(1) μ is *locally finite* in case for each $x \in X$ there is an open set U containing x with $\mu(U) < \infty$.

(2) μ is *outer regular* in case for each $E \in \mathcal{M}$,

$$\mu(E) = \inf\{\mu(U)\ :\ U \text{ open}\ ,\ E \subset U\ \} \tag{4.1}$$

(3) μ is *inner regular* in case for each $E \in \mathcal{M}$

$$\mu(E) = \sup\{\mu(K)\ :\ K \text{ compact}\ ,\ K \subset E\ \} \tag{4.2}$$

Observe that if μ is a locally finite regular Borel measure on X and K is a compact subset of X, for each $x \in K$ there is an open set U_x containing x such that $\mu(U_x)$ is finite. Since K is compact, there are points $x_1, \dots, x_n \in K$ so that $K \subset \cup_{j=1}^n U_{x_j}$. Then

$$\mu(K) \subset \sum_{j=1}^n \mu(U_{x_j}) < \infty\ .$$

That is, whenever μ is locally finite, $\mu(K) < \infty$ for all compact K. Conversely if X is a locally compact topological space, and μ is a Borel measure on X with the property that $\mu(K) < \infty$ for all compact K, then μ is locally finite. Consequently, if X is the countable union of compact sets, then any locally finite Borel measure μ on X is σ-finite. In particlar, every locally finite Borel measure on $\mathbb{R}$ is σ-finite.

We shall show that every locally finite Borel measure μ on $\mathbb{R}$ is generated by a Stieltjes premeasure m_F for some monotone non-decreasing, right continuous function F from $\mathbb{R}$ to $\mathbb{R}$. First note that by the remarks made just above, $\mu((a, b]) < \infty$ for all finite half open intervals $(a, b]$.

4.2 THEOREM (Locally finite Borel measures on $\mathbb{R}$ are generated by Stieltjes premeasures)**.** *Let μ be a locally finite measure defined on $\mathcal{B}_{\mathbb{R}}$, the Borel σ-algebra of $\mathbb{R}$. Define a function $F(x)$ on $\mathbb{R}$ by*

$$F(x) = \begin{cases} \mu((0,x]) & x \geq 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0 \ , \end{cases} \tag{4.3}$$

noting that the function is well defined since F is locally finite.

Then F is right continuous and monotone increasing. Let m_F be the Stieltjes premeasure on $\mathcal{A}$, the dyadic half open interval algebra, that is generated by F, and let μ_F is its Caratheodory extension restricted to $\sigma(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$ Then $\mu = \mu_F$. Consequently, every Borel measure on $\mathbb{R}$ is the restriction to $\mathcal{B}_{\mathbb{R}}$ of a measure obtained from a Stieltjes premeasure via the Caratheodory Extension Theorem.

Proof. The fact that F is monotone non-decreasing is an immediate consequence of the monotonicity property of measures. Next, for all $x \geq 0$

$$(0,x] = \cap_{n\in\mathbb{N}}(0, x+1/n]$$

and $\mu((0,x+1]) = F(x+1) < \infty$ since $[0,x+1/n]$ is compact so that $\mu([0,x+1]) < \infty$. Therefore, by continuity from above,

$$\lim_{n\to\infty} \mu((0,x+1/n]) = \mu((0,x]) \ ,$$

which means that $\lim_{n\to\infty} F(x+1/n) = F(x)$. Since F is monotone non-decreasing, this means that F is right continuous at each $x \geq 0$.

For $x < 0$ and $n > -1/x$, $F(x+1/n) = -\mu((x+1/n,0])$. Since $\cup_{n>-1/x}(x+1/n,0] = (x,0]$, it follows by continuity from below that $\lim_{n\to\infty} F(x+1/n) = F(x)$ for $x < 0$, and then since F is monotone decreasing, this means that F is right continuous at each $x < 0$. We have now shown that F is right continuous as well as monotone non-decreasing.

Now let m_F be the Stieltjes premeasure on $\mathcal{A}$, the dyadic half open interval algebra, that is generated by F, and let μ_F is its Caratheodory extension restricted to $\sigma(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$. The fact that $\mu_F = \mu$ now follows from Theorem 1.11 since m_F is σ-finite. □

We have just seen one way to construct an outer measure that extends any locally finite Borel measure μ on $\mathbb{R}$: Define the monotone right continuous function F by (4.3), and the Stieltjes premeaure m_F. Then let μ_F^* be defined as in Theorem 1.6 in terms of covering of an arbitrary set $E \subset \mathbb{R}$ with countably many sets in the dyadic half-open interval algebra $\mathcal{A}$,

There is another way to construct this same outermeasure that is more direct. Kowing that the two procedures give the same outer measure gives useful information about the original measure.

4.3 THEOREM. *Let μ be a locally finite Borel measure on $\mathbb{R}$. Let $(X, \mathcal{M}_\mu, \overline{\mu})$ denote its completion, and let μ^* be the outer masure induced by the premeasure m_F where F is defined by (4.3). Then for all $E \subset X$,*

$$\mu^*(E) = \inf\{\mu(U) \ : \ E \subset U \quad \text{and} \quad U \text{ open }\} \ , \tag{4.4}$$

and for all $B \subset \mathcal{M}_\mu$,

$$\overline{\mu}(B) = \inf\{\mu(U) \ : \ B \subset U \quad \text{and} \quad U \text{ open }\} \ . \tag{4.5}$$

Proof. In Theorem 1.6, μ^* is defined in terms of covering of an arbitrary set $E \subset \mathbb{R}$ with countably many sets in the dyadic half-open interval algebra $\mathcal{A}$. Since each set $A \in \mathcal{A}$ is a finite disjoint union of dyadic half open intervals, and since any countable union of finite unions is a countable union, it is the same to cover E by countably many dyadic half open intervals. Then since μ is the restriction of μ^* to $\mathcal{B}_\mathbb{R}$ and $m_F((a,b]) = \mu((a,b])$,

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu((a_n, b_n]) \ : a_n, b_n \text{ dyadic rational} \quad \text{and} \quad E \subset \bigcup_{n=1}^{\infty}(a_n, b_n] \right\} . \tag{4.6}$$

We are now ready to prove (4.4). Fix $E \subset \mathbb{R}$. If $\mu^*(E) = \infty$, we may take $U = \mathbb{R}$. Suppose that $\mu^*(E) < \infty$. Pick $\epsilon > 0$. Since the open sets are Borel, $\mu(U) = \mu^*(U) \geq \mu^*(E)$ for all open sets U with $E \subset U$. Thus it suffices to show that there exists an open set U with $E \subset U$ such that $\mu(U) \leq \mu^*(E) + \epsilon$, or, since $\mu^* = \mu^*_F$, that $\mu(U) \leq \mu^*_F(E) + \epsilon$

By the definition of μ_F, there exists a sequence $\{(a_j, b_j]\}_{j\in\mathbb{N}}$ of half open dyadic intervals such that $E \subset \cup_{j\in\mathbb{N}}(a_j, b_j]$ and

$$\mu(E) + \frac{\epsilon}{2} \geq \sum_{j=0}^{\infty} m_F((a_j, b_j]) \ .$$

Since F is right continuous, for each $j \in \mathbb{N}$, we may choose $c_j > b_j$ so that $m_F((b_j, c_j]) < \epsilon 2^{-j-1}$. Let $U = \cup_{j=1}^{\infty}(a_j, c_j)$. Then

$$\mu(U) = \mu^*_F(U) \leq \sum_{j=1}^{\infty} \mu^*_F((a_j, c_j]) \leq \sum_{j=1}^{\infty} \mu^*_F((a_j, b_j]) + \epsilon 2^{-j-1} \leq \mu^*_F(E) + \epsilon \ .$$

Finally (4.5) is true since $\overline{\mu}$ is the restriction of μ^* to $\mathcal{M}_\mu$, according to Theorem 4.5. □

4.4 DEFINITION (μ-measurable sets)**.** Let μ be a locally finite Borel measure on $\mathbb{R}$ and ket μ^* be the outer measure given by (**??**). Let $\mathcal{M}_\mu$ be the Caratheodory σ-algebra of μ^*, and note that this contans $\mathcal{B}_\mathbb{R}$. Let $\overline{\mu}$ be the restriction μ^* to $\mathcal{M}_\mu$, and note that this is an extension of μ to the larger σ-algebra $\mathcal{M}_\mu$, which is called the *σ-algebra of μ-measurable sets*. In particular, when μ is Lebesgue meausre, $\mathcal{M}_\mu$ is the σ-algebra of Lebesgue measurable sets.

4.5 THEOREM. *Let μ be a locally finite Borel measure on $\mathbb{R}$, and let $(X, \mathcal{M}_\mu, \overline{\mu})$ be the completion of $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu)$ Then $\overline{\mu}$ is both inner and outer regular. In particular, μ is both inner and outer regular on $\mathcal{B}_\mathbb{R}$.*

Proof. For any $E \in \mathcal{M}_\mu$, let $E_j = E \cap (j, j+1]$. Suppose that for any $\epsilon > 0$, we can find a compact set $K_j \subset E_j$ such that

$$\mu(K_j) = \overline{\mu}(K_j) > \overline{\mu}(E_j) - \epsilon 2^{-|j|+2} \ . \tag{4.7}$$

Then for each $N \in \mathbb{N}$, define $K^{(N)} = \cup_{j=-N}^{N} K_j$. Then each $K^{(N)}$ is compact and contained in E, and

$$\mu(K^{(N)}) = \overline{\mu}(K^{(N)}) \geq \overline{\mu}(\cup_{j=-N}^{N} E_j) - \epsilon \ .$$

By continuity from below, $\lim_{N\to\infty} \overline{\mu}(\cup_{j=-N}^{N} E_j) = \overline{\mu}(E)$. Thus,

$$\lim_{N\to\infty} \mu(K^{(N)}) \geq \overline{\mu}(E) - \epsilon \ .$$

Since $\epsilon > 0$ is arbitrary, this would prove that ν is inner regular. Therefore, it remains to show that we can find a compact subset K_j of E_j so that (4.7) is true.

Let $F_j = [j, j+1]\backslash E_j$, and let U_j be an open set containing F_j such that

$$\mu(U_j) < \overline{\mu}(F_j) + \epsilon 2^{-|j|+2} .$$

Let $K_j = [j, j+1]\backslash U_j$. Then K_j is closed and bounded, and hence is compact. Moreover, since $F_j \subset U_j$, and since $E_j \subset [j, j+1]$,

$$K_j = [j, j+1]\backslash U_j \subset [j, j+1]\backslash F_j = E_j .$$

Finally,

$$\mu(K_j) \geq \mu([j, j+1]) - \mu(U_j) \geq \mu([j, j+1]) - \overline{\mu}(F_j) - \epsilon 2^{-|j|+2} = \mu(E_j) - \epsilon 2^{-|j|+2} .$$

□

4.6 THEOREM. *Let μ be any locally finite Borel measure on $\mathbb{R}$. For all $E \in \mathcal{B}$ with $\mu(E) < \infty$, and all $\epsilon > 0$, there is a set $A \in \mathcal{A}$, the dyadic half open interval algebra, such that*

$$\mu(E\Delta A) < \epsilon .$$

Proof. Let $E_j = E \cap (-j, j]$. By continuity from below, $\lim_{j\to\infty} \mu(E_j) = \mu(E)$, and so there exists j such that $\mu(E\backslash E_j) < \epsilon/2$.

Define $\tilde{\mu}$ by $\tilde{\mu}(F) = \mu(E_j \cap F)$ for all $F \in \mathcal{B} = \sigma(\mathcal{A})$. Then $\tilde{\mu}$ is a finite measure on $\sigma(\mathcal{A})$, and then by a theorem we proved using the Monotone Class Theorem, there exits $A \in \mathcal{A}$ such that $\tilde{\mu}(A\Delta E_j) < \epsilon/2$. We can replace A by $A \cap (j, j+1]$ without affecting the value of $\tilde{\mu}(A\Delta E_j)$, and then since both A and E_j are subsets of $(j, j+1]$,

$$\mu((A\Delta E_j) = \tilde{\mu}(A\Delta E_j) < \epsilon/2 .$$

Finally, we have

$$\mu(A\Delta E) \leq \mu(A\Delta E_j) + \mu(E_j\Delta E) < \epsilon .$$

□

The previous theorem has many applications. Our first will be to prove tht continuous compactly supported functions are dense in $L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ for all locally finite Borel measures μ on $\mathbb{R}$. In fact, we shall prove somewhat more:

4.7 THEOREM. *Let μ be a locally finite Borel measure on $\mathbb{R}$. Then the set $C_c^\infty(\mathbb{R})$ of compactly supported and inifnitely differentiable functions is dense in $L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$.*

Proof. We know that the simple functions are dense in $L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$. Thus, it suffices to show that for all $E \in \mathcal{B}_{\mathbb{R}}$ with $\mu(E) < \infty$ and all $\epsilon > 0$, we can find a function $\varphi \in C_c^\infty(\mathbb{R})$ such that $\|1_E - \varphi\|_1 < \epsilon$. By Theorem 4.6, there is a set $A \in \mathcal{A}$, the half open interval algebra, such that $\|1_E - 1_A\|_1 < \epsilon/2$. Then for some $m \in \mathbb{N}$ and $-\infty \leq a_1 < b_1 < a_2 \dots a_m < b_m \leq \infty$,

$$1_A = \sum_{j=1}^m 1_{(a_j, b_j]} .$$

Suppose that $a_1 = \infty$. By continuity from below, $\lim_{n\to\infty} \mu((-n, b_1]) = \mu((-\infty, b_1])$, and hence we may replace a_1 by $-n$ for sufficiently large n, and still maintain $\|1_E - 1A\|_1 < \epsilon/2$. A similar arument shows that it is no loss of generality to assume that $b_m < \infty$. Thus, we may suppose that 1_A is supported in a compact interval. It is now east to "round the corners" on each $1_{(a_j,b_j]}$ to prove the theorem. □

We close with a theorem on a particularly important locally finite Borel measure, namely Lebesgue measure. We shall prove the translation invariance of Lebesgue measure on $\mathbb{R}$.

For fixed $a \in \mathbb{R}$, define the function $\tau_a : \mathbb{R} \to \mathbb{R}$ by $\tau_a(x) = x - a$. Clearly this is continuous, and hence Borel measurable. If $E \in \mathcal{B}$, then 1_E is a Borel measurable function. Since compositions of measurable functions are measurable, $1_E \circ \tau_a$ is measurable. But

$$1_E \circ \tau_a = 1_{E+a}$$

where

$$E + a = \{x + a \ : \ x \in E\ \} \ .$$

This shows that $E+a \in \mathcal{B}$ whenever E in B, and *vice-versa*. In this sense, $\mathcal{B}$ is translation invariant. We have proved part of the following theorem:

4.8 THEOREM (Translation invariance of Lebesgue measure on $\mathbb{R}$)**.** *Let μ^* denote Lebesgue measure on $\mathbb{R}$. For any $E \subset \mathbb{R}$, $\mu^*(E + a) = \mu^*(E)$. Consequently, if μ denotes Lebesgue measure on $\mathcal{B}$, then For any $E \in \mathcal{B}$ and $a \in \mathbb{R}$, $E + a \in \mathcal{B}$ and*

$$\mu(E + a) = \mu(E)\ .$$

Proof. Let m denote the Lebesgue premeasure. It is clear form the definition of m that $m(A + a) = m(A)$. If $\{A_n\}_{n\in\mathbb{N}}$ is any sequence in $\mathcal{A}$, the dyadic half open interval algebra, such that $E \subset \cup_{n=1}^\infty A_n$, then $E + a \subset \cup_{n=1}^\infty (A_n + a)$ and *vice-versa*. By the translation invariance of Lebesgue premeasure,

$$\sum_{n=1}^\infty m(A_n) = \sum_{n=1}^\infty m(A_n + a)$$

and hence, taking the infimum over all such coverings, $\mu^*(E) = \mu^*(E + a)$.

The rest of the theorem now follows from what we have noted above about the translation invariance of $\mathcal{B}$, and the fact that for all $E \in \mathcal{B}$, $\mu(E) = \mu^*(E)$. □

5 Exercises

1. Let A be a subset of $[0, 1]$. Let μ^* denote Lebesgue outer measures on $[0, 1]$. Show that an arbitrary set $A \subset [0, 1]$ is Lebesgue measurable if and only if

$$\mu^*(A) + \mu^*(A^c) = 1$$

where A^c is the complement of A.

2. Let μ^* denote Lebesgue outer measure on $\mathbb{R}$. Let A and B be any two subsets of $\mathbb{R}$ that are separated by a positive distance d. That is, if $x \in A$ and $y \in B$, then $|x - y| \geq d > 0$. Show that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)\ .$$

3. Let μ be Lebesgue measure on $\mathbb{R}$. Let $\mathcal{F}$ be a countable family of Lebesgue measurable real valued functions f on $\mathbb{R}$, and let $E \subset \mathbb{R}$ be a measurable set with $\mu(E) < \infty$. Suppose that for each $x \in E$, there is a $M_x < \infty$ so that $f(x) \leq M_x$ for all $f \in \mathcal{F}$. Show that for each $\epsilon > 0$, there exists a closed set $F \subset E$ and an $M < \infty$ such that $\mu(E \cap F^c) \leq \epsilon$ and $f(x) \leq M$ for all $x \in F$ and $f \in \mathcal{F}$.

4. Let μ be Lebesgue measure on $\mathbb{R}$. Given functions $f_n : [0,1] \to [-1,1]$ such that $\lim_{n\to\infty} \int_{(a,b)} f_n \mathrm{d}\mu = 0$ for all $0 \leq a < b \leq 1$, show that for every Lebesgue measurable set $E \subset [0,1]$,

$$\lim_{n\to\infty} \int_E f_n \mathrm{d}\mu = 0 \ ,$$

and then, more generally that for every Lebesgue integrable function on $[0,1]$,

$$\lim_{n\to\infty} \int_{[0,1]} f_n f \mathrm{d}\mu = 0 \ ,$$

5. Let μ be Lebesgue measure on $[0,1]$. Let $f \in L^1([0,1])$. Show that for all $\epsilon > 0$, there exists a compact $K \subset [0,1]$ such that $\mu(K) < 1 - \epsilon$ and a polynomial p such that $|f(x) - p(x)| < \epsilon$ for all $x \in$.

6. Let μ be Lebesgue measure on $[0,1]$. Let $f \in L^1([0,1])$. Show that if

$$\int_{[0,1]} f(x) x^n \mathrm{d}\mu(x) = 0$$

for all $n \in \mathbb{N}$, then $f = 0$ a.e.

7. Let μ be Lebesgue measure on $[0,1]$. Let A, B, E and F be subsets of $[0,1]$. Suppose that F and F are Lebesgue measurable, $A \subset E$, $B \subset F$ and $\mu(E) + \mu(F) = 1$. Show that A and B are Lebesgue measurable.

8. Let $E \subset \mathbb{R}$ be non-Lebesgue measurable. Let $A \subset \mathbb{R}$ be a set of Lebesgue measure zero. Show that $E \cap A^c$ is not Lebesgue measurable.