Notes on Integration

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Contents

1 Introduction

1.1 Measurable functions with values in $[0, \infty]$

The extended non-negative real numbers are the one point compactification of $[0, \infty)$: We adjoin a "point at infinity", denoted by ∞, and equip $[0, \infty) \cup \{\infty\}$ with smallest topology containing all open sets in $[0, \infty)$ in the usual metric topology, together with all sets of the form $(b, \infty) \cup \{\infty\}$, $b \in [o, \infty)$, which we usually denote by $(b, \infty]$. Likewise, we denote $[0, \infty) \cup \{\infty\}$ by $[0, \infty]$. It is easy to see that with this topology $[0,\infty]$ is compact, and we write $\mathcal{B}_{[0,\infty]}$ to denote the corresponding Borel σ-algebra.

Let a measurable space $(X,\mathcal{M})$ be a given. Then whenver we refer to a $[0,\infty]$ valued function on X as being measurable, we shall mean that it is measurable from $(X,\mathcal{M})$ to $([0,\infty],\mathcal{B}_{[0,\infty]})$.

We extend addition and multiplication from $[0,\infty)$ to $[0,\infty]$ through the rules $a+\infty=\infty+a=\infty$ for all $a\in[0,\infty]$ and $0\cdot\infty=\infty\cdot 0=0$, and finally, $a\cdot\infty=\infty\cdot a=\infty$ for all $a\in(0,\infty]$.

1.1 DEFINITION (L^+). *Let $(X,\mathcal{M})$ be a measurable space. Then $L^+(X,\mathcal{M})$ is the set of all measurable functions f on X with values in $[0,\infty]$. If the measurable space $(X,\mathcal{M})$ is clear from the context, we simply write L^+ in place of $L^+(X,\mathcal{M})$.*

1.2 Simple functions

1.2 DEFINITION (Simple functions). *Let $(X,\mathcal{M})$ be a measurable space. Then the* simple functions *on $(X,\mathcal{M})$ are the complex values functions that are $(X,\mathcal{M})-(\mathbb{C},\mathcal{B}_{\mathbb{C}})$ measurable,and which take on only finitely many values in $\mathbb{C}$. That is $\{f(x)\ : x\in X\}$ is a finite subset of $\mathbb{C}$.*

Let f be a simple function on some measurable space $(X,\mathcal{M})$. Let $\{z_1,\dots,z_k\}$ be the set of values of f, so that

$$\{f(x)\ : x\in X\}=\{z_1,\dots,z_k\}\ .$$

Define $E_j=f^{-1}(\{z_j\})$ which belongs to $\mathcal{M}$ since f is measurable. Then

$$f(x)=\sum_{j=1}^{k} z_j 1_{E_j}\ . \tag{1.1}$$

This is the *standard representation* of the simple function f. Any finite linear combination of indicator functions of sets in $\mathcal{M}$ is a simple function: If $\{F_1,\dots,F_m\}\subset\mathcal{M}$, and $\{y_1,\dots,m\}\subset\mathbb{C}$, then the function

$$g=\sum_{\ell=1}^{m} y_\ell 1_{F_\ell}$$

is a simple function: It is measuable since the indicator functions are measurable, and linear combination of complex valued measuralbe functions are measurable. Moreover, g assumes at most 2^m values:

$$g(x)=\sum_{\ell=1}^{m} y_\ell 1_{F_\ell}(x)\ ,$$

and $1_{F_\ell}(x)\in\{0,1\}$ for each x, so $g(x)$ can only be the sum of the numbers in one of the 2^m subsets of $\{y_1,\dots,m\}$

It is clear from this discussion that any convex combination of simple functions is again a simple function. That is the, the ste of simple functions, equipped with the usual algebraic operations, contitutes a vector space over $\mathbb{C}$.

1.3 Integration in L^+

Let $(X,\mathcal{M},\mu)$ be a measure space. Let φ be a simple function in L^+. That is, φ is a simple function, each of whose values is in $[0,\infty)$. (Note that simple functions, by definition, take there values in $\mathbb{C}$ which does not include ∞. Thus, the intersection of L^+ and the set of simple functions consists of measuarable functions with vlaues in $[0,\infty)$.

1.3 DEFINITION (Integral of a simple function in L^+)**.** *Let φ be a simple function in $L^+(X,\mathcal{M})$. Let μ be a measure on $(X,\mathcal{M})$. Let*

$$\varphi = \sum_{j=1}^{k} a_k 1_{E_j}$$

be the standard representation of φ, so that, in particular, $E_i \cap E_j = \emptyset$ when $i \neq j$. Then the integral of φ with respect to μ is given by

$$\int_X \varphi \mathrm{d}\mu = \sum_{j=1}^{k} a_j \mu(E_j) \ .$$

Notice that if

$$\varphi = \sum_{\ell=1}^{m} b_\ell 1_{F_\ell}$$

is any other way of writing φ in which $F_i \cap F_j = \emptyset$ whenever $i \neq j$, then each b_ℓ is one of the values a_j of g, and

$$E_j = \bigcup \{ \ E_\ell \ : \ b_\ell = a_j \ \} \ .$$

Consequently, since μ is additive,

$$\sum_{\ell=1}^{m} b_\ell \mu(F_\ell) = \sum_{j=1}^{k} a_j \left(\sum_{\{\ell \ : \ b_\ell = a_j\}} \mu(F_\ell) \right) = \sum_{j=1}^{k} a_j \mu(E_j) = \int_X \varphi \mathrm{d}\mu \ .$$

Thus, we can compute the integral of a simple function using any representation of it as a linear combination of indicator functions of *disjoint* measurable sets. The next lemma will show that even the restriction of disjointness is superfluous:

1.1 LEMMA. *Let $(X,\mathcal{M},\mu)$ be a measure space. Let φ and ψ be simple functions in $L^+(X,\mathcal{M})$. Then:*

(1) If $\varphi \geq \psi$, then $\int_X \varphi \mathrm{d}\mu \geq \int_X \psi \mathrm{d}\mu$.

(2) For all $s,t \in [0,\infty)$, $\int_X (s\varphi + t\psi)\mathrm{d}\mu = s\int_X \psi \mathrm{d}\mu + t\int_X \psi \mathrm{d}\mu$.

Proof. Let $\varphi = \sum_{j=1}^{k} a_j 1_{E_j}$ and $\psi = \sum_{\ell=1}^{m} b_\ell 1_{F_j}$ be the standard representation of φ and ψ. Define $G_{j,\ell} = E_j \cap F_\ell$. Then, since the sets in the standard representation are disjoint, $G_{j,\ell} \cap G_{j',\ell'} = \emptyset$ unless $\ell = \ell'$ and $j = j'$. Clearly,

$$\varphi = \sum_{\ell=1}^{m} \sum_{j=1}^{k} a_j 1_{G_{j,\ell}} \qquad \text{and} \qquad \psi = \sum_{\ell=1}^{m} \sum_{j=1}^{k} b_\ell 1_{G_{j,\ell}} \ .$$

From this representation, it is clear that $\varphi \geq \psi$ if and only if $a_, \geq b_\ell$ whenver $G_{j,\ell} \neq \emptyset$. Since we may use any representation in terms of linear combinations of indicator functions of disjoint sets, follows from the fact that the integrals are given by

$$\int_X \varphi \mathrm{d}\mu - \int_X \psi \mathrm{d}\mu = \sum_{\ell=1}^{m} \sum_{j=1}^{k} (a_j - b_\ell)\mu(G_{j,\ell}) \geq 0 \ ,$$

and this proves *(1)*.

Likewise

$$\begin{aligned}\int_X (s\varphi + t\psi)\mathrm{d}\mu &= \sum_{\ell=1}^{m}\sum_{j=1}^{k}(sa_j + tb_\ell)\mu(G_{j,\ell})\\ &= s\sum_{\ell=1}^{m}\sum_{j=1}^{k}a_j\mu(G_{j,\ell}) + t\sum_{\ell=1}^{m}\sum_{j=1}^{k}b_\ell\mu(G_{j,\ell})\\ &= s\int_X \varphi\mathrm{d}\mu + t\int_X \psi\mathrm{d}\mu\ .\end{aligned}$$

□

1.4 DEFINITION. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f \in L^+(X, \mathcal{M})$. Then the* integral of f with respect to μ *is dfined by*

$$\int_X f\mathrm{d}\mu = \sup\left\{ \int_X \varphi\mathrm{d}\mu\ :\ \varphi \in L^+ \quad \text{is simple, and} \quad \varphi \le f \right\}\ . \tag{1.2}$$

Note that $\int_X f\mathrm{d}\mu$ is always well-defined, although the value may be $+\infty$.

1.2 LEMMA. *Let Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f, g\ in L^+$ with $f \le g$. Then*

$$\int_X f\mathrm{d}\mu \le \int_X g\mathrm{d}\mu\ .$$

Proof. The class of simple functions in L^+ dominated by g contains the class of simple functions in L^+ dominated by f. □

The next theorem is a simple but important consequence of the "continuity from below" property of measures.

1.3 THEOREM (Lebesgue Monotone Convergence Theorem)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in L^+ such that $f_{n+1} \ge f_n$ for all $n \in \mathbb{N}$. Let $f = \lim_{n\to\infty} f_n$, which is an element of L^+. Then*

$$\int_X f\mathrm{d}\mu = \lim_{n\to\infty}\int_X f\mathrm{d}\mu\ .$$

Proof. Since $f_n \le f_{n+1} \le f$ for all $n \in \mathbb{N}$, Lemma 1.2 implies that

$$\int_X f_n\mathrm{d}\mu \le \int_X f_{n+1}\mathrm{d}\mu \le \int_X f\mathrm{d}\mu$$

for all $n \in \mathbb{N}$. Consequently, $\displaystyle\lim_{n\to\infty}\int_X f_n\mathrm{d}\mu$ exists, and

$$\lim_{n\to\infty}\int_X f_n\mathrm{d}\mu \le \int_X f\mathrm{d}\mu\ . \tag{1.3}$$

It remains to show that

$$\lim_{n\to\infty}\int_X f_n\mathrm{d}\mu \ge \int_X f\mathrm{d}\mu\ . \tag{1.4}$$

Let φ be any simple function in L^+ such that $\varphi \le f$. Let α be any number in $(0,1)$. Define

$$F_n = \{ \ x \ : \ f_n(x) > \alpha\varphi(x) \ \} \ .$$

Since $f_{n+1} \ge f_n$ for all $n \in \mathbb{N}$, $F_n \subset F_{n+1}$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} f_n(x) = f(x) > \alpha\varphi$, $\cup_{n=1}^{\infty} F_n = X$.

Let $\varphi = \sum_{j=1}^{k} a_j 1_{E_j}$ be the standard representation of φ. Since $f_n \ge 1_{F_n} f_n \ge \alpha 1_{F_n}\varphi$, Lemma 1.2 implies that

$$\int_X f_n \mathrm{d}\mu \ge \int_X 1_{F_n} f_n \mathrm{d}\mu \ge \alpha \int_X 1_{F_n}\varphi \mathrm{d}\mu \ . \tag{1.5}$$

However $\alpha 1_{F_n}\varphi = \sum_{j=1}^{k} \alpha a_j 1_{E_j\cap F_n}$ is simple, and hence

$$\int_X \alpha 1_{F_n}\varphi \mathrm{d}\mu = \alpha \sum_{j=1}^{k} a_j \mu(E_j \cap F_n) \ .$$

By continuity from below, $\lim_{n\to\infty} \mu(E_j \cap F_n) = \mu(E_j)$, and hence

$$\lim_{n\to\infty} \int_X \alpha 1_{F_n}\varphi \mathrm{d}\mu = \alpha \int_X \varphi \mathrm{d}\mu \ .$$

Combining this with (1.5), $\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu \ge \alpha \int_X \varphi \mathrm{d}\mu$, and then since $\alpha \in (0,1)$ is arbitrary, $\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu \ge \int_X \varphi \mathrm{d}\mu$. Since this is true for all simple φ in L^+ with $\varphi < f$, (1.4) now follows from the definition of the integral. □

1.4 LEMMA. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f, g \in L^+$ and let s, t in$(0,1)$. Then*

$$\int_X (sf + ts)\mathrm{d}\mu = s \int_X f \mathrm{d}\mu + t \int_X g \mathrm{d}\mu \ .$$

Proof. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ ge a sequence of simple functions in L^+ that increases pointwise to f, and let $\{\psi_n\}_{n\in\mathbb{N}}$ ge a sequence of simple functions in L^+ that increases pointwise to g. We have seen ealrier that such sequences exist. Then $\{(s\varphi_n + t\psi_n)\}_{n\in\mathbb{N}}$ is a sequence of simple functions in L^+ that increases pointwise to $sf + tg$. By Theorem 1.3 and then Lemma 1.1,

$$\begin{aligned}
\int_X (sf + ts)\mathrm{d}\mu &= \lim_{n\to\infty} \int_X (s\varphi_n + t\psi_n)\mathrm{d}\mu \\
&= \lim_{n\to\infty} \left(s \int_X \varphi_n \mathrm{d}\mu + t \int_X \psi_n \mathrm{d}\mu \right) \\
&= s \lim_{n\to\infty} \int_X \varphi_n \mathrm{d}\mu + t \lim_{n\to\infty} \int_X \psi_n \mathrm{d}\mu = s \int_X f \mathrm{d}\mu + t \int_X g \mathrm{d}\mu \ .
\end{aligned}$$

□

The following important thoerem is used so frequently that it is known as Fatou's Lemma.

1.5 THEOREM (Fatou's Lemma)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in L^+. The*

$$\int_X (\liminf_{n\to\infty} f_n)\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n \mathrm{d}\mu \ .$$

Proof. Define the function g by $g_m(x) = \inf_{n\ge m}\{f_n\}$. Since each f_n is measurable, so is g_m, so that $g_m\ inL^+$. Clearly, $g_{m+1} \ge g_m$ for all $m \in \mathbb{N}$. Moreover, by definition, $\lim_{m\to\infty} g_m = \liminf_{n\to\infty} f_n$. Thus, by Theorem 1.3,

$$\int_X (\liminf_{n\to\infty} f_n)\mathrm{d}\mu = \lim_{n\to\infty} \int_X g_m \mathrm{d}\mu \ .$$

By Lemma 1.2,

$$\int_X g_m \le \inf_{n\ge m} \left\{ \int_X f_n \mathrm{d}\mu \right\} \ .$$

Therefore,

$$\lim_{n\to\infty} \int_X g_m \mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n \mathrm{d}\mu \ .$$

□

2 Integration of complex valued measurable functions

2.1 Integrable functions

2.1 DEFINITION. *Let $(X, \mathcal{M}, \mu)$ be a measure space. A real valued measurable function f on $(X, \mathcal{M})$ is* integrable *in case $\int_X |f|\mathrm{d}\mu < \infty$, and then we define*

$$\int_X f\mathrm{d}\mu = \int_X f_+\mathrm{d}\mu - \int_X f_-\mathrm{d}\mu$$

where $f = f_+ - f_-$ is the decomposition of f into its positive and negative parts. A complex valued measurable function f on $(X, \mathcal{M})$ is integrable *in case its real and imaginary parts are itegrable, and then we define*

$$\int_X f\mathrm{d}\mu = \int_X \Re f\mathrm{d}\mu + i\int_X \Im f\mathrm{d}\mu \ .$$

Note that if f and g are integrableon $(X, \mathcal{M}, \mu)$, and $h = f + g$, then $h_+ - h_- = f + g = f_+ + g_+ - f_- - g_-$, so that

$$h_+ + f_- + g_- = h_- + f_+ + g_+ \ .$$

Unsing the additivity of integration on L^+ proved in Lemma 1.4, we have

$$\int_X h_+\mathrm{d}\mu + \int_X f_-\mathrm{d}\mu + \int_X g_-\mathrm{d}\mu = \int_X h_-\mathrm{d}\mu + \int_X f_+\mathrm{d}\mu + \int_X g_+\mathrm{d}\mu \ .$$

Raranging tems we have that

$$\int_X (f+g)\mathrm{d}\mu = \int_X f\mathrm{d}\mu + \int_X g\mathrm{d}\mu \ .$$

Next, let f be a real valued integrable function on $(X,\mathcal{M},\mu)$, and let $a\in\mathbb{R}$, af is integrable. Moreover, if $a>0$, $(af)_+ = af_+$ and $(af)_- = af_-$, and hence by Lemma 1.4

$$\int_X af\mathrm{d}\mu = a\int_X f_+\mathrm{d}\mu - a\int_X f_-\mathrm{d}\mu = a\int_X f\mathrm{d}\mu\ .$$

Likewise, if $a<0$, $(af)_+ = |a|f_-$ and $(af)_- = |a|f_+$, and then by Lemma 1.4 once more,

$$\int_X af\mathrm{d}\mu = |a|\int_X f_-\mathrm{d}\mu - |a|\int_X f_+\mathrm{d}\mu = a\int_X f\mathrm{d}\mu\ .$$

Altogether, we have proved:

2.1 THEOREM. *Let $(X,\mathcal{M},\mu)$ be a measure space. The set of complex valued integrable functions on $(X,\mathcal{M},\mu)$ is a vector space over $\mathbb{C}$ when equaipped with the usual algebraic operations. Motover, the map*

$$f\mapsto \int_X f\mathrm{d}\mu$$

is a linear functional on this vector space.

2.2 THEOREM. *Let $(X,\mathcal{M},\mu)$ be a measure space, and let f be a complex valued integrable function on $(X,\mathcal{M},\mu)$. Then*

$$\left|\int_X f\mathrm{d}\mu\right| \le \int_X |f|\mathrm{d}\mu \tag{2.1}$$

and

$$\int_X |f|\mathrm{d}\mu = 0 \quad\iff\quad \mu(\{x\ :\quad f(x)\neq 0\ \}) = 0\ . \tag{2.2}$$

Proof. If f is real valued, then

$$\left|\int_X f\mathrm{d}\mu\right| = \left|\int_X f_+\mathrm{d}\mu - \int_X f_-\mathrm{d}\mu\right| \le \int_X f_+\mathrm{d}\mu + \int_X f_-\mathrm{d}\mu = \int_X |f|\mathrm{d}\mu\ .$$

For f complex valued, let $e^{i\theta}$ be such that $e^{i\theta}\displaystyle\int_X |f|\mathrm{d}\mu \ge 0$. Then

$$\left|\int_X f\mathrm{d}\mu\right| = e^{i\theta}\int_X f\mathrm{d}\mu = \int_X e^{i\theta}f\mathrm{d}\mu = \Re\left(\int_X e^{i\theta}f\mathrm{d}\mu\right) = \int_X \Re(e^{i\theta}f)\mathrm{d}\mu$$

By the first part, $\int_X \Re(e^{i\theta}f)\mathrm{d}\mu \le \int_X |\Re(e^{i\theta}f)|\mathrm{d}\mu \le \int_X |f|\mathrm{d}\mu$, and thus (2.1) is proved.

Next, suppose that $\displaystyle\int_X |f|\mathrm{d}\mu = 0$. Let $E_n = \{x\ :\ |f(x)| > 1/n\ \}$. Then $E_n\in\mathcal{M}$, and $n^{-1}1_{E_n}\le |f|$, Therefore, $\mu(E_n) = \displaystyle\int_X 1_{E_n}\mathrm{d}\mu \le n\int_X |f|\mathrm{d}\mu = 0$. Then

$$\mu(\{x\ :\quad f(x)\neq 0\ \}) = \mu\left(\bigcup_{n=1}^\infty E_n\right) \le \sum_{j=1}^\infty \mu(E_n) = 0\ .$$

Conversely, suppose that $\mu(\{x\ :\quad f(x)\neq 0\ \}) = 0$. If $\varphi = \displaystyle\sum_{j=1}^n a_j 1_{F_j}$ is any simple function in L^+ with $\varphi\le|f|$, it follows that $\mu(F_j)=0$ for $j=1,\dots,m$, and hence $\displaystyle\int_X \varphi\mathrm{d}\mu = 0$. Therefore, $\displaystyle\int_X |f|\mathrm{d}\mu = 0$. □

2.2 DEFINITION (Almost everywhere convergence). *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f and f be functions on X, Then $f = g$* almost everywhere with respect to μ *in case there is $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $f(x) = g(x)$ for all $x \in E^c$. We often abbreviate this by writing $f = g$ a.e. μ.*

By the previous thoerem, changing an integrable function on a set of measure zero does not change its integral. Thus, when f and g are integrable functions on $(X, \mathcal{M}, \mu)$

$$f = g \text{ .e. } \mu \quad \Rightarrow \int_X f\mathrm{d}\mu = \int_X g\mathrm{d}\mu \ .$$

2.3 DEFINITION (Almost everywhere convergence). *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of complex valued functions on X, and let f be a complex values functions on X. We say that $\{f_n\}_{n\in\mathbb{N}}$* converges to f almost everywhere *with respect to μ if there is a set $E \in \mathcal{M}$ such that $\mu(E) = 0$, and for all $x \in E^c$, $\lim_{n\to\infty} f_n(x) = f(x)$. We often abbreviate this by writing $f_n \to f$ a.e. μ.*

Note that if $f_n \to f$ a.e. μ, and each f_n is measuaable, we may define $\tilde{f}_n = 1_{E^c} f_n$ and $\tilde{f} = 1_E f$. Then $\lim_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)$ for all x, and since $\tilde{f}_n$ is measurable for each n, so is $\tilde{f}$.

2.2 The Lebesgue Dominated Convergence Theorem

2.3 THEOREM (Lebesgue Dominated Convergence Theorem). *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of complex integrable functions on X, and suppose that: for some $g \in L^+(X, \mathcal{M}, \mu)$ such that $\displaystyle\int_X g\mathrm{d}\mu < \infty$ and that $\lim_{n\to\infty} f_n(x) =: f(x)$ for all x. Then*

$$\int_X f\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n\mathrm{d}\mu \ . \tag{2.3}$$

Proof. Since $f = \lim_{n\to\infty} f_n$, f is mesurable. Since $|f| = \lim_{n\to\infty} |f_n| \le g$, $\int_X |f|\mathrm{d}\mu < \infty$, so f is integrable. Note that $g + f_n \ge 0$ for all n and $\lim_{n\to\infty}(g + f_n)(x) = g(x) + f(x)$. By Fatou's Lemma,

$$\int_X g\mathrm{d}\mu + \int_X f\mathrm{d}\mu = \int_X (g+f)\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X (g+f_n)\mathrm{d}\mu = \int_X g\mathrm{d}\mu + \liminf_{n\to\infty} \int_X f_n\mathrm{d}\mu \ .$$

This shows that

$$\int_X f\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n\mathrm{d}\mu \ . \tag{2.4}$$

Now note that $g - f_n \ge 0$ for all n and $\lim_{n\to\infty}(g - f_n)(x) = g(x) - f(x)$. By Fatou's Lemma,

$$\int_X g\mathrm{d}\mu - \int_X f\mathrm{d}\mu = \int_X (g-f)\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X (g-f_n)\mathrm{d}\mu = \int_X g\mathrm{d}\mu - \liminf_{n\to\infty} - \int_X -f_n\mathrm{d}\mu \ .$$

This shows that

$$\int_X f\mathrm{d}\mu \ge \limsup_{n\to\infty} \int_X f_n\mathrm{d}\mu \ . \tag{2.5}$$

Together, (2.4) and (2.5) imply (2.3). $\square$

2.4 Remark. If as sequence $\{f_n\}_{n\in\mathbb{N}}$ is such that it only converges to f almost everywhere, but the conditions to the Lebesgue Dominated Convergence Theorem are otherwise met, we may redefine each f_n and f to be, say, zero on the exceptional set of non-convergence. The thoerem then applies to the modified functions, but since the modified functions have the same integrals, the conclusion applies to the original fucntions. By an large in integration thoery, the distinction be convergence almost everywhere convergence and pointewise convergence is immaterial. However, the concept of equivalence almost everywhere is crucial for what comes next.

3 The metric space $L^1(X, \mathcal{M}, \mu)$

Let $(X, \mathcal{M}, \mu)$ be a measure space. We write $f \sim g$ in case $f = g$ a.e. μ. It is then easy to check that $\sim$ is an *equivalence relation* on the set of complex valued measurable functions on $(X, \mathcal{M}, \mu)$. That is, for all measurable f, g and h, $f \sim f$ (reflexivity), $f \sim g \iff g \sim f$ (symmetry) and whenver $f \sim g$ and $g \sim h$, then $f \sim h$ (transitivity).

Note that if f and g are integrable, then

$$f \sim g \quad \iff \quad \int_X |f - g|\mathrm{d}\mu = 0 \ . \tag{3.1}$$

Let f be measurable on $(X, \mathcal{M}, \mu)$, and let $\{\tilde{f}\}_\sim$ donte the equivalence class of f in the set of measurable functions. If moreover f in sintegrable, so are all representatives of the equivalence class of f, and they all have the same integral. That is, the integral is a function on the set of equivalence classes of integrable functions, and its value is given by integrating any representative of the equivalence class.

3.1 DEFINITION ($L^1(X, \mathcal{M}, \mu)$)**.** *Let $L^1(X, \mathcal{M}, \mu)$ denote the set of equivlaenceclasses of integrable functions on $(X, \mathcal{M}, \mu)$ under the equivalence realtion $\sim$ of equivalence a.e. μ. The function on $L^1(X, \mathcal{M}, \mu)$ given by*

$$\{f\}_\sim \mapsto \int_X |f|\mathrm{d}\mu$$

is called the norm *on $L^1(X, \mathcal{M}, \mu)$, and is denoted by $\|\{f\}_\sim\|_1 = \int_X |f|\mathrm{d}\mu$.*

3.1 Remark. The notation $\|\{f\}_\sim\|_1$ is terribly cumbersome. We shall refer to elements of $L^1(X, \mathcal{M}, \mu)$ as if they were integrable functions f and not equivalence classes of integrable functions $\{f\}_\sim$. Thus, we shall generally write $\|f\|_1$ in place of $\|\{f\}_\sim\|_1$, which is harmless since the choice or representative is does not affect the value of any integrals with respect to μ.

What then is the point of definiing $L^1(X, \mathcal{M}, \mu)$ as a set of equivlaence classes of integrable functions instead of simply as the set of integrable functions? It is so the we may use the $L^1(X, \mathcal{M}, \mu)$ norm to define a matric on $L^1(X, \mathcal{M}, \mu)$.

3.2 THEOREM. *For $\{f\}_\sim, \{g\}_\sim \in L^1(X, \mathcal{M}, \mu)$, define $d_{L^1}(\{f\}_\sim, \{g\}_\sim)$ by*

$$d_{L^1}(\{f\}_\sim, \{g\}_\sim) = \|\{f - g\}_\sim\|_1 \ .$$

Then d_{L^1} is a metric on $L^1(X, \mathcal{M}, \mu) \times L^1(X, \mathcal{M}, \mu)$, called the L^1 metric*.*

Proof. Clearly $d_{L^1}(\{f\}_\sim, \{g\}_\sim) = d_{L^1}(\{g\}_\sim, \{f\}_\sim) \geq 0$, and there is equality if and only if $f = g$ a.e. μ, which means that $\{f\}_\sim, \{g\}_\sim$. Finally, if f, g and h are integrable, for all x,

$$|f(x) = h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| \ ,$$

by the triangle inequality in $\mathbb{C}$, and then integrating we obtain

$$\int_X |f - h| \mathrm{d}\mu \leq \int_X |f - g| \mathrm{d}\mu + \int_X |g - h| \mathrm{d}\mu$$

which shows that $d_{L^1}(\{f\}_\sim, \{h\}_\sim) \leq d_{L^1}(\{f\}_\sim, \{g\}_\sim) + d_{L^1}(\{g\}_\sim, \{h\}_\sim)$. □

Henceforth, we suprress the equivalence class notation, and treat the elements of $L^1(X, \mathcal{M}, \mu)$ as if they were functions. This will cause no confusion; on the contrary it will make everything easier to read.

3.3 Remark (The Minkowski inequality and the triangle inequality)**.** The triangle inequlaity, applied to $f, 0, -h \in L^1(X, \mathcal{M}, \mu)$, and using our abbreviated notation, says that

$$d_{L^1}(f, -h) \leq d_{L^1}(f, 0) + d_{L^1}(0, -h) \ ,$$

which, written out in terms of the norm, becomes the *Minkowski inequalaity*

$$\|f + h\|_1 \leq \|f\|_1 + \|h\|_1 \tag{3.2}$$

which is valid for all $f, h \in L^1(X, \mathcal{M}, \mu)$: It is simply another ay of expressing the triangle inequality. Indeed,

$$\|f - h\|_1 = \|(f - g) + (g - h)\|_1 \leq \|f - g\|_1 + \|g - h\|_1$$

which shows how the Minkowski inequality implies the triangle inequality, and is not simply a special case of it.

3.1 The completeness of $L^1(X, \mathcal{M}, \mu)$

3.4 THEOREM. *The metric space $L^1(X, \mathcal{M}, \mu)$ equipped with the L^1 metric is complete. Moreover, if $\{f_n\}_{n\in\mathbb{N}}$ is any Cauchy sequence in $L^1(X, \mathcal{M}, \mu)$, there exists a subseqence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges a.e. μ.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^1(X, \mathcal{M}, \mu)$. Define an increasing sequence of integera $\{n_k\}_{k\in\mathbb{N}}$ by requiring that

$$m > n_k \quad \Rightarrow \quad \|f_m - f_{n_k}\|_1 \leq 2^{-k} \ .$$

In particular, $\|f_{n_{k+1}} - f_{n_k}\|_1 \leq 2^{-k}$ for all $k \in \mathbb{N}$. Furthermore, for all $k \geq 2$,

$$f_{n_k} = f_{n_1} + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j}) \ .$$

Define

$$F_k = |f_{n_1}| + \sum_{j=1}^{k-1} |f_{n_{j+1}} - f_{n_j}| \ .$$

By the Minkowski inequality and a simple induction,

$$\|F_k\|_1 \leq \|f_{n_1}\|_1 + \sum_{j=1}^{k-1} \|f_{n_{j+1}} - f_{n_j}\|_1 \leq\leq \|f_{n_1}\|_1 + 1 \ .$$

Let $F = \lim_{k\to\infty} F_k$. by the Lebesgue Monotone Convergence Theorem,

$$\int_X F\mathrm{d}\mu = \lim_{k\to\infty} \int_X F_k\mathrm{d}\mu = \lim_{k\to\infty} \|F_k\|_1 \leq\leq \|f_{n_1}\|_1 + 1 < \infty \ .$$

In particular, define $E = \{x \ : \ F(x) = \infty\}$, and then itfollows that $\mu(E) = 0$. Moreover, on E^c, $\sum_{j=1}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)|$ consverges, which means that $\sum_{j=1}^{\infty} (f_{n_{j+1}}(x) - f_{n_j}(x))$ converges absolutely. Thus

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) = f_{n_1}(x) \lim_{k\to\infty} \left(\sum j = 1^k (f_{n_{j+1}}(x) - f_{n_j}(x)) \right)$$

exists on E^c. Morover $|f_{n_k}| \leq F_k \leq F$ for all k, and consequently, on E^c, $|f| \leq F$. Define $f = 0$ on E^c, so that is defined everywhere on X; evidently $f \in L^1(X, \mathcal{M}, \mu)$. Then

$$|f_{n_k} - f| \leq |f_{n_k}| + |f| \leq 2F \in L^1(X, \mathcal{M}, \mu)$$

and $\lim_{k\to\infty}(f_{n_k} - f) = 0$ a.e. μ. It now follows by the Lebesgue Dominated Convergence Thoerem that $\lim_{k\to\infty} \|f_{n_k} - f\|_1 = 0$.

So far, we have shown the existence of the subsequence that converges a.e. μ, and we have shown that this subsequence converges in the metric. But in any metric space, whenever any subsequence of a Cauchy seuqence converges, the whole sequence converges to the same limit. Hence $\lim_{n\to\infty} \|f_n - f\|_1 = 0$. □

3.2 Chebychev's Inequality

3.5 THEOREM (Chebychev's Inequality). *Let $f \in L^1(X, \mathcal{M}, \mu)$. Then for all $\lambda > 0$,*

$$\mu\left(\{ \ x \ : \ |f(x)| \geq \lambda \ \}\right) \leq \frac{\|f\|_1}{\lambda} \ . \tag{3.3}$$

Proof. Let $E = \{ \ x \ : \ |f(x)| \geq \lambda \ \}$. Then $\lambda 1_E \leq |f|$ and so

$$\int_X \lambda 1_E \mathrm{d}\mu \leq \int_X |f|\mathrm{d}\mu \ .$$

□

If $\{f_n\}$ is a convergent seuqence in $L^1(X, \mathcal{M}, \mu)$ with limit f, then $\lim_{n\to\infty} \|f_n - f\|_1 = 0$. Then for all $\lambda > 0$,

$$\mu\left(\{ \ x \ : \ |f_n(x) - f(x)| \geq \lambda \ \}\right) \leq \frac{\|f_n - f\|_1}{\lambda} \to 0 \ . \tag{3.4}$$

This leads to the notion of *convergence in measure*, which makes sense not only for sequences of integrable functions, but also sequences of measurable functions.

3.3 Convergence in measure

3.2 DEFINITION (Convergence in measure)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of complex valued measurable functions, and let f be a complex valued measurable function. We say that $\{f_n\}$* converges to f in measure *in case for all $\epsilon > 0$,*

$$\lim_{n\to\infty} \mu\left(\{\ x\ :\ |f_n(x) - f(x)| \geq \epsilon\ \}\right) = 0\ .$$

We say that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure *in case for all $\epsilon > 0$, there is an N_ϵ so that*

$$m, n \geq N_\epsilon \quad \Rightarrow \quad \mu\left(\{\ x\ :\ |f_n(x) - f_m(x)| \geq \epsilon\ \}\right) < \epsilon\ .$$

3.6 THEOREM. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of complex valued integrable functions. If $\lim_{n\to\infty} f_n = f$ in the L^1 metric, then $\lim_{n\to\infty} f_n = f$ in measure.*

Proof. This is immediate from (3.4) and the definition of convergence in measure. □

Convergence in measure does not imply pointwise convergence. Indeed it is possible for a seuqence $\{f_n\}_{n\in\mathbb{N}}$ to converge to f in measure, but to converge at no point, since the exception set on which $f_n(x)$ is not close to $f(x)$ can move around as n changes, coming back to cover the whole space infinitely many times.

3.1 EXAMPLE (Convergence in measure without convergence at any point)**.** *Let μ be Lebesue measure on $[0, 1)$. For $x \in \mathbb{R}$, let $[x]$ denote the fractional part of x. Our goal is to construct a sequence of measuarable sets $\{E_n\}_{n\in\mathbb{N}}$ such that for each $x \in [0, 1)$, $x \in E^n$ for infinitely many values of n, and $x \in E_n^c$ for infnitely many (other) values of n, and also such that $\lim_{n\to\infty} \mu(E_n) = 0$. Then if we define $f_n = 1_{E_n}$, we have that $\lim_{n\to\infty} f_n(x)$ does not exist for any $x \in [0, 1)$. However, the sequence $\{f_n\}$ converges to zero in measure since $\mu(\{\ x :\ |f_n(x)| > \epsilon\}) \leq \mu(E_n)$ for all $\epsilon > 0$.*

The idea is to "sweep back and forth" across $[0, 1)$ with a succession of shorter and short intervals. Here is one way to do this. For all $n \in \mathbb{N}$, define E_n by

$$E_n = \{[y]\ :\ y \in [\ln(n), \ln(n+1))\ \}\ .$$

which is clearly a Borel set. Since each $x \in [0, 1)$ satisfies $x = [y]$ for infinitely many $y \in [0, \infty)$, and since $\cup_{n=1}^\infty [\ln(n), \ln(n+1)] = [0, \infty)$, it is clear that $x \in E_n$ for infinitely many n. Moreover, since $\mu([\ln(n), \ln(n+1))) = \ln(1 + 1/n) \leq 1/n$, when $x \in (2/n, 1 - 2/n)$ and $x \in E_n$, E_n is an interval of length less than $1/n$, and E_{n+1} is a disjoint interval immediately to its right, so that $x \notin E_{n+1}$. It is also clear the $0 \in E_n$ for infinitely many n, and obviously $0 \notin E_n$ when $1/2 \in E_n$ and $n > 2$.

In the previous example, the measures of the sets E_n converged to zero, but rather slowly. Had they converged to zero fast enough that $\sum_{n=1}^\infty \mu(E_n) < \infty$, then it would have been impossible for all x to have belonged to E_n for infinitely many n. In fact, the set of such x would be a null set:

3.7 THEOREM (Borel-Cantelli Lemma)**.** *Let $(X, \mathcal{M}, \mu)$ be a mesure space. Let $\{E_n\}_{n\in\mathbb{N}}$ be any sequence of sets in $\mathcal{M}$. Let*

$$F_m = \bigcup_{n=m}^{\infty} E_n\ . \tag{3.5}$$

If $\displaystyle\sum_{n=1}^{\infty} \mu(E_n) < \infty$,

$$\lim_{m\to\infty} \mu(F_m) = 0 \quad \text{and} \quad \mu\left(\bigcap_{m\in\mathbb{N}} F_m\right) = 0 \ . \tag{3.6}$$

Proof. By subadditivity, $\mu(F_m) \le \sum_{n=m}^{\infty} \mu(E_n)$, and under the summability condition, the first part of (3.6) is then immediate. Next, $F_{n+1} \subset F_n$ for all n, so that the second part of (3.6) is a consquence of the first and conitnuity from above. □

3.8 Remark. The set $\displaystyle\bigcap_{m\in\mathbb{N}} F_m$ consists of those x such that $x \in E_n$ for infinitely many n. Hence, the summability condition implies that for μ a.e. x, $x \in E_n$ for only finitely many n. This is one half of the Borel-Cantelli Lemma in probability theory. The other half is a converse for the case $\mu(X) = 1$ that involves the notion of "independent" measuarable sets, that is not relevant ot the ideas bening developed in this section.

Theorem 3.7 has the following consequence:

3.9 THEOREM. *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of measuable function on* $(X, \mathcal{M}, \mu)$ *that is Cauchy in measure. Then there is a subsequence* $\{f_{n_k}\}$ *such that* $\lim_{k\to\infty} f_{n_k}(x)$ *exists a.e.* μ*. Moreover, there is a measurable function* f *such that* $\lim_{n\to\infty} f_n = f$ *in measure.*

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure, for all $k \in \mathbb{N}$, there is an n_k such that

$$n, m \ge n_k \quad \Rightarrow \quad \mu\Big(\{\ x\ :\ |f_n(x) - f_m(x)| \ge 2^{-k}\ \}\Big) < 2^{-k} \ .$$

Since we can choose the n_k successively to ensure that $\{n_k\}_{k\in\mathbb{N}}$ is a monotone increasing sequence, we define

$$E_k = \{\ x\ :\ |f_{n_{k+1}}(x) - f_{n_k}(x)| \ge 2^{-k}\ \} \quad \text{and} \quad F_\ell = \bigcup_{k=\ell}^{\infty} E_k \ .$$

Then by construction, $\mu(E_k) < 2^{-k}$, and hence by Theorem 3.7, the set of x that belong to E_k for infinitely many k is a set of measure zero. But if $x \in E_k$ for only finitely many k, then clealry $\{f_{n_k}(x)\}_{k\in\mathbb{N}}$ is a Cauchy seuqence in $\mathbb{C}$. It therefore has a limit $f(x)$. Defining $f(x) = 0$ on the exception null set, f is measurable and and f_{n_k} converges to f a.e. μ.

Moreover, if $x \notin F_\ell$,

$$|f_{n_\ell}(x) - f(x)| \le \sum_{j=\ell}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)| \le \sum_{j=\ell}^{\infty} 2^{-j} = 2^{1-\ell} \ .$$

Hence, for all $\epsilon > 0$, there is an $\ell \in N$, so that

$$k \ge \ell \quad \Rightarrow \quad \{x\ :\ |f_{n_k}(x) - f(x)| > \epsilon\ \} \subset F_\ell \ .$$

Theorem 3.7 implies that $\lim_{\ell\to\infty} \mu(F_\ell) = 0$, and hence $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to f in measure. But then since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure, $\{f_n\}_{n\in\mathbb{N}}$ must also converge to f in measure. □

Together, Theroema 3.6 and 3.9 give a second (but closely related) proof of the fact that an a.e. convergent subsequence can be extracted from every L^1 convergent seuqence.

Finally, let us obverse that if $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$, then a.e. convergence implies uniform convergence on the complement of an arbitrailiy small set, and clearly this implies convergence in measure. However, when $\mu(X) = \infty$, a.e. convergence does not imply convergence in measure:

3.2 EXAMPLE (Convergence almost everywhere without convergence in mmeasure)**.** *Let* $X = [0, \infty)$*, and let* μ *be Lebesgue measure on* X*. Define*

$$f_n(x) = (1 - e^{-x})^n \ .$$

Then $\lim_{n\to\infty} f_n(x) = 0$ *for all* x*. However, by Bernouli's inequality,* $f_n(x) \geq 1 - ne^{-x}$*, so that for all* $\epsilon > 0$*,* $\{x \ : \ f_n(x) > \epsilon\}$ *is an inteval of inifnite length. so that* $\{f_n\}_{n\in\mathbb{N}}$ *does not converge to* 0 *in measure.*

4 Uniform integrability

4.1 Concentration proprties of integrable functions

There is an important sense in which integrable functions are "almost bounded", "almost supported on sets of finite measure" and "cannot concentrate mass on too small a set". The first theorem in this section makes this precise.

4.1 THEOREM (Concentration Properties of Integrable Functions)**.** *Let* f *be an integrable function on* $(X, \mathcal{M}, \mu)$*. Then, for all* $\epsilon > 0$*:*

(1) There is a $\lambda < \infty$ *so that*

$$\int_{\{x\ :\ |f(x)|>\lambda\}} |f(x)| \mathrm{d}\mu \leq \epsilon \ . \tag{4.1}$$

(2) There is a set $A \in \mathcal{M}$ *with* $\mu(A) < \infty$ *so that*

$$\int_{A^c} |f(x)| \mathrm{d}\mu \leq \epsilon \ . \tag{4.2}$$

(3) There is a $\delta > 0$ *so that for all* $E \in \mathcal{M}$*, whenever* $\mu(E) < \delta$

$$\int_E |f(x)| \mathrm{d}\mu \leq \epsilon \ . \tag{4.3}$$

Proof. For $n \in \mathbb{N}$, define $f_n = f1_{\{x\ :\ |f(x)|\geq n\}}$. Then $|f_n| \leq |f|$ for all n, and $|f_n| \to 0$ a.e., Therefore, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\{x\ :\ |f(x)|\geq n\}} |f| \mathrm{d}\mu = 0 \ ,$$

and this proves (4.1).

Next, for $n \in \mathbb{N}$, define let $B_n = \{x \ : \ |f(x)| \geq 1/n\}$. By Chebychev's inequality,

$$\mu(B_n) \leq n\|f\|_1 \ .$$

Next, $1_{B_n^c}|f| \leq |f|$ and $1_{B_n^c}|f| \leq 1/n$ for all $n \in \mathbb{N}$. The latter inequality shows that $1_{B_n^c}|f| \to 0$ a.e., and then the former allows us to apply the Dominated Convergence Theorem to show that

$$\lim_{n\to\infty} \int_{B_n^c} |f|\mathrm{d}\mu = 0 \ .$$

This shows that with B_n in place of A, (4.2) is true for all sufficiently large n, and this proves (4.2).

Finally, since we have proved (4.1), we know there is an $n \in \mathbb{N}$ so that

$$\int_{\{x\ :\ |f(x)|>n\}} |f(x)|\mathrm{d}\mu \leq \frac{\epsilon}{2} \ . \tag{4.4}$$

Then, for any $E \in \mathcal{M}$,

$$\begin{aligned}
\int_E |f|\mathrm{d}\mu &= \int_{E\cap\{x\ :\ |f(x)|>n\}} |f|\mathrm{d}\mu + \int_{E\cap\{x\ :\ |f(x)|\leq n\}} |f|\mathrm{d}\mu \\
&\leq \int_{\{x\ :\ |f(x)|>n\}} |f|\mathrm{d}\mu + \int_E n\mathrm{d}\mu \\
&\leq \frac{\epsilon}{2} + n\mu(E) \ .
\end{aligned}$$

Thus, provided n is chosen so that (4.4) is satisfied, and then we set $\delta = \epsilon/(2n)$, (4.3) is satisfied. □

We now turn to the followiing question: Consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ of integrable functions such that $f_n \to f$, either almost everyhwhere or in measure. What else is required to ensure that $f_n \to f$ in L^1?

According to Vitali's Theorem that we state and prove below, the answer is that the properties *(2)* and *(3)* listed in Theorem 4.1 must hold *uniformly* for all the functions in the sequence, and $\{\|f_n\|_1\}_{n\in\mathbb{N}}$ must be uniformly bounded.

4.1 DEFINITION (Uniform Integrability). *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $\mathcal{F}$ a set of measurable functions on X. Then $\mathcal{F}$ is* uniformly integrable *in case*

(1) There is a $C < \infty$ such that for all $f \in \mathcal{F}$,

$$\int_X |f|\mathrm{d}\mu \leq C \ . \tag{4.5}$$

(2) For all $\epsilon > 0$, there is a set $A_\epsilon \in \mathcal{M}$ so that for all $f \in \mathcal{F}$,

$$\int_{A_\epsilon^c} |f|\mathrm{d}\mu \leq \epsilon \ . \tag{4.6}$$

(3) For all $\epsilon > 0$, there is a $\delta_\epsilon > 0$ so that when $E \in \mathcal{M}$ and $\mu(E) \leq \delta_\epsilon$, then for all $f \in \mathcal{F}$,

$$\int_E |f|\mathrm{d}\mu \leq \epsilon \ . \tag{4.7}$$

4.1 EXAMPLE. *Let g be a non-negative integrable function, and let $\mathcal{F}$ be the set of measurable functions satisfying*

$$|f| \leq g \ .$$

Then by Theorem 4.1, $\mathcal{F}$ is uniformly integrable.

Indeed, given $\epsilon > 0$ let A_ϵ and δ_ϵ be such that $\mu(A_\epsilon) < \infty$, $\int_{A_\epsilon^c} |g|\mathrm{d}\mu < \epsilon$, and $\mu(E) < \delta_\epsilon \Rightarrow \int_E |g|\mathrm{d}\mu < \epsilon$. Since $|f| \leq |g|$, the same A_ϵ and δ_ϵ work for each f in $\mathcal{F}$, and of course $\int |f|\mathrm{d}\mu \leq \int g\mathrm{d}\mu =: C$ for all $f \in \mathcal{F}$.

4.2 Vitali's Theorem

4.2 THEOREM (Vitali's Theorem)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\mathcal{F}$ be a uniformly integrable set of functions on $(X, \mathcal{M}, \mu)$. Let $\{f_n\}$ be a sequence of functions in $\mathcal{F}$ and suppose that $\lim_{n\to\infty} f_n = f$ either almost everywhere or in measure. Then*

$$\lim_{n\to\infty} \int_X |f_n - f| \mathrm{d}\mu = 0 \ . \tag{4.8}$$

Conversely, Suppose that $\{f_n\}$ is any sequence of integrable functions and that (4.8) holds. Then the set $\mathcal{F}$ consisting of the functions f_n in the sequence, together with the limit f, is uniformly integrable

Proof. Fix $\epsilon > 0$, and let C, A_ϵ and δ_ϵ be such that (4.5), (4.6) and (4.7) hold for all g in $\mathcal{F}$, and each f_n in our sequence. By Fatou's Lemma, or its analog for convergence in measure,

$$\int_X |f| \mathrm{d}\mu \le \liminf_{n\to\infty} \int_X |f_n| \mathrm{d}\mu \le C \ ,$$

$$\int_{A_\epsilon^c} |f| \mathrm{d}\mu \le \liminf_{n\to\infty} \int_{A_\epsilon^c} |f_n| \mathrm{d}\mu < \epsilon$$

and

$$\int_E |f| \mathrm{d}\mu \le \liminf_{n\to\infty} \int_E |f_n| \mathrm{d}\mu$$

so that $\mu(E) \le \delta_\epsilon \Rightarrow \int_E |f| \mathrm{d}\mu \le \epsilon$.

Now use (4.6) in the definition of uniform integrability to reduce the proof to that of the special case in which $\mu(X) < \infty$:

$$\begin{aligned}\int_X |f_n - f| \mathrm{d}\mu &= \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu + \int_{A_\epsilon^c} |f_n - f| \mathrm{d}\mu \\ &= \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu + \int_{A_\epsilon^c} (|f_n| + |f|) \mathrm{d}\mu\epsilon \\ &\le \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu + 2\epsilon \ . \end{aligned} \tag{4.9}$$

It therefore suffices to show that

$$\lim_{n\to\infty} \int_{A_\epsilon} |f_n - f| \mathrm{d}\mu = 0 \ . \tag{4.10}$$

Suppose first that $f_n \to f$ a.e. Since $\int_X |f| \mathrm{d}\mu \le C$, f is finite almost everywhere. By Egoroff's Theorem, there is a subset $E \subset A_\epsilon$ with $\mu(E) \le \delta_\epsilon$, and such that $f_n \to f$ uniformly on $A_\epsilon \backslash E$. Thus,

$$\begin{aligned}\int_{A_\epsilon} |f_n - f| \mathrm{d}\mu &\le \int_E |f_n - f| \mathrm{d}\mu + \int_{A_\epsilon \backslash E} |f_n - f| \mathrm{d}\mu \\ &\le \int_E (|f_n| + |f|) \mathrm{d}\mu + \mu(A_\epsilon) \sup\{ \ |f_n(x) - f(x)| \ : \ x \in A_\epsilon \backslash E \ \} \\ &\le 2\epsilon + \mu(A_\epsilon) \sup\{ \ |f_n(x) - f(x)| \ : \ x \in A_\epsilon \backslash E \ \} \end{aligned} \tag{4.11}$$

Since $\epsilon > 0$ is arbitrary and $\lim_{n\to\infty} \sup\{ \ |f_n(x) - f(x)| \ : \ x \in A_\epsilon \backslash E \ \} = 0$, we have proved (4.10) in this case.

Next, suppose that $f_n \to f$ in measure. Fix $\eta > 0$, and define $E_\eta(n) = \{ \ x \ : \ |f_n(x) - f(x)| > \eta \ \}$. Since $f_n \to f$ in measure, $\lim_{n\to\infty} \mu(E_\eta(n)) = 0$. Now observe that for all n such that $\lim_{n\to\infty} \mu(E_\eta(n)) \le \delta_\epsilon$,

$$\begin{aligned}
\int_{A_\epsilon} |f_n - f| \mathrm{d}\mu &\le \int_{E_\eta(n)} |f_n - f| \mathrm{d}\mu + \int_{A_\epsilon \backslash E_\eta(n)} |f_n - f| \mathrm{d}\mu \\
&\le \int_{E_\eta(n)} (|f_n| + |f|) \mathrm{d}\mu + \mu(A_\epsilon)\eta \\
&\le 2\epsilon + \mu(A_\epsilon)\eta
\end{aligned} \tag{4.12}$$

Since $\epsilon, \eta > 0$ are arbitrary, this proves (4.10) in this case as well.

Now we prove the converse part of the thoerem. For any set B,

$$\int_B |f_n| \mathrm{d}\mu \le \int_B |f| \mathrm{d}\mu + \int_B |f_n - f| \mathrm{d}\mu \le \int_B |f| \mathrm{d}\mu + \int_X |f_n - f| \mathrm{d}\mu \ .$$

For any fixed $\epsilon > 0$, choose N_ϵ so that

$$n > N_\epsilon \Rightarrow \int_X |f_n - f| \mathrm{d}\mu < \epsilon/2 \ .$$

We then have that for all $n > N_\epsilon$,

$$\int_B |f_n| \mathrm{d}\mu \le \int_B |f| \mathrm{d}\mu + \epsilon/2 \ .$$

Since $\{f\}$ itself is uniformly integrable, there is a number $\tilde{\delta}_\epsilon > 0$ so that

$$\mu(B) \le \tilde{\delta}_\epsilon \Rightarrow \int_B |f| \mathrm{d}\mu \le \epsilon/2 \ .$$

Hence, for all $n > N_\epsilon$,

$$\mu(B) \le \tilde{\delta}_\epsilon \Rightarrow \int_B |f_n| \mathrm{d}\mu \le \epsilon \ .$$

Finally, using the fact that for each $n \le N_\epsilon$, $\{f_n\}$ is uniformly integrable, there is a $\delta_\epsilon^{(n)} > 0$ so that

$$\mu(B) \le \delta_\epsilon^{(n)} \Rightarrow \int_B |f_n| \mathrm{d}\mu \le \epsilon \ .$$

Define

$$\delta_\epsilon = \min\{\delta_\epsilon^{(1)}, \delta_\epsilon^{(2)}, \ldots, \delta_\epsilon^{(N_\epsilon)}, \tilde{\delta}_\epsilon\} \ .$$

Since the minimum of a *finite* set of strictly positive numbers is strictly positive, we have that $\delta_\epsilon > 0$ Also,

$$\mu(B) \le \delta_\epsilon \Rightarrow \int_B |f_n| \mathrm{d}\mu \le \epsilon$$

for all n and for f as well. Thus, condition (4.6) is satisfied. The other two condtions are easily proved in the same way. □

Vitali's Theorem implies a generalized form of the Dominated Convergence Thoerem:

4.3 THEOREM (A Generalized Dominated Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of measurable functions on $(X, \mathcal{M}, \mu)$, and let $\{g_n\}$ be a sequence of integrable functions on $(X, \mathcal{M}, \mu)$ such that for sime $g \in L^1(X, \mathcal{M}, \mu)$, $\lim_{n\to\infty} \|g_n - g\|_1 = 0$.*

Suppose that

$$|f_n| \leq |g_n|$$

a.e. for all n, and that for some f, $f_n \to f$ either a.e. or in measure. Then

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)| \mathrm{d}\mu = 0 \ .$$

Proof. By the converse to Vitali's Theroem, the sets $\{g_n\}_{n\in\mathbb{N}}$ is uniformly integrable. But then since $|f_n| \leq |g_n|$ for all n, $\{f_n\}_{n\in\mathbb{N}}$ is also uniformly integrable with the same C, A_ϵ and δ_ϵ as $\{g_n\}_{n\in\mathbb{N}}$. Then the first psrt of Vitali's Theorem yields $f_n \to f$ in L^1. □

4.4 Remark. The special case in which $g_n = g$ for all n gives us the Dominated Converge Theorem since

$$\left| \int_X f_n \mathrm{d}\mu - \int_X f \mathrm{d}\mu \right| \leq \int_X |f_n(x) - f(x)| \mathrm{d}\mu \ .$$

4.3 Simple conditions that imply uniform integrability

Not all applications of Vitali's Theorem involve a dominating function. A situtation that frequently arrises in applications is that one has a sequence of functions $\{f_n\}$ for which one has an *a priori* bound on

$$\int_X \phi(|f_n|) \mathrm{d}\mu$$

for some function ϕ that grows faster than linearly at infinity; for example $\phi(t) = t \log_+(t)$ or $\phi(t) = t^2$.

4.5 THEOREM (Integral Limits on Concentration)**.** *Let ϕ be a monotone increasing function on $[0, \infty)$ with values in $[0, \infty)$ such that*

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \infty \ .$$

Then for any measure space $(X, \mathcal{M}, \mu)$ and any $C > 0$, let $\mathcal{F}_C$ be the set of functions satisfying

$$\int_X \phi(|f|) \mathrm{d}\mu \leq C \ .$$

Then

$$\lim_{\delta\to 0} \left(\sup \left\{ \int_E |f| \mathrm{d}\mu \ \middle| \ \mu(E) \leq \delta \ , \ f \in \mathcal{F}_C \ \right\} \right) = 0 \ . \tag{4.13}$$

In particular, if $\mu(X) < \infty$, $\mathcal{F}_C$ is uniformly integrable.

Proof. Let E be any measurable set, and f any memeber of $\mathcal{F}_C$. Then for any $a > 0$, let

$$B_a = \{x \mid |f(x)| > a \} \ ,$$

and let a_0 be such that $\phi(a)$ is strictly positive for $a > a_0$. Then since ϕ is monotone increasing, for all $a > a_0$,

$$\begin{aligned}
\int_E |f| \mathrm{d}\mu &= \int_{E\cap B_a} |f| \mathrm{d}\mu + \int_{E\cap B_a^c} |f| \mathrm{d}\mu \\
&\leq \int_{E\cap B_a} |f| \frac{\phi(|f|)}{\phi(a)} \mathrm{d}\mu + \int_{E\cap B_a^c} a \mathrm{d}\mu \\
&\leq \int_X |f| \frac{\phi(|f|)}{\phi(a)} \mathrm{d}\mu + \int_E a \mathrm{d}\mu \\
&\leq \frac{C}{\phi(a)} + a\mu(E) \ .
\end{aligned}$$

Now given $\epsilon > 0$, choose a so that $C/\phi(a) < \epsilon/2$, and then choose $\delta_\epsilon = \epsilon/(2a)$. It then follows that

$$\mu(E) < \delta_\epsilon \Rightarrow \int_E |f| \mathrm{d}\mu < \epsilon$$

and f was an arbirary member of $\mathcal{F}_C$. Since $\epsilon > 0$ was arbitrary, this proves (4.13), which is another way of stating condition *(3)* in the definition of uniform integrability.

Now suppose $\mu(X) < \infty$. Let a_1 be such that $\phi(t) \geq t$ for all $t \geq a_1$. Then for $f \in \mathcal{F}_C$,

$$\begin{aligned}
\int_X |f| \mathrm{d}X &\leq \int_{\{|f| \leq a_1\}} |f| \mathrm{d}\mu + \int_{\{|f| \geq a_1\}} |f| \mathrm{d}\mu \\
&\leq \int_{\{|f| \leq a_1\}} a_1 \mathrm{d}\mu + \int_{\{|f| \geq a_1\}} \phi(|f|) \mathrm{d}\mu \\
&\leq a_1\mu(X) + C,
\end{aligned}$$

so that *(1)* is satisfied. Finally, for *(2)*, we can simply take $A_\epsilon = X$; the second requirement in the definition of uniform integrability is vacuous in case $\mu(X) < \infty$. □

5 Divisibility for non-atomic measures

Definition (Atom) Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $A \in \mathcal{M}$ is an *atom* if and only if $\mu(A) > 0$ and whenever $B \subset A$ and $B \in \mathcal{B}$, either $\mu(B) = 0$ or $\mu(B) = A$. In other words, a measurable set A is an atom if and only if it has strictly positive measure, but it cannot be divided into two measurable sets B and $A\backslash B$ with strictly positive measure.

The measure space $(X, \mathcal{M}, \mu)$ is *diffuse* if and only if § contains no atoms. The measure space $(X, \mathcal{M}, \mu)$ is *purely atomic* if and only if $\mathcal{O}$ is a countable union of atoms, and a set of measure zero.

We have not yet constructed Lebesgue measure, but we have asserted the existence of a counably additive measure μ in $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ such that $\mu((a,b)) = b - a$ for all $a < b$. Let $E \in \mathcal{B}_\mathbb{R}$ such that $\mu(E) > 0$. Then for all $t > 0$, $\mu(E \cap (t,t)) \leq 2t$ so that $\lim_{t\to 0} \mu(E \cap (t,t)) = 0$. But by continuity from below, $\lim_{t\to\infty} \mu(E \cap (t,t)) = \mu(E)$. Moreover, by continuity from above an below, the function $\varphi(t)$ given by $\varphi(t) = \mu(E \cap (t,t))$ is continuous on $(0,\infty)$. Hence, by the Intermediate Value Theorem, for *any* $0 < \lambda < \mu(E)$, there is a a $t \in (0,\infty)$ such that $\mu(E \cap (t,t)) = \lambda$. This show that Lebesgue measure is diffuse. It seems to show more,since we only needed to show that for *some* $\lambda \in (0,\mu(E))$, there is a Borel set $F \subset E$, such that $\mu(F) = \lambda$. However, *every* diffuse masure has this stronger property:

5.1 THEOREM (Lyapunov's Theorem)**.** *Let $(X,\mathcal{M},\mu)$ be a diffuse measure space. with $\mu(X) < \infty$. Then for every a with $0 \leq a \leq \mu(X)$, there exists and $F \in \mathcal{M}$ such that $\mu(F) = a$.*

5.2 LEMMA. *Let $(X,\mathcal{M},\mu)$ be a diffuse measure space with $\mu(X) < \infty$. Then for every $\delta > 0$, there exists a set $C \in \mathcal{M}$ with $0 < \mu(C) < \delta$.*

Proof. Since X is not an atom, we may divide into two measurable sets A_1 and A_1^c both with strictly positive measure. Relabeling the sets if need be, we may assume without loss of generality that $0 < \mu(A_1) \leq \mu(X)/2$.

Next, since A_1 is not an atom, we may decompose it into two sets A_2 and $A_1\backslash A_2$, and we may again assume that $0 < \mu(A_2) \leq \mu(A_1)/2$. Proceeding in this way, we produce a nested sequence $\{A_k\}$ of measurable sets with $0 < \mu(A_k) \leq \mu(\mathcal{O})/2^k$. □

5.3 LEMMA. *Let $(X,\mathcal{M},\mu)$ be a diffuse measure space with $\mu(X) < \infty$. Then for every $\delta > 0$, there exists and $A \in X$ such that*

$$|\mu(A) - \mu(X)/2| < \delta \ .$$

Proof. We may assume without loss of generality that $\mu(X) = 1$. Define c by

$$c = \sup\{\ \mu(A)\ :\ A \in \mathcal{M}\ ,\quad \text{and}\quad \mu(A) \leq 1/2\ \}\ .$$

If $c = 1/2$, then the assertion is clearly true. Hence we must show that $c < 1/2$ is impossible.

Therefore, suppose that $c < 1/2$. Then the interval $(c, 1-c)$ is excluded from the range of μ: *There are no sets A with $\mu(A)$ in this interval.* We will show this is impossible.

For any $0 < \epsilon < 1/2 - c$, choose A such that $c - \epsilon \leq \mu(A) \leq c$. Let b be defined by

$$b = \sup\{\ \mu(B)\ :\ B \in \mathcal{M}\ ,\ B \supset A\ \ \text{and}\ \ \mu(B) \leq 1/2\ \}\ .$$

Note that $b \leq c$ on account of th additional constraint.

Construct a sequence $\{B_m\}$ of measurable sets as follows: For each $m \in \mathbb{N}$, choose a set $B_m \in \mathcal{M}$ so that

$$B_m \supset A \qquad \text{and} \qquad (1 - 1/m)b \leq \mu(B_m) \leq b\ .$$

Next, define C_n by

$$C_n = \bigcup_{m=1}^{n} B_m\ .$$

Clearly, for all n

$$C_n \supset A \qquad \text{and} \qquad (1 - 1/n)b \leq \mu(C_n)\ .$$

We claim that $\mu(C_n) \leq b$ for all n. To see this, note that this is true for $n = 1$. Next, suppose that $\mu(C_n) \leq b$. $C_{n+1} = C_n \cup B_{n+1}$ and therefore,

$$\begin{aligned} \mu(C_{n+1}) &= \mu(C_n) + \mu(B_{n+1}) - \mu(C_n \cap B_{n+1}) \\ &\leq b + b - \mu(A) \\ &\leq 2b - (c - \epsilon) \\ &\leq c + \epsilon \end{aligned}$$

where we have used $b \leq c$ in the last two lines.

Since $\epsilon < 1/2 - c$, then $c + \epsilon < 1/2$. It follows that $\mu(C_{n+1}) < 1/2$. Then, since $C_{n+1} \supset A$, $\mu(C_{n_1}) \leq b$ by the definition of b.

We now have an increasing sequence of sets, and so if $C = \cup_{n=1}^{\infty}$, $\mu(C) = b$ by continuity from below, and of course $C \supset A$.

However, by the first lemma, there is a measurable set $F \subset C^c$ with $\mu(F) < 1/2 - b$. Then $C \cup F$ is a set containing A, but with a measure in $(b, 1/2)$, and this is impossible. This contradiction implies that $c = 1/2$. □

Proof of Lyapunov's Theorem. We may assume without loss of generality that $\mu(X) = 1$. Then, for any fixed $\epsilon > 0$, and any $k \in \mathbb{N}$, we can repeatedly make approximate bisections to divide X into 2^k disjoint measurable sets A_j, $j = 1, \ldots, 2^k$, such that

$$(1 - \epsilon)2^{-k} \leq \mu(A_j) \leq (1 + \epsilon)2^{-k} \ .$$

Taking unions of these sets, we can find, for any dyadic rational number $m2^{-k}$ a measurable set B with

$$(1 - \epsilon)m2^{-k} \leq \mu(B) \leq (1 + \epsilon)m2^{-k} \ .$$

Since the dyadic rationals are dense, and since $\epsilon > 0$ is arbitrary, we can find measurable sets whose measure is arbitrarily close to any number a with $0 \leq a \leq 1$.

Now given $a > 0$, chose a measurable set A_1 with

$$a/2 < \mu(A_1) < a \ .$$

Then, in the complement of A_1, choose A_2 so that

$$(a - \mu(A_1))/2 < \mu(A_2) < (a - \mu(A_1)) \ .$$

Let $B_2 = A_1 \cup A_2$. Then in the complement of B_2, choose A_3 so that

$$(a - \mu(B_2))/2 < \mu(A_3) < (a - \mu(B_2)) \ .$$

Iterating this procedure produces a sequence of disjoint measurable sets $\{A_j\}$ such that

$$a(1 - 2^{-k}) \leq \mu\left(\bigcup_{j=1}^{k} A_j\right) \leq a \ .$$

Then with $A = \cup_{j=1}^{\infty} A_j$, $\mu(A) = a$. □

We now show that ona diffuse measure space, one of the three conditions in the definition of uniform intergrability is redundant, so that ony two conditions need be checked in this case.

5.4 THEOREM (Uniform integrability in diffuse spaces). *Let $(X, \mathcal{M}, \mu)$ be a diffuse measure space Let $\mathcal{F}$ be a set of measurable functions on X such that (1) For all $\epsilon > 0$, there is a set $A_\epsilon \in \mathcal{M}$ so that for all $f \in \mathcal{F}$,*

$$\int_{A_\epsilon^c} |f| \mathrm{d}\mu \le \epsilon \ . \tag{5.1}$$

(2) For all $\epsilon > 0$, there is a $\delta_\epsilon > 0$ so that when $E \in \mathcal{M}$ and $\mu(E) \le \delta_\epsilon$, then for all $f \in \mathcal{F}$,

$$\int_E |f| \mathrm{d}\mu \le \epsilon \ . \tag{5.2}$$

The $\mathcal{F}$ unifiormly integrable.

Proof. It suffices to show that when *(1)* and *(2)* are satisified, there is a finite C such that for all $f \in \mathcal{F}$,

$$\int_X |f| \mathrm{d}\mu \le C \ .$$

First, by *(1)*, we may choose a set A with $\mu(A) < \infty$ so that $\int_{A^c} |f| \mathrm{d}\mu < 1$ for all $f \ in \mathcal{F}$. Then by *(2)*, we may choose a $\delta > 0$ so that for all $f \ in \mathcal{F}$ so that for all $E \in \mathcal{M}$ with $\mu(E) < \delta$, $\int_E |f| \mathrm{d}\mu < 1$ for all $f \in \mathcal{F}$.

Let N be the least integer greater than $\mu(A)/\delta$. Since $(X, \mathcal{M}, \mu)$ is diffuse, we may partition A into N measurable subsets $\{E_1, \dots, E_N\}$ of equal measure. Then for all $f \in \mathcal{F}$,

$$\int_X |f| \mathrm{d}\mu = \int_{A^c} |f| \mathrm{d}\mu + \sum_{n=1}^{N} \int_{E_n} |f| \mathrm{d}\mu \le N + 1 \ .$$

□

6 Exercises

1. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of sets in $\mathcal{M}$. Let E be the set of all x belonging to E_n for infinitely many values of n. Show that if $\mu(E) = 0$, then $\lim_{n \to \infty} \mu(E_n) = 0$. Is this still true if the condiiton that $\mu(X) < \infty$ is dropped?

2. Let μ be Lebesgue measure on $\mathbb{R}$. Let $f \in L^1(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu)$, and let g be a bounded real vlaued Borel mesurable function on $\mathbb{R}$ that is conitnuous at eaxh x outside a set E with $\mu(E) = 0$. For $t > 0$, define $F(t) = \int_\mathbb{R} f(x) g(tx) \mathrm{d}\mu(x)$. Show that $F(t)$ depends contnuously on t.

3. Let μ be Lebesgue measure on $\mathbb{R}$. Let $E \subset \mathbb{R}$ be measuarable, and suppose that there is an $r \in (0, 1)$ such that

$$\mu(E \cap I) \le r\mu(I)$$

for all open intervals I. Show that $\mu(E) = 0$.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of integrable functions such that $\lim_{n\to\infty} f_n = 0$ in measure and such that $\sup_{n\in\mathbb{N}}\{\|f_n\|_1\} < \infty$. Show that

$$\lim_{n\to\infty} \int_X \sqrt{f_n}\mathrm{d}\mu = 0 \ .$$

5. Let μ be Lebesgue measure on $[0,1]$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of integrable functions on $[0,1]$ with values in $[0,\infty)$ such that

$$\int_{(1/n,1]} f_n\mathrm{d}\mu \leq \frac{1}{n} \ .$$

Let $g = \sup_{n\in\mathbb{N}}\{f_n\}$. Show that g is not integrable.

6. Let μ be Lebesgue measure on $\mathbb{R}$. Let $f \in L^1(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu)$ and for $n \in \mathbb{N}$, define $f_n(x) = f(x)\sin^n(x)$. Show that

$$\lim_{n\to\infty} \int_\mathbb{R} f_n\mathrm{d}\mu = 0 \ .$$

7. Let μ be Lebesgue measure on $\mathbb{R}$. Let $\{f_n\}_{n\in\mathbb{N}}$ and let $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of integrable functions. Suppose that they converge a.e. to f and g resepctively. Finally, suppose that each g_n is non-negative and that $\lim_{n\to\infty}\int_\mathbb{R} g_n\mathrm{d}\mu = \int_\mathbb{R} g\mathrm{d}\mu$. Prove that

$$\lim_{n\to\infty} \int_\mathbb{R} f_n\mathrm{d}\mu = \int_\mathbb{R} f\mathrm{d}\mu \ .$$

8. Let $(X, \mathcal{M}, \mu)$ b a measure space. Let f be a non-negative measurable function, and suppose that for every non-negative integrable funcitons g on $(X, \mathcal{M}, \mu)$,

$$\int_X fg\mathrm{d}\mu < \infty \ .$$

Show that thre is a constant M such that $f \leq M$ a.e.

9. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. Let $\{E_k\}_{k\in\mathbb{N}}$ be a sequence of measurable subsets of E. Suppose that

$$\mu\left(\cap_{n=1}^\infty \cup_{k=m}^\infty E_k\right) = 0 \ .$$

Show that for all integrable functions f,

$$\lim_{k\to\infty} \int_{E_k} f\mathrm{d}\mu = 0 \ .$$