Final Exam, Math 501 Fall 2014

December 13, 2014

Instructions

Do five problems from among the six posed below. The exam is due at noon, Dec 18, Eastern Standard Time. Circle the numbers of the five that you **do** want graded.

1. **2.** **3.** **4.** **5.** **6.**

1. Let (X, d) be a metric space. If every real-valued continuous function f on X has a maximum, does this mean that X is compact? Prove your answer is correct.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f \in L^1(X, \mathcal{M}, \mu)$ with $f > 0$ a.e. Let $E \in \mathcal{M}$ be such that

$$\int_E f \mathrm{d}\mu < \infty \ .$$

Show that

$$\lim_{k\to\infty} \int_E f^{1/k} \mathrm{d}\mu(x) = \mu(E)$$

3. Let $\{f_k\}$ be a sequence of non-negative Lebesgue integrable functions on $[0, 1]$. Let f be a non-negative Lebesgue integrable function on $[0, 1]$, and suppose that $\lim_{k\to\infty} f_k = f$ in measure. Suppose in addition that

$$\lim_{k\to\infty} \int_{[0,1]} f_k \mathrm{d}m = \int_{[0,1]} f \mathrm{d}m \ ,$$

where m is Lebesgue measure.

Prove that for every Borel set $E \subset [0, 1]$,

$$\lim_{k\to\infty} \int_E f_k \mathrm{d}m = \int_E f \mathrm{d}m \ .$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$.. Let $\{f_n\}$ be a sequence of functions in $L^1(X, \mathcal{M}, \mu)$, and let $f \in L^1(X, \mathcal{M}, \mu)$. Suppose that

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0 \ .$$

Suppose also that

$$\sup_{n\in\mathbb{N}} \int_X e^{|f_n|} \mathrm{d}\mu < \infty \ .$$

Show that for all $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \int_X |f_n|^k \mathrm{d}\mu = \int_X |f|^k \mathrm{d}\mu \ .$$

5. Let $(X, \mathcal{M})$ be a measurable space. Let μ be a measure on $\mathcal{M}$, and let $\{\nu_n\}_{n\in\mathbb{N}}$ be a sequence of measures on $\mathcal{M}$.

(a) Define the set function λ on $\mathcal{M}$ by

$$\lambda(E) = \sum_{n=1}^{\infty} \nu_n(E) \ .$$

Show that λ is a measure.

(b) With λ as in part **(a)**, show that $\nu_n \perp \mu$ for all n if and only if $\lambda \perp \mu$.

(c) With λ as in part **(a)**, show that $\nu_n << \mu$ for all n if and only if $\lambda << \mu$.

6. Let E be a Borel set in $\mathbb{R}^n$. Let m denote Lebesgue measure on $\mathbb{R}^n$. Define the function $D_E(x)$ on $\mathbb{R}^n$ by

$$D_E(x) = \lim_{r\to 0} \frac{m(E \cap B_r((x))}{m(B_r(x))} \qquad (*)$$

wherever this limit exists. Define $D_E(x) = 0$ at all other x.

(a) Explain why $D_E(x)$ is well-defined and measurable.

(b) Show that for almost every $x \in E$, $D_E(x) = 1$ and for almost every $x \in E^c$, $D_E(X) = 0$.

(c) For all $\alpha \in (0, 1)$, find an example of E and x such that $D_E(x) = \alpha$.

(d) Find an x and an E such that the limit in $(*)$ does not exist.