ROBIN'S THEOREM, PRIMES, AND A NEW ELEMENTARY REFORMULATION OF THE RIEMANN HYPOTHESIS

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Let $\sigma(n)$ denote the sum of the divisors of n, and for $n > 1$ set

$$G(n) := \frac{\sigma(n)}{n \log \log n}.$$

In 1913 Gronwall [3], [4, Theorem 323] found that the maximal order of G is

$$\limsup_{n\to\infty} G(n) = e^{\gamma},$$

where γ is Euler's constant. In 1915 Ramanujan [5, 6] proved that if the Riemann Hypothesis (RH) is true, then $G(n) < e^{\gamma}$ for all large n. In 1984 Robin [7] sharpened this by showing that

$$\text{RH} \iff G(n) < e^{\gamma} \quad (n > 5040).$$

Recently Geoffrey Caveney, Jean-Louis Nicolas and I [2] used Robin's theorem to prove that *the RH holds if and only if* 4 *is the only composite number* N *satisfying*

$$G(N) \geq \max\left(G(N/p), G(aN)\right)$$

for all prime factors p *of* N *and all multiples* aN *of* N. An alternate proof of one step depends on two properties of superabundant numbers derived from those of Alaoglu and Erdős [1] in 1944.

References

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