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Modules Which are Isomorphic to Submodules of Each Other

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R. T. BUMBY

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Suppose we have two modules A, B which are isomorphic to submodules of each other: how close must they be to being isomorphic? The discussion which follows may be considered as an answer to this question. We shall prove that two injective modules so related are isomorphic, and we shall draw some conclusions about the general case from this. Since our result is really a property of the category of modules and not of modules per se we will not need to identify the ring of operators or the side on which they operate.

Theorem. *If $A \supseteqq B$, both injective, and if there is a monomorphism $\varphi : A \to B$, then $A \simeq B$.*

Proof. We note the fact: an injective module is a direct summand of every module containing it. Thus we can find a module H such that

$$A = H \oplus B.$$

Now $A = H \oplus B \supseteqq H \oplus \varphi(A) = H \oplus \varphi(H) \oplus \varphi(B) \supseteqq \dots$. Thus

$$A \supseteqq P = H \oplus \varphi(H) \oplus \varphi(\varphi(H)) \oplus \dots.$$

Then $P \cap B = \varphi(H) \oplus \varphi(\varphi(H)) \oplus \dots$, i.e. $P \cap B = \varphi(P)$.

Let Q be a maximal essential extension of $P \cap B$ in B. Since B is injective, Q is injective. Hence $B = Q \oplus K$ for some K. Also $A = H \oplus (Q \oplus K) = (H \oplus Q) \oplus K$. This implies that $H \oplus Q$ is injective. We use the familiar diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d, "\varphi"'] & H \oplus Q \arrow[dl, "\bar{\varphi}"] \\ & Q & \end{tikzcd}$$

to define a homomorphism $\bar{\varphi}$. The definition of Q allows us to conclude that $\bar{\varphi}$ is an isomorphism. Finally, the map $\bar{\varphi} \oplus 1 : (H \oplus Q) \oplus K \to Q \oplus K$ is the required isomorphism.

Corollary 1. *If A and B are any two modules which are isomorphic to submodules of each other, their injective hulls are isomorphic.*

Proof. Let $\bar{A}$, $\bar{B}$ be the injective hulls of A, B, respectively. Then if $\varphi : A \to B$ is a monomorphism so must be any map $\bar{\varphi} : \bar{A} \to \bar{B}$ which extends it (the injectivity of $\bar{B}$ guarantees the existence of maps $\bar{\varphi}$). Likewise if we have $\psi : B \to A$, a monomorphism, we obtain a monomorphism $\bar{\psi} : \bar{B} \to \bar{A}$ extending it. Now apply the theorem with $\bar{A}$, $\bar{\psi}(\bar{B})$, $\bar{\psi}\bar{\varphi}$ in the roles of A, B, φ.

Now that we know that $\bar{A} \simeq \bar{B}$ we may embed A and B as essential submodules of the same injective module. The monomorphisms $\varphi : A \to B$, $\psi : B \to A$ extend to monic endomorphisms of this injective module.

Definition. *A quasi-injective module is a module which is a fully invariant submodule of its injective hull.*

Remark. This is not the usual definition, but it is a very useful defining property (see Johnson and Wong [2]).

Definition. *The quasi-injective hull of a module A is the smallest fully invariant submodule of the injective hull $\bar{A}$ of A which contains A.*

Lemma. *If B is a fully invariant submodule of an injective module E, if A is any submodule of E, and if there is a monomorphism $\varphi : A \to B$, then $A \subseteqq B$.*

Proof. Since φ is a monomorphism, there is $\varphi^* : \varphi(A) \to A$ which is an inverse of φ. Then φ^* extends to an endomorphism (still called φ^*) of E, since E is injective. Since B is fully invariant, we have

$$B \supseteqq \varphi^*(B) \supseteqq \varphi^*(\varphi(A)) = A .$$

Corollary 2. *If A, B are isomorphic to submodules of each other, then their quasi-injective hulls are isomorphic.*

Proof. By the discussion following Corollary 1 we may assume that A and B are essential submodules of the injective module E. Let $\tilde{A}$, $\tilde{B}$ denote the quasi-injective hulls of A, B in E. The existence of a monomorphism $\varphi : A \to B$ implies $A \subseteqq \tilde{B}$, and since $\tilde{A}$ is the smallest fully invariant submodule of E containing A, we have $\tilde{A} \subseteqq \tilde{B}$. Likewise $\tilde{B} \subseteqq \tilde{A}$, hence $\tilde{A} = \tilde{B}$.

Thus it appears likely that the work done on quasi-injective modules and their endomorphism rings (see [1]) may be applicable to the study of modules related as in the title of this paper.

Remark. The properties of injective modules which we used are properties possessed by every object in the category of sets. In this category direct sums are disjoint unions so our proof reduces to a familiar proof of the equivalence theorem of set theory.

Bibliography

[1] C. Faith and Y. Utumi, Quasi-injective modules and their endomorphism rings. Arch. Math. **15**, 166–176 (1964).

[2] R. E. Johnson and E. T. Wong, Quasi-injective modules and irreducible rings. J. London Math. Soc. **36**, 260–268 (1961).

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Anschrift des Autors:

R. T. Bumby, Department of Mathematics, College of Arts and Sciences
Rutgers, The State University, New Brunswick (N.J.) 08903, USA