Some Variational Problems
with Lack of Compactness

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0. Introduction. In these lectures I propose to describe various nonlinear elliptic equations (or systems) which admit a variational structure. Their common feature is that the standard variational techniques (minimization, critical point theory) do not apply in a straightforward way because of *lack of compactness* or *lack of the Palais–Smale condition* (which is a form of compactness). Such a difficulty seems to be inherent to a number of problems in geometry and physics in view of their invariance under some transformations (dilations, etc. . . .). Technically, the lack of compactness arises because the Sobolev inequality—or some isoperimetric inequality—occurs with its *limiting exponent*.

Most of the results below have been obtained in collaboration, either with L. Nirenberg or with J.-M. Coron. I start with a simple, but very instructive example.

PROBLEM (I). Let $\Omega \subset \mathbf{R}^N$ be a (smooth) bounded domain with $N \geqslant 3$. We look for a function $u: \bar{\Omega} \to \mathbf{R}$ satisfying

$$\text{(I)} \qquad \begin{cases} -\Delta u = u^p + \lambda u & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p \leqslant (N + 2)/(N - 2)$ and $\lambda \in \mathbf{R}$. The *subcritical* case, i.e., $1 < p < (N + 2)/(N - 2)$, has been extensively studied—see e.g. the review article by P. L. Lions [42]. It is easy to establish

PROPOSITION 0. *Suppose* $1 < p < (N + 2)/(N - 2)$. *Then, for each* $\lambda \in (-\infty, \lambda_1)$ *there is a solution of* (I). *There is no solution of* (I) *for* $\lambda \geqslant \lambda_1$.

Here, λ_1 denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary condition.

PROOF. Set

$$(1) \qquad S_\lambda = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{p+1}=1}} \left\{ \int |\nabla u|^2 - \lambda \int u^2 \right\}.$$

Since $p + 1 < 2N/(N - 2)$ it follows that $H_0^1(\Omega) \subset L^{p+1}(\Omega)$ with *compact* injection. As a consequence the infimum in (1) is achieved. Indeed let (u_j) be a minimizing sequence, so that

$$(2) \qquad \begin{aligned} u_j \in H_0^1, \quad & \|u_j\|_{p+1} = 1, \\ \int |\nabla u_j|^2 - \lambda u_j^2 & = S_\lambda + o(1). \end{aligned}$$

Thus u_j is bounded in H_0^1 and we may assume that $u_j \rightharpoonup u$ weakly in H_0^1 and *strongly* in L^{p+1}. Passing to the limit in (2) we obtain

$$\int |\nabla u|^2 - \lambda \int u^2 \leqslant S_\lambda \quad \text{and} \quad \|u\|_{p+1} = 1.$$

Hence u is a minimizer for (1). Moreover, we may assume that $u \geqslant 0$ in Ω—otherwise we replace u by $|u|$. Writing the Euler equation for (1), we obtain

$$-\Delta u - \lambda u = \mu u^p \quad \text{on } \Omega$$

for some Lagrange multiplier μ—and, in fact, $\mu = S_\lambda$ (just multiply the equation by u). If $\lambda < \lambda_1$, then $S_\lambda > 0$ and we may "stretch out" μ by playing on the difference of homogeneities. Finally, the strong maximum principle implies that $u > 0$ on Ω. In order to see that (1) has no solution when $\lambda \geqslant \lambda_1$ it suffices to multiply (I) by ϕ_1, the eigenfunction of $-\Delta$ corresponding to λ_1.

REMARK 1. Under some appropriate restrictions (either on Ω or on p) one can assert that there is a continuous *branch* of solutions (λ, u) of (I) which bifurcates from $(\lambda_1, 0)$ and which covers the interval $(-\infty, \lambda_1)$; see Figure 1.
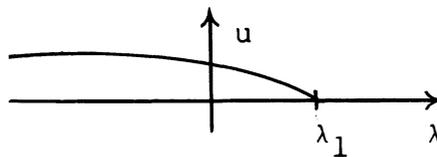

FIGURE 1

This follows from the abstract results of P. Rabinowitz [48] combined with a priori estimates for the solutions of (I) (see H. Brezis–R. Turner [17], D. de Figueiredo–P. L. Lions–R. Nussbaum [26]). The question of *uniqueness* is essentially open even if Ω is a ball—in which case all solutions of (I) are radial by the results of Gidas–Ni–Nirenberg [28]. When Ω is a ball the solution is known to be unique in various special cases: $\lambda = 0$ (via a simple scaling argument), or $\lambda \geqslant 0$ and $p \leqslant N/(N - 2)$ (see Ni–Nussbaum [44]).

We now turn to the *limiting case* $p = (N + 2)/(N - 2)$. This case is especially interesting because it resembles the *Yamabe problem* in differential geometry: Find a function u satisfying

$$-4\frac{(N - 1)}{(N - 2)}\Delta u = u^{(N+2)/(N-2)} - R(x)u \quad \text{on } M,$$

$$u > 0 \qquad\qquad\qquad\qquad \text{on } M.$$

Here M is an N-dimensional Riemannian manifold and $R(x)$ is the scalar curvature—for further details we refer to Th. Aubin [**4, 5, 6**], J. Kazdan [**37, 72**] and to R. Schoen [**64**] for the complete solution of Yamabe's conjecture. *The argument used in the proof of Proposition* 0 *fails in the limiting case.* Indeed $u_j \rightharpoonup u$ weakly in H_0^1 and weakly in L^{p+1}. However, u_j *need not converge strongly in* L^{p+1}. Passing to the limit in (2), we find only

$$\int |\nabla u|^2 - \lambda \int u^2 \leqslant J_\lambda \quad \text{and} \quad \|u\|_{p+1} \leqslant 1,$$

since *the unit sphere in* L^{p+1} *is not closed for the weak* H^1 *topology. Thus the weak limit u of a minimizing sequence u need not be a minimizer. This is the fundamental difficulty in all the variational problems with lack of compactness.*

In fact, we shall see in §1 that if $\lambda \leqslant 0$ the infimum in (1) is never achieved. When Ω is starshaped this follows easily from:

THEOREM (POHOŽAEV [**47**]). *Suppose* Ω *is starshaped and* $p = (N + 2)/(N - 2)$. *Then there is no solution of* (I) *with* $\lambda \leqslant 0$.

Pohožaev's Theorem is a direct sequence of *Pohožaev's identity*: If

$$(3) \qquad \begin{cases} -\Delta u = g(u) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and g is a continuous function, then

$$(4) \qquad \left(1 - \frac{N}{2}\right)\int_\Omega g(u)u + N\int_\Omega G(u) = \frac{1}{2}\int_{\partial\Omega}(x \cdot n)\left(\frac{\partial u}{\partial n}\right)^2,$$

where $G(u) = \int_0^u g(s)\,ds$ and n is the outward normal (the proof of Pohožaev's identity is elementary: multiply (3) by u and by $\sum x_i \partial u/\partial x_i$).

In the special case where $g(u) = u^p + \lambda u$, Pohožaev's identity reduces to

$$(5) \qquad \lambda\int_\Omega u^2 = \int_{\partial\Omega}(x \cdot n)\left(\frac{\partial u}{\partial n}\right)^2$$

and therefore $\lambda > 0$ if Ω is starshaped (since $x \cdot n > 0$ a.e. on $\partial\Omega$).

Pohožaev's interesting nonexistence result had a disastrous impact: most authors carefully avoided the "taboo" limiting case by invoking Pohožaev as an excuse! It turns out that *in the limiting case the lower-order terms—which are irrelevant in the subcritical case—play here a very important role* and they "help" to establish the existence by "lowering the infimum".

The main results from [**6**] concerning (I) are the following.

THEOREM 1. *Let Ω be any bounded domain in $\mathbf{R}^N$ with $N \geqslant 4$, and let $p = (N + 2)/(N - 2)$. Then, for every $\lambda \in (0, \lambda_1)$ there is a solution of* (I). *Moreover the infimum in* (1) *is achieved.*

The case of dimension three is much more difficult and we have a satisfactory result only when Ω is a ball:

THEOREM 2. *Let Ω be a ball in $\mathbf{R}^3$ and let $p = 5$. Then for every $\lambda \in (\lambda_1/4, \lambda_1)$ there is a solution of* (I); *moreover, the infimum in* (1) *is achieved. There is no solution of* (I) *for $\lambda \notin (\lambda_1/4, \lambda_1)$.*

The striking difference between the cases $N \geqslant 4$ and $N = 3$ remains a mystery. In §1 I will describe some of the ingredients which enter in the proofs of Theorems 1 and 2. The best constant S for the Sobolev inequality plays an important role. The proof of existence involves two independent steps:

Step 1. Show that if $S_\lambda < S$, then the infimum in (1) is achieved.

Step 2. Show that indeed $S_\lambda < S$ under some appropriate restrictions on λ or on Ω.

More precisely, we prove that *below the level S some form of compactness is restored*. This approach, which has been introduced by Th. Aubin in [4], is also used by R. Schoen [64]. As we shall see, Step 1 is straightforward, while Step 2 involves a heavy technical machinery. Alternative proofs have been obtained by Atkinson–Peletier [66] and McLeod–Norbury [67].

PROBLEM (II). Let $\Omega \subset \mathbf{R}^N$ be a (smooth) bounded domain with $N \geqslant 3$. We look for a function $u: \overline{\Omega} \to \mathbf{R}$ satisfying

$$
\text{(II)} \qquad
\begin{cases}
-\Delta u = u^p + \mu u^q & \text{on } \Omega, \\
u > 0 & \text{on } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

where $p = (N + 2)/(N - 2)$, $1 < q < p$ and $\mu \in \mathbf{R}$. Pohožaev's identity applied to a solution of (II) says that

$$
\text{(6)} \qquad \mu\left(1 - \frac{N}{2} + \frac{N}{q + 1}\right)\int_\Omega u^{q+1} = \frac{1}{2}\int_{\partial\Omega}(x \cdot n)\left(\frac{\partial u}{\partial n}\right)^2
$$

and therefore $\mu > 0$ when Ω is starshaped. The main results from [16] concerning (II) are the following.

THEOREM 3. *Let Ω be any bounded domain in $\mathbf{R}^N$ with $N \geqslant 4$. Then for every $\mu > 0$ there is a solution of* (II).

Again, dimension three is much more delicate:

THEOREM 4. *Let $\Omega \subset \mathbf{R}^3$ be any bounded domain and let $p = 5$. We distinguish two cases*:

  (i) *if $3 < q < 5$, then for every $\mu > 0$ there is a solution of* (II),

  (ii) *if $1 < q \leqslant 3$, then for every μ large enough there is a solution of* (II).
Moreover, (II) *has no solution for $\mu > 0$ small when Ω is strictly starshaped.*

Problem (II) cannot be solved by a simple minimizing argument; one has to use more sophisticated variational tools. The solutions of (II) correspond to nonzero critical points of the functional

$$F(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega (u^+)^{p+1} - \frac{\mu}{q+1} \int_\Omega (u^+)^{q+1}$$

on the space $H_0^1(\Omega)$. All the assumptions of the Mountain Pass Lemma of Ambrosetti and Rabinowitz (see [1] and Lemma 7 below) are satisfied, *except for the Palais–Smale* (PS) *condition.* Usually—when the (PS) condition is satisfied—one can assert that the constant
$$c = \operatorname{Inf} \operatorname{Sup} F$$
(Inf Sup over appropriate sets) is a *critical value* of F. Here, however, one cannot conclude that c is a critical value because the (PS) condition fails. Again, our approach for existence involves essentially two steps:

Step 1. Show that if $c < (1/N)S^{N/2}$ then c is indeed a critical value. More precisely we show that *below the "magic" level* $(1/N)S^{N/2}$ *the* (PS) *condition is restored.*

Step 2. Show that indeed $c < (1/N)S^{N/2}$ under some appropriate conditions (on μ, Ω, q).

PROBLEM (III) (RELLICH'S CONJECTURE). Let $\Gamma \subset \mathbf{R}^3$ be a Jordan curve with $\Gamma \subset B_R$—a ball of radius R. Let $H > 0$ be a fixed constant. We look for a surface Σ of constant mean curvature H spanned by Γ (i.e., $\partial \Sigma = \Gamma$).

For example, if Γ is a *circle of radius R* and $HR < 1$, there are two such surfaces: the small spherical "bubble" Σ_1 and the large spherical "bubble" Σ_2 (see Figure 2).
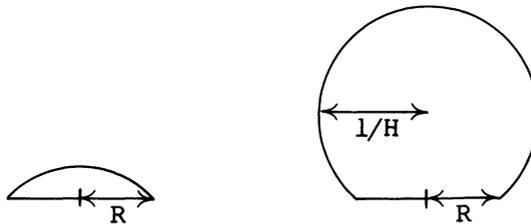

FIGURE 2

Incidentally, it is not known whether these are the only solutions (see Remark 11).

If $HR = 1$ there is one solution (the hemisphere) and if $HR > 1$ there is no solution (this intuitive fact has been proved rigorously by E. Heinz [31]). A conjecture attributed to Rellich—and publicized more recently by S. Hildebrandt and L. Nirenberg—asserts that a similar result holds for any (smooth) Jordan curve Γ. By a solution Σ we mean a generalized surface parametrized on the unit disk with possible self-intersections, multiple coverings, etc. More precisely, let

$$\Omega = \left\{ (x, y) \in \mathbf{R}^2, \ x^2 + y^2 < 1 \right\}.$$

We look for a surface Σ of the form $\Sigma = u(\Omega)$ where $u: \overline{\Omega} \to \mathbf{R}^3$ satisfies the system

$$(\mathrm{III_P}) \qquad \begin{cases} \Delta u = 2Hu_x \wedge u_y & \text{on } \Omega, \\ u_x^2 - u_y^2 = u_x \cdot u_y = 0 & \text{on } \Omega, \\ u(\partial\Omega) = \Gamma. \end{cases}$$

The last two conditions are called the *Plateau conditions*. We may also consider the same system with the usual *Dirichlet condition*:

$$(\mathrm{III_D}) \qquad \begin{cases} \Delta u = 2H_x \wedge u_y & \text{on } \Omega, \\ u = \gamma & \text{on } \partial\Omega, \end{cases}$$

where $\gamma: \partial\Omega \to \mathbf{R}^3$ is a given smooth map such that $\gamma(\partial\Omega) \subset B_R$. The system $(\mathrm{III_D})$ has no simple geometric interpretation; however, it is easier to handle than $(\mathrm{III_P})$ and it involves essentially the same difficulties as $(\mathrm{III_P})$.

The main results of [9] concerning (III) are the following.

THEOREM 5. *Suppose $HR < 1$; then there exist at least two solutions of* $(\mathrm{III_P})$.

THEOREM 6. *Suppose $HR < 1$ and γ is not a constant; then there exist at least two solutions of* $(\mathrm{III_D})$.

Here are the principal ideas which enter in the proof of Theorem 6. A first solution $\underline{u}$ (the counterpart of the small bubble) of $(\mathrm{III_D})$ had been obtained by S. Hildebrandt [32] (following some earlier work of E. Heinz [30]). It plays the same role as the zero solution in Problems (I) and (II). We look for a second solution $\bar{u}$ of the form $\bar{u} = \underline{u} - v$ which leads to the following system for v:

$$(7) \qquad \begin{cases} \mathscr{L}v \equiv -\Delta v + 2H\underline{u}_x \wedge v_y + 2Hv_x \wedge \underline{u}_y = 2Hv_x \wedge v_y & \text{on } \Omega, \\ v \not\equiv 0 & \text{on } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Problem (7) has a variational structure which resembles (I) because of its *lack of compactness*. I will explain in §3 how to adapt some of the methods used for Problem (I). We shall see that *below some magic level—related to the best constant for an isoperimetric inequality—a kind of compactness is restored*.

PROBLEM (IV) (LARGE HARMONIC MAPS). Let $\Omega = \{(x, y) \in \mathbf{R}^2; \ x^2 + y^2 < 1\}$ and let S^2 be the unit sphere in $\mathbf{R}^3$. We look for a mapping $u: \overline{\Omega} \to \mathbf{R}^3$ satisfying

$$(\mathrm{IV}) \qquad \begin{cases} -\Delta u = u|\nabla u|^2 & \text{on } \Omega, \\ u \in S^2 & \text{on } \Omega, \\ u = \gamma & \text{on } \partial\Omega, \end{cases}$$

where $\gamma: \partial\Omega \to S^2$ is given.

Problem (IV) has a variational structure; namely, the solutions of (IV) correspond to the critical points of the functional

$$E(u) = \int |\nabla u|^2,$$

subject to the constraint

$$u \in \mathscr{E} = \left\{ u \in H^1(\Omega; S^2); \; u = \gamma \text{ on } \partial\Omega \right\}.$$

The solutions of (IV) are *harmonic maps* with the prescribed boundary condition $u = \gamma$ on $\partial\Omega$. It is clear that there is some $\underline{u} \in \mathscr{E}$ which is an *absolute minimum* of E on $\mathscr{E}$, that is,

$$E(\underline{u}) \leqslant E(v) \qquad \forall v \in \mathscr{E}.$$

In [27], Giaquinta and Hildebrandt have raised the question whether (IV) admits other solutions besides $\underline{u}$. They had observed that for some special γ's (such that $\gamma(\partial\Omega)$ is a circle) there are indeed at least two solutions of (IV). The main result of [10] concerning (IV) is the following.

THEOREM 7. *Suppose γ is not a constant. Then there is some $\bar{u} \neq \underline{u}$ which is a solution of* (IV).

The method—which will be presented in §4—is the following. The set $\mathscr{E}$ is not connected and we split $\mathscr{E}$ into its connected components $\mathscr{E} = \bigcup_{k \in Z} \mathscr{E}_k$. This is done using degree theory. Then we try to minimize E on each $\mathscr{E}_k$. The *main difficulty is that the sets $\mathscr{E}_k$ are not closed for the weak H^1 topology* (the degree is continuous for the strong H^1 topology but not for the weak H^1 topology). Here again the same methodology applies:

Step 1. We prove that if $\mathrm{Inf}_{\mathscr{E}_k} E$ is less than some "magic" level, then $\mathrm{Inf}_{\mathscr{E}_k} E$ is achieved (and even some form of compactness is restored).

Step 2. We prove that indeed $\mathrm{Inf}_{\mathscr{E}_k} E$ is less than the magic level for at least two different k's.

Finally I should mention that there are many other problems where a similar lack of compactness also occurs. Here are some of them:

Best constants in Hardy–Littlewood, Sobolev, traces and related inequalities, see Th. Aubin [3], Talenti [57], E. Lieb [40], P. L. Lions [43], P. Cherrier [19].

Best constants in inequalities involving analytic functions, see Jacobs [34].

Variants of the Yamabe problem and questions involving the scalar curvature, see Bourguignon [8], Jerison–Lee [35], P. L. Lions [43], A. Bahri–J.-M. Coron [65].

Problems related to the Yang–Mills equations, see K. Uhlenbeck [61], C. Taubes [59], S. Sedlacek [50].

Of course a lack of compactness also occurs when a problem is posed in all of $\mathbf{R}^N$ and is translation invariant. This leads to other kinds of difficulties (not related to the critical exponents but rather to the noncompactness of $\mathbf{R}^N$), see e.g. P. L. Lions [42] for a detailed analysis and an extensive bibliography; see also Brezis–Lieb [14] for some vector field equations and C. Taubes [58, 60] for the Yang–Mills–Higgs equations on $\mathbf{R}^3$.

1. Problem (I). Sobolev's inequality asserts that there is some constant $\alpha > 0$ such that

$$\| \nabla\phi \|_2^2 \geqslant \alpha \| \phi \|_{2^*}^2 \qquad \forall \phi \in H_0^1(\Omega) = \left\{ \phi \in L^{2^*}(\Omega), \nabla\phi \in L^2, \phi = 0 \text{ on } \partial\Omega \right\},$$

where $2^* = 2N/(N-2)$; *the best Sobolev constant is by definition*

(8)
$$S = \inf_{\substack{\phi \in H_0^1(\Omega) \\ \|\phi\|_{2^*}=1}} \|\nabla\phi\|_2^2.$$

We recall some known facts about S:

(a) S is independent of Ω and depends only on N, since the ratio $\|\nabla\phi\|_2/\|\phi\|_{2^*}$ is invariant under dilations.

(b) When $\Omega = \mathbf{R}^N$ the infimum in (8) is achieved by the function

$$U(x) = \frac{C}{\left(1 + |x|^2\right)^{(N-2)/2}}$$

or, after scaling, by any of the functions

(9)
$$U_\varepsilon(x) = \frac{C_\varepsilon}{\left(\varepsilon + |x|^2\right)^{(N-2)/2}}, \qquad \varepsilon > 0,$$

where C and C_ε are some normalization constants. In addition, the functions $U_\varepsilon(x - x_0)$ $(x_0 \in \mathbf{R}^N)$ are the *only* functions for which the infimum in (8) is achieved. More precisely, a minimizer U of (8) (in $\mathbf{R}^N$) must satisfy the Euler equation

(10)
$$-\Delta U = SU^p \quad \text{in } \mathbf{R}^n \text{ with } \quad p = (N+2)/(N-2)$$
$$\text{and} \quad U > 0 \quad \text{in } \mathbf{R}^n,\ U \in L^{2^*},\ \nabla U \in L^2.$$

However, all the solutions of (10) are known to be of the form (9).

For these facts we refer to Th. Aubin [3], G. Talenti [57], E. Lieb [40], P. L. Lions [43], Gidas–Ni–Nirenberg [29].

(c) As a direct consequence of (a) and (b) we find that the best constant in (8) is *never achieved* except when $\Omega = \mathbf{R}^N$.

The following quantity plays an important role in the solution of Problem (I):

(11)
$$S_\lambda = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{2^*}=1}} \left\{ \int |\nabla u|^2 - \lambda u^2 \right\}.$$

Our first lemma provides a fundamental estimate for S_λ:

LEMMA 1. *Assume $N \geqslant 4$, then*

(12)
$$S_\lambda < S \qquad \forall \lambda > 0,$$

(13)
$$S_\lambda = S \qquad \forall \lambda \leqslant 0.$$

In other words, the graph of the function $\lambda \mapsto S_\lambda$ looks as in Figure 3.

FIGURE 3

The proof of Lemma 1 is rather technical and we refer to [**16**] for details. The idea is the following. If the infimum in (8) were achieved by some function ϕ we could use it as a testing function in (11) and the conclusion would be trivial. Since it is not achieved, we try instead

$$u_\varepsilon(x) = \zeta(x)U_\varepsilon(x) \qquad (\text{assuming } 0 \in \Omega),$$

where $\zeta \in \mathscr{D}(\Omega)$ is any fixed function such that $\zeta \equiv 1$ near 0, so that as $\varepsilon \to 0$ the function $U_\varepsilon(x)$ "concentrates" near $x = 0$ where ζ has "no influence". We set

$$Q_\lambda(u) = \frac{\int |\nabla u|^2 - \lambda u^2}{\|u\|_{2*}^2}.$$

A careful expansion as $\varepsilon \to 0$ shows that

$$Q_\lambda(u_\varepsilon) = \begin{cases} S + O(\varepsilon^{(N-2)/2}) - \lambda K\varepsilon & \text{if } N \geqslant 5, \\ S + O(\varepsilon) - \lambda K\varepsilon|\log \varepsilon| & \text{if } N = 4, \end{cases}$$

for some constant K, which depends only on N. The conclusion of Lemma 1 follows easily. If we try the same argument when $N = 3$ we find

$$Q_\lambda(u_\varepsilon) = S + A\varepsilon^{1/2} - B\lambda\varepsilon^{1/2} + O(\varepsilon),$$

where $A > 0$, $B > 0$ are constants (depending on ζ). We may only conclude that

$$S_\lambda \leqslant S \qquad \forall \lambda \in \mathbf{R},$$
$$S_\lambda = S \qquad \forall \lambda \leqslant 0.$$

We postpone for a little while the analysis of the case $N = 3$, which is rather delicate.

Theorem 1 follows easily from Lemma 1 and our next lemma:

LEMMA 2. *Assume $N \geqslant 3$ and let (u_j) be a minimizing sequence for* (11). *Then for some subsequence still denoted by (u_j), we have either*
  (a) $u_j \to u$ *strongly in H_0^1 and u is a minimizer for* (11), *or*
  (b) $u_j \rightharpoonup 0$ *weakly in H_0^1.*
Moreover, if $S_\lambda < S$, then case (a) *occurs.*

The argument is rather simple and I will describe two different proofs.
FIRST PROOF. By definition of (u_j) we have

(14) $$\int |\nabla u_j|^2 - \lambda \int u_j^2 = S_\lambda + o(1),$$

(15) $$\int |u_j|^{p+1} = 1 \qquad (p = (N+2)/(N-2)).$$

It follows that (u_j) is bounded in H_0^1, and we may always assume that $u_j \rightharpoonup u$ weakly in H_0^1; we write

$$u_j = u + v_j$$

with $v_j \rightharpoonup 0$ weakly in H_0^1 and in L^{p+1}, $v_j \to 0$ strongly in L^q, $q < 2*$ and a.e.

We deduce from (14) that

$$(16) \qquad \int |\nabla u|^2 + \int |\nabla v_j|^2 - \lambda \int u^2 = S_\lambda + o(1).$$

On the other hand, we may use the following:

LEMMA 3 (BREZIS–LIEB [13]). *Suppose* $u \in L^{p+1}$, (v_j) *is bounded in* L^{p+1} *and* $v_j \to 0$ *a.e. Then*

$$(17) \qquad \int |u + v_j|^{p+1} = \int |u|^{p+1} + \int |v_j|^{p+1} + o(1).$$

Combining (15) and (17), we obtain

$$(18) \qquad 1 = \int |u|^{p+1} + \int |v_j|^{p+1} + o(1),$$

and by convexity

$$(19) \quad 1 \leqslant \left[\int |u|^{p+1} + \int |v_j|^{p+1} \right]^{2/(p+1)} + o(1) \leqslant \|u\|_{p+1}^2 + \|v_j\|_{p+1}^2 + o(1).$$

Using (16) and (19) we find

$$(20) \quad \int |\nabla u|^2 - \lambda \int u^2 + \int |\nabla v_j|^2 \leqslant S_\lambda \left[\|u\|_{p+1}^2 + \|v_j\|_{p+1}^2 \right] + o(1).$$

However, we have

$$(21) \qquad \int |\nabla u|^2 + - \lambda \int u^2 \geqslant S_\lambda \|u\|_{p+1}^2$$

and

$$(22) \qquad S_\lambda \|v_j\|_{p+1}^2 \leqslant \frac{S_\lambda}{S} \int |\nabla v_j|^2 \leqslant \int |\nabla v_j|^2$$

(by definition of S_λ, S and since $S_\lambda \leqslant S$). We deduce from (20), (21), (22) that several *equalities* hold, namely

$$\int |\nabla u|^2 - \lambda \int u^2 = S_\lambda \|u\|_{p+1}^2,$$

$$(23) \qquad \int |\nabla v_j|^2 = \frac{S_\lambda}{S} \int |\nabla v_j|^2 + o(1),$$

$$(24) \qquad \left[\int |u|^{p+1} + \int |v_j|^{p+1} \right]^{2/(p+1)} = \|u\|_{p+1}^2 + \|v_j\|_{p+1}^2 + o(1).$$

If $S_\lambda < S$, from (23) we obtain $\int |\nabla v_j|^2 = o(1)$, and thus $u_j \to u$ strongly in H_0^1; hence u is a minimizer.

Therefore we may assume that $S_\lambda = S$ and also $u \neq 0$ (otherwise (b) holds). We set

$$t_j = \|v_j\|_{p+1}^{p+1} \Big/ \|u\|_{p+1}^{p+1}$$

so that, by (24),

$$\left(1 + t_j\right)^{2/(p+1)} = 1 + t_j^{2/(p+1)} + o(1).$$

This implies that $t_j = o(1)$ since the function $1 + t^{2/(p+1)} - (1 + t)^{2/(p+1)}$ is increasing for $t \in [0, \infty)$. Hence we conclude again that $u_j \to u$ strongly in H_0^1.

SECOND PROOF. By definition of S_λ we have

$$(25) \quad \int \left|\nabla\left(u_j + \phi\right)\right|^2 - \lambda \int \left(u_j + \phi\right)^2 \geq S_\lambda \|u_j + \phi\|_{p+1}^2 \qquad \forall \phi \in H_0^1.$$

On the other hand, we have by convexity

$$\left|u_j + \phi\right|^{p+1} - \left|u_j\right|^{p+1} \geq (p+1)\left|u_j\right|^{p-1}u_j\phi,$$

and therefore

$$(26) \qquad \|u_j + \phi\|_{p+1}^2 \geq \left[\left(1 + (p+1)\int |u_j|^{p-1}u_j\phi\right)^+\right]^{2/(p+1)}.$$

Combining (25) and (26) and passing to the limit, we obtain

$$S_\lambda + 2\int \nabla u \nabla \phi + \int |\nabla\phi|^2 - 2\lambda \int u\phi - \lambda \int \phi^2$$
$$\geq S_\lambda\left[\left(1 + (p+1)\int |u|^{p-1}u\phi\right)^+\right]^{2/(p+1)}.$$

Replacing t by $t\phi$, we find, as $t \to 0$,

$$\int \nabla u \nabla \phi - \lambda \int u\phi = S_\lambda \int |u|^{p-1}u\phi \qquad \forall \phi \in H_0^1;$$

that is,

$$-\Delta u - \lambda u = S_\lambda |u|^{p-1}u,$$

and, in particular,

$$(27) \qquad\qquad \int |\nabla u|^2 - \lambda \int u^2 = S_\lambda \|u\|_{p+1}^{p+1}.$$

Combining (21) and (27), we obtain that either $\|u\|_{p+1} \geq 1$ or $\|u\|_{p+1} = 0$. In the first case we must have $\|u\|_{p+1} = 1$ (since $\|u\|_{p+1} \leq 1$); it follows that $u_j \to u$ strongly in H_0^1 since $\int |\nabla u_j|^2 = S_\lambda + \lambda \int u^2 + o(1) = \int |\nabla u|^2 + o(1)$ (by (27)). Finally, it is easy to see that $S_\lambda < S$ implies that case (a) occurs. Indeed, we have

$$\int |\nabla u_j|^2 - \lambda \int u_j^2 = S_\lambda + o(1)$$
$$\geq S\|u_j\|_{p+1}^2 - \lambda \int u^2 + o(1) = S - \lambda \int u^2 + o(1),$$

and therefore

$$\lambda \int u^2 \geq S - S_\lambda > 0.$$

REMARK 2. A variant of the first proof was originally due to E. Lieb—who uses a similar device in [40]; it was later simplified by the introduction of Lemma 3. The strong convergence was originally pointed out by F. Browder, at an earlier stage when the argument was less transparent. The second proof is inspired by a device used in [14].

REMARK 3. Suppose (u_j) is a minimizing sequence for (11) and $u_j \rightharpoonup 0$ weakly in H_0^1. Then much more can be said about (u_j): there exist a sequence $\varepsilon_j \to 0$ ($\varepsilon_j > 0$) and a bounded sequence (a_j) in $\mathbf{R}^N$ such that

$$\left\| u_j - \varepsilon_j^{-(N-2)/2} \omega\left(\frac{\cdot - a_j}{\varepsilon_j} \right) \right\|_{H^1} \to 0$$

where $\omega(x) = C/(1 + |x|^2)^{(N-2)/2}$ and $\|\omega\|_{p+1} = 1$.

In other words, (u_j) "concentrates" near a point where it resembles an extremal function (for the Sobolev inequality) with a high peak. Indeed, in view of Lemma 2, we have $S_\lambda = S$ and $\int |\nabla u_j|^2 = S + o(1)$, $\|u_j\|_{p+1} = 1$. Therefore (u_j) is a *minimizing sequence for the Sobolev inequality*, and the conclusion follows from the results of P. L. Lions [43].

We now turn to the analysis of the case $N = 3$.

LEMMA 4. *Suppose $\Omega \subset \mathbf{R}^3$ is strictly starshaped (i.e. $x \cdot n \geqslant \alpha > 0 \ \forall x \in \partial\Omega$). Then, there is some $\underline{\lambda} > 0$ depending on Ω such that* (I) *has no solution for $\lambda \leqslant \underline{\lambda}$. When Ω is a ball we may take $\underline{\lambda} = \lambda_1/4$.*

SKETCH OF THE PROOF. Suppose u is a solution of (I) with $\lambda \geqslant 0$. By Pohožaev's identity (5) we have

$$\lambda \int_\Omega u^2 = \frac{1}{2} \int_{\partial\Omega} (x \cdot n)\left(\frac{\partial u}{\partial n} \right)^2 \geqslant \frac{\alpha}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} \right)^2 \geqslant b \left(\int_{\partial\Omega} \frac{\partial u}{\partial n} \right)^2$$

$$= b \left(\int_\Omega \Delta u \right)^2 = b \left(\int_\Omega |\Delta u| \right)^2 \geqslant c \int_\Omega u^2.$$

Here we have used the fact that $-\Delta u = u^5 + \lambda u \geqslant 0$ and also that $(\Delta)^{-1}$ is a bounded operator from L^1 into L^2.

When Ω is a ball there is no solution of (I) for $\lambda \leqslant \lambda_1/4$. The argument is somewhat tricky and I refer to [16] for the details. Here is the main idea. Suppose u is a solution of (I); we know from Gidas–Ni–Nirenberg [28] that u must be radial and we write $u(r)$ with $0 < r < 1$ (assuming Ω is the unit ball). First one proves that

$$(28) \quad \int_0^1 u^2\left(\lambda\psi' + \frac{1}{4}\psi''' \right) r^2 dr = \frac{2}{3} \int_0^1 u^6\left(r\psi - r^2\psi' \right) dr + \frac{1}{2} |u'(1)|^2 \psi(1)$$

for every smooth function such that $\psi(0) = 0$. Identity (28) is a sharpening of Pohožaev's identity—which corresponds to the special choice $\psi(r) = r$. Next, we assume by contradiction that $0 < \lambda \leqslant \pi^2/4$ (we recall that $\lambda_1 = \pi^2$ for the unit ball). In (28) we choose $\psi(r) = \sin(\sqrt{4\lambda}\,r)$, so that $\lambda\psi' + \frac{1}{4}\psi''' \equiv 0$, while $r\psi - r^2\psi' > 0$ on $(0, 1)$ and $\psi(1) \geqslant 0$, which is absurd.

The following lemma allows us to conclude the proof of Theorem 2.

LEMMA 5. *Suppose $\Omega \subset \mathbf{R}^3$ is any bounded domain. Then there is some $\lambda^* \in (0, \lambda_1)$, depending on Ω, such that*

$$S_\lambda = S \quad for \ \lambda \leqslant \lambda^*,$$
$$S_\lambda < S \quad for \ \lambda > \lambda^*.$$

When Ω is a ball we have $\lambda^* = \lambda_1/4$.

In other words, the graph of the function $\lambda \mapsto S_\lambda$ looks as in Figure 4. Note the difference between the cases $N = 3$ and $N \geqslant 4$.
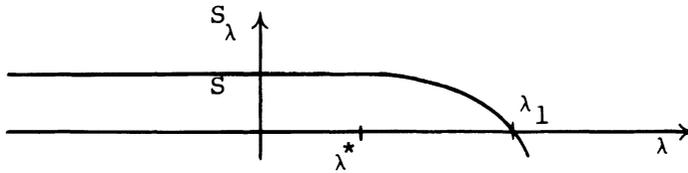

FIGURE 4

REMARK 4. It follows from Lemma 5 and Lemma 2 that for $\lambda > \lambda^*$, the infimum in (11) is achieved and it provides (after stretching) a solution of (I). On the other hand, if $\lambda < \lambda^*$, then the infimum for S_λ (i.e. (11)) is *not* achieved. Indeed, suppose by contradiction that it is achieved by some u_0. Let μ be such that $\mu > \lambda$. Then $S_\mu < S_\lambda = S$, since we may use u_0 as a testing function when computing S_μ. This is absurd since $S_\mu = S$ for $\mu \leqslant \lambda^*$.

We do not know if there can be solutions of (I) when Ω is starshaped and $\lambda < \lambda^*$; obviously, if there are any they would correspond to critical points of the functional $\int |\nabla u|^2 - \lambda \int u^2$ on the sphere $\|u\|_{p+1} = 1$—but they would not be minima. When Ω is a ball we know (see Lemma 4) that there is no solution of (I) for $\lambda \leqslant \lambda^* = \lambda_1/4$. On the other hand, when Ω is an annulus, there are solutions of (I) for *all* $\lambda \in (-\infty, \lambda_1)$.

Incidentally, we do not know whether for some domains Ω the infimum in (11) is achieved when $\lambda = \lambda^*$; this is an interesting open problem (if Ω is a ball it is not achieved since there is no solution of (I) when $\lambda = \lambda^*$.

SKETCH OF THE PROOF OF LEMMA 5. We start with the case where Ω *is the unit ball*. If we try the same argument as in the proof of Lemma 1 we find, as $\varepsilon \to 0$,

$$Q_\lambda(u_\varepsilon) = S + A\varepsilon^{1/2} - B\lambda\varepsilon^{1/2} + O(\varepsilon),$$

where A and B depend on ζ. We conclude that $S_\lambda < S$ for $\lambda > A/B$. Therefore one should try to minimize the ratio A/B. The optimal choice of ζ corresponds to $\zeta(x) = \cos(\pi|x|/2)$, i.e.,

$$u_\varepsilon(x) = \cos(\pi|x|/2)\big(\varepsilon + |x|^2\big)^{1/2}.$$

A careful expansion as $\varepsilon \to 0$ (see [16]) shows that

$$Q_\lambda(u_\varepsilon) = S + \left(\tfrac{1}{4}\pi^2 - \lambda\right)K\varepsilon^{1/2} + O(\varepsilon)$$

where $K > 0$ is a constant. Therefore we see that $S_\lambda < S$ for $\lambda > \pi^2/4$.

On the other hand, we must have $S_\lambda = S$ for $\lambda \leqslant \pi^2/4$. Otherwise—if $S_\lambda < S$ —we would deduce from Lemma 2 that there is a solution of (I); this is impossible, by Lemma 4.

We turn now to the case of a *general bounded domain* Ω. Let $\tilde{\Omega}$ be a ball with $\Omega \subset \tilde{\Omega}$. Given $u \in H_0^1(\Omega)$, we extend it by 0 outside Ω. The previous analysis shows that

$$\int_{\tilde{\Omega}} |\nabla u|^2 \geqslant S\|u\|_{L^6(\tilde{\Omega})}^2 + \frac{\lambda_1(\tilde{\Omega})}{4}\|u\|_{L^2(\tilde{\Omega})}^2$$

and hence there is some $\delta > 0$ such that

$$\int_\Omega |\nabla u|^2 \geqslant S\|u\|_{L^6(\Omega)}^2 + \delta\|u\|_{L^2(\Omega)}^2 \qquad \forall u \in H_0^1(\Omega).$$

(Alternatively, one could also use symmetrization as in [16].) This implies that $S_\delta \geqslant S$; but on the other hand $S_\lambda \leqslant S$ $\forall \lambda \in \mathbf{R}$ (see the proof of Lemma 1). Therefore we conclude that $S_\delta = S$.

ADDITIONAL PROPERTIES. OPEN PROBLEMS.

1. *Uniqueness.* It is not known—even when Ω is a ball—whether the solution of (I) is unique, except for λ near λ_1, where the standard bifurcation analysis gives uniqueness.

2. *Regularity. Further estimates.* So far, we were concerned only with the existence of weak H^1 solutions of (I). It can be shown that H^1 solutions of (I) are smooth. This is a consequence of the following.

LEMMA 6 (BREZIS–KATO [12]). *Suppose $a(x) \in L^{N/2}(\Omega)$ and $u \in H_0^1(\Omega)$ satisfies*

$$-\Delta u = au \quad in \ \Omega.$$

Then $u \in L^q(\Omega)$ for all $q < \infty$.

In the case of Problem (I) we use Lemma 6 with $a = u^{p-1} + \lambda \in L^{N/2}$ (since $(p - 1)(N/2) = 2N/(N - 2)$). Therefore, we conclude that $\Delta u \in L^t(\Omega)$ for all $t < \infty$, and thus $u \in W^{2,t}(\Omega)$ for all $t < \infty$, etc.... *It is, however, surprising to find out that there is no estimate of the type*

$$\|u\|_\infty \leqslant F\big(\|u\|_{H^1}, |\lambda|\big)$$

for the solutions of (I), where F is bounded on bounded sets. To see this, suppose $N \geqslant 4$ and let u_λ be a solution of (I) with $\lambda \in (0, \lambda_1)$ corresponding to a minimum v_λ of (11). More precisely, we have

$$\int |\nabla v_\lambda|^2 - \lambda \int v_\lambda^2 = S_\lambda, \qquad \|v_\lambda\|_{p+1} = 1$$

and some multiple of v_λ, namely $u_\lambda = S_\lambda^{1/(p-1)} v_\lambda$, satisfies (I). It follows that

$$\| \nabla u_\lambda \|_2^2 \leqslant S_\lambda^{2/(p-1)} \left(S_\lambda + \lambda \int v_\lambda^2 \right)$$

and, in particular, u_λ *remains bounded in H_0^1 as $\lambda \to 0$.*

On the other hand, we claim that

$$\lim_{\lambda \downarrow 0} \| u_\lambda \|_q = \infty \quad \text{for any } q > 2^*.$$

Indeed suppose that $\| u_{\lambda_n} \|_q \leqslant C$ for some sequence $\lambda_n \to 0$ and some $q > 2^*$. It follows from (I) that (u_{λ_n}) is bounded in $W^{2,q/p}$ and therefore (u_{λ_n}) is relatively compact in H_0^1 since $p/q - 1/N < 1/2$. Thus (v_{λ_n}) is also relatively compact in H_0^1 and we may assume that $v_{\lambda_n} \to v$ strongly in H_0^1 with

$$\int |\nabla v|^2 = S, \qquad \| v \|_{p+1} = 1;$$

but that is absurd.

3. *Continuous branches of solutions.* It follows from the abstract bifurcation theory (see P. Rabinowitz [48]) that there is a continuous branch $\mathscr{C}$ of solutions (λ, u) of (I) which emanates from $(\lambda_1, 0)$ and which goes to infinity in $\mathbf{R} \times L^\infty$. It would be interesting to prove, for example, that if $N \geqslant 4$, then $\mathscr{C}$ covers at least the interval $(0, \lambda_1)$. *This is related to the open problem of finding L^∞ estimates for all solutions of* (I) *when $\lambda > 0$ stays bounded away from* 0. Even H^1 estimates are not known—but as we have pointed out above, they would not imply L^∞ estimates.

Presumably, the bifurcation diagrams, when Ω is a ball, look as in Figure 5.

FIGURE 5

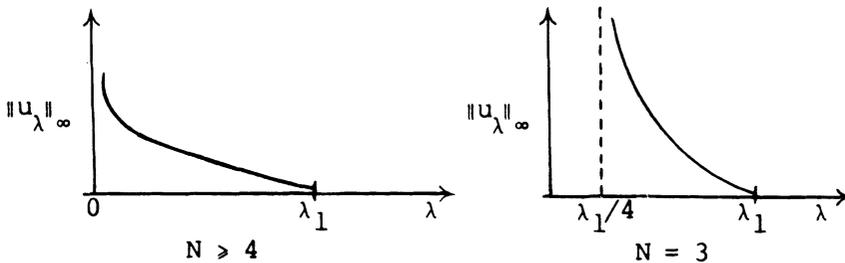4. *The case of an annulus.* Suppose Ω is an annulus, say

$$\Omega = \{ x \in \mathbf{R}^N; \ a < |x| < b \} \quad \text{with } N \geqslant 4.$$

We claim that for every $\lambda > 0$, small enough, Problem (I) admits both radial and nonradial solutions. Indeed, set

$$(29) \qquad \Sigma_\lambda = \inf_{\substack{u \in H_0^1(\Omega) \\ u \text{ radial} \\ \|u\|_{p+1} = 1}} \left\{ \int |\nabla u|^2 - \lambda \int u^2 \right\}.$$

Since the *injection* $H_0^1 \subset L^{p+1}$ *is compact for radial functions* we see that the infimum in (29) is achieved for all $\lambda \in \mathbf{R}$ and—after stretching—it provides a solution of (I) for all $\lambda \in (-\infty, \lambda_1)$. The functions $\lambda \mapsto \Sigma_\lambda$ and $\lambda \mapsto S_\lambda$ are continuous and, moreover, $S_0 < \Sigma_0$ (since the best Sobolev constant is not achieved). It folows that $S_\lambda < \Sigma_\lambda$ for $\lambda > 0$ small enough. *We deduce that the infimum in* (11) *is achieved by some nonradial function.* We do not know whether these nonradial solutions occur by "secondary bifurcation" from the "branch" of radial solution. Also, it is plausible that there is some constant $\lambda_c \in (0, \lambda_1)$ such that

$$\Sigma_\lambda = S_\lambda \quad \text{for } \lambda \geqslant \lambda_c,$$

$$\Sigma_\lambda > S_\lambda \quad \text{for } \lambda < \lambda_c,$$

and that a secondary bifurcation occurs at λ_c. One may guess the diagrams in Figure 6. Other results concerning *symmetry breaking* for positive solutions of semilinear elliptic equations have been obtained by C. Coffman [20] and J. Smoller–A. Wasserman [52].
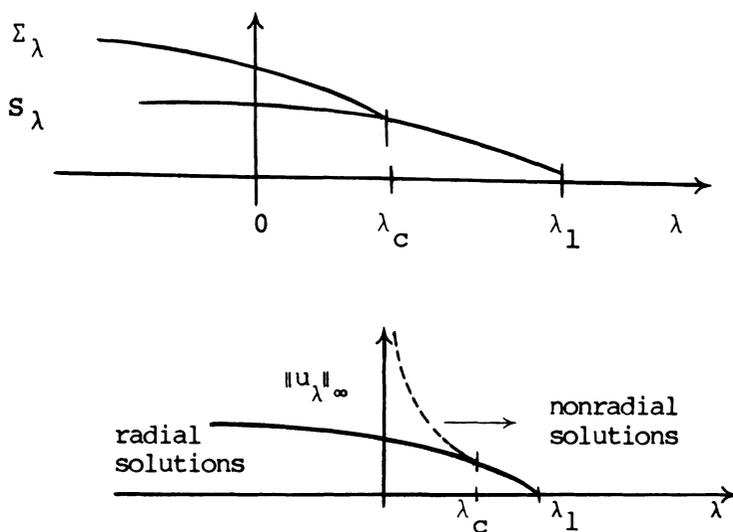

FIGURE 6

5. *Domains with topology.* Suppose Ω is not contractible in itself to a point. It is a very interesting open problem to determine whether there exist solutions of (I) for all $\lambda \in (-\infty, \lambda_1)$; but some exciting partial results have been obtained recently by J.-M. Coron [22] and J.-M. Coron–A. Bahri [69].

More generally, it is tempting to conjecture that if Ω is not contractible to a point the problem

$$\begin{cases} -\Delta u = u^p + a(x)u & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a solution under the single assumption $\int |\nabla\phi|^2 - a(x)\phi^2 \geqslant \delta \int |\nabla\phi|^2$, $\forall \phi \in H_0^1$, with $\delta > 0$.

6. *Solutions with variable sign.* Some results concerning the problem

$$\begin{cases} -\Delta u = |u|^{p-1}u + \lambda u & \text{on } \Omega, \\ u \not\equiv 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

have been obtained recently by Cerami–Fortunato–Struwe [18], Struwe [56], Capozzi–Fortunato–Palmieri [68] and D. Zhang [70].

7. *Improved Sobolev inequalities with best constants.* Let $\Omega \subset \mathbf{R}^3$ be (any) bounded domain. The conclusion of Lemma 5 may be read as follows. There is a (best) constant $\lambda^* > 0$ depending on Ω, such that

$$\|\nabla u\|_2^2 \geqslant S\|u\|_6^2 + \lambda^*\|u\|_2^2 \qquad \forall u \in H_0^1(\Omega).$$

Suppose now that $\Omega \subset \mathbf{R}^N$ with $N \geqslant 4$. In view of Lemma 1 there is *no* inequality of the type

$$\|\nabla u\|_2^2 \geqslant S\|u\|_{2*}^2 + \delta\|u\|_2^2 \qquad \forall u \in H_0^1(\Omega),$$

with $\delta > 0$. However one can still exhibit a "bonus term", namely we have

$$\|\nabla u\|_2^2 \geqslant S\|u\|_{2*}^2 + \delta[u]_{N/(N-2)}^2 \qquad \forall \in H_0^1(\Omega)$$

for some $\delta > 0$, where $[\ ]_q$ denotes the weak (Marcinkiewicz) q-norm; see [15]. We recall that in $\mathbf{R}^N$ the Sobolev inequality

$$\|\nabla u\|_2^2 \geqslant S\|u\|_{2*}^2$$

is strict except if u belongs to the set $\mathscr{E}$ of extremal functions, i.e., u is of the form $u(x) = CU_\varepsilon(x - x_0)$. In particular, the inequality is strict if u has its support in a bounded domain. It is an interesting question to estimate from below the quantity

$$\|\nabla u\|_2^2 - S\|u\|_{2*}^2$$

by an expression which involves the "distance from u to $\mathscr{E}$".

That question recalls the Bonnesen-type isoperimetric inequality (see Osserman [46]); it provides an estimate from below for $L^2 - 4\pi A$, where A is the area of a convex set in $\mathbf{R}^2$, bounded by a curve of length L.

I should also mention that an improvement of the Hardy–Littlewood inequality (with the best constant) for functions with support in a fixed ball has been obtained and used by Daubechies–Lieb [24].

8. *Equations with variable coefficients.* It is fairly easy to obtain results comparable to Theorem 1 for the problem

$$\begin{cases} -\Delta u = u^p + a(x)u & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

However, the problem

$$\begin{cases} -\Delta u = a(x)u^p + \lambda u & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $a(x)$ smooth on $\overline{\Omega}$ and $a(x) \geqslant a > 0$ seems more delicate and largely open (except for some special cases, see e.g. A. Bahri–J.-M. Coron [65] and Escobar–Schoen [71]).

2. Problem (II). The main abstract tool which we shall use is the following variant of the Mountain Pass Lemma of Ambrosetti–Rabinowitz [1]:

LEMMA 7. *Let F be a C^1 function on a Banach space E. Assume*

there are constants $\rho > 0$ and $R > 0$ such that

(30)
$$F(u) \geqslant \rho \text{ for all } u \in E \text{ with } \|u\| = R$$

and

(31) $$F(0) = 0 \text{ and } F(v_0) \leqslant 0 \text{ for some } v_0 \in E \text{ with } \|v_0\| > R.$$

Set

$$M = \left\{ p \in C([0,1]; E); \ p(0) = 0, \ p(1) = v_0 \right\}$$

and

(32) $$c = \inf_{p \in M} \max_{t \in [0,1]} F(p(t)) \geqslant \rho.$$

Then there exists a sequence (u_j) in E such that $F(u_j) \to c$ and $F'(u_j) \to 0$ in E^*.

REMARK 5. I emphasize that the (PS) condition is not among the assumptions of Lemma 7. When the (PS) condition is assumed we may conclude that c is a critical value, i.e., there exists some $u \in E$ such that $F(u) = c$ and $F'(u) = 0$. This is the standard version of the Ambrosetti–Rabinowitz Lemma. Indeed, the (PS) condition says that (u_j) is relatively compact whenever $F(u_j)$ is bounded and $F'(u_j) \to 0$.

Recently J. P. Aubin and I. Ekeland [2] have found a very elegant proof of the Mountain Pass Lemma. Their proof fits very well to our situation and I will sketch it briefly. It relies on *Ekeland's minimization principle*.

LEMMA 8. (*See* [25] *or* [2].) *Let M be a complete metric space and let $f: M \to (-\infty, +\infty]$ be a l.s.c. function.*

Assume $c = \text{Inf}_M f > -\infty$. Then for every $\varepsilon > 0$ there exists a $p_\varepsilon \in M$ such that

(33) $$c \leqslant f(p_\varepsilon) \leqslant c + \varepsilon$$

and

(34) $$f(p) - f(p_\varepsilon) + \varepsilon d(p, p_\varepsilon) \geqslant 0 \qquad \forall p \in M.$$

SKETCH OF THE PROOF OF LEMMA 7. Let $M = \{ p \in C([0,1]; E); \ p(0) = 0, p(1) = v_0 \}$ with its usual sup norm. We consider the mapping $f: M \to \mathbf{R}$ defined by

$$f(p) = \max_{t \in [0,1]} F(p(t))$$

so that

$$c = \operatorname*{Inf}_{p \in M} f(p) \geqslant \rho.$$

We deduce from Lemma 8 that there is a $p_\varepsilon \in M$ satisfying (33) and (34); in other words $p_\varepsilon([0,1])$ is almost an "optimal path" joining 0 to v_0. A careful analysis of (33) and (34) (see [2]) shows that there exists $t_\varepsilon \in [0,1]$ such that

$$f(p) = c \leqslant F(p_\varepsilon(t_\varepsilon)) = \operatorname*{Max}_{t \in [0,1]} F(p_\varepsilon(t)) \leqslant c + \varepsilon$$

and

$$\left\| F'(p_\varepsilon(t_\varepsilon)) \right\| < \varepsilon.$$

The conclusion of Lemma 7 follows easily by choosing $\varepsilon_j = 1/j$ and $u_j = p_{\varepsilon_j}(t_{\varepsilon_j})$.

In order to prove Theorems 3 and 4 we introduce on the space $E = H_0^1(\Omega)$ the C^1 functional

$$(35) \qquad F(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{p+1} \int (u^+)^{p+1} - \frac{\mu}{q+1} \int (u^+)^{q+1}.$$

In general F does not satisfy the (PS) condition (see Remark 6 below). However, the following lemma *makes up for the lack of the* (PS) *condition.*

LEMMA 9. *Suppose* $\mu \geqslant 0$ *and let* (u_j) *be a sequence in* H_0^1 *such that*

$$(36) \qquad\qquad F(u_j) \to c \text{ and } F'(u_j) \to 0 \quad in \ H^{-1},$$

with

$$(37) \qquad\qquad c < (1/N)S^{N/2}.$$

Then (u_j) *is relatively compact in* H_0^1.

REMARK 6. In view of Lemma 9 we can say that the "(PS)$_c$ *condition*" holds when c is less than the critical level $(1/N)S^{N/2}$. It is very easy to see that the (PS)$_c$ condition does not hold at the level $c = (1/N)S^{N/2}$. It suffices to choose a sequence of the form $u_\varepsilon(x) = S^{1/(p-1)}\zeta(x)SU_\varepsilon(x)$, so that $F(u_\varepsilon) \to (1/N)S^{N/2}$, $F'(u_\varepsilon) \to 0$ in H^{-1} as $\varepsilon \to 0$; moreover $u_\varepsilon \rightharpoonup 0$ weakly in H_0^1 but u_ε does not converge to 0 strongly in H_0^1.

PROOF OF LEMMA 9. We rewrite (36) as

$$(38) \quad \frac{1}{2}\int |\nabla u_j|^2 - \frac{1}{p+1}\int (u_j^+)^{p+1} - \frac{\mu}{q+1}\int (u_j^+)^{q+1} = c + o(1)$$

and

$$(39) \qquad\qquad -\Delta u_j = (u_j^+)^p + \mu(u_j^+)^q + \varepsilon_j,$$

with $\varepsilon_j \to 0$ in H^{-1}. Multiplying (39) by u_j, we find

$$(40) \qquad \int |\nabla u_j|^2 = \int (u_j^+)^{p+1} + \mu\int (u_j^+)^{q+1} + \langle \varepsilon_j, u_j \rangle.$$

Combining (38) and (40), one shows easily that (u_j) is bounded in H_0^1. We may therefore assume that $u_j \rightharpoonup u$ weakly in H_0^1 and u satisfies

$$-\Delta u = (u^+)^p + \mu(u^+)^q \quad in \ \Omega.$$

Hence we also have

(41)
$$\int |\nabla u|^2 = \int (u^+)^{p+1} + \mu \int (u^+)^{q+1}.$$

We write

$$u_j = u + v_j.$$

A variant of Lemma 3 (see [13]) leads to

(42)
$$\int (u_j^+)^{p+1} = \int (u^+)^{p+1} + \int (v_j^+)^{p+1} + o(1).$$

Combining (38) with (42) and (40) with (42) we obtain

(43)
$$F(u) + \frac{1}{2}\int |\nabla v_j|^2 - \frac{1}{p+1}\int (v_j^+)^{p+1} = c + o(1)$$

and
(44)
$$\int |\nabla u|^2 + \int |\nabla v_j|^2 = \int (u^+)^{p+1} + \int (v_j^+)^{p+1} + \mu \int (u^+)^{q+1} + o(1).$$

Then, using (41), we deduce that

(45)
$$\int |\nabla v_j|^2 = \int (v_j^+)^{p+1} + o(1).$$

We may therefore assume that

$$\int |\nabla v_j|^2 \to k, \qquad \int (v_j^+)^{p+1} \to k.$$

By Sobolev's inequality, we have

$$\int |\nabla v_j|^2 \geqslant S\|v_j^+\|_{p+1}^2,$$

and at the limit we have $k \geqslant Sk^{2/(p+1)}$. It follows that *either* $k = 0$ *or* $k \geqslant S^{N/2}$. We shall now see that $k \geqslant S^{N/2}$ is excluded—which implies that $k = 0$, i.e., $u_j \to u$ strongly in H_0^1 and the proof of Lemma 9 will be concluded. Suppose $k \geqslant S^{N/2}$. Passing to the limit in (43) we obtain

$$F(u) + \frac{1}{N}k = c,$$

and with assumption (37) we find
(46)
$$F(u) < 0.$$

On the other hand we have, by definition of F and in view of (41),

$$F(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int (u^+)^{p+1} - \frac{\mu}{q+1}\int (u^+)^{q+1}$$

$$= \frac{1}{N}\int (u^+)^{p+1} + \mu\left(\frac{1}{2} - \frac{1}{q+1}\right)\int (u^+)^{q+1} \geqslant 0,$$

a contradiction of (46).

SKETCH OF THE PROOF OF THEOREM 3. The functional F defined by (35) obviously satisfies (30) if $R > 0$ is small enough. Also, there are plenty of v_0's satisfying (31): it suffices to pick any $w_0 \equiv 0$, $w_0 \not\equiv 0$ and then $v_0 = kw_0$ satisfies (31) when k is large enough. In general, when the (PS) condition holds, the choice of v_0 is irrelevant. Here, however the situation is different since c *defined by* (31) *depends* on v_0 and we would like to achieve $c < (1/N)S^{N/2}$. The choice of v_0 becomes a delicate matter. Clearly, it suffices to find a $w_0 \in H_0^1$ such that $w_0 \geqslant 0$, $w_0 \not\equiv 0$ and

$$(47) \qquad\qquad \sup_{t \geqslant 0} F(tw_0) < \frac{1}{N}S^{N/2}.$$

Indeed we may then take $v_0 = kw_0$ with $k > 0$ so large that $k\|w_0\| > R$ and $F(kw_0) < 0$; also we have

$$c \leqslant \sup_{t \in [0,1]} F(tv_0) \leqslant \sup_{t \geqslant 0} F(tw_0) < \frac{1}{N}S^{N/2}.$$

It is not clear that one can find a w_0 satisfying (47). For example, if $\mu = 0$ an easy computation shows that

$$\sup_{t \geqslant 0} F(tw_0) = \frac{1}{N}\left(\frac{\|\nabla w_0\|_2^2}{\|w_0\|_{p+1}^2} \right)^{N/2} > \frac{1}{N}S^{N/2}.$$

This computation suggests, however, that the term μu^q "helps" to lower c and that w_0 should be chosen close to the extremal function for the Sobolev inequality. Therefore we set, as in the proof of Lemma 1,

$$w_\varepsilon(x) = \zeta(x)U_\varepsilon(x).$$

A careful expansion as $\varepsilon \to 0$ (see [16]) shows that

$$\sup_{t \geqslant 0} F(tw_\varepsilon) \leqslant \frac{1}{N}S^{N/2} + O(\varepsilon^{(N-2)/2}) - \mu K\varepsilon^\alpha,$$

where $K > 0$ is a constant and $\alpha = ((N + 2) - q(N - 2))/4 < (N - 2)/2$. The conclusion follows by choosing $\varepsilon > 0$ small.

SKETCH OF THE PROOF OF THEOREM 4. When $N = 3$ the argument above leads to

$$\sup_{t \geqslant 0} F(tw_\varepsilon) \leqslant \frac{1}{3}S^{3/2} + O(\varepsilon^{1/2}) - \mu K\varepsilon^\alpha$$

for some constants $K > 0$ and α such that

$$\begin{aligned} 0 < \alpha < 1/2 & \quad \text{if } 3 < q < 5, \\ \alpha = 1/2 & \quad \text{if } q = 3, \\ \alpha \geqslant 1/2 & \quad \text{if } 1 < q < 3. \end{aligned}$$

This concludes the proof of Theorem 4 for $3 < q < 5$.

When $1 < q \leqslant 3$ the argument is different. We fix *any* $w_0 \geqslant 0$, $w_0 \not\equiv 0$. In order to emphasize the μ dependence I write now F_μ instead of F. I claim that

$$(48) \qquad \sup_{t \geqslant 0} F_\mu(tw_0) \to 0 \quad \text{as } \mu \to \infty.$$

Indeed, the sup in (48) is achieved for some $t = t_\mu$ such that

$$t_\mu \int |\nabla w_0|^2 - t_\mu^p \int w_0^{p+1} - \mu t_\mu^q \int w_0^{q+1} = 0$$

and thus

$$t_\mu \leqslant \left[\frac{1}{\mu} \frac{\int |\nabla w_0|^2}{\int w_0^{q+1}} \right]^{1/(q-1)}.$$

Hence $\sup_{t \geqslant 0} F_\mu(tw_0) \leqslant \frac{1}{2} t_\mu^2 \int |\nabla w_0|^2 \to 0$ as $\mu \to \infty$. It follows that for μ large enough, condition (47) is satisfied and (II) has a solution. We establish now that (II) has no solution when Ω is strictly starshaped, $1 < q \leqslant 3$ and $\mu > 0$ is small. Indeed, suppose u is a solution of (II). Pohožaev's identity (6) leads to

$$\mu \int_\Omega u^{q+1} \geqslant a \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} \right)^2 \geqslant b \left(\int_{\partial\Omega} \frac{\partial u}{\partial n} \right)^2 = b \left(\int_\Omega \Delta u \right)^2 = b \left(\int_\Omega |\Delta u| \right)^2.$$

Next, we use the fact that Δ^{-1} is a bounded operator from L^1 into L_w^3 (weak L^3) and that $|\Delta u| \geqslant u^5$. Hence we find

$$(49) \qquad \mu \int_\Omega u^{q+1} \geqslant c[u]_3^2,$$

$$(50) \qquad \mu \int_\Omega u^{q+1} \geqslant c\|u\|_5^{10}.$$

By Hölder and interpolation we have

$$(51) \qquad \|u\|_{q+1} \leqslant C\|u\|_r \leqslant C[u]_3^a \|u\|_5^{1-a}$$

with $r = 6(q+1)/(q+3)$, $a/3 + (1-a)/5 = 1/r$, i.e., $a = (9-q)/r(q+1)$ (note that $r \geqslant q+1$ and $3 < r \leqslant 5$ since $1 < q \leqslant 3$). Combining (49), (50), and (51), we find

$$\|u\|_{q+1} \leqslant C \left(\mu \int u^{q+1} \right)^{a/2 + (1-a)/10}.$$

But $a/2 - (1-a)/10 = 1/(q+1)$ and thus we conclude that $\mu \geqslant \mu_0 > 0$.

ADDITIONAL PROPERTIES. OPEN PROBLEMS.

1. Instead of (II) we now consider the more general problem

$$(52) \qquad \begin{cases} -\Delta u = \lambda u^p + f(x, u) & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$, $p = (N + 2)/(N - 2)$, $f\colon \Omega \times \mathbf{R}_+ \to \mathbf{R}_+$ and $f(x,0) = 0$. Using the same method as above one can prove (see [**16**]) that (52) has a solution under the following assumptions.

$$(53) \qquad f(x, u) = o(u^p) \quad \text{as } u \to +\infty;$$

$$(54) \qquad \begin{cases} -\Delta - f_u(x,0) \quad \text{is positive, i.e.,} \\ \int |\nabla\phi|^2 - f_u(x,0)\phi^2 \geq \delta \int |\nabla\phi|^2 \quad \text{with } \delta > 0, \forall\phi \in H_0^1; \end{cases}$$

$$(55) \qquad f(x, u) \geq \delta u^q \quad \text{with } \delta > 0, q \geq 1, \forall x \in \Omega, \forall u \geq 0.$$

When $N \geq 5$ it suffices to assume, instead of (55),

$$(55') \qquad f(x_0, u_0) > 0 \text{ for some } x_0 \in \Omega \text{ and some } u_0 > 0.$$

This result may be used in order to solve the problem: Find u satisfying

$$(56) \qquad \begin{cases} -\Delta u = \lambda(1 + u)^p & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ and $p = (N + 2)/(N - 2)$.

Then there exists a constant $\bar{\lambda} > 0$ such that problem (56) admits:

(a) at least two solutions for every $\lambda \in (0, \bar{\lambda})$,

(b) one solution for $\lambda = \bar{\lambda}$,

(c) no solution for $\lambda > \bar{\lambda}$.

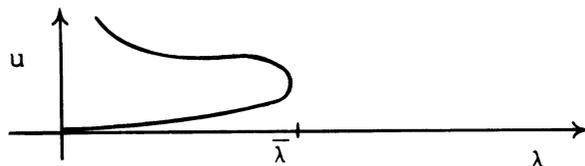In other words, $\bar{\lambda}$ is a turning point, and we have the diagram in Figure 7.

FIGURE 7

Indeed it is already known (see [**38**, **23**]) that there is a constant $\bar{\lambda}$ such that problem (56) admits:

(a) a minimal solution $\underline{u}$ for every $\lambda \in (0, \bar{\lambda})$ and that $-\Delta - \lambda p(1 + \underline{u}(x))^{p-1}$ is positive,

(b) one solution for $\lambda = \bar{\lambda}$,

(c) no solution for $\lambda > \bar{\lambda}$.

We look for a second solution of (56) of the form

$$u = \underline{u} + v$$

so that (56) becomes

$$(57) \qquad \begin{cases} -\Delta u = \lambda(1 + \underline{u} + v)^p - \lambda(1 + \underline{u})^p & \text{on } \Omega, \\ v > 0 & \text{on } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, we write

$$\lambda(1 + \underline{u} + v)^p - \lambda(1 + \underline{u})^p \equiv \lambda v^p + f(x, v)$$

and it is easy to check that (53), (54), and (55) hold.

2. There are still many unsolved problems[1] concerning Problem (II) even when Ω is a *ball*, $N = 3$ and $p = 5$. Numerical computations due to O. Bristeau (at INRIA) suggest that:

(i) *When $q = 3$ there is some μ_0 such that*
    (a) for $\mu > \mu_0$ there is a unique solution of (II),
    (b) for $\mu \leqslant \mu_0$ there is no solution of (II)
[when Ω is the unit ball, it seems that $\mu_0 = 8/\sqrt{3}\,\pi$];

(ii) *when $1 < q < 3$ there is some μ_0 such that*
    (a) for $\mu > \mu_0$ there are two solutions of (II),
    (b) for $\mu = \mu_0$ there is one solution of (II),
    (c) for $\mu < \mu_0$ there is no solution of (II).

3. Problem (III) (Rellich's conjecture). I will explain in detail how to handle the Dirichlet problem (III$_D$) and then I will say a few words about the Plateau problem (III$_P$).

The existence of a first solution $\underline{u}$ of (III$_D$)—the"*small solution*— is due to Hildebrandt [**32, 33**]. His argument is the following. The solutions of (III$_D$) are the critical points of the functional

$$\Phi(u) = \int |\nabla u|^2 + \frac{4H}{3} \int u \cdot u_x \wedge u_y$$

subject to the constraint

$$u \in H^1_\gamma = \left\{ u \in H^1(\Omega; \mathbf{R}^3); \ u = \gamma \quad \text{on } \partial\Omega \right\}.$$

Unfortunately, $\inf_{u \in H^1_\gamma} \Phi(u) = -\infty$. Therefore, we consider instead the "variational inequality"

$$(58) \qquad\qquad A = \inf_{\substack{u \in H^1_\gamma \\ \|u\|_\infty \leqslant R'}} \Phi(u),$$

where R' is a fixed constant with $R < R' < 1/H$. It is easy to see that the infimum in (58) is achieved. Indeed, let (u^j) be a minimizing sequence. Since

$$|a \wedge b| \leqslant |a||b| \leqslant \tfrac{1}{2}\left(|a|^2 + |b|^2\right)$$

and since $HR'/3 < \tfrac{1}{2}$ it follows that (u^j) is bounded in H^1. We may assume that $u^j \rightharpoonup \underline{u}$ weakly in H^1 and $u^j \to \underline{u}$ a.e. Writing

$$u^j = \underline{u} + v^j,$$

we obtain

$$(59) \quad \int |\nabla \underline{u}|^2 + \int |\nabla v^j|^2 + \frac{4H}{3} \int u^j \cdot \left(\underline{u}_x + v^j_x\right) \wedge \left(\underline{u}_y + v^j_y\right) = A + o(1).$$

---

[1] Very recently Atkinson and Peletier [**66**] have partially answered the problem described below.

By dominated convergence we have

$$u^j \wedge \underline{u}_x \to \underline{u} \wedge \underline{u}_x \quad \text{and} \quad u^j \wedge \underline{u}_y \to \underline{u} \wedge \underline{u}_y \quad \text{strongly in } L^2,$$

and thus

$$\int u^j \cdot \underline{u}_x \wedge v_y^j = o(1), \qquad \int u^j \cdot v_x^j \wedge \underline{u}_y = o(1).$$

Since

$$\frac{4H}{3} \left| \int u^j \cdot \left(v_x^j \wedge v_y^j \right) \right| \leqslant \frac{2}{3} \int |\nabla v^j|^2,$$

we deduce from (59) that $\underline{u}$ is a minimizer for (58).

Next, one shows (using $HR' < 1$) that $-\Delta |\underline{u}|^2 \leqslant 0$ on Ω and the maximum principle implies that

$$\sup_\Omega |\underline{u}| \leqslant \sup_{\partial\Omega} |\underline{u}| \leqslant R.$$

Therefore $\underline{u}$ satisfies $\Phi'(\underline{u}) = 0$ (and not just the variational inequality), that is, $\underline{u}$ is a solution of $(\mathrm{III}_\mathrm{D})$.

Using the fact that the second variation of Φ is $\geqslant 0$ we obtain

$$\int |\nabla\phi|^2 + 4H \int \underline{u} \cdot \phi_x \wedge \phi_y \geqslant 0 \qquad \forall \phi \in H_0^1.$$

A more delicate argument (see [9]) shows that $\underline{u}$ is a strict minimum, i.e., there is a constant δ such that

$$(60) \qquad \int |\nabla\phi|^2 + 4H \int \underline{u} \cdot \phi_x \wedge \phi_y \geqslant \delta \int |\nabla\phi|^2 \qquad \forall \phi \in H_0^1.$$

Then, we look for a second solution $\underline{u}$ of $(\mathrm{III}_\mathrm{D})$ of the form

$$\bar{u} = \underline{u} - v$$

and we are led to the system

$$(7) \qquad \begin{cases} \mathscr{L}v \equiv -\Delta v + 2H\left(\underline{u}_x \wedge v_y + v_x \wedge \underline{u}_y \right) = 2Hv_x \wedge v_y & \text{on } \Omega, \\ v \not\equiv 0 & \text{on } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

which has again a variational structure.

The linear operator $\mathscr{L}$ is selfadjoint and is the Fréchet derivative of the functional $\frac{1}{2}(\mathscr{L}v, v)$ with

$$(\mathscr{L}v, v) = \int |\nabla v|^2 + 4H \int \underline{u} \cdot v_x \wedge v_y,$$

while the nonlinear term $v_x \wedge v_y$ is the Fréchet derivative of the volume functional $\frac{1}{3}Q(v)$ where

$$(61) \qquad Q(v) = \int v \cdot v_x \wedge v_y.$$

In view of the analogy between (7) and (I) (a linear selfadjoint operator versus a nonlinear potential operator which is homogeneous) it is tempting to try the following program. Consider

$$(62) \qquad\qquad J = \operatorname*{Inf}_{\substack{v \in H_0^1 \\ Q(v)=1}} (\mathscr{L}v, v)$$

and suppose that the Inf in(62) is achieved by some v^0. Then we have

$$\mathscr{L}v^0 = \mu v_x^0 \wedge v_y^0$$

where μ is a Lagrange multiplier—and clearly $\mu = J$. Thus $v = (J/2H)v^0$ satisfies (7).

There are several difficulties with this program:

1. It is not clear that $Q(v)$, defined by (61), makes sense for all $v \in H_0^1$, since $v \notin L^\infty$ in general. But, as we shall see, it is indeed possible to extend Q by continuity to all of H_0^1.

2. The function Q, which is continuous for the strong H_0^1 topology, is *not* continuous for the weak H^1 topology and the set $\{v \in H_0^1;\ Q(v) = 1\}$ is *not* closed for the weak H_0^1 topology. This generates the same difficulty as in Problem (I) and we overcome it with the same strategy. The inequality

$$|Q(v)|^{2/3} \leqslant C \int |\nabla v|^2 \qquad \forall v \in H_0^1$$

takes the place of the Sobolev inequality, and we have a best constant

$$S = \operatorname*{Inf}_{\substack{v \in H_0^1 \\ Q(v)=1}} \int |\nabla v|^2.$$

Then we split the argument into two parts:

Step 1. Show that if $J < S$, then the infimum in (62) is achieved,

Step 2. Show that indeed $J < S$: the additional term $4H \int \underline{u} \cdot v_x \wedge v_y$ "helps" to lower the infimum.

Here are some of the technical details. We start with a very useful estimate (see Lemma A.1 in [9]):

LEMMA 10. *Let* $\phi, \psi \in H^1(\Omega; \mathbf{R})$ *and let v be the solution of*

$$\begin{cases} \Delta v = \phi_x \psi_y - \psi_x \phi_y & \text{on } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $v \in L^\infty \cap H^1$ and

$$(63) \qquad\qquad \|v\|_\infty \leqslant C \|\nabla \phi\|_2 \|\nabla \psi\|_2,$$

$$(64) \qquad\qquad \|\nabla v\|_2 \leqslant C \|\nabla \phi\|_2 \|\nabla \psi\|_2.$$

REMARK 7. The conclusion of Lemma 10 is somewhat surprising. Indeed the function $f = \phi_x \psi_y - \psi_x \phi_y$ lies in L^1 and *not better*! Therefore u could have a logarithmic singularity; however, this does not happen, because the special form of f produces some cancellations. Also note that (64) is a direct consequence of

(63) since

$$\int |\nabla v|^2 \leqslant \|v\|_\infty \int |\phi_x\psi_y - \psi_x\phi_y| \leqslant \|v\|_\infty \|\nabla\phi\|_2 \|\nabla\psi\|_2.$$

LEMMA 11. *There is a constant C such that*

(65) $\quad \left| \int u \cdot \phi_x \wedge \phi_y \right| \leqslant C\|\nabla u\|_2 \|\nabla\phi\|_2^2 \quad \forall u \in \mathscr{D}(\Omega; \mathbf{R}^3), \forall \phi \in H^1(\Omega; \mathbf{R}^3).$

PROOF. Indeed, we introduce the solution v of the problem

$$\begin{cases} \Delta v = \phi_x \wedge \phi_y & \text{on } \Omega, \\ v = 0 & \text{on } \partial\Omega; \end{cases}$$

we write

$$\int u.\phi_x \wedge \phi_y = \int u\Delta v = -\int \nabla u \cdot \nabla v$$

and we use (64).

We set

$$R(u, v) = \int u \cdot v_x \wedge v_y \quad \text{for } u \in \mathscr{D}(\Omega; \mathbf{R}^3),\, v \in H^1(\Omega; \mathbf{R}^3)$$

and extend R by continuity to $H_0^1 \times H^1$ with the help of Lemma 11. Clearly we have

$$|R(u, v)| \leqslant C\|\nabla u\|_2 \|\nabla v\|_2^2 \quad \forall u \in H_0^1, \quad \forall v \in H^1$$

and

$$|R(u, v + w) - R(u, v) - R(u, w)| \leqslant C\|\nabla u\|_2 \|\nabla v\|_2 \|\nabla w\|_2$$
$$\forall u \in H_0^1, \quad \forall v, w \in H^1.$$

We set

$$Q(u) = R(u, u) \quad \text{for } u \in H_0^1,$$

and obviously Q is continuous on H_0^1 for the strong H_0^1 topology. However, one should be careful with the meaning of R and Q. If $u \in H_0^1$, the function $u \cdot u_x \wedge u_y$ need not be integrable in the Lebesgue sense and $Q(u)$ is a kind of *improper integral* of $u \cdot u_x \wedge u_y$.

An easy integration by parts shows that

(66) $\quad Q(u + v) = Q(u) + Q(v) + 3R(u, v) + 3R(v, u) \quad \forall u, v \in H_0^1.$

Our next two lemmas describe some useful properties of R and Q under weak limits.

LEMMA 12. *Suppose* (v^j) *is a sequence in* H_0^1 *such that* $v^j \rightharpoonup 0$ *weakly in* H_0^1. *Then*

(67) $\qquad\qquad R(u, v^j) \to 0 \qquad \forall u \in H_0^1$

and also

(68) $\qquad\qquad \int u \cdot v_x^j \wedge v_y^j \to 0 \qquad \forall u \in H^1 \cap L^\infty.$

REMARK 8. The conclusion of Lemma 12 is rather surprising. Indeed v_x^j and v_y^j converge to 0 weakly in L^2; therefore their products are dangerous!

PROOF. We establish for example (67); for the proof of (68) see [9]. Given $\varepsilon > 0$ there is some $\tilde{u} \in \mathscr{D}(\Omega; \mathbf{R}^3)$ such that

$$\| u - \tilde{u} \|_{H^1} < \varepsilon.$$

We write

$$\int \tilde{u} \cdot v_x^j \wedge v_y^j = \frac{1}{2} \int \tilde{u} \cdot \left(v^j \wedge v_y^j \right)_x + \tilde{u} \cdot \left(v_x^j \wedge v^j \right)_y$$

$$= \frac{1}{2} \int v^j \cdot \tilde{u}_x \wedge v_y^j + v^j \cdot v_x^j \wedge \tilde{u}_y \to 0$$

since $v^j \to 0$ strongly in L^2. On the other hand we have

$$\left| R(u, v^j) - R(\tilde{u}, v^j) \right| \leqslant C \| \nabla u - \nabla \tilde{u} \|_2 \| \nabla v^j \|_2^2 \leqslant C\varepsilon \| \nabla v^j \|_2^2$$

and therefore $\limsup_{j \to \infty} |R(u, v^j)| \leqslant C\varepsilon$, $\forall \varepsilon > 0$, i.e. $\lim_{j \to \infty} R(u, v^j) = 0$.

As a direct consequence of (66) and (67) we have

LEMMA 13. *Suppose (v^j) is a sequence in H_0^1 such that $v^j \rightharpoonup$ weakly in H_0^1. Then we have for every $u \in H_0^1$*

$$Q(u + v^j) = Q(u) + Q(v^j) + o(1).$$

REMARK 9. Note the analogy between Lemma 3 and Lemma 13.

We deduce from (65) the following important inequality:

(69) $$|Q(v)|^{2/3} \leqslant C \| \nabla v \|_2^2 \qquad \forall v \in H_0^1,$$

which has some *striking similarities with the Sobolev inequality*. In particular, let us consider the best constant for (69) which is, by definition,

(70) $$S = \underset{\substack{v \in H_0^1(\Omega) \\ Q(v)=1}}{\text{Inf}} \int_\Omega |\nabla v|^2,$$

so that we have

(71) $$|Q(v)|^{2/3} \leqslant (1/S) \| \nabla v \|_2^2 \qquad \forall v \in H_0^1$$

[this S has nothing to do with the best Sobolev constant of §1].

We now describe some properties of S:

(a) Because of scale invariance, S is unchanged if we replace the unit disk Ω by any domain in $\mathbf{R}^2$;

(b) In *all of* $\mathbf{R}^2$ the infimum (70) is achieved provided we replace H_0^1 by $\{ v \in L^\infty;\ \nabla v \in L^2 \text{ and } v \to 0 \text{ at infinity} \}$, and one *extremal function* is

$$U(x, y) = \frac{C}{1 + x^2 + y^2} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

which is essentially a stereographic projection from $\mathbf{R}^2$ onto S^2. The *exact value of S* is

(72) $$S = (32\pi)^{1/3}.$$

Any dilate of U, namely

$$U_\varepsilon(x, y) = \frac{C_\varepsilon}{\left(\varepsilon^2 + x^2 + y^2\right)} \begin{pmatrix} x \\ y \\ \varepsilon \end{pmatrix}, \qquad \varepsilon > 0,$$

is also an extremal function for (70).

Incidentally, *all the extremal functions for* (70) *are known.* More precisely, any extremal function for (70) must satisfy the equation

$$(73) \qquad\qquad -\Delta U = S U_x \wedge U_y \quad \text{in } \mathbf{R}^2$$

and all the solutions of (73) with $\int |\nabla U|^2 < \infty$ are described in [**11**].

(c) The best constant in (70) is *never achieved in any bounded domain* Ω. This can be proved in two different ways:

Method 1. Suppose U is an extremal function for (70) and extend U by 0 outside Ω. Then U is an extremal function for (70) in $\mathbf{R}^2$. However, they are all known and satisfy $\text{Supp } U = \mathbf{R}^2$ (see [**11**])—a contradiction.

Method 2. We can always assume that Ω *is a disk* (otherwise fix a large disk containing Ω and extend U by 0 outside Ω). We must have

$$(74) \qquad\qquad \begin{cases} -\Delta U = S U_x \wedge U_y & \text{on } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases}$$

and also $U \not\equiv 0$ since $Q(U) = 1$. Then we obtain a contradiction with

LEMMA 14 (WENTE [63]). *Suppose U satisfies* (74). *Then we have $U \equiv 0$.*

REMARK 10. Wente's Lemma may be viewed as the "counterpart" of Pohožaev's Theorem for Problem (III). It also shows that the assumption $\gamma \not\equiv C$ in Theorem 6 is essential. When $\gamma \equiv C$, the only solution of (III$_D$) is $u \equiv C$.

(d) *Inequality* (71) *is also related to the classical isoperimetric inequality*

$$(75) \qquad\qquad V^{2/3} \leqslant \frac{1}{(36\pi)^{1/3}} A,$$

where V denotes the volume of a set in $\mathbf{R}^3$ bounded by a surface of area A, and equality holds only for spheres.

Suppose $\phi \in H_0^1(\Omega; \mathbf{R}^3)$; then $\phi(\Omega)$ is a closed surface enclosing a set of (algebraic) volume

$$V = \tfrac{1}{3} Q(\phi).$$

On the other hand, its area is given by

$$A = \int \left| \phi_x \wedge \phi_j \right|.$$

We deduce from (75) that

$$(76) \qquad\qquad |Q(\phi)|^{2/3} \leqslant \frac{1}{(4\pi)^{1/3}} \int |\phi_x \wedge \phi_y|,$$

which implies (71) with $S = (32\pi)^{1/3}$ since $|\phi_x \wedge \phi_y| \leqslant \frac{1}{2}|\nabla\phi|^2$. Equality in (76) holds whenever $\phi(\Omega)$ is a sphere. However, equality in (71) holds only if $\phi(\Omega)$ is a sphere *and* $|\phi_x \wedge \phi_y| = \frac{1}{2}|\nabla\phi|^2$, i.e., $\phi_x^2 - \phi_y^2 = \phi_x \cdot \phi_y = 0$. But this is impossible —except if $\phi(\Omega)$ is a point—since there is no conformal map from the disk onto a sphere.

We may now sketch the proof of Theorem 6. It is divided into two steps.

STEP 1. Let J be defined by (62) and assume

$$(77) \qquad\qquad\qquad\qquad J < S.$$

Then the infimum in (62) is achieved. More precisely, every minimizing sequence for (62) is relatively compact in H_0^1.

PROOF. The argument resembles the proof of Lemma 2. Let (v^j) be a minimizing sequence for (62), i.e.,

$$(78) \qquad\qquad\qquad (\mathscr{L}v^j, v^j) = J + o(1),$$

$$(79) \qquad\qquad\qquad Q(v^j) = 1.$$

It follows from (60) that (v^j) is bounded in H_0^1 and we may assume that $v^j \rightharpoonup v$ weakly in H_0^1. We write

$$v^j = v + w^j$$

and we deduce from Lemma 13 that

$$1 = Q(v^j) = Q(v) + Q(w^j) + o(1).$$

Thus we have

$$(80) \qquad\qquad 1 \leqslant |Q(v)|^{2/3} + |Q(w^j)|^{2/3} + o(1).$$

On the other hand, we obtain from (68) that

$$(81) \qquad\qquad (\mathscr{L}v^j, v^j) = (\mathscr{L}v, v) + \int |\nabla w^j|^2 + o(1).$$

Combining (78), (80), and (81) we find

$$(\mathscr{L}v, v) + \int |\nabla w^j|^2 \leqslant J\left[|Q(v)|^{2/3} + |Q(w^j)|^{2/3}\right] + o(1).$$

However, we know that

$$(\mathscr{L}v, v) \geqslant J|Q(v)|^{2/3}$$

and

$$\int |\nabla w^j|^2 \geqslant S|Q(w^j)|^{2/3}.$$

Hence we conclude that $\int |\nabla w^j|^2 = o(1)$; therefore $v^j \to v$ *strongly* in H_0^1 and v is a minimizer for (62).

STEP 2. We claim that $J < S$. The argument resembles the proof of Lemma 1. Here, one uses the fact that $\gamma \not\equiv C$, which implies $\underline{u} \not\equiv C$. Therefore there is some point $(x_0, y_0) \in \Omega$ where $\nabla\underline{u}(x_0, y_0) \neq 0$. We choose an orthonormal basis

$\{i, j, k\}$ of $\mathbf{R}^3$ such that $\alpha = u_x \cdot i + u_y \cdot j < 0$ at (x_0, y_0). We consider the ratio

$$R(v) = \frac{(\mathscr{L}v, v)}{|Q(v)|^{2/3}}$$

and the function v_ε defined in the basis $\{i, j, k\}$ by

$$v_\varepsilon(x, y) = \phi(x, y)U_\varepsilon(x - x_0, y - y_0),$$

where $\phi \in \mathscr{D}(\Omega)$ is any fixed function with $\phi \equiv 1$ near (x_0, y_0). A careful expansion as $\varepsilon \to 0$ shows that

$$R(v_\varepsilon) = S + \alpha S H \varepsilon + O(\varepsilon^2 |\log \varepsilon|)$$

and the conclusion follows, since $\alpha < 0$.

The proof of Theorem 5 is somewhat technical and I will only describe its main steps. Given any $\gamma\colon \partial\Omega \to \mathbf{R}^3$ with $\gamma(\partial\Omega) \subset B_R$ and $HR < 1$ we have obtained a small solution $\underline{u}_\gamma$ and a large solution $\bar{u}^\gamma$ of the Dirichlet problem (III$_D$).

An easy computation shows that

$$\Phi(\bar{u}^\gamma) = \Phi(\underline{u}_\gamma) + \frac{1}{12H^2}J_\gamma^3,$$

where

$$J_\gamma = \inf_{\substack{v \in H_0^1 \\ Q(v)=1}} \left\{ \int |\nabla v|^2 + 4H \int \underline{u}_\gamma \cdot v_x \wedge v_y \right\}.$$

[We recall that $\Phi(u) \equiv \int |\nabla u|^2 + (4H/3)\int u \cdot u_x \wedge u_y$.]

One can prove that the small solution $\underline{u}_\gamma$ is *unique* (see [9]), and therefore the expression

$$A(\gamma) \equiv \Phi(\bar{u}^\gamma)$$

is defined without ambiguity even though $\bar{u}^\gamma$ need not be unique.

Next, one "slides" γ along the given Jordan curve Γ by introducing the family

$$\mathscr{C} = \left\{ \gamma\colon \partial\Omega \to \mathbf{R}^3;\ \gamma(\partial\Omega) = \Gamma \text{ and } \gamma \text{ is "nondecreasing"} \right\}.$$

One proves (see [9]) that $\mathrm{Inf}_{\gamma \in \mathscr{C}} A(\gamma)$ is achieved by some γ^0 and that $\bar{u}^{\gamma^0}$ is a "large" solution of the Plateau problem (III$_P$).

REMARK 11. It is a very interesting open problem to determine whether, or under what conditions (on γ, resp. Γ), there exist *other solutions* of (III) besides $\underline{u}$ and $\bar{u}$.

REMARK 12. It would also be interesting to show that given γ (resp. Γ) there is some *critical constant* $H^* > 0$ such that Problem (III$_D$) (resp. (III$_P$)) admits

(a) at least two solutions for every $H \in (0, H^*)$,

(b) one solution for $H = H^*$,

(c) no solution for $H > H^*$.

REMARK 13. Many works have been devoted to the question of surfaces of constant mean curvature spanned by a given curve Γ, see e.g. [**30**, **31**, **51**, **53**, **54**, **55**, **62**] and their references.

4. Problem (IV) (large harmonic maps). We recall that we look for critical points of the functional

$$E(u) = \int |\nabla u|^2$$

subject to the constraint

$$u \in \mathscr{E} = \left\{ u \in H^1(\Omega; S^2); u = \gamma \text{ on } \partial\Omega \right\}.$$

Clearly the absolute minimum $\underline{u}$ of E on $\mathscr{E}$ exists, i.e., $\underline{u} \in \mathscr{E}$ and $E(\underline{u}) \leqslant E(v)$ $\forall v \in \mathscr{E}$. First we split $\mathscr{E}$ into its (connected) components $\mathscr{E} = \bigcup_{k \in \mathbf{Z}} \mathscr{E}_k$ using degree theory. Since we deal with H^1 *maps in two dimensions they need not be continuous and their degree is not defined in a standard way.* However, according to Schoen–Uhlenbeck [**49**], we know that H^1 maps from S^2 into S^2 do have a well-defined degree.

Given two elements $u_1, u_2 \in \mathscr{E}$ we "transport" them on S^2 and "glue" them into a single map ϕ from S^2 into S^2: we write $S^2 = S_N^2 \cup S_S^2$, the northern hemisphere and the southern hemisphere, and we identify S_N^2, S_S^2, and Ω by stereographic projection; note that u_1, u_2 glue well on the equator since $u_1 = u_2$ $= \gamma$ on $\partial\Omega$. Hence $\phi \in H^1(S^2, S^2)$ and it makes sense to talk about $\deg \phi$. There is even an analytic expression for $\deg \phi$ (see [**45**]), namely,

$$\deg \phi = (1/4\pi)[Q(u_1) - Q(u_2)],$$

where $Q(u) = \int_\Omega u \cdot u_x \wedge u_y$.

For each $k \in \mathbf{Z}$ we set

$$\mathscr{E}_k = \left\{ u \in \mathscr{E}; (1/4\pi)[Q(u) - Q(\underline{u})] = k \right\}$$

($\underline{u}$ does not play any special role—we could fix instead any element in $\mathscr{E}$). Clearly $\mathscr{E} = \bigcup_{k \in \mathbf{Z}} \mathscr{E}_k$ and each $\mathscr{E}_k$ is both open and closed for the strong H^1 topology; moreover, $\mathscr{E}_k \neq \varnothing$ $\forall k$ and $\underline{u} \in \mathscr{E}_0$.

In order to find critical points of E in $\mathscr{E}$ it is tempting to consider, for $k \neq 0$,

$$(82) \qquad\qquad J_k = \operatorname*{Inf}_{\mathscr{E}_k} E.$$

However, there is a difficulty with this program. Indeed, suppose (u^j) is a minimizing sequence for (82) so that (u^j) is bounded in H^1 and we may assume that $u^j \rightharpoonup u$ weakly in H^1. However, $\mathscr{E}_k$ is *not closed for the weak H^1 topology* since $Q(u)$ is not continuous for the weak H^1 topology. Hence it may well happen that $u \neq \mathscr{E}_k$. This does indeed happen each time $\operatorname{Inf}_{\mathscr{E}_k} E$ is not achieved —a frequent situation (see Remark 15). We use again the same strategy as in Problems (I) and (III) and we divide the argument in two parts.

Step 1. We assume that for some $k \in \mathbf{Z}$

$$(83) \qquad\qquad J_k < J_0 + 8\pi.$$

Then the infimum in (82) is achieved.

PROOF. Let (u^j) be a minimizing sequence for (82), so that

(84)
$$\int |\nabla u^j|^2 = J_k + o(1),$$

(85)
$$Q(u^j) - Q(\underline{u}) = 4\pi k.$$

We may assume that

$$u^j \rightharpoonup \bar{u} \text{ weakly in } H^1,$$

$$u^j \to \bar{u} \text{ strongly in } L^2 \text{ and a.e.}$$

Clearly we have by lower semicontinuity

$$\int |\nabla \bar{u}|^2 \leqslant J_k,$$

and the main difficulty is to show that $\bar{u} \in \mathscr{E}_k$, i.e.,

(86)
$$Q(\bar{u}) - Q(\underline{u}) = 4\pi k.$$

As always, we write

$$u^j = \bar{u} + v^j$$

so that $v^j \rightharpoonup 0$ weakly in H_0^1, and by (84) we have

(87)
$$\int |\nabla \bar{u}|^2 + \int |\nabla v^j|^2 = J_k + o(1).$$

On the other hand, we have

$$Q(u^j) = \int u^j \cdot (\bar{u}_x + v_x^j) \wedge (\bar{u}_y + v_y^j)$$

$$= \int \bar{u} \cdot \bar{u}_x \wedge \bar{u}_y + \int u^j \cdot v_x^j \wedge v_y^j + o(1),$$

since products of the form $u^j \wedge \bar{u}_x$ and $u^j \wedge \bar{u}_y$ converge strongly in L^2 by dominated convergence. Thus we obtain

(88)
$$|Q(u^j) - Q(\bar{u})| \leqslant \frac{1}{2} \int |\nabla v^j|^2 + o(1).$$

Combining (85), (86), (88), and (87), we find

$$|Q(\underline{u}) - Q(\bar{u}) + 4\pi k| \leqslant \frac{1}{2} \int |\nabla v^j|^2 + o(1)$$

$$= \frac{1}{2} \left[J_k - \int |\nabla \bar{u}|^2 \right] + o(1).$$

Using assumption (83), we conclude that

$$|Q(\underline{u}) - Q(\bar{u}) + 4\pi k| < \frac{1}{2} \left[J_0 - \int |\nabla \bar{u}|^2 \right] + 4\pi \leqslant 4\pi,$$

since J_0 is the absolute minimum of E on $\mathscr{E}$. Finally, we note that

$$(1/4\pi)[Q(\underline{u}) - Q(\bar{u})] \in \mathbf{Z}$$

and thus $Q(\underline{u}) - Q(\bar{u}) + 4\pi k = 0$, so that $\bar{u} \in \mathscr{E}_k$.

Step 2. We claim that indeed $J_k < J_0 + 8\pi$ *either* for $k = 1$ *or* $k = -1$.

SKETCH OF THE PROOF. It suffices to find a function $v \in \mathscr{E}$ such that

(89) $$|Q(v) - Q(\underline{u})| = 4\pi$$

and

(90) $$\int |\nabla v|^2 < \int |\nabla \underline{u}|^2 + 8\pi.$$

The construction of v is explicit and technical. Again it involves a *blow-up* near a point $(x_0, y_0) \in \Omega$ where $\nabla \underline{u}(x_0, y_0) \neq 0$ (the assumption that γ is nonconstant enters here). Set

$$D_\epsilon = \left\{ (x, y) \in \mathbf{R}^2;\ (x - x_0)^2 + (y - y_0)^2 < \epsilon^2 \right\}.$$

One chooses some $v^\epsilon \in \mathscr{E}$ such that

(a) $$v^\epsilon = \underline{u} \text{ on } \Omega \setminus D_{2\epsilon},$$

(b) $$v^\epsilon(x, y) = \frac{2\lambda}{\lambda^2 + r^2} \begin{pmatrix} x - x_0 \\ y - y_0 \\ -\lambda \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{in } D_\epsilon,$$

where $\lambda = c\epsilon^2$, $r^2 = (x - x_0)^2 + (y - y_0)^2$, and c is a constant to be fixed. A careful expansion as $\epsilon \to 0$ shows that

$$(1/4\pi)|Q(v^\epsilon) - Q(\underline{u})| = 1$$

and

$$\int |\nabla v^\epsilon|^2 = \int |\nabla u|^2 + 8\pi - \alpha\epsilon^2 + o(\epsilon^2)$$

with $\alpha > 0$ (for an appropriate choice of c).

REMARK 14. When $\gamma = C$ is a constant, it is known (see Lemaire [39]) that $u \equiv C$ is the only solution of Problem (IV). This result may be viewed as the "counterpart" of Pohožaev's Theorem and Wente's Lemma (Lemma 14).

REMARK 15. We consider now the special case where

(91) $$\gamma(x, y) = \begin{pmatrix} Rx \\ Ry \\ \sqrt{1 - R^2} \end{pmatrix} \quad \text{with } 0 < R < 1.$$

Then there are two *explicit* solutions of (IV), namely,

$$\underline{u}(x, y) = \frac{2\lambda}{\lambda^2 + r^2} \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

and

$$\bar{u}(x, y) = \frac{2\mu}{\mu^2 + r^2} \begin{pmatrix} x \\ y \\ -\mu \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $r^2 = x^2 + y^2$, while $\lambda = 1/R + ((1/R^2) - 1)^{1/2}$ and $\mu = 1/R - ((1/R^2) - 1)^{1/2}$. It is known (see [10]) that:

(a) $\mathrm{Min}_{\mathscr{E}} E$ is achieved only at $\underline{u}$,

(b) $\bar{u} \in \mathscr{E}_{-1}$ and $\mathrm{Min}_{\mathscr{E}_{-1}} E$ is achieved only at $\bar{u}$,

(c) $\mathrm{Inf}_{\mathscr{E}_k} E$ is *not* achieved except when $k = 0$ and $k = -1$.

However, there could possibly be other critical points (nonminimal solutions) on $\mathscr{E}_k$. This is an interesting open problem, even for general γ—not just the special γ given by (91).

REMARK 16. A result comparable to Theorem 7 has been obtained independently by Jost [36].

REMARK 17. Benci and Coron have recently considered an extension of Problem (IV): Find a mapping u: $\Omega \to \mathbf{R}^{n+1}$ (Ω is still the unit disk in $\mathbf{R}^2$) such that

$$(92) \qquad \begin{cases} -\Delta u = u |\nabla u|^2 & \text{on } \Omega, \\ u \in S^n & \text{on } \Omega, \\ u = \gamma & \text{on } \partial\Omega. \end{cases}$$

They prove (see [7] and [21]) that if γ is nonconstant there exist at least two solutions of (92). The argument is quite different from the one described here since $\mathscr{E}$ is connected when $n \geqslant 3$.

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